Stochastic Analysis of a Repairable System with Three Units and Two Repair Facilities

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Abstract

This paper investigates the availability characteristics and the reliability of a three-dissimilar-unit repairable system with two different repair facilities. Under some practical assumptions, we obtain the explicit expressions of the state probabilities of the system, and then the explicit expressions of the following performance measures of the system:
(1) the pointwise and steady-state availability;
(2) the pointwise and steady-state failure frequency;
(3) the pointwise and steady-state renewal frequency;
(4) the reliability and the mean time to system failure.

1 Introduction

The study of repairable systems is an important topic in reliability. There is an extensive literature on availability characteristics of repairable systems with two or three components under varying assumptions on the failures and repairs, see [7]-[11], [13]-[15] and the references therein. In most of these papers, exponential distributions are assumed for some system variables and only one repair facility is considered for mathematical convenience. Methods used in the existing literature dealing with non-Markov systems involving many general random variables include the *Regenerative Point Technique* (RPT) [6, 7, 15] and the *Supplementary Variables Method* (SVM) [8]-[11], [13, 14]. In order to use the RPT, one has to correctly formulate and solve a system of Markov renewal equations, usually using an analytical method which is difficult for a non-Markov repairable system with only a few renewal points. By using the SVM, on the other hand, we can readily obtain all differential equations in terms of the state transition diagram of the model. However, it is still not easy to solve these differential equations because they usually involve some functions to be determined if there are at least two hazard rate functions involved in one of the equations. Therefore, the issues of how to formulate and solve the system of Markov renewal equations when the RPT is used and how to specify those undetermined functions when the SVM is used are gradually becoming interesting and important in the analysis of stochastic models [11, 13, 14].

In this paper, we investigate a three-dissimilar-unit model with two different repair facilities to show how to obtain the undetermined functions when the SVM is used. This model is one of the important models we often encounter in reliability applications and is also a difficult one to analyze if there are some random variables having a general distribution. For the model considered here, some of the system equations have two hazard functions involved and the process of using the SVM then involves some functions which need to be determined. There is no general method to deal with this problem and in many cases it is almost impossible to derive explict results from the system equations (see [11], [13] and [14]). Here, by decomposing the undetermined functions into the product of two independent functions, we successfully obtain the solution of the system. Furthermore, we obtain explicit expressions for some main availability indices of the system and the system's reliability. In the next two sections, we first define the system, derive the system equations, introduce two undetermined functions and decompose each of them into a product of two independent functions. We then obtain the explicit solution of the system and the expressions of the system's state probabilities. The explicit results of some main availability indices of the system , and also an example, are given in Section 4. In the last section, by decomposing an undetermined function into the product of two simple functions, we derive the explicit expression of the system's reliability.

2 Description of the System

2.1 Assumptions

The system we consider here consists of three dissimilar units, called unit 1, 2 and 3, and two different repair facilities, say repair facility 1 and 2, respectively. The system is operating if and only if unit 3 and at least one of unit 1 and unit 2 are working. Repair facility 1 can repair either failed unit 1 or unit 2, and repair facility 2 is responsible to repairing only unit 3. Other assumptions are given as follows:

- Initially, the system with three new units begins to operate. The system fails if and only if one of the following two situations happen: i) both unit 1 and unit 2 fail; ii) unit 3 fails.
- When the system fails, the unit(s) which is (are) in working order, if any, will be temporarily halted and the uptime of the unit(s) will be accumulated after the system reoperates.
- If both unit 1 and unit 2 fail, the one that fails later has to wait for repair until the repair of the unit which failed earlier is completed.
- A repaired unit is as good as a new one.
- The uptime X_i and the repairtime Y_i of unit i $(i = 1, 2, 3)$ are assumed to be general continuous random variables with distribution functions $F_i(x)$ and $G_i(x)$ and density functions $f_i(x)$ and $g_i(x)$, respectively. Let $\lambda_i(x)$ and $\mu_i(x)$ be the hazard rate functions of X_i and Y_i respectively. The following relationship is clear.

  $$P(X_i \leq x) = F_i(x) = \int_0^x f_i(t)dt = 1 - exp\left(-\int_0^x \lambda_i(t)dt\right)$$

  and

  $$P(Y_i \leq y) = G_i(y) = \int_0^y g_i(t)dt = 1 - exp\left(-\int_0^y \mu_i(t)dt\right).$$

- All random variables X_i and Y_i, $i = 1, 2, 3$, are independent.

2.2 State equations of the system

Since the system consists of three units and two different repair facilities, it is almost impossible to derive any explicit results if all of the random variables involved are generally distributed. Therefore, we further assume $\lambda_i(t) = \lambda_i$ $(i = 1, 2, 3)$, which means that the uptime of the three units are exponentially distributed. We shall find that even in this case the problem is still complicated. Fortunately, by decomposing each of the two undetermined functions into the product of two independent functions respectively, we obtain the explict expressions for the probabilities of system states and expressions of other interesting measures.

Define a stochastic process $S(t)$ that takes values from state space

$$J = \{000, 001, 010, 100, 011, 101, 110, 111\}.$$

In the state space, state 110 (111) represents the situation where both unit 1 and unit 2 have failed, unit 1 (2) is being repaired and unit 2 (1) is waiting for repair, whereas unit 3 is suspended because the system is down. For any of the other states (i_1, i_2, i_3), $i_j = 1$ represents that unit j is in repair, and $i_j = 0$ represents that unit j is working if the system is in operation or unit j is in working order but temporarily halted if the system is down.

Since there are still some general random vairables involved, $S(t)$ is not a Markov process. For each unit i $(i = 1, 2, 3)$, by introducing the elapsed repair time $Y_i(t)$ at time t and following a standard probabilistic argument (for example, see Cox [5] or Chaudhry and Templeton [2]), we can show that the process $\{S(t), Y_1(t), Y_2(t), Y_3(t)\}$ forms a Markov process with state space

$$J^* = \{000, (001, z), (010, y), (100, x), (011, y, z), (101, x, z), (110, x), (111, y) | 0 \le x, y, z < \infty\},$$

where x, y and z arethe value taken by $Y_1(t), Y_2(t)$ and $Y_3(t)$ respectively. Transitions among the states are shown in Figure 1.

