A CONSTRUCTIVE METHOD FOR FINDING β-INVARIANT MEASURES FOR TRANSITION MATRICES OF $M/G/1$ TYPE

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In this paper, we study the transition matrix of $M/G/1$ type. The radius of convergence is discussed, conditions on the α-classification of the states are obtained, and expressions of the β-invariant measure are constructed. The censoring technique is generalized to deal with nonnegative matrices, which may be neither stochastic nor substochastic. This allows us to prove a factorization result for the discounted transition matrix. This factorization provides a unified algorithmic approach for expressing the β-invariant measure for transition matrices with a block-structure, including the matrix of $M/G/1$ type.

Keywords: β-invariant measures, duality, factorizations, $M/G/1$ type, quasi-stationary distributions, radius of convergence.

1 Introduction

We consider an irreducible aperiodic Markov chain $\{X_n; n = 1, 2, \dots\}$ of $M/G/1$ type, whose transition matrix P is partitioned into block-form:

$$P = \begin{bmatrix} D_1 & D_2 & D_3 & D_4 & \cdots \\ D_0 & C_1 & C_2 & C_3 & \cdots \\ & C_0 & C_1 & C_2 & \cdots \\ & & C_0 & C_1 & \cdots \\ & & & \ddots & \ddots \end{bmatrix}, \tag{1}$$

where D_1 is a matrix of size $m_0 \times m_0$, all C_i are square matrices of finite size m, the sizes of the other block-entries are determined accordingly and all empty entries are zero. P is assumed to be stochastic or strictly substochastic.

By strictly substochastic, we mean that $P \geq 0$, $Pe \leq e$ and $Pe \neq e$, where e is a column vector of ones.

The state space of the above block-partitioned Markov chain can be expressed as $S = \cup_{i=0}^{\infty} L_i$, where $L_0 = \{(0,j); j = 0,1,2,\ldots,m_0\}$ and $L_i = \{(i,j); j = 0,1,2,\ldots,m\}$ for $i \geq 1$. In state (i,j), variable i is called the *level* and variable j, the *phase*. Therefore, L_i is the set of all states at level i. For convenience, we write $L_{\leq i} = \cup_{k=0}^{i} L_k$ and $L_{\geq i}$ for the complement of $L_{\leq(i-1)}$.

Let α be the radius of convergence for the transition matrix $P = (p_{(i,r),(j,s)})$. We know that $\alpha = \sup\{z > 0; \sum_{n=0}^{\infty} z^n p_{(i,r),(j,s)}^{(n)} < \infty\} \geq 1$, where $p_{(i,r),(j,s)}^{(n)}$ is the n-step transition probability and α is independent of states (i,r) and (j,s).

A nonnegative nonzero row vector π is said to be an invariant measure of P if $\pi = \pi P$. For $0 < \beta \leq \alpha$, a nonnegative nonzero row vector π is said to be a β-invariant measure of P if $\pi = \pi \beta P$. Call βP the discounted transition matrix at rate β. Then, a β-invariant measure is simply an invariant measure of the discounted matrix. It follows from the definition that a 1-invariant measure is simply an invariant measure.

For the transition matrix P of $M/G/1$ type, we are interested in

a) the radius of convergence α;

b) the α-classification of the process if $\alpha > 1$; and

c) β-invariant measures for $0 < \beta \leq \alpha$.

There are a number of reasons why the above items are of interest.

1) It is well-known that $\pi = (\pi_i)$ is a quasistationary distribution if and only if for some $\beta > 1$ π is a β-invariant measure satisfying $\sum_i \pi_i < \infty$. The study of quasistationary behavior of a Markov chain is not only theoretically important, but also finds interesting and important applications in many areas, including biology (Scheffer 1951, Holling 1973, Pakes 1987 and Pollett 1987), chemistry (Oppenheim, Shuler and Weiss 1977, Parsons and Pollett 1987 and Pollett 1988), and telecommunications (Schrijner 1995), among others.

