Stochastic Scheduling on a Repairable Machine with Erlang Uptime Distribution

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Abstract

A set of jobs is to be processed on a machine which is subject to breakdown and repair. When the processing of a job is interrupted by a machine breakdown, the processing later resumes at the point at which the breakdown occurred. We assume that the machine uptime is Erlang distributed and that processing and repair times follow general distributions. Simple permutation policies on both machine parameters and the processing distributions are given which minimize the weighted number of tardy jobs, weighted flow times, and the weighted sum of the job delays.

Key words: Stochastic scheduling, breakdowns, Erlang uptimes, Non-preemptive resume model.

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1 Introduction

In this paper, we consider the problem of optimally scheduling a set of jobs on a single machine which is susceptible to occasional breakdowns. When a breakdown occurs, the job currently being processed is halted until the machine is repaired; then the job is resumed at the point at which it was interrupted. Scheduling problems of this sort have been studied thoroughly in the case of reliable machines and in cases where job processing times and due dates are known in advance. However, such assumptions are not very realistic; machines are often subject to unpredictable breakdowns and possibly lengthy repairs. Furthermore, processing times and deadlines may be more adequately modelled as random variables.

Single-machine stochastic scheduling models incorporating breakdowns and repairs were first studied by Glazebrook [14]. Subsequently, many authors have investigated stochastic problems on unreliable or repairable machines (see references [1]-[7], [10]-[25]). However, almost all of the results are based on one or more strong assumptions.

Allahverdi and Mittenthal [3, 4, 5], and Birge *et al.* [6] obtained some powerful results in the case of deterministic processing times. In case of exponential uptimes, it is always possible to transform a scheduling problem on an unreliable machine to a problem on a reliable machine by extending the processing time of a particular job to the total time that the job occupies the machine. For some examples, see Glazebrook [14], Pinedo and Rammouz [23], Allahverdi and Mittenthal [3], and Li and Cao [18]. Under the assumption of exponential processing times, some simple policies have been obtained (see, for example, Pinedo and Rammouz [23], Chang *et al.* [7], Du and Pinedo [10], and Li and Cao [17]), but these are usually based on some stronger conditions, and the policies obtained are not related to the machine parameters. Few results have been obtained when specific distributional assumptions have not been made about the machine uptime.

In stochastic scheduling problems where specific distributional assumptions have not been made about processing requirements, or machine uptimes and downtimes, optimal processing strategies could depend upon both the elapsed time since the last repair period and the elapsed processing time of the current job. Therefore, it would seem very difficult to find optimal index policies which depend on uptime and downtime distributions as well as the processing time distribution.

In view of this, we investigate a single machine scheduling model in which the machine is subject to breakdowns with Erlang uptimes and general repair times. Each job's processing time is also a random variable with a general distribution. This model would seem to be as general as can be realistically considered, if one seeks a schedule which depends on the machine parameters as well as the processing distribution. Li and Glazebrook [19] have considered the completely general case, but their results are not related to the machine parameters.

In order to proceed, we restrict our attention to the class of simple recourse strategies. Under such a strategy, the sequence of job completion times in a schedule is fixed, but the completion times may be postponed as a result of the machine breakdowns (see Allahverdi and Mittenthal [4, 5] or Frenk [12]). Further assumptions and notation are as follows:

A set of jobs $J = \{1, 2, \cdots, N\}$ ($N < \infty$) is to be processed on a single machine which is subject to breakdown and repair. Job j has weight $h_j \geq 0$, which is basically a priority factor, denoting the importance of job j relative to the other jobs in the system. For example, this weight may represent the actual cost of keeping the job in the system. This cost could be a holding or inventory cost; it could also be the amount of value already added to the job. All jobs have a common due date d which is a random variable with an exponential distribution and rate r; this index represents the committed shipping or completion date (the date the jobs are promised to the customer). The completion of a job after its due date is allowed, but a penalty is incurred.

The jth job has a random processing requirement X_j with distribution function $F_j(\cdot)$, having hazard rate function $\mu_j(\cdot)$ and finite expectation $1/\mu_j$. The machine is available for processing from time 0 until the first breakdown occurs at time U_1. The machine then takes time D_1 to be repaired, while no processing takes place. The repair having been completed, the machine is again available for processing from time $U_1 + D_1$ until time $U_1 + D_1 + U_2$, and so on. The machine up times $U_1, U_2, \cdots$, are independent random variables with an Erlang distribution function $U(\cdot)$ having m phases and where each phase is exponentially distributed with parameter α. That is,

$$U(t) = 1 - e^{-\alpha t} \sum_{k=0}^{m-1} \frac{(\alpha t)^k}{k!}.$$

The repair times $D_1, D_2, \cdots$ are also independent and identically distributed but with general distribution function $D(\cdot)$, having hazard rate $\beta(\cdot)$ and finite expectation $1/\beta$.

All of the above random variables are assumed to be mutually independent.

Under the above assumptions, we shall consider the so-called non-preemptive resume model; that is, if a job is being processed when a breakdown occurs, the processing time is cumulative after the machine is repaired. Our goal is to find schedules, depending on both machine parameters and the processing distributions, for the jobs which minimize one of several possible objectives. The objectives we consider are the weighted flow time, the weighted numbers of tardy jobs, and the weighted sum of job tardiness. General results are obtained which can be simplified for certain special cases already in the literatures [18], [19] and [23]. In addition, some new results are obtained for special cases not yet considered in the literature.

2 A Relation Between Completion Time and System Parameters

Completion times are important random variables in scheduling theory. Since our goal is to optimize performance characteristics related to job completion time, it is the purpose of this section to find a useful relationship between the completion time of job n and the parameters of the system.