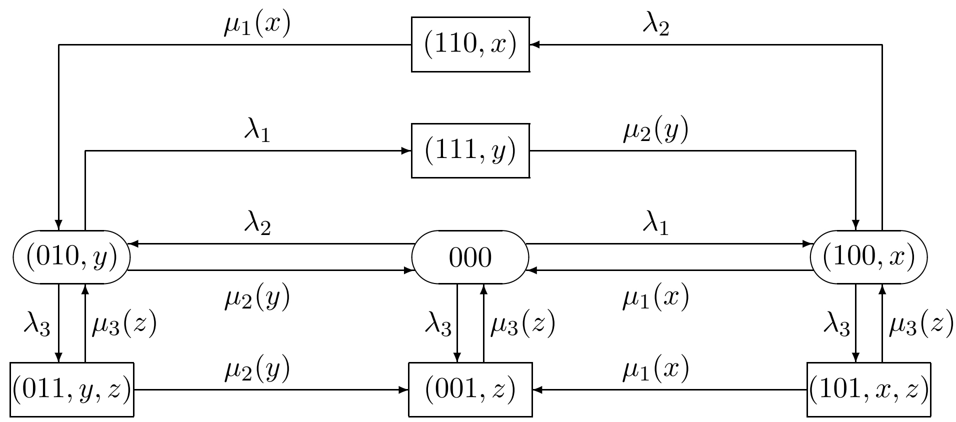

Figure 1: The state transition diagram.

Now, we define the following state probabilities:

$$\begin{aligned}
P_{000}(t) &= P\{S(t)=000\};\\
P_{100}(t,x)dx &= P\{S(t)=100, x\le Y_1(t)<x+dx\};\\
P_{010}(t,y)dy &= P\{S(t)=010, y\le Y_2(t)<y+dy\};\\
P_{001}(t,z)dz &= P\{S(t)=001, z\le Y_3(t)<z+dz\};\\
P_{101}(t,x,z)dxdz &= P\{S(t)=101, x\le Y_1(t)<x+dx, z\le Y_3(t)<z+dz\},\ x>z;\\
P_{011}(t,y,z)dydz &= P\{S(t)=011, y\le Y_2(t)<y+dy, z\le Y_3(t)<z+dz\},\ y>z;\\
P_{110}(t,x)dx &= P\{S(t)=110, x\le Y_1(t)<x+dx\};\\
P_{111}(t,y)dy &= P\{S(t)=111, y\le Y_2(t)<y+dy\}.
\end{aligned}$$

By a standard probabilistic argument, for example see [2] or [5], one can derive the following differential equations:

$$\left[\frac{d}{dt}+\lambda_1+\lambda_2+\lambda_3\right]P_{000}(t) = \int_0^\infty P_{100}(t,x)\mu_1(x)dx + \int_0^\infty P_{010}(t,y)\mu_2(y)dy + \int_0^\infty P_{001}(t,z)\mu_3(z)dz; \tag{2.1}$$

$$\left[\frac{\partial}{\partial t}+\frac{\partial}{\partial x}+\mu_1(x)+\lambda_2+\lambda_3\right]P_{100}(t,x) = \int_0^x P_{101}(t,x,z)\mu_3(z)dz; \tag{2.2}$$

$$\left[\frac{\partial}{\partial t}+\frac{\partial}{\partial y}+\lambda_1+\mu_2(y)+\lambda_3\right]P_{010}(t,y) = \int_0^y P_{011}(t,y,z)\mu_3(z)dz; \tag{2.3}$$

$$\left[\frac{\partial}{\partial t}+\frac{\partial}{\partial z}+\mu_3(z)\right]P_{001}(t,z) = \int_z^\infty P_{101}(t,x,z)\mu_1(x)dx + \int_z^\infty P_{011}(t,y,z)\mu_2(y)dy; \tag{2.4}$$

$$\left[\frac{\partial}{\partial t}+\frac{\partial}{\partial x}+\frac{\partial}{\partial z}+\mu_1(x)+\mu_3(z)\right]P_{101}(t,x,z) = 0,\quad x>z; \tag{2.5}$$

$$\left[\frac{\partial}{\partial t}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}+\mu_2(y)+\mu_3(z)\right]P_{011}(t,y,z) = 0,\quad y>z; \tag{2.6}$$

$$\left[\frac{\partial}{\partial t}+\frac{\partial}{\partial x}+\mu_1(x)\right]P_{110}(t,x) = \lambda_2 P_{100}(t,x); \tag{2.7}$$

$$\left[\frac{\partial}{\partial t}+\frac{\partial}{\partial y}+\mu_2(y)\right]P_{111}(t,y) = \lambda_1 P_{010}(t,y) \tag{2.8}$$

with boundary conditions:

$$P_{100}(t,0) = \lambda_1 P_{000}(t) + \int_0^\infty P_{111}(t,y)\mu_2(y)dy; \tag{2.9}$$

$$P_{010}(t,0) = \lambda_2 P_{000}(t) + \int_0^\infty P_{110}(t,x)\mu_1(x)dx; \tag{2.10}$$

$$P_{001}(t,0) = \lambda_3 P_{000}(t); \tag{2.11}$$

$$P_{101}(t,x,0) = \lambda_3 P_{100}(t,x); \tag{2.12}$$

$$P_{011}(t,y,0) = \lambda_3 P_{010}(t,y); \tag{2.13}$$

$$P_{110}(t,0) = 0; \tag{2.14}$$

$$P_{111}(t,0) = 0; \tag{2.15}$$

and the only non-zero initial condition:

$$P_{000}(0) = 1. \tag{2.16}$$

2.3 Notations

In this subsection, we summarize some notations which will be used throughout the rest of the paper.

The complement of a probability distribution function $G(\cdot)$, the convolution of two functions f and g, and the Laplace transform of a function $P(\cdot)$ are respectively denoted by

$$\begin{aligned} \bar{G}(\cdot) &= 1 - G(\cdot), \\ f(y) * g(y) &= \int_0^y f(t)g(y-t)dt, \end{aligned}$$

and

$$P^*(s) = \int_0^\infty e^{-st} P(t)dt.$$

Next, we let

$$L_1(s) = \frac{\lambda_2 + \lambda_1\lambda_2 \int_0^\infty e^{-sy} g_1(y) \left(\int_0^y e^{-(\lambda_2+\lambda_3)t} C_{01}(t)dt \right) dy}{\lambda_1 + \lambda_1\lambda_2 \int_0^\infty e^{-sy} g_2(y) \left(\int_0^y e^{-(\lambda_1+\lambda_3)t} C_{02}(t)dt \right) dy},$$

$$L_2(s) = \frac{1}{\lambda_1} \left[1 - \lambda_1 L_1(s) \int_0^\infty e^{-sy} g_2(y) \left(\int_0^y e^{-(\lambda_1+\lambda_3)t} C_{02}(t)dt \right) dy \right]$$