2) When the entries π_i in π cannot be summed, the concept of the β-invariant measure is a generalization of invariant measures for a nonergodic chain (Derman 1955, Harris 1957, Latouche, Pearce and Taylor 1998, Gail, Hantler and Taylor 1998, Zhao, Li and Braun 1998). In this case, π can still be interpreted probabilistically in terms of the movement of particles

whose initial states are governed by Poisson distributions (Derman 1955 and Kelly 1983). Also, π can be used to define a time-reversed matrix or dual matrix, which has important applications (Kelly 1979, Ramaswami 1990, Asmussen and Ramaswami 1990, Bright 1996 and Zhao, Li and Alfa 1999).

3) It is well known how important the Perron-Frobenius Theorem is in the theory of finite nonnegative matrices. The decay parameter $\frac{1}{\alpha}$ of P can be considered the Perron-Frobenius eigenvalue of the nonnegative matrix P and an α-invariant measure of P a Perron-Frobenius eigenvector of P.

It is believed that the study of quasistationary behavior was originated by Yaglom (1947). Since then, significant advances in the theory of quasistationarity have been made through the efforts of many researchers. A detailed review on the literature can be found in the Ph.D. dissertation of Schrijner (1995). This study has also been successfully advanced to consider transition matrices with a block-structure since Kijima (1993) made a break through on the determination of the radius of convergence for Markov chains of $GI/M/1$ type and $M/G/1$ type without boundaries. For transition matrices with a block-structure, studies have been centered on obtaining probabilistic measures to express the radius of convergence and quasistationary distributions, including classifications of the states in terms of these measures. People are searching for expressions which are numerically preferable. Results on quasi-birth-and-death (QBD) processes can be found in Kijima (1993), Makimoto (1993), Bean *et al.* (1997), and Bean, Pollett and Taylor (1998, 2000). Some preliminary results on the expressions for the matrices of $GI/M/1$ type and $M/G/1$ type were obtained in Li (1997). A survey on quasistationary distributions of Markov chains arising from queueing processes was given by Kijima and Makimoto (1999).

In this paper, we will study the matrix of $M/G/1$ type with boundary blocks as defined in (1). The issue on the radius of convergence will be addressed by combining a result (see Lemma 5) obtained by Kijima (1993) and the boundary treatment based on censoring. For the case without boundaries, the matrix is always α-transient. With the presence of the boundary, the matrix can be either α-transient or α-recurrent. Conditions on classifications of the transient states will also be discussed in this paper. For the matrix of $M/G/1$ type, we have not noticed the existence of an expression for the β-invariant measure in the literature. We will provide a constructive way of expressing such a measure.

The technique used in this paper to study the radius of convergence and conditions on classifications of the transient states is based on censoring. A censored process is also referred as the imbedded process. For any subset, as the censoring set, of the state space of a process, the censored process is the process obtained by watching the original process only when it travels in the censoring set. This technique has been successfully used in studying many other aspects of block-structured stochastic or strictly substochastic matrices (for example, see Grassmann and Heyman 1990, Latouche 1993, Zhao, Li and Braun 1998, 2001, Zhao, Li and Alfa 1999, Latouche and Ramaswami 1999, and Zhao 2000). In order to use the censoring technique to deal with the issue on the β-invariant measure, we need to generalize results on stochastic or strictly substochastic matrices to that on nonnegative matrices.

What we will use to obtain expressions for the β-invariant measure is the method of factorization, where $I - \beta P$ is factorized into the product of an upper triangular matrix and a lower triangular matrix. We shall call it the RG-factorization, since the factors in the factorization involve the R- and G-measures, two key probabilistic measures in our study, which will be defined later. This factorization may be viewed as an LU-factorization for the infinite matrix $I - \beta P$. The procedure of obtaining a solution for the β-invariant measure can be considered a generalization of using an LU-factorization to solve a finite system of linear equations. Expressions for the β-invariant measure are different according to the classification of the states and the value of β. When we use the factorization technique, it is a key how to associate the middle factor or the diagonal matrix with either the upper triangular or the lower triangular matrix. Our study will give a way to successively identify two different sets of solutions for the β-invariant measure. When $\beta = 1$, an equivalent form of this factorization was obtained and studied by Heyman (1995), Zhao, Li and Braun (1997, 2000) and Zhao (2000). In Li (1997), the matrix $I - \beta P$ was factored into an equivalent form of the RG-factorization without using the R-measure. There are three possible difficulties when using the RG-factorization on infinite matrices. