Aperiodic Tilings

Charles Starling

Carleton University

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Overview

Motivation and History

- Tiling the plane
- Quasicrystals

Aperiodic Tilings

- Constructing aperiodic tilings
- Spaces of tilings

Tiling the plane

A tiling is a cover of $\mathbb{R}^2$ (or more generally $\mathbb{R}^n$) by polygons.

Question 1: given a finite set of polygons, can they tile the plane?

Single parallelogram ✓

Single triangle ✓

Regular hexagon ✓

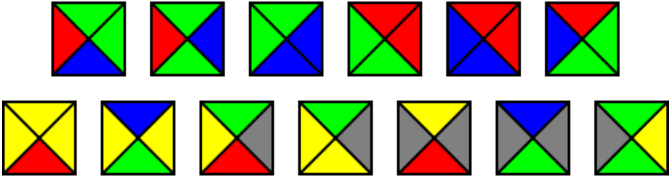
4-gon ✓

These can all tile the plane periodically.

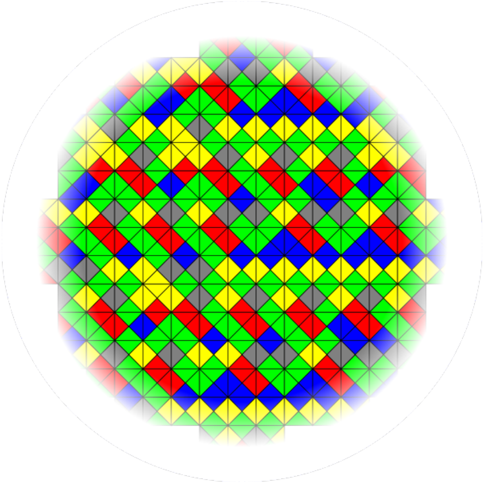
Tiling the plane

Question 2: are there any sets of polygons which can only tile the plane aperiodically?

Dominos — square tiles with colored sides, indicating allowed adjacencies.



Tiling the plane



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Tiling the plane

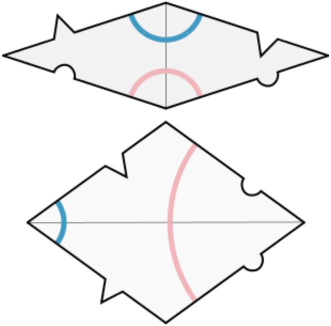
Conjecture (Wang 1961) — if a finite set of square dominos can tile the plane, then they can tile it periodically.

False — Berger (1966) found a set of 20426 dominos which only tile the plane aperiodically!

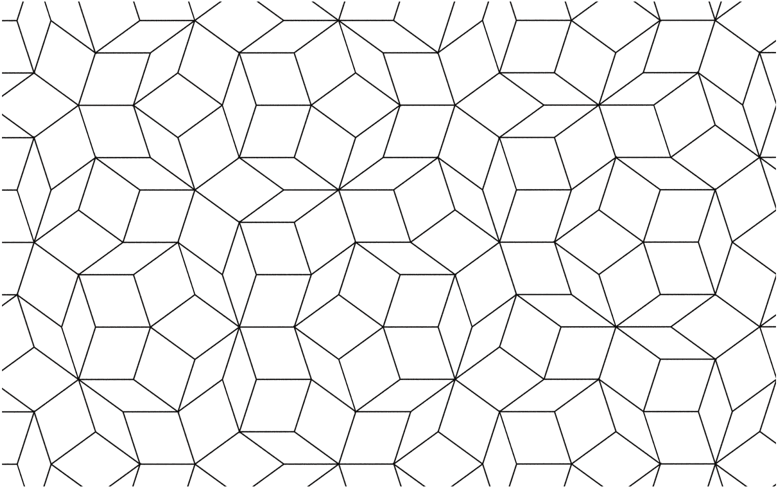
Since then, smaller so-called "aperiodic sets" have been found. (The 13 tiles on the previous slide).

Penrose tilings

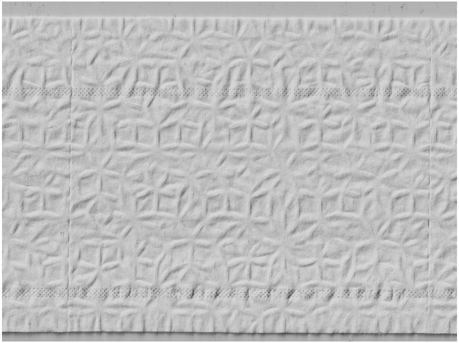
Sir Roger Penrose (1974)



Penrose tilings



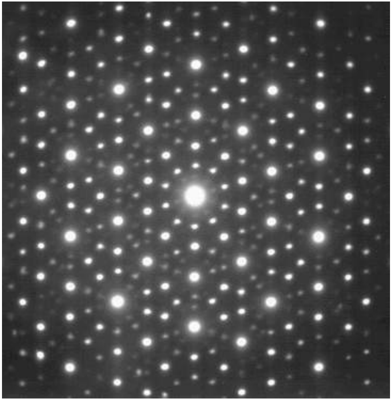
Example: Penrose Tiling



"So often we read of very large companies riding rough-shod over small businesses or individuals. But when it comes to the population of Great Britain being invited by a multi-national to wipe their bottoms on what appears to be the work of a Knight of the Realm without his permission, then a last stand must be made."

Quasicrystals

In 1984, Shechtman et al discovered an alloy with the following diffraction pattern.



The strong peaks mean that the atoms must be configured in an orderly way.

This diffraction pattern has 5-fold rotational symmetry.

"The most interesting thing about 5 is that it is not 3, 4, or 6"

– John Hunton.

Quasicrystals

Only 3-, 4-, and 6-fold rotational symmetry is allowed for diffraction patterns of periodic crystals.