We first set out some notation which will be used throughout this paper. Let

- EY represent the expectation of a random variable Y.
- $C_{\pi(n)}$ represent the completion time of the nth job $(n = 1, 2, \cdots, N)$ under the scheduling policy π; π is a permutation of the vector $(1, 2, \cdots, N)$.
- $\omega_i = \exp\left(\frac{2\pi i}{m}\sqrt{-1}\right)$ $(i = 1, 2, \cdots, m)$, the mth roots of unity.
- $\phi_i(s) = \alpha + s - \alpha\omega_i \sqrt[m]{Ee^{-sD}}$ $\quad (i = 1, 2, \cdots, m)$, where D is a random variable having the repair time distribution.
- $\gamma_i(s) = s\left(Ee^{-sD}\right)^{\frac{1-i}{m}}$ $\quad (i = 1, 2, \cdots, m-1)$ and $\gamma_m(s) = \left[s + \alpha(1 - Ee^{-sD})\right]\left(Ee^{-sD}\right)^{\frac{1-m}{m}}$.
- $\mathbf{e} = (1, 1, \cdots, 1)$, $\mathbf{e}_i = (0, \cdots, 0, 1, 0, \cdots, 0)$ and $\mathbf{e}^\tau$ represent the transpose of $\mathbf{e}$.
- $\pi_{(n;i,j)}$ denote a scheduling policy in which job i and job j have been assigned to the nth and $(n+1)$st positions, respectively, $(n = 1, 2, \cdots, N-1; i, j = 1, 2, \cdots, N)$. $\pi_{(n;j,i)}$ is the policy which is identical to $\pi_{(n;i,j)}$ except for a switch in the nth and $(n+1)$st positions.

For the purpose of considering the completion time of the nth job $(n = 1, 2, \cdots, N)$ in any given policy π, we define $S_n(t)$ to be the state of the system at time t before the nth job has been completed. This stochastic process can be in any of the following states

$$\{*\} \bigcup \{(i, k) : i = 0, 1, 2, \cdots, m; k = 1, 2, \cdots, n\}$$

where $*$ denotes the absorbing state corresponding to the completion of the nth job, and (i, k) denotes the state in which the machine is processing the kth job while the uptime is in phase i $(i = 1, 2, \cdots, m)$; phase 0 corresponds to machine downtime. That is, state $(0, k)$ corresponds to the machine being repaired after having broken down while the kth job is being processed.

Clearly, the process $S_n(t)$ is not Markovian, but it can be extended to a Markov process with the use of supplementary variables (e.g. Cox[9], Chaudhry and Templeton[8]). To do this, we denote the elapsed processing time of the kth job at t by $V_k(t)$, and the elapsed repair time at t by $V(t)$. Then $\{(S_n(t), V_k(t), V(t)); t \geq 0, k = 1, 2, \cdots, n\}$ is a Markov process.

Set

$$p_{i,k}(t,x)dx = P\{S_n(t) = (i,k), x \leq V_k(t) < x + dx | S_n(0) = (1,1), V_1(0) = 0\}$$
$$p_{0,k}(t,x,y)dy = P\{S_n(t) = (0,k), V_k(t) = x, y \leq V(t) < y + dy | S_n(0) = (1,1), V_1(0) = 0\}$$

where $i = 1, 2, \cdots, m$ and $k = 1, 2, \cdots, n$. Notice that the kth job in the schedule π is the job whose processing time has hazard function $\mu_{\pi(k)}$.

Lemma 2.1 *The system is governed by the following differential equations:*

$$\left[\frac{\partial}{\partial t} + \frac{\partial}{\partial x} + \alpha + \mu_{\pi(k)}(x)\right] p_{1,k}(t,x) = \int_0^\infty p_{0,k}(t,x,y)\beta(y)dy, \ \ k = 1, 2, \cdots, n \quad (1)$$
$$\left[\frac{\partial}{\partial t} + \frac{\partial}{\partial x} + \alpha + \mu_{\pi(k)}(x)\right] p_{i,k}(t,x) = \alpha p_{i-1,k}(t,x) \ \ i = 2, \cdots, m; \ k = 1, \cdots, n \quad (2)$$
$$\left[\frac{\partial}{\partial t} + \frac{\partial}{\partial y} + \beta(y)\right] p_{0,k}(t,x,y) = 0 \quad k = 1, 2, \cdots, n \quad (3)$$

with boundary conditions

$$p_{i,1}(t,0) = 0 \quad (i = 1, 2, \cdots, n) \quad (4)$$
$$p_{i,k}(t,0) = \int_0^\infty p_{i,k-1}(t,x)\mu_{\pi(k-1)}(x)dx, \ \ i = 1, 2, \cdots, m; k = 2, \cdots, n \quad (5)$$
$$p_{0,k}(t,x,0) = \alpha p_{m,k}(t,x), \quad k = 1, 2, \cdots, n \quad (6)$$

and initial conditions

$$p_{i,k}(0,x) = \delta_{i,1}\delta_{k,1}\delta(x), \quad i = 1, 2, \cdots, m; \ \ k = 1, 2, \cdots, n \quad (7)$$
$$p_{0,k}(0,x,y) = 0 \quad k = 1, 2, \cdots, n \quad (8)$$

where $\delta(x)$ denotes the Dirac distribution and $\delta_{i,k}$ denotes the Kronecker delta.

Some details of the proof are supplied in the appendix to this paper.

The previous lemma allows us to find a relation between the Laplace transform of the completion times and the Laplace transform of the processing times and downtimes, a result which is crucial to most of the scheduling policies provided later.

Lemma 2.2 *For $n = 1, 2, \cdots, N,$ and $s \geq 0,$*

$$Ee^{-sC_{\pi(n)}} = 1 - \frac{1}{m} \sum_{i=1}^{m} \sum_{k=1}^{m} \frac{\gamma_i(s)\omega_k^{1-i}}{\phi_k(s)} \left[1 - E \exp \left(-\phi_k(s) \sum_{j=1}^{n} X_{\pi(j)} \right) \right]$$

The proof is supplied in the appendix.

As a special case, we can obtain the following result, previously obtained in [23] .

Corollary 2.3 *When the uptime is exponentially distributed with rate α, then for $n = 1, 2, \cdots, N$, and $s \geq 0$,*

$$Ee^{-sC_{\pi(n)}} = E \exp \left(-[s + \alpha(1 - Ee^{-sD})] \sum_{j=1}^{n} X_{\pi(j)} \right)$$

Proof: This follows directly from Lemma 2.2, since when $m = 1$, $\gamma_1(s) = \phi_1(s) = s + \alpha(1 - Ee^{-sD})$.