and

$$\begin{aligned} L_3(s) &= [s + \lambda_1 + \lambda_2 + \lambda_3 - \lambda_3 g_3^*(s)]\, L_2(s) - \left[\int_0^\infty e^{-(s+\lambda_2+\lambda_3)y} g_1(y) C_{01}(y)dy \right. \\ &\quad \left. + \lambda_3 \int_0^\infty e^{-sz} g_3(z) \left(\int_0^\infty e^{-st} D_1(t)[\bar{G}_1(t) - \bar{G}_1(t+z)]dt \right) dz \right] \end{aligned}$$

$$-L_1(s)\left[\int_0^\infty e^{-(s+\lambda_1+\lambda_3)y}g_2(y)C_{02}(y)dy\right.$$
$$\left.+\lambda_3\int_0^\infty e^{-sz}g_3(z)\left(\int_0^\infty e^{-st}D_2(t)[\bar{G}_2(t)-\bar{G}_2(t+z)]dt\right)dz\right],$$

where

$$D_1(u)=L^{-1}\left[\frac{1}{\lambda_2+\eta[1+\lambda_3\bar{G}_3^*(\eta)]}\right]$$

and

$$D_2(u)=L^{-1}\left[\frac{1}{\lambda_1+\eta[1+\lambda_3\bar{G}_3^*(\eta)]}\right]$$

are the inverses of the Laplace transforms, and

$$C_{01}(t)=1+\lambda_3\int_0^y e^{(\lambda_2+\lambda_3)t}g_3(t)*D_1(t)dt$$

and

$$C_{02}(t)=1+\lambda_3\int_0^y e^{(\lambda_1+\lambda_3)t}g_3(t)*D_2(t)dt.$$

We also introduce

$$N_1(s)=\int\limits_0^\infty e^{-(s+\lambda_2+\lambda_3)x}\left(1+\lambda_2\int\limits_0^x e^{\lambda_2 z}g_2(z)*D_3(z)dz\right)f_1(x)dx\quad,\quad(s\geq 0)$$

and

$$N_2(s)=\int\limits_0^\infty e^{-(s+\lambda_2+\lambda_3)x}\left(1+\lambda_2\int\limits_0^x e^{\lambda_2 z}g_2(z)*D_3(z)dz\right)\bar{F}_1(x)dx\quad,\quad(s\geq 0),$$

where

$$D_3(x)=L^{-1}\left[\frac{1}{s[1+\lambda_2\bar{G}_2^*(s)]}\right].$$

These will be used in Section 5.

3 The solution of the equations

By taking the Laplace transform of (2.5) and (2.6) on t, we can obtain

$$P_{101}^*(s,x,z)=e^{-sx}\bar{G}_1(x)\bar{G}_3(z)H_1(s,x-z)\tag{3.17}$$

and

$$P_{011}^*(s,y,z)=e^{-sy}\bar{G}_2(y)\bar{G}_3(z)H_2(s,y-z),\tag{3.18}$$

respectively, where $H_i(s,u)$ $(i=1,2)$ are two functions to be determined. Substituting (3.17) into the Laplace tansform of (2.2) yields

$$\left[\frac{\partial}{\partial x}+s+\mu_1(x)+\lambda_2+\lambda_3\right]P^*_{100}(s,x)=\bar{G}_1(x)e^{-sx}\int_0^x g_3(z)H_1(s,x-z)dz.$$

Thus,

$$P^*_{100}(s,x)=e^{-(s+\lambda_2+\lambda_3)x}\bar{G}_1(x)\left[\int_0^x e^{(\lambda_2+\lambda_3)u}g_3(u)*H_1(s,u)du+C_1(s)\right], \tag{3.19}$$

where $C_1(s)$ is a function of s to be determined. Similarly, substituting (3.18) into the Laplace transform of (2.3) yields

$$P^*_{010}(s,y)=e^{-(s+\lambda_1+\lambda_3)y}\bar{G}_2(y)\left[\int_0^y e^{(\lambda_1+\lambda_3)u}g_3(u)*H_2(s,u)du+C_2(s)\right], \tag{3.20}$$

where $C_2(s)$ is a function of s to be determined. Substituting (3.17) and (3.18) into the Laplace transform of (2.4), we can obtain

$$\begin{aligned} P^*_{001}(s,z) &= e^{-sz}\bar{G}_3(z)\left[\int_0^\infty e^{-su}H_1(s,u)\left[\bar{G}_1(u)-\bar{G}_1(u+z)\right]du\right.\\ &\quad\left.+\int_0^\infty e^{-su}H_2(s,u)\left[\bar{G}_2(u)-\bar{G}_2(u+z)\right]du+C_3(s)\right], \end{aligned} \tag{3.21}$$

where $C_3(s)$ is a function of s to be determined. By using the Laplace transforms of (2.7) and (2.8), and also the results in (3.19) and (3.20), we have

$$P^*_{110}(s,x)=\lambda_2 e^{-sx}\bar{G}_1(x)\int_0^x e^{-(\lambda_2+\lambda_3)u}C_1(s,u)du \tag{3.22}$$

and

$$P^*_{111}(s,y)=\lambda_1 e^{-sy}\bar{G}_2(y)\int_0^y e^{-(\lambda_1+\lambda_3)u}C_2(s,u)du, \tag{3.23}$$

where

$$C_1(s,u)=\int_0^y e^{(\lambda_2+\lambda_3)u}g_3(u)*H_1(s,u)du+C_1(s)$$

and

$$C_2(s,u)=\int_0^y e^{(\lambda_1+\lambda_3)u}g_3(u)*H_2(s,u)du+C_2(s).$$

Using equations (2.1), (3.19), (3.20) and (3.21), and noticing (2.16), we know that

$$\begin{aligned} P^*_{000}(s) &= \frac{1}{s+\lambda_1+\lambda_2+\lambda_3}\left[1+\int_0^\infty e^{-(s+\lambda_2+\lambda_3)x}g_1(x)C_1(s,x)dx\right.\\ &\quad\left.+\int_0^\infty e^{-(s+\lambda_1+\lambda_3)y}g_2(y)C_2(s,y)dy+\int_0^\infty e^{-sz}g_3(z)C_3(s,z)dz\right], \end{aligned} \tag{3.24}$$

where

$$\begin{aligned} C_3(s,z) &= \int_0^\infty e^{-su} H_1(s,u)\left[\bar{G}_1(u) - \bar{G}_1(u+z)\right] du \\ &\quad + \int_0^\infty e^{-su} H_2(s,u)\left[\bar{G}_2(u) - \bar{G}_2(u+z)\right] du + C_3(s). \end{aligned}$$

Therefore, to obtain the explict solution for equations (3.17) to (3.24), we have to determine $H_i(s,u)$ and $C_j(s)$ $(i = 1,2;\ j = 1,2,3)$. In general, if we can obtain the undetermined functions, we then have explicit solutions of the system. Unfortunately, in most cases, we can not obtain explicit expressions of the undetermined functions. However, for the case discussed here, the explicit expressions can be obtained. To this end, we first introduce the following relationship between $H_i(s,u)$, and $C_i(s)$ $(i = 1,2)$, and afterwards the expressions of the $C_i(s)$ $(i = 1,2,3)$.