Shechtman had discovered quasicrystals, a discovery for which he was ridiculed and fired.

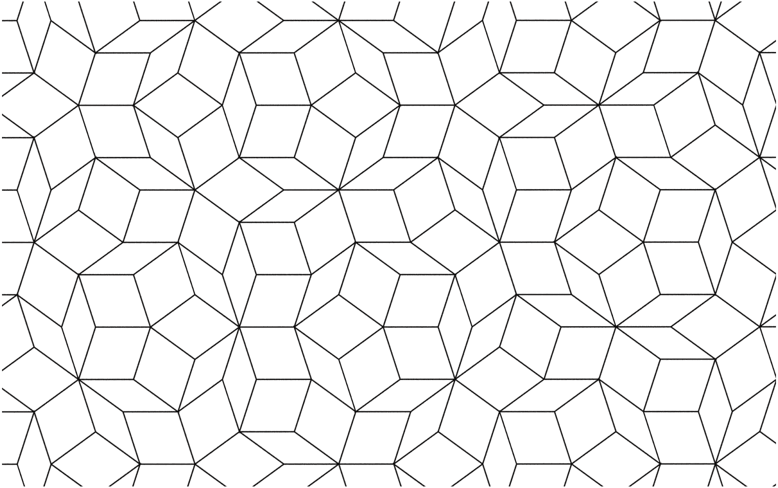
"Danny Shechtman is talking nonsense. There is no such thing as quasicrystals, only quasi-scientists,"

– unattributed.

In 2011 he was given the Nobel Prize in Chemistry for his discovery.

Mathematical models?

Quasicrystals



Tilings

Definition

A **tiling** T of $\mathbb{R}^2$ is a countable set $T = \{t_1, t_2, \dots\}$ of subsets of $\mathbb{R}^2$, called **tiles** such that

- Each tile is homeomorphic to the closed ball (they are usually polygons),
- $t_i \cap t_j$ has empty interior whenever $i \neq j$, and
- $\cup_{i=1}^{\infty} t_i = \mathbb{R}^2$.

- If T is a tiling, $x \in \mathbb{R}^2$, $T + x$ is the tiling formed by translating every tile in T by x.
- T is **aperiodic** if $T + x \neq T$ for all $x \in \mathbb{R}^2 \setminus \{0\}$.

Tilings

There are uncountably many Penrose tilings, even up to tranlsation.

However, all Penrose tilings look similar locally.

For any $r > 0$, there are only a finite number of patches of radius r possible in Penrose tilings — finite local complexity.

For any patch P, there is an $R > 0$ such that every ball of radius R contains a copy of P — repetitivity.

Aperiodicity + Finite local complexity + Repetitivity = Aperiodic order

Substitution rules

One common way of creating aperiodic tilings is through substitution rules.

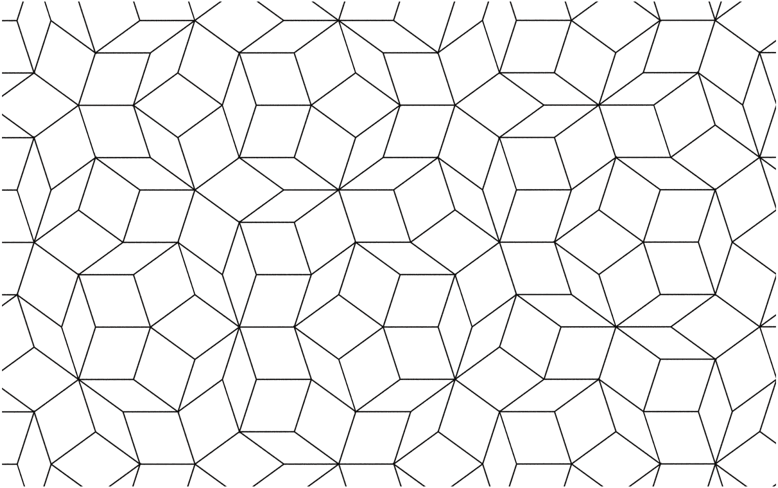
A substitution rule is:

Finite set of tiles

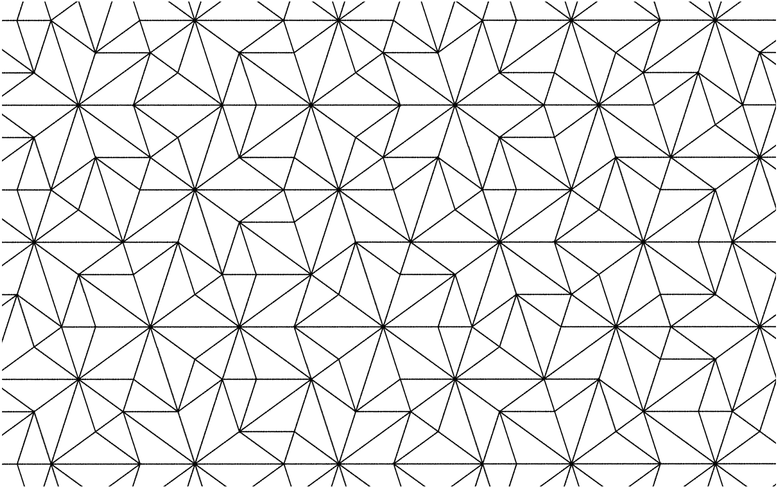
+ rule ω for subdividing them into smaller copies