Remark. We note that when the uptime has a more general form it is not always possible to obtain the Laplace transform of $C_{\pi(n)}$ in terms of system parameters. Even if the Laplace transform of the completion times can be found, it is not always convenient for obtaining optimal scheduling policies. In the case discussed, as we will see in later sections, the above result gives an exceedingly useful relation.

3 The Optimal Schedule for the Weighted Flow Time

In this section, we shall find schedules which minimize the expectation of the total weighted completion times, $\sum_k h_{\pi(k)} C_{\pi(k)}$. This quantity gives an indication of the total holding or inventory costs incurred by a schedule. The sum of the completion times is often referred to as the flow time while the total weighted completion time is referred to as the weighted flow time. Some simple permutation policies have been found, under the assumption that the machine's uptime is exponentially distributed (see [13], [18] and [22], for example). However, as far as we know, results depending on parameters of both processing and the machine for the case of nonexponential uptime have not been reported. Here, we shall consider this often-studied scheduling problem in the sense of expectation. Frequently, however, the policies that minimize the objective in expectation minimize the objective stochastically as well (see pp. 181, [22]).

We have the following necessary and sufficient condition:

Theorem 3.1 *The scheduling policy* $\pi_{(n;i,j)}$ *is better than the policy* $\pi_{(n;j,i)}$ *if and only if*

$$\begin{aligned} & \mu_j h_j \left\{1 + \frac{\alpha}{m\beta} + \frac{\alpha\mu_i}{m\beta}\sum_{l=1}^{m-1}\frac{\omega_l}{\phi_l}[1 - Ee^{-\phi_l X_i}]E\exp\left[-\phi_l\left(\sum_{k=1}^{n-1}X_{\pi(k)} + X_j\right)\right]\right\} \\ \leq \quad & \mu_i h_i \left\{1 + \frac{\alpha}{m\beta} + \frac{\alpha\mu_j}{m\beta}\sum_{l=1}^{m-1}\frac{\omega_l}{\phi_l}[1 - Ee^{-\phi_l X_j}]E\exp\left[-\phi_l\left(\sum_{k=1}^{n-1}X_{\pi(k)} + X_i\right)\right]\right\} \end{aligned}$$

where $\phi_l = \phi_l(0) = \alpha(1-\omega_l) \quad (l = 1, 2, \cdots, m-1)$.

Proof: According to Lemma 2.2, we know that for any policy π the expected completion time of the nth job is:

$$EC_{\pi(n)} = \left(1 + \frac{\alpha}{m\beta}\right)\sum_{i=1}^{n}\frac{1}{\mu_{\pi(i)}} + \frac{\alpha}{m\beta}\sum_{i=1}^{m-1}\omega_i d_{in}^{\pi}$$

where $d_{in}^{\pi} = \frac{1}{\phi_i}\left[1 - E\exp\left(-\phi_i\sum_{k=1}^{n}X_{\pi(k)}\right)\right]$. Therefore, by direct calculation, we obtain:

$$\begin{aligned} & E\left(\sum_{k=1}^{N}h_{\pi_{(n;i,j)}(k)}C_{\pi_{(n;i,j)}(k)}\right) - \left(\sum_{k=1}^{N}h_{\pi_{(n;j,i)}(k)}C_{\pi_{(n;j,i)}(k)}\right) \\ = \quad & -h_i\left\{\left(1 + \frac{\alpha}{m\beta}\right)\frac{1}{\mu_j} + \frac{\alpha}{m\beta}\sum_{l=1}^{m-1}\frac{\omega_l}{\phi_l}[1 - Ee^{-\phi_l X_j}]E\exp\left[-\phi_l\left(\sum_{k=1}^{n-1}X_{\pi(k)} + X_i\right)\right]\right\} \\ & +h_j\left\{\left(1 + \frac{\alpha}{m\beta}\right)\frac{1}{\mu_i} + \frac{\alpha}{m\beta}\sum_{l=1}^{m-1}\frac{\omega_l}{\phi_l}[1 - Ee^{-\phi_l X_i}]E\exp\left[-\phi_l\left(\sum_{k=1}^{n-1}X_{\pi(k)} + X_j\right)\right]\right\} \end{aligned}$$

Thus, the Theorem is proved.

It should be pointed out that the above necessary and sufficient condition is dependent on the sequence before the nth job in a schedule, so it is impossible to get an index optimal policy related to both the parameters of the processing times and the machine. However, since the above condition has a nice structure, some optimal schedule policies can still be found as follows:

Theorem 3.2 *The scheduling policy* $\pi_0 = (1, 2, \cdots, N)$ *is optimal if for* $i < j$ *and* $t \geq 0$

$$H_C(t; i, j) \geq H_C(t; j, i)$$

where

$$H_C(t; i, j) = h_i\mu_i\left[1 + \frac{\alpha\mu_j}{m\beta}\int_0^{\infty}h_C(x+t)P(X_i \leq x \leq X_i + X_j)dx\right]$$

and the quantity

$$h_C(x) = me^{-\alpha x}\sum_{k=1}^{\infty}\frac{(\alpha x)^{km-1}}{(km-1)!}, \qquad x \geq 0$$

is only dependent on the parameters of the uptime distribution of the machine.

Proof: Let $EWFT[\pi]$ represent the expected sum of the weighted flow times under policy π. We shall prove that $EWFT[\pi_0] \le EWFT[\pi]$ for every policy π. Without loss of generality, suppose there exists a policy π and an n as well as $i < j$ satisfying $\pi(n) = j$ and $\pi(n+1) = i$. We next show that $EWFT[\pi_{(n;i,j)}] \le EWFT[\pi_{(n;j,i)}]$. In fact, by the identity:

$$\sum_{l=1}^{m-1} \omega_l e^{-\phi_l x} = -1 + me^{-\alpha x} \sum_{k=1}^{\infty} \frac{(\alpha x)^{km-1}}{(km-1)!}$$

and

$$\begin{aligned}
& h_i \left\{ \left(1 + \frac{\alpha}{m\beta}\right) \frac{1}{\mu_j} + \frac{\alpha}{m\beta} \sum_{l=1}^{m-1} \frac{\omega_l}{\phi_l} [1 - Ee^{-\phi_l X_j}] E \exp\left[-\phi_l \left(\sum_{k=1}^{n-1} X_{\pi(k)} + X_i\right)\right]\right\} \\
= \quad & h_i \left\{ \frac{1}{\mu_j} + \frac{\alpha}{m\beta} \int_0^{\infty} h_C(x) P\left(\sum_{k=1}^{n-1} X_{\pi(k)} + X_i \le x \le \sum_{k=1}^{n-1} X_{\pi(k)} + X_i + X_j\right) dx \right\} \\
= \quad & \frac{1}{\mu_i \mu_j} \int_0^{\infty} H_C(t; i, j) dP\left(\sum_{k=1}^{n-1} X_{\pi(k)} \le t\right)
\end{aligned}$$

as well as the result in Theorem 3.1, we see that $EWFT[\pi_{(n;i,j)}] \le EWFT[\pi_{(n;j,i)}]$. Thus, the theorem follows from a pairwise interchange argument (see [26]).

The following consequence was previously found in [23].

Corollary 3.3 *The expected weighted flow time is minimized by the schedule where the jobs are arranged in decreasing order of $h_i\mu_i$, provided one of the following conditions holds:*

1. $m = 1$*;*
2. *the job processing times are all exponentially distributed.*

Proof: 1. This follows directly from Theorem 3.2, since $H_C(t; i, j) = h_i\mu_i \left(1 + \frac{\alpha}{\beta}\right)$ if $m = 1$.

2. This follows directly from Theorem 3.2. However, we offer an alternative proof to show how to use the conclusion in Theorem 3.1 to obtain the result for this special case. Note that, in this case,

$$\begin{aligned}
& \mu_i h_i \left\{ 1 + \frac{\alpha}{m\beta} + \frac{\alpha\mu_j}{m\beta} \sum_{l=1}^{m-1} \frac{\omega_l}{\phi_l} [1 - Ee^{-\phi_l X_j}] E \exp\left[-\phi_l \left(\sum_{k=1}^{n-1} X_{\pi(k)} + X_i\right)\right]\right\} \\
= \quad & \mu_i h_i \left\{ 1 + \frac{\alpha}{m\beta} \int_0^{\infty} \left[\sum_{l=1}^{m} \omega_l e^{-\phi_l x}\right] dP\left(\sum_{k=1}^{n-1} X_{\pi(k)} + X_i + X_j \le x\right)\right\}
\end{aligned}$$

using the corresponding equation with j and i interchanged together with

$$\sum_{l=1}^{m} \omega_l e^{-\phi_l x} = m e^{-\alpha x} \sum_{k=1}^{\infty} \frac{(\alpha x)^{km-1}}{(km-1)!} \geq 0$$

We see that the condition in Theorem 3.1 holds if $\mu_j h_j \leq \mu_i h_i$. A pairwise interchange argument completes the proof.

If we are interested only in optimal policies which are independent of the machine parameters we can get the following result previously obtained in [19].

Corollary 3.4 *The policy $\pi_0 = (1, 2, \cdots, N)$ minimizes the expected weighted flow time if*

$$h_j P(X_j \leq t \leq X_i + X_j) \leq h_i P(X_i \leq t \leq X_i + X_j)$$

holds for $i < j$ and $t \geq 0$.

Proof: This is a straightforward application of Theorem 3.2, since the above condition implies $h_i \mu_i \geq h_j \mu_j$.

4 The Optimal Schedule for the Weighted Number of Tardy Jobs

In this section, we shall consider the problem of minimizing the expectation of the weighted number of tardy jobs, $\sum_k h_{\pi(k)} U_{\pi(k)}$. Here,

$$U_{\pi(k)} = \begin{cases} 1, & \text{if } C_{\pi(k)} > d \\ 0, & \text{if } C_{\pi(k)} \leq d \end{cases}$$

where d, the common due date of the jobs, is an exponential random variable with rate r. This quantity is an often-used objective in practice since it is a measure that can be recorded very easily. It is a more general cost function than the one studied in Section 3. Now, the cost is discounted at a rate of r per unit time. That is, if the jth job is not completed by time t, an additional expected cost $h_j r e^{-rt} dt$ is incurred over the period $[t, t + dt]$. If job j is completed at time t, the total expected cost incurred over the period $[0, t]$ is $h_j(1 - e^{-rt})$.

In the present paper, we shall find optimal schedules which minimize the expected value of this objective. Some simple permutation policies have been found previously in the case of exponential uptime (see [13] and [18], for example), but in the nonexponential case, no results have been reported.

Since

$$EU_{\pi_{(n;i,j)}(k)} = 1 - E\exp[-rC_{\pi_{(n;i,j)}(k)}]$$

holds for $k = 1, 2, \cdots, N$, a direct calculation using Lemma 2.2 gives the following necessary and sufficient condition:

Theorem 4.1 *The scheduling policy $\pi_{(n;i,j)}$ is better than the policy $\pi_{(n;j,i)}$ if and only if*

$$\begin{aligned} & h_j \sum_{l=1}^{m}\sum_{p=1}^{m} \frac{\omega_l^{1-p}\gamma_p(r)}{\phi_l(r)} \left(1 - Ee^{-\phi_l(r)X_i}\right) E\exp\left[-\phi_l(r)\left(\sum_{k=1}^{n-1} X_{\pi(k)} + X_j\right)\right] \\ \leq \quad & h_i \sum_{l=1}^{m}\sum_{p=1}^{m} \frac{\omega_l^{1-p}\gamma_p(r)}{\phi_l(r)} \left(1 - Ee^{-\phi_l(r)X_j}\right) E\exp\left[-\phi_l(r)\left(\sum_{k=1}^{n-1} X_{\pi(k)} + X_i\right)\right] \end{aligned} \tag{9}$$