Lemma 3.1 $H_i(s,u)$ $(i = 1,2)$ *can be expressed as a product of* $C_i(s)$ *and a function of* u. *In fact, we have*

$$H_i(s,u) = \lambda_3 C_i(s) D_i(u), \quad i = 1,2.$$

Proof: Substitute (3.18) and (3.20) into the Laplace transform of (2.13), we have

$$e^{(\lambda_1+\lambda_3)y} H_2(s,y) = \lambda_3 \left(\int_0^y e^{(\lambda_1+\lambda_3)u} g_3(u) * H_2(s,u) du + C_2(s) \right).$$

Thus, $H_2(s,0) = \lambda_3 C_2(s)$ and

$$(\lambda_1+\lambda_3) H_2(s,y) + \frac{\partial}{\partial y} H_2(s,y) = \lambda_3 g_3(y) * H_2(s,y).$$

By taking the Laplace transform of the above equation on y, we obtain

$$(\lambda_1+\lambda_3) H_2^*(s,\eta) + \eta H_2^*(s,\eta) - H_2(s,0) = \lambda_3 g_3^*(\eta) H_2^*(s,\eta).$$

Therefore,

$$H_2^*(s,\eta) = H_2(s,0)\left[\lambda_1 + \lambda_3 + \eta - \lambda_3 g_3^*(\eta)\right]^{-1},$$

which is equivalent to $H_2(s,u) = \lambda_3 C_2(s) D_2(u)$. Similarly, we can prove that $H_1(s,u) = \lambda_3 C_1(s) D_1(u)$.

■

Lemma 3.2 *The functions* C_i $(i = 1,2,3)$ *are determined as* $C_1(s) = 1/L_3(s)$, $C_2(s) = L_1(s)/L_3(s)$, *and* $C_3(s) = L_2(s)/L_3(s)$.

Proof: Substituting equation (2.21), (2.24) into (2.11) and using the result in Lemma 3.1 lead to

$$\begin{aligned}
& [s+\lambda_1+\lambda_2+\lambda_3-\lambda_3 g_3^*(s)]\, C_3(s) \\
= \quad & \lambda_3+\lambda_3 C_1(s)\left[\int_0^\infty e^{-(s+\lambda_2+\lambda_3)x} g_1(x) C_{01}(x)dx\right. \\
& \quad \left. +\lambda_3 \int_0^\infty e^{-sz} g_3(z)\left(\int_0^\infty e^{-su} D_1(u)[\bar{G}_1(u)-\bar{G}_1(u+z)]du\right) dz\right] \\
& +\lambda_3 C_2(s)\left[\int_0^\infty e^{-(s+\lambda_1+\lambda_3)y} g_2(y) C_{02}(y)dy\right. \\
& \quad \left. +\lambda_3 \int_0^\infty e^{-sz} g_3(z)\left(\int_0^\infty e^{-su} D_2(u)[\bar{G}_2(u)-\bar{G}_2(u+z)]du\right) dz\right].
\end{aligned}$$

By using equation (2.9), (2.19), (2.23), (2.10), (2.20), and (2.22), as well as the result in Lemma 3.1, we have

$$C_1(s)=\lambda_1 P_{000}(s)+\lambda_1 C_2(s)\int_0^\infty e^{-sy} g_2(y)\left(\int_0^y e^{-(\lambda_1+\lambda_3)u} C_{02}(u)du\right) dy$$

and

$$C_2(s)=\lambda_2 P_{000}(s)+\lambda_2 C_1(s)\int_0^\infty e^{-sx} g_1(x)\left(\int_0^x e^{-(\lambda_2+\lambda_3)u} C_{01}(u)du\right) dy.$$

Then, the result is straightforward by simple calculations.

■

For convenience, we summarize above explicit solutions of the system as follows.

Theorem 3.3 *The Laplace transforms of the explicit solutions of the system are given by*

$$\begin{aligned}
P_{011}^*(s,y,z) &= \lambda_3 e^{-sy}\bar{G}_2(y)\bar{G}_3(z) D_2(y-z) C_2(s), \\
P_{101}^*(s,x,z) &= \lambda_3 e^{-sx}\bar{G}_1(x)\bar{G}_3(z) D_1(x-z) C_1(s), \\
P_{100}^*(s,x) &= e^{-(s+\lambda_2+\lambda_3)x}\bar{G}_1(x) C_{01}(x) C_1(s), \\
P_{010}^*(s,y) &= e^{-(s+\lambda_1+\lambda_3)y}\bar{G}_2(y) C_{02}(y) C_2(s), \\
P_{001}^*(s,z) &= e^{-sz}\bar{G}_3(z) C_3(s,z), \\
P_{110}^*(s,x) &= \lambda_2 e^{-sx}\bar{G}_1(x)\int_0^x e^{-(\lambda_2+\lambda_3)u} C_{01}(u)du C_1(s), \\
P_{111}^*(s,y) &= \lambda_1 e^{-sy}\bar{G}_2(y)\int_0^y e^{-(\lambda_1+\lambda_3)u} C_{02}(u)du C_2(s), \\
P_{000}^*(s) &= \frac{1}{s+\lambda_1+\lambda_2+\lambda_3}\left[1+C_1(s)\int_0^\infty e^{-(s+\lambda_2+\lambda_3)x} g_1(x) C_{01}(x)dx\right. ,
\end{aligned}$$

$$+C_2(s)\int_0^\infty e^{-(s+\lambda_1+\lambda_3)y} g_2(y)C_{02}(y)dy + \int_0^\infty e^{-sz} g_3(z)C_3(s,z)dz\Big],$$

where $D_i(t)$, $C_{0i}(t)$ *(i=1,2) and* $C_j(s)$, $j = 1,2,3$, *are given in the section 2.3 and the Lemma 2.2, respectively, and*

$$\begin{aligned} C_3(s,z) &= \lambda_3 C_1(s)\int_0^\infty e^{-sx}D_1(x)\left[\bar{G}_1(x)-\bar{G}_1(x+z)\right]dx \\ &\quad +\lambda_3 C_2(s)\int_0^\infty e^{-sy}D_2(y)\left[\bar{G}_2(y)-\bar{G}_2(y+z)\right]dy + C_3(s). \end{aligned}$$