+ scaling factor $\lambda > 1$ to make the smaller copies the original size

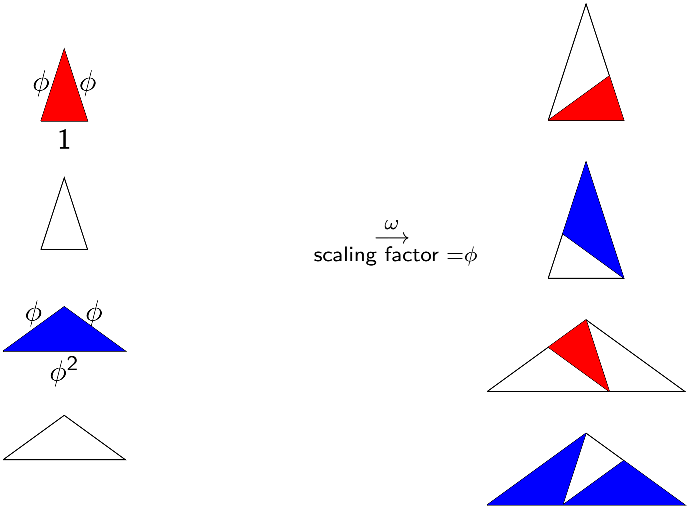
Substitution rules



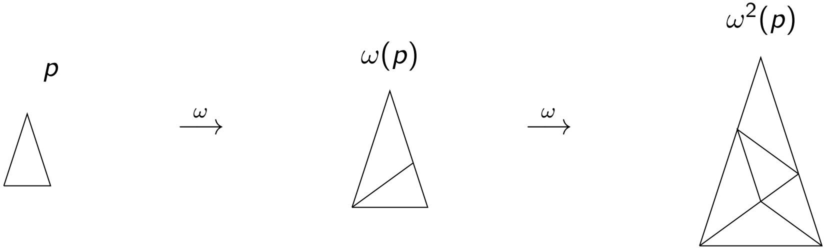
Substitution rules



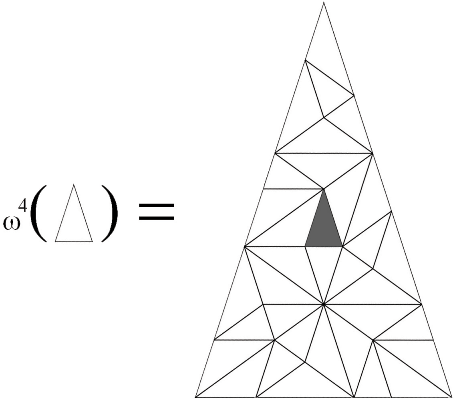
Example: Penrose Tiling



Example: Penrose Tiling



Producing a Tiling from a Substitution Rule



$\{p\} \subset \omega^4(p) \subset \omega^8(p)$

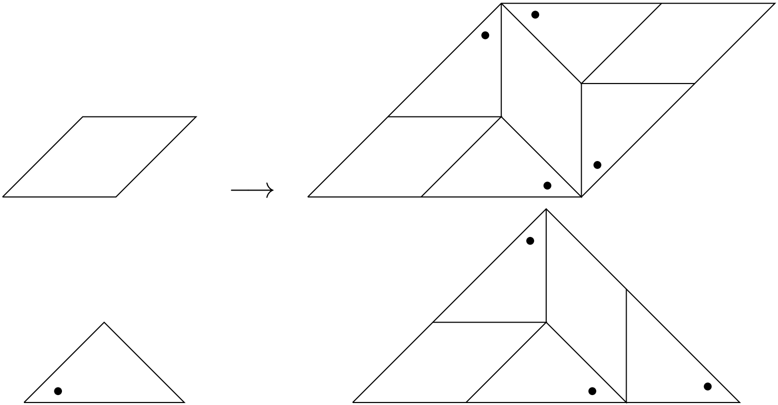
$\omega^{4n}(p) \subset \omega^{4(n+1)}(p)$

Then

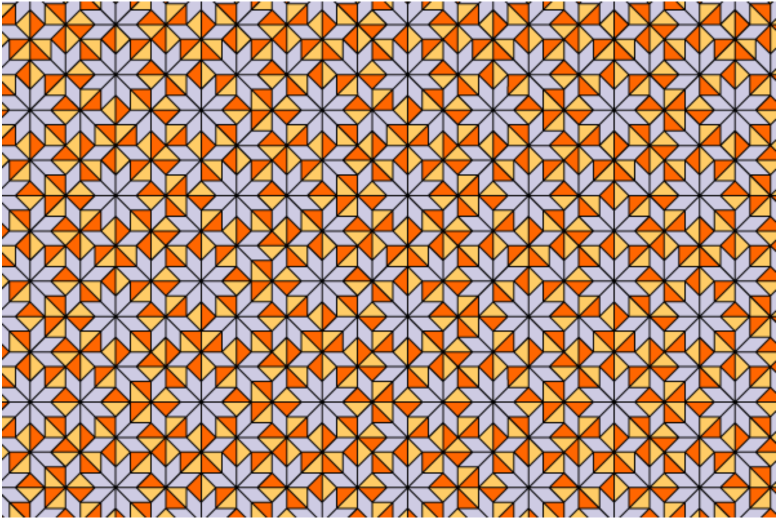
$$T = \bigcup_{n=1}^{\infty} \omega^{4n}(p)$$

is a tiling.