We note here that it is not possible to obtain a general index optimal policy since the above necessary and sufficient condition depends on the sequence before the nth job. However, one optimal scheduling policy can still be found :

Theorem 4.2 *The policy $\pi_0 = (1, 2, \cdots, N)$ is optimal, if for $i < j$ and $t \geq 0$,*

$$H_U(t; i, j) \geq H_U(t; j, i)$$

where $H_U(t; i, j) = h_i \int_0^\infty h_U(x + t)P(X_i \leq x \leq X_i + X_j)dx$ and the quantity

$$h_U(x) = me^{-(\alpha+r)x} \sum_{p=1}^{m}\sum_{k=0}^{\infty} \frac{\left(\alpha x \sqrt[m]{Ee^{-rD}}\right)^{km+p-1} \gamma_p(r)}{(km + p - 1)!} \quad , \quad x \geq 0$$

is only dependent on the parameters of the machine.

Proof: The left-hand-side of (9) can be written as

$$\begin{aligned} & h_j \sum_{l=1}^{m}\sum_{p=1}^{m} \frac{\omega_l^{1-p}\gamma_p(r)}{\phi_l(r)} E\left[\exp\left(-\phi_l(r)(X_j + \sum_{k=1}^{n-1} X_{\pi(k)})\right) - \exp\left(-\phi_l(r)(\sum_{k=1}^{n-1} X_{\pi(k)} + X_j + X_i)\right)\right] \\ = \quad & h_j \int_0^\infty \left[\sum_{l=1}^{m}\sum_{p=1}^{m} \gamma_p(r)\omega_l^{1-p} e^{-\phi_l(r)x}\right] P\left(\sum_{k=1}^{n-1} X_{\pi(k)} + X_j \leq x \leq \sum_{k=1}^{n-1} X_{\pi(k)} + X_i + X_j\right) dx \\ = \quad & \int_0^\infty H_U(t; j, i) dP\left(\sum_{k=1}^{n-1} X_{\pi(k)} \leq t\right) \end{aligned}$$

The result now follows from a pairwise interchange argument and Theorem 4.1.

The following special cases can be found in [23].

Corollary 4.3 *When $m = 1$, the schedule which minimizes the expected weighted number of tardy jobs is in decreasing order of $\frac{h_i Ee^{-\phi X_i}}{1-Ee^{-\phi X_i}}$, where $\phi = r + \alpha(1 - Ee^{-rD})$.*

Proof: This follows from Theorem 4.2 upon noting that

$$H_U(t; i, j) = e^{-\phi t} h_i Ee^{-\phi X_i} \left(1 - Ee^{-\phi X_j}\right)$$

or, more simply, by using Theorem 4.1 directly and a pairwise interchange argument.

Corollary 4.4 *If the processing times are all exponential, the schedule which minimizes the expected weighted number of tardy jobs is in decreasing order of $h_i \mu_i$*

Proof: This is a straightforward application of Theorem 4.2. Alternatively, this result can be obtained by using the same idea as in Corollary 3.3 and

$$\sum_{l=1}^{m} \omega_l^{1-p} e^{-\phi_l(r)x} \geq 0 \quad , \qquad x \geq 0$$

Theorem 4.2 yields a simple optimal policy related only to the parameters of the processing time distribution as in [19].

Corollary 4.5 *The policy $\pi_0 = (1, 2, \cdots, N)$ minimizes the expected weighted number of tardy jobs if*

$$h_j P(X_j \leq t \leq X_i + X_j) \leq h_i P(X_i \leq t \leq X_i + X_j)$$

holds for any $i < j$ and $t \geq 0$.

5 The Optimal Schedule for the Sum of Weighted Tardiness

In this section, we shall consider schedules which minimize the expectation of the sum of weighted tardiness, $\sum_k h_{\pi(k)} T_{\pi(k)}$, where $T_{\pi(k)} = \max\{0, C_{\pi(k)} - d\}$.

Many papers have dealt with this scheduling problem in the deterministic case, but few have considered the stochastic case. Pinedo [21] was the first to consider this scheduling problem. He assumed that the processing time of job i is an exponential random variable with rate μ_i, the weight associated with job i is h_i and job i is due at time d_i, which is a random variable. By using a compatibility condition, i.e., $\mu_i h_i \geq \mu_k h_k$ implies $d_i \leq_{st} d_k$, he proved that the policy that minimizes this objective is the one where the jobs are arranged in decreasing order of $h_i\mu_i$. However, when the machine is unreliable, the question is more complicated. The case in which uptimes are arbitrary has been considered by [23] in case of exponential processing times. We now consider the case of arbitrary processing times with Erlang uptimes.

We have the following necessary and sufficient condition:

Theorem 5.1 *The scheduling policy $\pi_{(n;i,j)}$ is better than the policy $\pi_{(n;j,i)}$ if and only if*

$$\begin{aligned} & h_j\Bigg\{\left(1+\frac{\alpha}{m\beta}\right)\frac{1}{\mu_i}+\frac{\alpha}{m\beta}\sum_{l=1}^{m-1}\frac{\omega_l}{\phi_l}[1-Ee^{-\phi_l X_i}]E\exp\left[-\phi_l\left(\sum_{k=1}^{n-1}X_{\pi(k)}+X_j\right)\right] \\ & -\frac{1}{mr}\sum_{l=1}^{m}\sum_{p=1}^{m}\frac{\omega_l^{1-p}\gamma_p(r)}{\phi_l(r)}\left(1-Ee^{-\phi_l(r)X_i}\right)E\exp\left[-\phi_l(r)\left(\sum_{k=1}^{n-1}X_{\pi(k)}+X_j\right)\right]\Bigg\} \\ \leq\ & h_i\Bigg\{\left(1+\frac{\alpha}{m\beta}\right)\frac{1}{\mu_j}+\frac{\alpha}{m\beta}\sum_{l=1}^{m-1}\frac{\omega_l}{\phi_l}[1-Ee^{-\phi_l X_j}]E\exp\left[-\phi_l\left(\sum_{k=1}^{n-1}X_{\pi(k)}+X_i\right)\right] \\ & -\frac{1}{mr}\sum_{l=1}^{m}\sum_{p=1}^{m}\frac{\omega_l^{1-p}\gamma_p(r)}{\phi_l(r)}\left(1-Ee^{-\phi_l(r)X_j}\right)E\exp\left[-\phi_l(r)\left(\sum_{k=1}^{n-1}X_{\pi(k)}+X_i\right)\right]\Bigg\} \end{aligned}$$

Proof: The argument follows the same lines as the proofs of Theorems 3.1 and Theorem 4.1, after one notes that

$$ET_{\pi_{(n;i,j)}(k)}=EC_{\pi_{(n;i,j)}(k)}-\frac{1}{r}\left[1-Ee^{-rC_{\pi_{(n;i,j)}(k)}}\right]$$

holds for $k=1,2,\cdots,N$.

Theorem 5.2 *The policy $\pi_0=(1,2,\cdots,N)$ is optimal if, for $i<j$ and $t\geq 0$,*

$$H_T(t;i,j)\geq H_T(t;j,i)$$

where $H_T(t;i,j)=h_i\int_0^\infty h_T(x+t)P(X_i\leq x\leq X_i+X_j)dx$ and the quantity

$$h_T(x)=1+\frac{\alpha}{\beta}e^{-\alpha x}\sum_{k=1}^{\infty}\frac{(\alpha x)^{km-1}}{(km-1)!}-\frac{1}{r}e^{-(\alpha+r)x}\sum_{k=0}^{\infty}\sum_{p=1}^{m}\frac{\left(\alpha x\sqrt[m]{Ee^{-rD}}\right)^{km+p-1}\gamma_p(r)}{(km+p-1)!}$$

$(x\geq 0)$ is only dependent on the parameters of the machine.

Proof: This follows from Theorem 5.1 and

$$\begin{aligned} & 1+\frac{\alpha}{m\beta}\sum_{k=1}^{m}\omega_k e^{-\phi_k x}-\frac{1}{mr}\sum_{p=1}^{m}\sum_{k=1}^{m}\gamma_p\omega_k^{1-p}e^{-\phi_k(r)x} \\ =\ & 1+\frac{\alpha}{\beta}e^{-\alpha x}\sum_{k=1}^{\infty}\frac{(\alpha x)^{km-1}}{(km-1)!}-\frac{1}{r}e^{-(\alpha+r)x}\sum_{k=0}^{\infty}\sum_{p=1}^{m}\frac{\left(\alpha x\sqrt[m]{Ee^{-rD}}\right)^{km+p-1}\gamma_p(r)}{(km+p-1)!} \end{aligned}$$

Theorem 5.1 also yields the following result

Corollary 5.3 *When $m = 1$, the optimal schedule arranges the jobs in the order* $(1, 2, \cdots, N)$ *if the compatibility conditions*

$$h_j\mu_j \le h_i\mu_i \quad and \quad \frac{h_j Ee^{-\phi X_j}}{1 - Ee^{-\phi X_j}} \ge \frac{h_i Ee^{-\phi X_i}}{1 - Ee^{-\phi X_i}}$$

hold for $i < j$, where $\phi = r + \alpha(1 - Ee^{-rD})$.

Theorem 5.2 yields three additional results, of which the first is new, and the others were ontained in [23] and [19].

Corollary 5.4 *When $m = 1$, the schedule which minimizes the expected sum of the weighted tardiness of the jobs arranges the jobs in the order* $(1, 2, \cdots, N)$ *if*

$$\begin{aligned} & \mu_i h_i \left[\left(1 + \frac{\alpha}{\beta}\right) r - e^{-\phi t} \mu_j Ee^{-\phi X_i} \left(1 - Ee^{-\phi X_j}\right) \right] \\ \ge \quad & \mu_j h_j \left[\left(1 + \frac{\alpha}{\beta}\right) r - e^{-\phi t} \mu_i Ee^{-\phi X_j} \left(1 - Ee^{-\phi X_i}\right) \right] \end{aligned}$$

holds for $i < j$ and $t \ge 0$.

Proof: Here, we note that

$$H_T(t; i, j) = \frac{h_i}{r\mu_j} \left[\left(1 + \frac{\alpha}{\beta}\right) r - e^{-\phi t} \mu_j Ee^{-\phi X_i} \left(1 - Ee^{-\phi X_j}\right) \right]$$

Corollary 5.5 *If the processing times are all exponential, the schedule which minimizes the expected sum of the weighted tardiness of the jobs arranges the jobs in decreasing order of $h_i\mu_i$.*

Corollary 5.6 *The policy $\pi_0 = (1, 2, \cdots, N)$ minimizes the expected sum of the weighted tardiness if*

$$h_j P(X_j \le t \le X_i + X_j) \le h_i P(X_i \le t \le X_i + X_j)$$

holds for any $i < j$ and $t \ge 0$.

6 Conclusions and Further Research

In this paper, we have studied a class of single unreliable machine stochastic scheduling problems, with general processing time, Erlang uptime and general downtime distributions. We have obtained some general optimal scheduling policies which are not index policies and which depend on both processing parameters and machine parameters, under the nonpreemptive resume model assumption. Some of the special cases provided are also new to the literature.

The method of supplementary variables has been used successfully here to find useful representations for the completion times. It should be noted that, because objective functions are often functions of completion times, much research focuses on completion times, a problem for which the method of supplementary variables seems well-suited.

It should be emphasized that finding optimal schedules which depend on machine parameters has proved difficult in cases where the uptime distribution is nonexponential. Thus, the extension to Erlang uptimes given here represents a significant step forward.

We also wish to point out here that the optimality conditions found in this paper are both necessary and sufficient. This is in contrast to other papers in the field which usually provide only sufficient conditions.

When conditions sufficient for optimal policies of simple structure fail, it is beneficial to understand the cost implications of implementing simple policies nevertheless. Li and Glazebrook [19] established an upper bound on the loss incurred when a processing policy is adopted under the simplifying assumption of an exponential processing requirement. Several suboptimality bounds for the nonexponential case shall be given in another paper.

One question for future research that was pointed out by the referee concerns the case of deterministic uptime. By relaceing α by $m\alpha_0$ and letting m become infinitely large, we can obtain the special cases of all our results for constant uptime α_0. Obtaining simple policies will require some effort and will be the subject of a subsequent paper in which numerical implementation of our policies will also be considered.

We have restricted our attention to the nonpreemptive resume model, but it would be interesting to see if similar results can be found for the nonpreemptive repeat model. In that case, the makespan would seem to be an important objective to study first. So far, we have found that if the uptime has an Erlang distribution, the Laplace transform of the completion time of the nth job in any given policy π possesses a nice structure, but optimality results are still sought.

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Appendix

Proof of Lemma 2.1:
The proof of these equations follows a standard probabilistic argument (for example, see [8] or [9]). Since these equations are the starting point for all other developments in the paper, we provide some details of the proof for (1) and (7). The arguments for (2), (3), (4), (5) and (6) are similar to that for (1), and equation (8) is immediate from the definition.

To obtain equation (1), consider the state of the system at t and at $t + \Delta t$. In order that $S_n(t + \Delta t) = (1, k)$ and $V_k(t + \Delta t) = x + \Delta t$ $(x > 0)$, it is necessary that

(a) at time t, $S_n(t) = (1, k)$ and $V_k(t) = x$. Within $(t, t + \Delta t)$, the uptime of the machine is still in phase 1 and the kth job has not been completed; or

(b) at time t, $S_n(t) = (0, k)$, and $V_k(t) = x$, $V(t) = y$ for any $y > 0$. Within $(t, t + \Delta t)$, the repair of the machine has been completed.

Using a conditioning argument, we have

$$p_{1,k}(t + \Delta t, x + \Delta t) = p_{1,k}(t, x)[1 - \alpha\Delta t - \mu_{\pi(k)}(x)\Delta t] + \int_0^\infty p_{0,k}(t, x, y)\beta(y)\Delta t dy + o(\Delta t)$$

Thus,

$$\begin{aligned} & p_{1,k}(t + \Delta t, x + \Delta t) - p_{1,k}(t, x + \Delta t) + p_{1,k}(t, x + \Delta t) - p_{1,k}(t, x) \\ = \quad & -p_{1,k}(t, x)[\alpha\Delta t + \mu_{\pi(k)}(x)\Delta t] + \int_0^\infty p_{0,k}(t, x, y)\beta(y)\Delta t dy + o(\Delta t) \end{aligned}$$

Dividing by Δt and letting $\Delta t \to 0$ gives equation (1).

Most of the equations given by (7) follow immediately from the model assuptions. The only exception needing more proof is $p_{1,1}(0, x) = \delta(x)$. That is, we must show

$$p_{1,1}(0, x) = 0$$

for all $x > 0$, while

$$\int_0^\infty p_{1,1}(0, x)dx = \int_0^\varepsilon p_{1,1}(0, x)dx = 1$$

for any $\varepsilon > 0$.

The first relation is obvious, because it is impossible for the machine to have processed the first job for a non-zero amount of time at time t=0. Next, by the definition of $p_{1,1}(0, x)$, it follows that

$$\int_0^\infty p_{1,1}(0, x)dx = \int_0^\infty P\{S_n(0) = (1, 1), x \leq V_1(0) < x + dx | S_n(0) = (1, 1), V_1(0) = 0\} = 1$$

Thus, together with the first relation, we can get the second relation.

Remark: $\int_0^\infty p_{1,1}(0,x)dx = 1$ can be also directly obtained from the general relation in the equation (17) used in the following proof of the Lemma 2.2.

Proof of Lemma 2.2:
Solving the Laplace transformed version of (3) with the boundary condition (6) gives

$$p^*_{0,k}(s,x,y) = \alpha e^{-sy}[1-D(y)]p^*_{m,k}(s,x), \quad k = 1,2,\cdots,n \tag{10}$$

where $p^*(s)$ denotes the Laplace transform of a function $p(t)$.

Using (1), (2), (7) and the corresponding definitions, we can obtain

$$\frac{\partial}{\partial x}\mathbf{p}^*_k(s,x) = A\mathbf{p}^*_k(s,x) - \mu_{\pi(k)}(x)\mathbf{p}^*_k(s,x), + \delta_{k1}\delta(x)e_1^\tau \quad (k = 1,2,\cdots,n) \tag{11}$$

where $\mathbf{p}_k(t,x) = (p_{1,k}(t,x), p_{2,k}(t,x), \cdots, p_{m,k}(t,x))^\tau$, and

$$A = \begin{pmatrix} -s-\alpha & 0 & \cdots & 0 & \alpha E e^{-sD} \\ \alpha & -s-\alpha & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -s-\alpha & 0 \\ 0 & 0 & \cdots & \alpha & -s-\alpha \end{pmatrix}_{m\times m}$$

Noting that $-\phi_1(s), -\phi_2(s), \cdots, -\phi_m(s)$ are the m different eigenvalues of matrix A, we can write

$$A = \Delta\left\{\left(Ee^{-sD}\right)^{\frac{1-i}{m}}\right\} T\,\Delta\{-\phi_i(s)\}\,T^{-1}\,\Delta\left\{\left(Ee^{-sD}\right)^{\frac{i-1}{m}}\right\} \tag{12}$$

where $\Delta\{x_i\} = \mathrm{diag}\{x_i\}$,

$$T = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \omega_1^{-1} & \omega_2^{-1} & \cdots & \omega_m^{-1} \\ \vdots & \vdots & \ddots & \vdots \\ \omega_1^{1-m} & \omega_2^{1-m} & \cdots & \omega_m^{1-m} \end{pmatrix} \quad \text{and} \quad T^{-1} = \frac{1}{m}\begin{pmatrix} 1 & \omega_1 & \cdots & \omega_1^{m-1} \\ 1 & \omega_2 & \cdots & \omega_2^{m-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \omega_m & \cdots & \omega_m^{m-1} \end{pmatrix}$$

Substituting (12) into (11) and solving, we have

$$\mathbf{p}^*_1(s,x) = [1-F_{\pi(1)}(x)]\,\Delta\left\{\left(Ee^{-sD}\right)^{\frac{1-i}{m}}\right\} T\,\Delta\{-\phi_i(s)x\}T^{-1}\,\Delta\left\{\left(Ee^{-sD}\right)^{\frac{i-1}{m}}\right\} U(x)e_1^\tau \tag{13}$$

and

$$\mathbf{p}^*_k(s,x) = [1-F_{\pi(k)}(x)]\,\Delta\left\{\left(Ee^{-sD}\right)^{\frac{1-i}{m}}\right\} T\,\Delta\{-\phi_i(s)x\}T^{-1}\,\Delta\left\{\left(Ee^{-sD}\right)^{\frac{i-1}{m}}\right\} L_k \tag{14}$$

where L_k $(k = 2,\cdots,n)$ is a column vector to be determined, and $U(x)$ is one if $x > 0$ and zero otherwise. Using boundary conditions (4), (5) and the above two results and defining $L_1 = e_1^\tau$, vectors L_k $(k = 2,\cdots,n)$ can be recursively determined as

$$L_k \;=\; \Delta\left\{\left(Ee^{-sD}\right)^{\frac{1-i}{m}}\right\} T\,\Delta\{E\exp[-\phi_i(s)X_{\pi(k-1)}]\}T^{-1}\,\Delta\left\{\left(Ee^{-sD}\right)^{\frac{i-1}{m}}\right\} L_{k-1}$$

$$
\begin{aligned}
&= \cdots\cdots \\
&= \Delta\left\{\left(Ee^{-sD}\right)^{\frac{1-i}{m}}\right\} T\, \Delta\left\{\left(e^{-\phi_i(s)x}\right) E\exp\left[-\phi_i(s)\sum_{j=1}^{k-1} X_{\pi(j)}\right]\right\} \times \\
&\quad T^{-1}\, \Delta\left\{\left(Ee^{-sD}\right)^{\frac{i-1}{m}}\right\} \mathbf{e}_1^{\tau} \qquad (15)
\end{aligned}
$$

Then substituting (15) into (14) gives us

$$
\begin{aligned}
\mathbf{p}_k^*(s,x) &= [1-F_{\pi(k)}(x)]\Delta\left\{\left(Ee^{-sD}\right)^{\frac{1-i}{m}}\right\} T\, \Delta\left\{\left(e^{-\phi_i(s)x}\right) E\exp\left[-\phi_i(s)\sum_{j=1}^{k-1} X_{\pi(j)}\right]\right\} \\
&\quad \times T^{-1}\, \Delta\left\{\left(Ee^{-sD}\right)^{\frac{i-1}{m}}\right\} \mathbf{e}_1^{\tau} \qquad (k=2,3,\cdots,n) \qquad (16)
\end{aligned}
$$

Since

$$
P(C_{\pi(n)} > t) = \sum_{k=1}^{n}\left[\sum_{i=1}^{m}\int_0^{\infty} p_{i,k}(t,x)dx + \int_0^{\infty}\int_0^{\infty} p_{0,k}(t,x,y)dxdy\right] \qquad (17)
$$

the results in equations (10), (13) and (16) can be combined to give

$$
\begin{aligned}
&\int_0^{\infty} e^{-st}P(C_{\pi(n)} > t)dt = \sum_{k=1}^{n}\left[\sum_{i=1}^{m}\int_0^{\infty} p_{i,k}^*(s,x)dx + \int_0^{\infty}\int_0^{\infty} p_{0,k}^*(t,x,y)dxdy\right] \\
&= \left[\mathbf{e} + \frac{\alpha}{s}(1-Ee^{-sD})\mathbf{e}_m\right]\sum_{k=1}^{n}\int_0^{\infty}\mathbf{p}_k^*(s,x)dx \\
&= \frac{1}{s}\left[s\mathbf{e} + \alpha(1-E^{-sD})\right]\Delta\left\{\left(Ee^{-sD}\right)^{\frac{1-i}{m}}\right\} T\Delta\left\{\frac{1}{-\phi_i(s)}\left[1-E\exp\left(-\phi_i(s)\sum_{j=1}^{n} X_{\pi(j)}\right)\right]\right\} \\
&\quad \times T^{-1}\Delta\left\{\left(Ee^{-sD}\right)^{\frac{i-1}{m}}\right\}\mathbf{e}_1^{\tau} \\
&= \frac{1}{s}\mathbf{e}\Delta\left\{\gamma_i(s)\right\} T\Delta\left\{\frac{1}{-\phi_i(s)}\left[1-E\exp\left(-\phi_i(s)\sum_{j=1}^{n} X_{\pi(j)}\right)\right]\right\} T^{-1}\Delta\left\{\left(Ee^{-sD}\right)^{\frac{i-1}{m}}\right\}\mathbf{e}_1^{\tau} \\
&= \frac{1}{sm}\sum_{i=1}^{m}\sum_{k=1}^{m}\frac{\gamma_i(s)\omega_k^{1-i}}{\phi_k(s)}\left[1-E\exp\left(-\phi_k(s)\sum_{j=1}^{n} X_{\pi(j)}\right)\right] \qquad (18)
\end{aligned}
$$

The last equation above follows since it is a special case of the formula

$$
\mathbf{e}\Delta\{y_i\}T\Delta\{d_i\}T^{-1}\Delta\{x_i\}\mathbf{e}^{\tau} = \frac{1}{m}\sum_{k=1}^{m} d_k\left(\sum_{i=1}^{m} y_i\omega_k^{-i}\right)\left(\sum_{j=1}^{m} x_j\omega_k^{j}\right)
$$

when $x_1 = 1$ and $x_i = 0$ $(i = 2,3,\cdots,m)$.

Now, by using the relationship:

$$
E[e^{-sC_{\pi(n)}}] = 1 - s\int_0^{\infty} e^{-st}P(C_{\pi(n)} > t)dt
$$

the proof is readily completed.

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