■

By Theorem 3.3, we can readily get the following important result:

Theorem 3.4 *Denote by* $P_\alpha(t) = P(S(t) = \alpha)$ *for* $\alpha \in J$, *then the Laplace transform of the explicit state probabilities of the system are given by*

$$\begin{aligned} P^*_{011}(s) &= \lambda_3\int_0^\infty\int_0^y e^{-sy}\bar{G}_2(y)\bar{G}_3(z)D_2(y-z)dydzC_2(s), \\ P^*_{101}(s) &= \lambda_3\int_0^\infty\int_0^x e^{-sx}\bar{G}_1(x)\bar{G}_3(z)D_1(x-z)dzdxC_1(s), \\ P^*_{100}(s) &= \int_0^\infty e^{-(s+\lambda_2+\lambda_3)x}\bar{G}_1(x)C_{01}(x)dxC_1(s), \\ P^*_{010}(s) &= \int_0^\infty e^{-(s+\lambda_1+\lambda_3)y}\bar{G}_2(y)C_{02}(y)dyC_2(s), \\ P^*_{001}(s) &= \int_0^\infty e^{-sz}\bar{G}_3(z)C_3(s,z)dz, \\ P^*_{110}(s) &= \lambda_2\int_0^\infty e^{-sx}\bar{G}_1(x)\left[\int_0^x e^{-(\lambda_2+\lambda_3)u}C_{01}(u)du\right]dxC_1(s), \\ P^*_{111}(s) &= \lambda_1\int_0^\infty e^{-sy}\bar{G}_2(y)\left[\int_0^y e^{-(\lambda_1+\lambda_3)u}C_{02}(u)du\right]dyC_2(s), \\ P^*_{000}(s) &= \frac{1}{s+\lambda_1+\lambda_2+\lambda_3}\Big[1 + C_1(s)\int_0^\infty e^{-(s+\lambda_2+\lambda_3)x}g_1(x)C_{01}(x)dx\ , \\ &\quad +C_2(s)\int_0^\infty e^{-(s+\lambda_1+\lambda_3)y}g_2(y)C_{02}(y)dy + \int_0^\infty e^{-sz}g_3(z)C_3(s,z)dz\Big]. \end{aligned}$$

■

Remark 3.5 If we denote by $P_\alpha = \lim_{t\to\infty} P_\alpha(t)$ the steady-state probability of being in state $\alpha \in J$, then P_α can be explicitly given by using Theorem 3.4 and the Terminal-Value Theorem of Laplace Transform (see [14], p402).

4 Availability analysis of the System

According to the probability analysis of the system in Section 3, we can obtain the transient and equilibrium availability characteristics of the system as follows.

4.1 Availability of the system

The availability of the system, denoted by $A(t)$, is the probability that the system is operating at time t.

Theorem 4.1 *The Laplace transform of $A(t)$ is explicitly given by*

$$\begin{aligned} A^*(s) &= P^*_{000}(s) + C_1(s) \int_0^\infty e^{-(s+\lambda_2+\lambda_3)x} \bar{G}_1(x) C_{01}(x) dx \\ &\quad + C_2(s) \int_0^\infty e^{-(s+\lambda_1+\lambda_3)y} \bar{G}_2(y) C_{02}(y) dy \end{aligned}$$

and the steady-state availability of the system, denoted by A, is

$$A = P_{000} + C_1 \int_0^\infty e^{-(\lambda_2+\lambda_3)x} \bar{G}_1(x) C_{01}(x) dx + C_2 \int_0^\infty e^{-(\lambda_1+\lambda_3)y} \bar{G}_2(y) C_{02}(y) dy$$

where $P_{000} = lim_{s\to 0} s P^*_{000}(s)$ and $C_i = lim_{s\to 0} s C_i(s)$ (i=1,2).

Proof: Based on the definition of the availability of the system and the fact that the system is operating if and only if the stochastic process S(t) is in state 000, 010 or 100, we know

$$A(t) = P_{000}(t) + \int_0^\infty P_{100}(t,x) dx + \int_0^\infty P_{010}(t,y) dy.$$

Thus, from Theorem 3.3 or Corollary 3.4, we can readily obtain the first result. The second result can be obtained by a simple use of the Terminal-Value Theorem of Laplace transform:

$$\begin{aligned} A &= \lim_{t\to\infty} A(t) = \lim_{s\to 0} sA^*(s) \\ &= P_{000} + C_1 \int_0^\infty e^{-(\lambda_2+\lambda_3)x} \bar{G}_1(x) C_{01}(x) dx + C_2 \int_0^\infty e^{-(\lambda_1+\lambda_3)y} \bar{G}_2(y) C_{02}(y) dy. \end{aligned}$$

■

4.2 Failure frequence of the system

Denoted by $m_f(t)$ the derivative of the expected number of failures of the system having occurred by time t. It is called the failure frequency [12] or the rate of occurrence of failures of the system [8] during [0, t). By using Theorem 2 and Corollary 2 in [8], we have

Theorem 4.2 *The Laplace transform of $m_f(t)$ is given by*

$$\begin{aligned} m_f^*(s) &= \lambda_2 C_1(s)\int_0^\infty e^{-(s+\lambda_2+\lambda_3)x}\bar{G}_1(x)C_{01}(x)dx \\ &\quad +\lambda_1 C_2(s)\int_0^\infty e^{-(s+\lambda_1+\lambda_3)y}\bar{G}_2(y)C_{02}(y)dy+\lambda_3 A^*(s) \end{aligned}$$

and the steady-state failure frequency of the system, denoted by m_f, is

$$\begin{aligned} m_f &= \lambda_2 C_1\int_0^\infty e^{-(\lambda_2+\lambda_3)x}\bar{G}_1(x)C_{01}(x)dx \\ &\quad +\lambda_1 C_2\int_0^\infty e^{-(\lambda_1+\lambda_3)y}\bar{G}_2(y)C_{02}(y)dy+\lambda_3 A. \end{aligned}$$

Proof: Based on the definition of $m_f(t)$ and Theorem 2 in [8], we know that

$$\begin{aligned} m_f(t) &= \lambda_3 P_{000}(t)+(\lambda_2+\lambda_3)\int_0^\infty P_{100}(t,x)dx+(\lambda_1+\lambda_3)\int_0^\infty P_{010}(t,y)dy \\ &= \lambda_3 A(t)+\lambda_2\int_0^\infty P_{100}(t,x)dx+\lambda_1\int_0^\infty P_{010}(t,y)dy. \end{aligned}$$

Thus, by Theorem 3.3, we have

$$\begin{aligned} m_f^*(s) &= \lambda_2 C_1(s)\int_0^\infty e^{-(s+\lambda_2+\lambda_3)x}\bar{G}_1(x)C_{01}(x)dx \\ &\quad +\lambda_1 C_2(s)\int_0^\infty e^{-(s+\lambda_1+\lambda_3)y}\bar{G}_2(y)C_{02}(y)dy+\lambda_3 A^*(s). \end{aligned}$$

By Corollary 2 in [8], we have

$$\begin{aligned} m_f &= \lim_{t\to\infty} m_f(t)=\lim_{s\to 0} s m_f^*(s) \\ &= \lambda_2 C_2\int_0^\infty e^{-(\lambda_2+\lambda_3)x}\bar{G}_1(x)C_{01}(x)dx \\ &\quad +\lambda_1 C_2\int_0^\infty e^{-(\lambda_1+\lambda_3)y}\bar{G}_2(y)C_{02}(y)dy+\lambda_3 A. \end{aligned}$$

■

4.3 Renewal frequency of the system

Denote by $m_r(t)$ the derivative of the expected number of renewals of the system having occurred by time t. It is called the renewal frequency. A renewal of the system means that the state of the system returns to the initial states. By using Theorem 2 and Corollary 2 in [8] again, we have

Theorem 4.3 *The Laplace transform of $m_r(t)$ is given by*

$$m_r^*(s) \quad = \quad (s+\lambda_1+\lambda_2+\lambda_3)P_{000}^*(s) - 1$$

and the steady-state renewal frequency of the system, denoted by m_r, is

$$m_r = (\lambda_1+\lambda_2+\lambda_3)P_{000}.$$

Proof: According to the definition of $m_r(t)$ and the results in [8], we know that

$$m_r(t) \quad = \quad \int_0^\infty P_{100}(t,x)\mu_1(x)dx + \int_0^\infty P_{010}(t,y)\mu_2(y)dy + \int_0^\infty P_{001}(t,z)\mu_3(z)dz.$$

Thus, by equations (2.1) and (2.16), we get

$$m_f^*(s) = (s+\lambda_1+\lambda_2+\lambda_3)P_{000}^*(s) - 1.$$

By Terminal-Value Theorem of the Laplace transform again, we have

$$m_f \quad = \quad (\lambda_1+\lambda_2+\lambda_3)P_{000}.$$

■

4.4 Example

Now, we consider an example when $\lambda_3 = 0$. In this case, the model in fact is transformed into a two-unit parellel system with one repair facility because $\lambda_3 = 0$ implies that unit 3 is always operating. By the previous notation, we can easily get $C_{01}(t) = C_{02}(t) = 1$ and the explicit expressions for $L_1(s)$, $L_2(s)$ and $L_3(s)$. For example:

$$\begin{aligned} L_3(s) \quad = \quad & \frac{s}{\Delta}\left\{\left(1+\lambda_1\bar{G}_1^*(s)\right)\left[\lambda_1+\lambda_2\left((s+\lambda_1)\bar{G}_2^*(s+\lambda_1) - s\bar{G}_2^*(s)\right)\right]\right. \\ & \left. +(s+\lambda_1)\left(\bar{G}_2^*(s)-\bar{G}_2^*(s+\lambda_1)\right)\left[(s+\lambda_2)\left(1+\lambda_1\bar{G}_1^*(s+\lambda_2)\right) - s\left(1+\lambda_1\bar{G}_1^*(s)\right)\right]\right\}, \end{aligned}$$

where $\Delta = \lambda_1\left[\lambda_1+\lambda_2\left((s+\lambda_1)\bar{G}_2^*(s+\lambda_1) - s\bar{G}_2^*(s)\right)\right]$. By the above expression and letting $\rho_i = \lambda_i/\mu_i$ for $i = 1, 2$, we have:

$$\begin{aligned} C_1 \quad &= \quad \lim_{s\to 0} sC_1(s) = \lim_{s\to 0}\frac{s}{L_3(s)} \\ &= \quad \frac{\lambda_1[1+\lambda_2\bar{G}_2^*(\lambda_1)]}{(1+\rho_1)[1+\lambda_2\bar{G}_2^*(\lambda_1)] + (1+\rho_2)[1+\lambda_1\bar{G}_1^*(\lambda_2)] - [1+\lambda_1\bar{G}_1^*(\lambda_2)][1+\lambda_2\bar{G}_2^*(\lambda_2)]} \end{aligned}$$

$$\begin{aligned} C_2 \quad &= \quad \lim_{s\to 0} sC_2(s) \\ &= \quad \frac{\lambda_2[1+\lambda_1\bar{G}_1^*(\lambda_2)]}{(1+\rho_1)[1+\lambda_2\bar{G}_2^*(\lambda_1)] + (1+\rho_2)[1+\lambda_1\bar{G}_1^*(\lambda_2)] - [1+\lambda_1\bar{G}_1^*(\lambda_2)][1+\lambda_2\bar{G}_2^*(\lambda_2)]}. \end{aligned}$$

Therefore, by using the results in Theorems 4.1, 4.2 and 4.3, we can obtain the following explicit result.

Corollary 4.4 *If $\lambda_3 = 0$, i.e., for the two-unit parallel system with one repair facility,*
(1) The steady-state availability of the system is

$$A = \frac{[1+\lambda_1\bar{G}_1^*(\lambda_2)][1+\lambda_2\bar{G}_2^*(\lambda_1)]}{(1+\rho_1)[1+\lambda_2\bar{G}_2^*(\lambda_1)]+(1+\rho_2)[1+\lambda_1\bar{G}_1^*(\lambda_2)]-[1+\lambda_1\bar{G}_1^*(\lambda_2)][1+\lambda_2\bar{G}_2^*(\lambda_2)]};$$

(2) The steady-state failure frequency of the system is

$$m_f = \frac{\lambda_1\lambda_2\bar{G}_2^*(\lambda_1)[1+\lambda_1\bar{G}_1^*(\lambda_2)]+\lambda_1\lambda_2\bar{G}_1^*(\lambda_2)[1+\lambda_2\bar{G}_2^*(\lambda_1)]}{(1+\rho_1)[1+\lambda_2\bar{G}_2^*(\lambda_1)]+(1+\rho_2)[1+\lambda_1\bar{G}_1^*(\lambda_2)]-[1+\lambda_1\bar{G}_1^*(\lambda_2)][1+\lambda_2\bar{G}_2^*(\lambda_2)]};$$

(3) The steady-state renewal frequency of the system is

$$m_r = \frac{\lambda_1[1+\lambda_2\bar{G}_2^*(\lambda_1)][1-\lambda_1\bar{G}_1^*(\lambda_2)]+\lambda_2[1+\lambda_1\bar{G}_1^*(\lambda_2)][1-\lambda_2\bar{G}_2^*(\lambda_1)]}{(1+\rho_1)[1+\lambda_2\bar{G}_2^*(\lambda_1)]+(1+\rho_2)[1+\lambda_1\bar{G}_1^*(\lambda_2)]-[1+\lambda_1\bar{G}_1^*(\lambda_2)][1+\lambda_2\bar{G}_2^*(\lambda_2)]}.$$

Proof: This is staightforward by directly using of the formulas in Theorems 4.1, 4.2 and 4.3.

5 Reliability of the System

In order to determine the reliability of the system, we assume that $\lambda_2(t) = \lambda_2$, $\lambda_3(t) = \lambda_3$ and the uptime of unit 1 is generally distributed. Now, we consider the system in a similar way as in Section 2, but with a different Markov process. At first, we define $*$ to be an absorbing state representing the first time to failure of the system. Let $S_r(t)$ be the state of the system at time t before its first failure. Obviously, the state space is $J_r = \{*, 0, 1, 2\}$, where state 0 represents the state that the three units are working, state 1 (2) represents the state that unit 1 (2) is under repair and the other two units are working.

Since X_1 and Y_i $(i = 1, 2, 3)$ are all general continuous random variables, the stochastic process $S_r(t)$ is not a Markov process. However, it can be extended to a vector Markov process by the same idea as that in [2] or [5]. To do this, denote by $X_1(t)$ the elapsed uptime of machine 1 at time t , and by $Y_i(t)$ the elapsed repair time of machine i $(i = 1, 2)$ at time t. it is then easy to see that $\{(S_r(t), X_1(t), Y_1(t), Y_2(t)), t \geq 0\}$ forms a Markov process taking values on

$$J_r^* = \{(0, x), (1, y), (2, x, z) | 0 \leq x, y, z < \infty\}.$$

The state transition diagram of the system among states can be shown as in Figure 2.

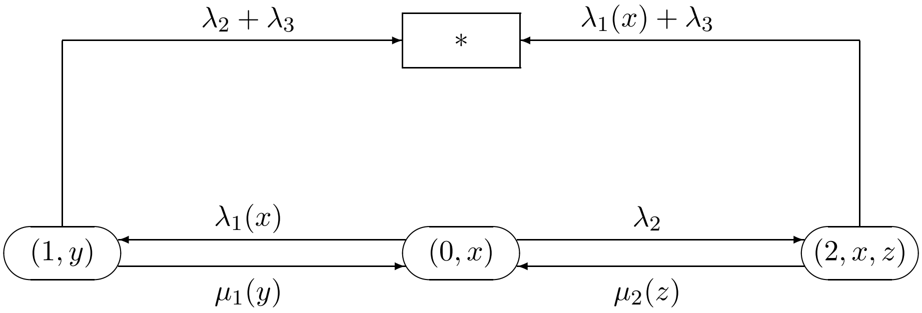

Figure 2: The state transition diagram.

Now, define:

$$\begin{aligned} P_0(t,x)dx &= P\{S_r(t)=0, x \le X_1(t) < x+dx | S_r(0)=0, X_1(0)=0\} \ , \\ P_1(t,y)dy &= P\{S_r(t)=1, y \le Y_1(t) < y+dy | S_r(0)=0, X_1(0)=0\} \ , \\ P_2(t,x,z)dxdz &= P\{S_r(t)=2, x \le X_1(t) < x+dx, z \le Y_2(t) < z+dz \\ & \qquad |S_r(0)=0, X_1(0)=0\}, \quad x > z. \end{aligned}$$

By using a standard probability argument again, we can derive the following differential equations:

$$\left[\frac{\partial}{\partial t} + \frac{\partial}{\partial x} + \lambda_1(x) + \lambda_2 + \lambda_3\right] P_0(t,x) = \int_0^x P_2(t,x,z)\mu_2(z)dz \ ; \tag{5.25}$$

$$\left[\frac{\partial}{\partial t} + \frac{\partial}{\partial y} + \mu_1(y) + \lambda_2 + \lambda_3\right] P_1(t,y) = 0 \ ; \tag{5.26}$$

$$\left[\frac{\partial}{\partial t} + \frac{\partial}{\partial x} + \frac{\partial}{\partial z} + \lambda_1(x) + \mu_2(z) + \lambda_3\right] P_2(t,x,z) = 0 \ , \ x > z \ ; \tag{5.27}$$

$$P_0(t,0) = \int_0^\infty P_1(t,y)\mu_1(y)dy \ ; \tag{5.28}$$

$$P_1(t,0) = \int_0^\infty P_0(t,x)\lambda_1(x)dx \ ; \tag{5.29}$$

$$P_2(t,x,0) = \lambda_2 P_0(t,x) \quad ; \tag{5.30}$$

with the initial condition $P_0(0,x) = \delta(t)$, where $\delta(x)$ is the Dirac delta function.

From (5.25), (5.26) and (5.27), as well as the initial condition, we have

$$P_1^*(s,y) = P_1^*(s,0)e^{-(s+\lambda_2+\lambda_3)y}\bar{G}_1(y); \tag{5.31}$$

$$P_2^*(s,x,z) = \bar{F}_1(x)\bar{G}_2(z)e^{-(s+\lambda_3)x}H(s,x-z); \tag{5.32}$$

$$P_0^*(s,x) = e^{-(s+\lambda_2+\lambda_3)x}\bar{F}_1(x)\left[\int_0^x e^{\lambda_2 z}H(s,z) * g_2(z)dz + U(x) + P_0^*(s,0)\right]; \tag{5.33}$$

where $H(s,z)$ is an undetermined function, and $U(x)$ is one if $x>0$ and zero otherwise. In order to determine the above function, we first introduce the following result.

Lemma 5.1

$$H(s,z)=\lambda_2[U(z)+P_0^*(s,0)]D_3(z).$$

Proof: By equations (5.30), (5.32) and (5.33), we know that

$$H(s,x)=e^{-\lambda_2 x}\lambda_2\left[\int_0^x e^{\lambda_2 z}H(s,z)*g_2(z)dz+U(x)+P_0^*(s,0)\right]. \tag{5.34}$$

By taking the Laplace transform of above equation on x, we have:

$$H^*(s,\eta)=\lambda_2\left[\frac{1}{\lambda_2+\eta}H^*(s,\eta)g_2^*(\eta)+\frac{1}{\lambda_2+\eta}\left(1+P_0^*(s,0)\right)\right],$$

i.e.,

$$H^*(s,\eta)=\lambda_2[1+P_0^*(s,0)]\left[\lambda_2+\eta-\lambda_2 g_2^*(\eta)\right]^{-1}=\lambda_2[1+P_0^*(s,0)]\left(\eta[1+\lambda_2\bar{G}_2^*(\eta)]\right)^{-1}.$$

Therefore, by noticing $H(s,0)=\lambda_2 P_0^*(s,0)$, Lemma 5.1 is proved.

■

Next, we determine $P_0^*(s,0)$ and $P_1^*(s,0)$ in equations (5.31) and (5.33).

Lemma 5.2

$$P_0^*(s,0) = N_1(s)g_1^*(s+\lambda_2+\lambda_3)\left[1-N_1(s)g_1^*(s+\lambda_2+\lambda_3)\right]^{-1}, \tag{5.35}$$

$$P_1^*(s,0) = N_1(s)\left[1-N_1(s)g_1^*(s+\lambda_2+\lambda_3)\right]^{-1}. \tag{5.36}$$

Proof: Substitute (5.31) and (5.33) into (5.28) and (5.29) respectively, and by using Lemma 5.1, we have

$$P_0^*(s,0)=P_1^*(s,0)g_1^*(s+\lambda_2+\lambda_3),$$

and

$$\begin{aligned} P_1^*(s,0) &= \int_0^\infty e^{-(s+\lambda_2+\lambda_3)x}f_1(x)\left[\int_0^x e^{\lambda_2 z}g_2(z)*H(s,z)dz+U(x)+P_0^*(s,0)\right]dx \\ &= [1+P_0^*(s,0)]N_1(s). \end{aligned}$$

Thus, the result is clear.

■

Now, we can derive the reliability of the system as follows.

Theorem 5.3 *The Laplace transform of the system's reliability, R(t), is given by*

$$R^*(s) = \frac{\bar{G}_1^*(s+\lambda_2+\lambda_3)N_1(s) + \lambda_2 \int_0^\infty e^{-(s+\lambda_3)t}\bar{F}_1(t)\bar{G}_2(t) * D_3(t)dt + N_2(s)}{1 - N_1(s)g_1^*(s+\lambda_2+\lambda_3)}$$

and the mean time to system failure is

$$MTSF = \frac{\bar{G}_1^*(\lambda_2+\lambda_3)N_1(0) + \lambda_2 \int_0^\infty e^{-\lambda_3 t}\bar{F}_1(t)\bar{G}_2(t) * D_3(t)dt + N_2(0)}{1 - N_1(0)g_1^*(\lambda_2+\lambda_3)}.$$

Proof: According to the definition of $R(t)$, we know

$$R(t) = \int_0^\infty P_0(t,x)dx + \int_0^\infty P_1(t,y)dy + \int_0^\infty \int_0^x P_2(t,x,z)dxdz$$

and

$$MTSF = \int_0^\infty R(t)dt = R^*(0).$$

Thus by using (5.31), (5.32), (5.33) and Lemma 5.1, as well as Lemma 5.2, we can readily get the results.

■

Corollary 5.4 *If* $\lambda_1(t) = \lambda_1$, *then*

$$R^*(s) = \frac{1 + \lambda_1\bar{G}_1^*(s+\lambda_2+\lambda_3) + \lambda_2\bar{G}_2^*(s+\lambda_1+\lambda_3)}{(s+\lambda_3) + \lambda_1(s+\lambda_2+\lambda_3)\bar{G}_1^*(s+\lambda_2+\lambda_3) + \lambda_2(s+\lambda_1+\lambda_3)\bar{G}_2^*(s+\lambda_1+\lambda_3)}$$

and

$$MTSF = \frac{1 + \lambda_1\bar{G}_1^*(\lambda_2+\lambda_3) + \lambda_2\bar{G}_2^*(\lambda_1+\lambda_3)}{\lambda_3 + \lambda_1(\lambda_2+\lambda_3)\bar{G}_1^*(\lambda_2+\lambda_3) + \lambda_2(\lambda_1+\lambda_3)\bar{G}_2^*(\lambda_1+\lambda_3)}.$$

Proof: Notice that for this case, $N_1(s) = \lambda_1 D_3^*(s+\lambda_1+\lambda_3)$, $N_2(s) = D_3^*(s+\lambda_1+\lambda_3)$, and $D_3^*(s) = [s(1+\lambda_2\bar{G}_2^*(s)]^{-1}$. Using Theorem 5.3 and simple calculations, we can obtain the result.

■

Remark 5.1: Since unit 1 and unit 2 possess a symmetric position in the system, if $\lambda_1(t) = \lambda_1$ and $\lambda_2(t) \neq$ constant, we can obtain a similar result to that in Theorem 5.3.

6 Conclusion

When we are investigating the availability performance and/or the system's reliability of some reliability model, we are often faced some undetermined functions arrising from using the supplementary variables method (SVM) [11], [13] and [14]. How to specify the undetermined functions is gradually becoming interesting and important in the analysis of stochastic models. Refer to [11], [13] and [14] for some interesting problems and open problems.

In this paper, for lighting how to obtain the undetermined functions when the SVM is used, we considered a three-dissimilar-unit model with two different repair facilities. The model discussed can be deemed as an extension of a two-unit parellel system, which is one of important models we often encounter in reliability applications and is also a difficult one to analyze if there are many random variables with general distributions involved (see [11]). Here, by decomposing an undetermined function into a product of two independent functions, we obtain the solution of the system succesfully. Furthermore, we obtain explicit expressions for some main availability measures of the system and the system's reliability.

There is no general method to deal with this kind of problems and in most cases it is almost impossible to derive explict results from the equations (see [13] and [14]). However, by using the idea in this paper, we may obtain the explicit expressions of undetermined function for a number of models.

Finally, we remark on fact that the recent developement in studying numerical inversions of Laplace transforms makes numerical solutions of our results possible, and some of them become trivial, for example, see [3] .

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