Octagonal Tiling

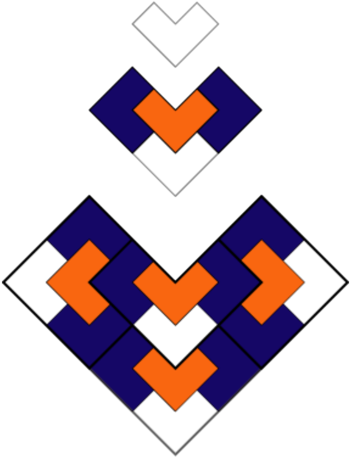


Octagonal Tiling



Tilings encyclopedia http://tilings.math.uni-bielefeld.de/substitution/ammann-beenker-rhomb-triangle/

Chair Tiling



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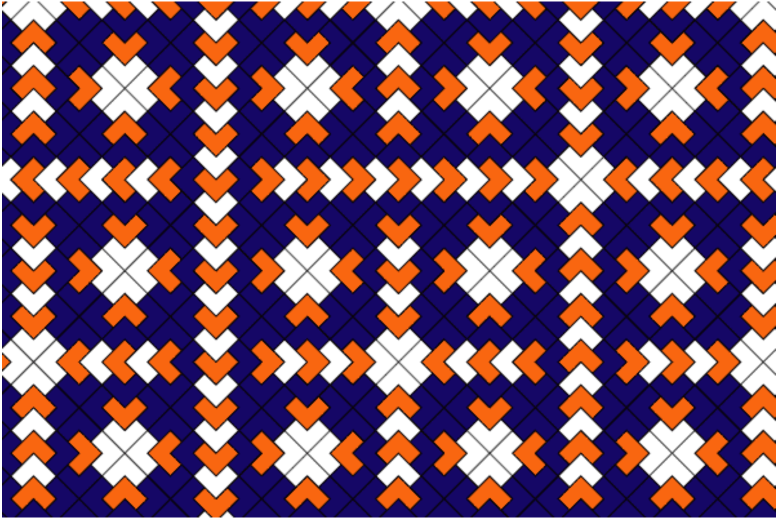
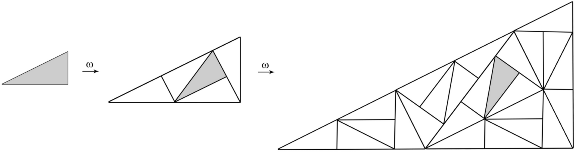
Chair Tiling



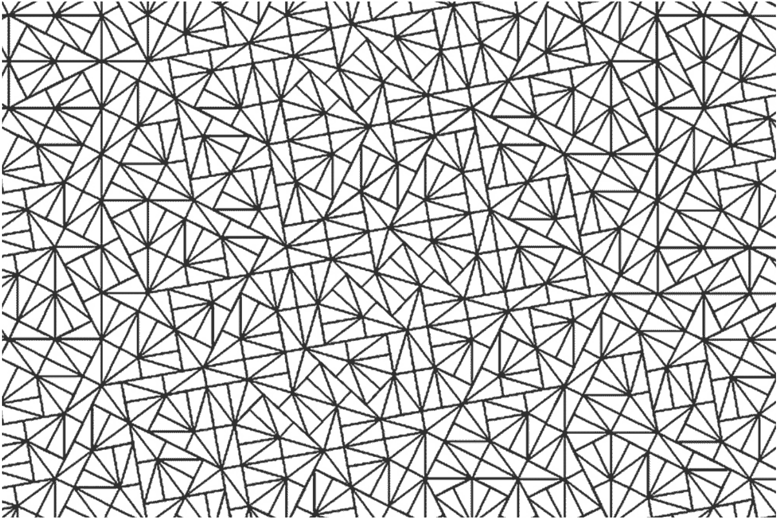
Tilings encyclopedia http://tilings.math.uni-bielefeld.de/substitution/chair/

Pinwheel Tiling



Prof. Michael Whittaker, michaelwhittaker.ca

Pinwheel Tiling



Prof. Michael Whittaker, michaelwhittaker.ca

Pinwheel Tiling — Federation Square, Melbourne

Weisstein, Eric W. "Aperiodic Tiling." http://mathworld.wolfram.com/AperiodicTiling.html

Substitution Tiling Properties

If the tiles meet full-edge to full-edge, then any tiling this way has finite local complexity

A substitution is **primitive** if there is an $n \in \mathbb{N}$ such that $\omega^n(p)$ contains a translate of q for any two tiles p and q.

If we form T from a primitive substitution, then T is repetitive.

Aperiodic when the substitution is “invertible”

Tiling Space

Given a Penrose tiling, any translate will also be.

$\rightsquigarrow$ Consider the set of all Penrose tilings.

This will have the translation action of $\mathbb{R}^2$ on it.

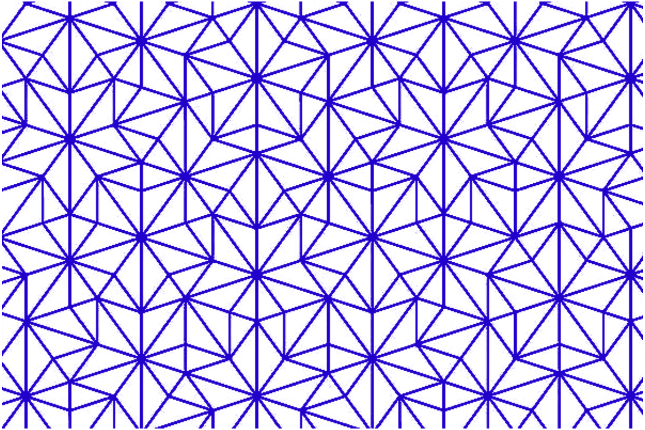
Given T, one obtains the set of all Penrose tilings by completing $T + \mathbb{R}^2$ in a metric.

The tiling metric satisfies the following: T_1 and T_2 are close if

1. $T_1 = T_2 + x$ for some small x.
2. T_1 agrees with T_2 exactly on a large ball around the origin, then disagrees elsewhere.

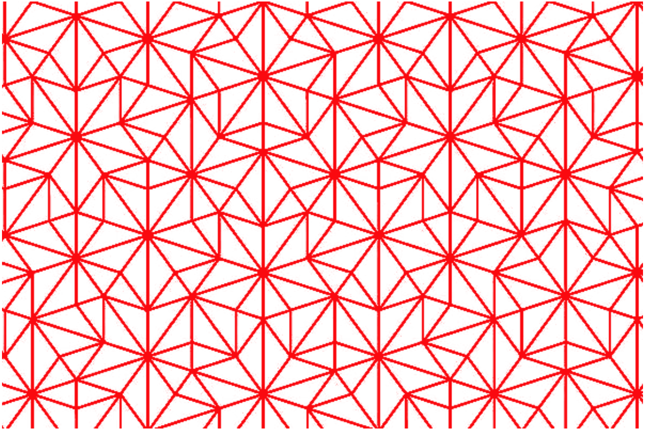
Or any combination of 1 and 2. In most cases, 1 looks like a disc while 2 is totally disconnected (in nice cases, a Cantor set).

Example: Penrose Tiling



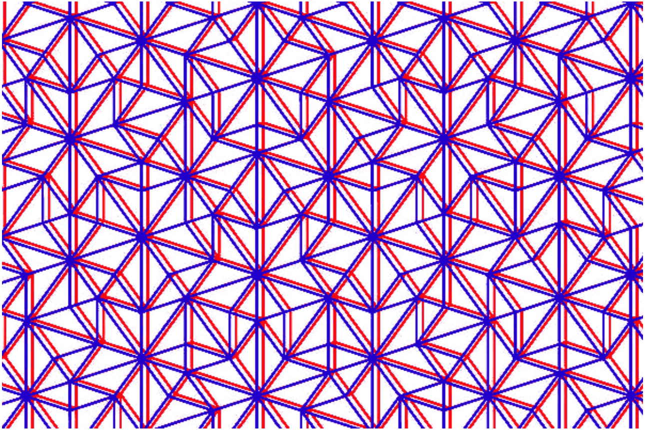
T_1

Example: Penrose Tiling



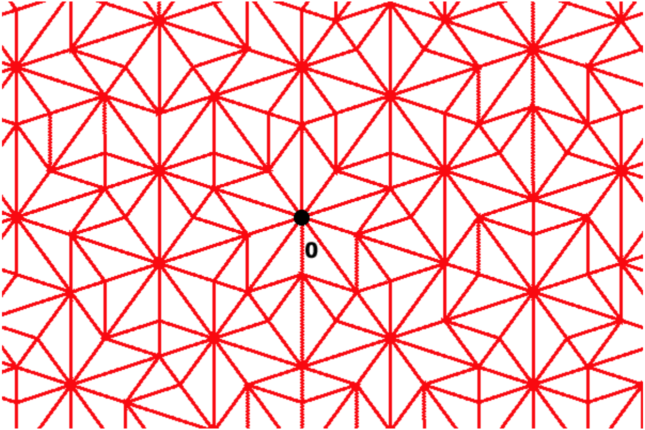
T_2

Example: Penrose Tiling



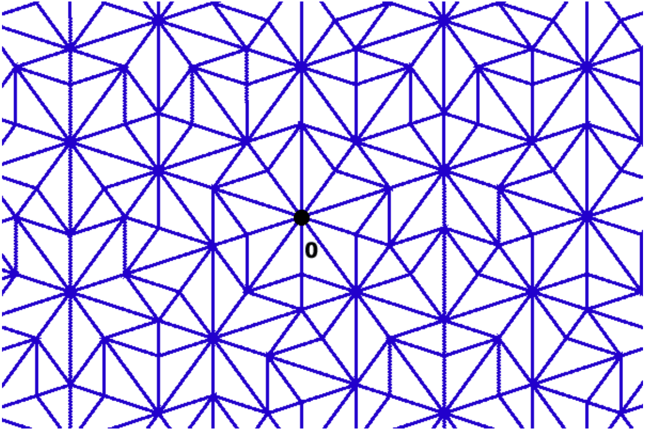
T_2 is a small shift of T_1 $\Rightarrow$ T_1 is close to T_2

Example: Penrose Tiling



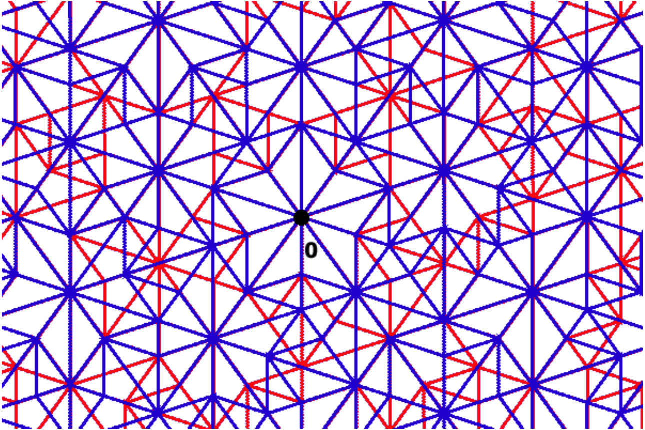
T_1

Example: Penrose Tiling



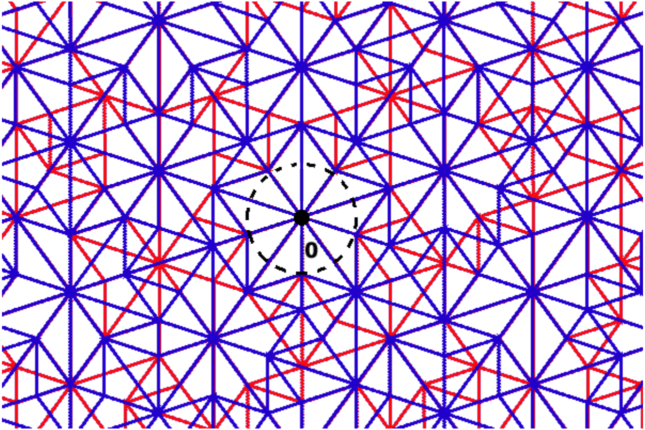
T_2

Example: Penrose Tiling



T_1 and T_2 agree around the origin, disagree elsewhere.

Example: Penrose Tiling



$$d(T_1, T_2) < (\text{radius of the ball above.})^{-1}$$

The Continuous Hull

The **continuous hull** of a tiling T, denoted Ω_T, is the completion of $T + \mathbb{R}^2 = \{T + x \mid x \in \mathbb{R}^2\}$ in the tiling metric. This is also called the **tiling space**.

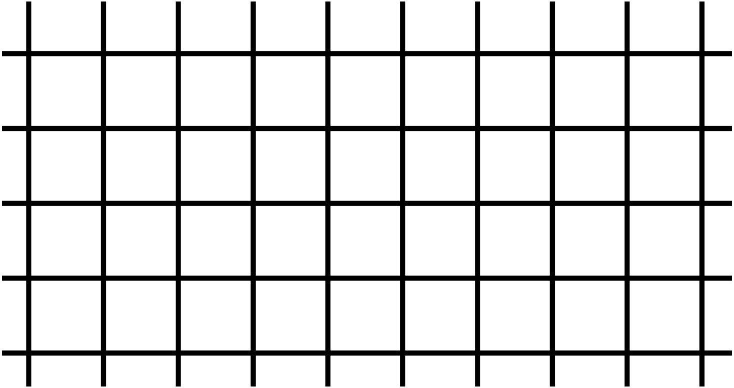
It's not obvious, but the elements of Ω_T are tilings.

Ω_T is the set of all tilings T' such that every patch in T' appears somewhere in T.

Finite local complexity $\implies \Omega_T$ compact. (Radin-Wolff)

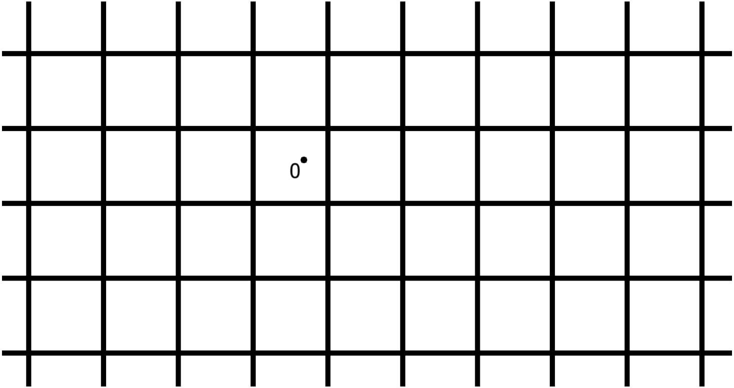
Repetitivity $\implies$ every orbit is dense in Ω_T. (Solomyak)

Example: Grid



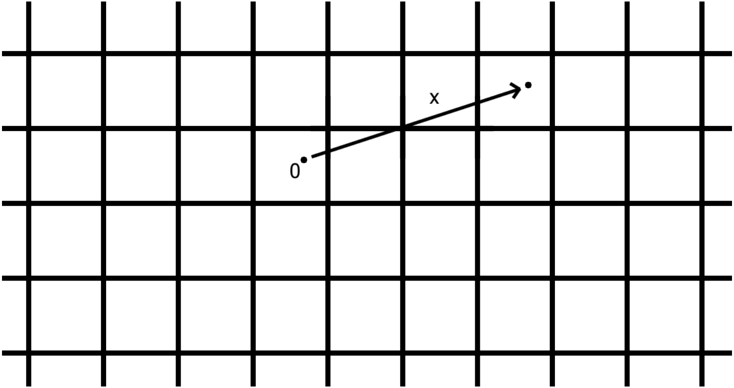
Infinite grid in $\mathbb{R}^2$

Example: Grid



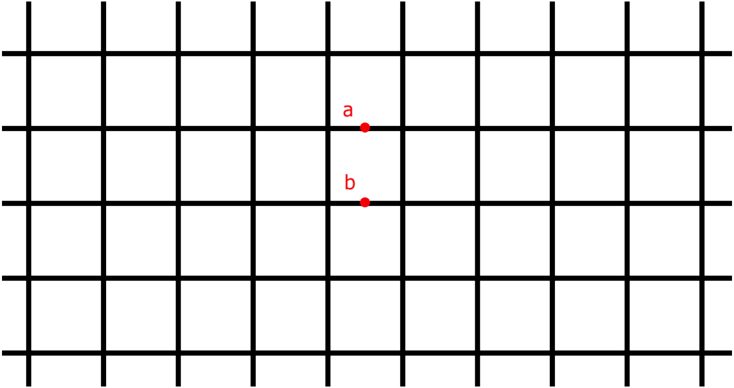
Placement of the origin in any square determines the tiling.

Example: Grid



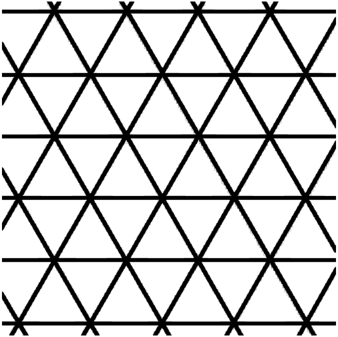
Placement of the origin in any square determines the tiling. $T = T - x$

Example: Grid



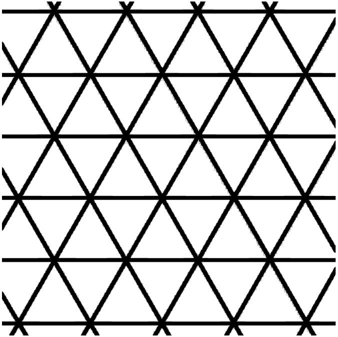
a and b are the same in the tiling space $\implies \Omega_T \cong \mathbb{T}^2$

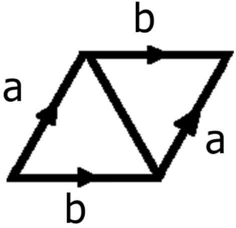
Example: Equilateral Triangles



Infinite tiling of the plane with equilateral triangles.

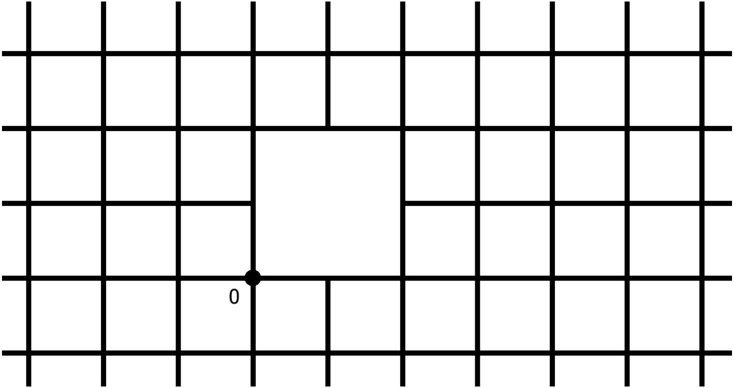
Example: Equilateral Triangles





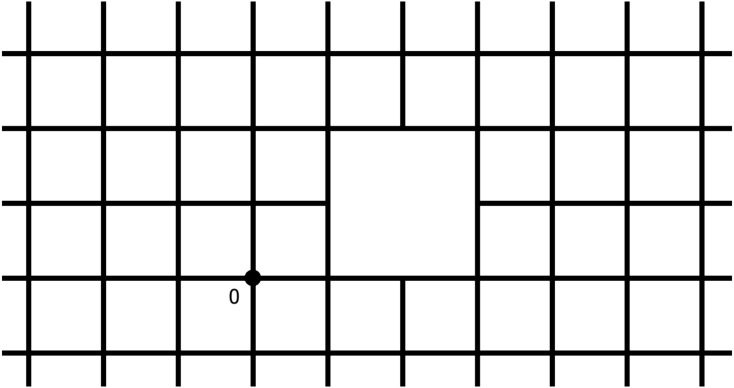
Space of "origin placements" $\Omega_T \cong \mathbb{T}^2$

Example: Modified Grid



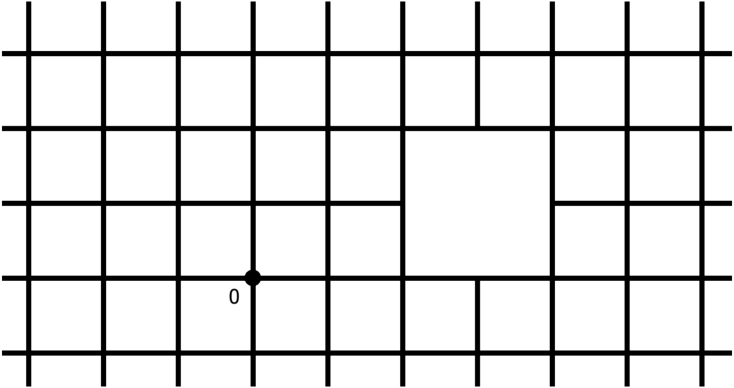
T, same as usual grid with a larger square at origin.

Example: Modified Grid



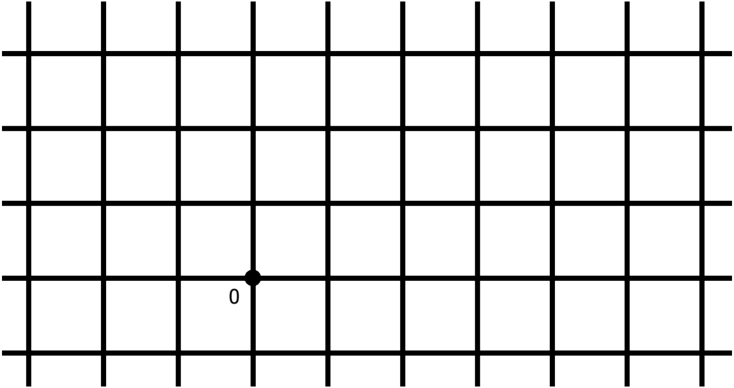
$T + (1, 0)$

Example: Modified Grid



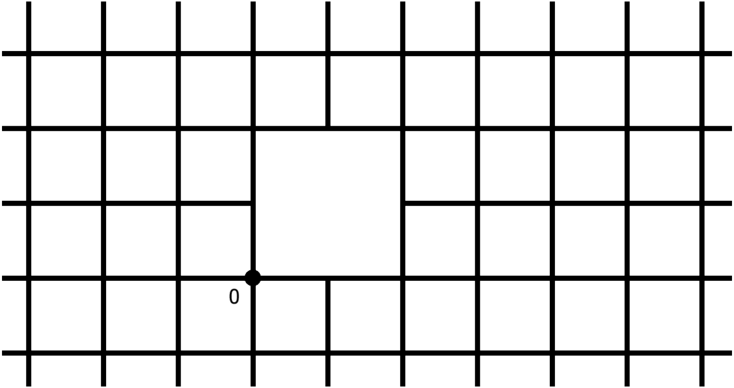
$T + (2, 0)$

Example: Modified Grid



$T + (51, 0)$

Example: Modified Grid



$T + (n, 0)$ is a Cauchy sequence converging to the periodic grid.

Aperiodicity

How can we tell when a substitution tiling is aperiodic?

The substitution ω induces a map $\omega : \Omega_T \to \Omega_T$.

One can show that this map is surjective and continuous.

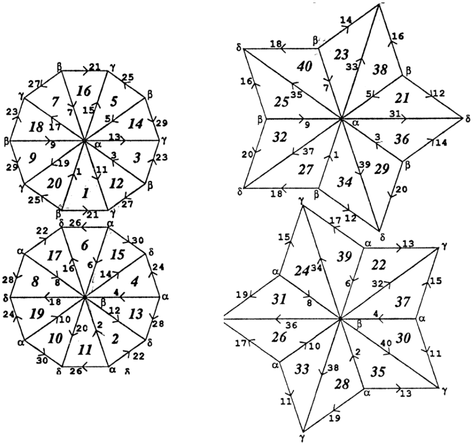
ω is injective $\Leftrightarrow$ every tiling in Ω_T is aperiodic. In this case, ω is a homeomorphism.

Approximating the tiling space

For periodic tilings, we made Ω_T by building a space out of the prototiles. We "glued them together" along their edges if those edges could touch in the tiling.

Idea: do this for aperiodic tilings $\rightarrow$ obtain a space Γ, but not Ω_T.

Approximating the tiling space



Γ for the Penrose tiling.

Jared E. Anderson and Ian F. Putnam. *Topological invariants for substitution tilings and their associated C^*-algebras.* Ergodic Theory Dynam. Systems, **18**(3):509–537, 1998.

Approximating the tiling space

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Idea: do this for aperiodic tilings $\rightarrow$ obtain a space Γ, but not Ω_T.

Anderson, Putnam (1998) – Γ approximates Ω_T in an appropriate sense (Ω_T is an **inverse limit** of such spaces).

(Ω_T, ω) has "local hyperbolic coordinates" – Smale space (chaos).

$(\Omega_T, \mathbb{R}^2)$ is a dynamical system, so we can form the **crossed product C*-algebra** $C(\Omega_T) \rtimes \mathbb{R}^2$.

- Its selfadjoint elements are "observables" of a particle moving through a quasicrystal. (Kellendonk, Bellisard)
- This C*-algebra is interesting in its own right – it is simple, nuclear, has a unique trace, real rank zero.
- The **K-theory** describes the spectrum of the quasicrystal.

T gives rise to an interesting **inverse semigroup**. This is a motivating example for so-called "noncommutative Stone duality" (Exel, Lawson)

The einstein problem

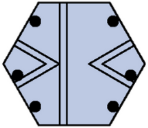
"ein" $=$ one

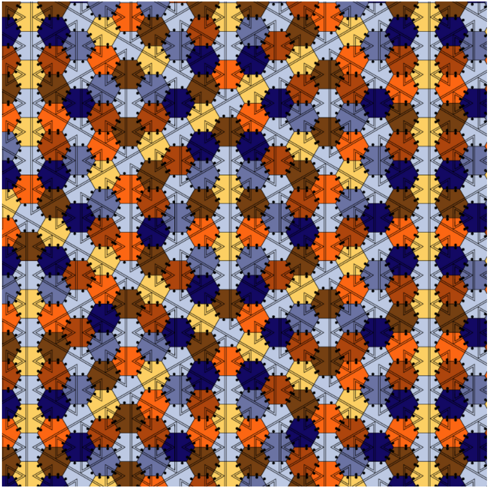
"stein" $=$ stone

The einstein problem: does there exist a single tile which can only tile the plane aperiodically?

The einstein problem

Taylor (2009)

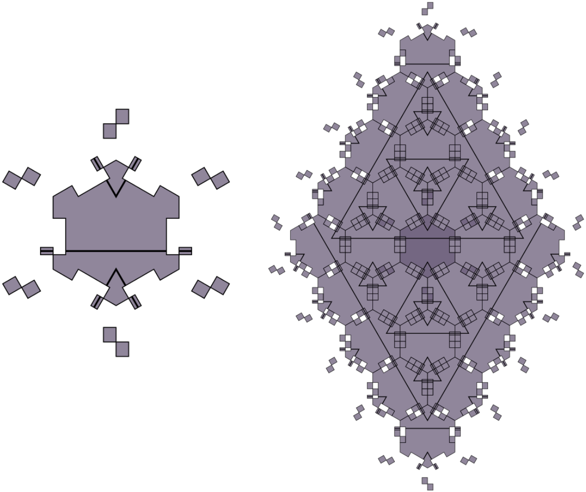




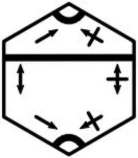
Tilings encyclopedia http://tilings.math.uni-bielefeld.de/substitution/hexagonal-aperiodic-monotile/

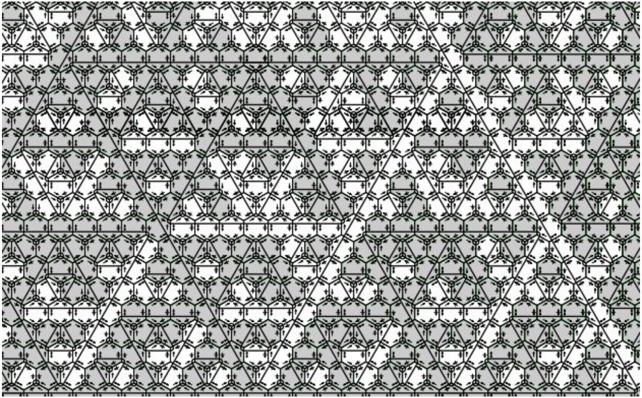
The einstein problem

Socolar-Taylor (2012)



Walton-Whittaker (2019)





J. Walton and M. Whittaker, *An aperiodic tile with edge-to-edge orientational matching rules*, arXiv 1907.10139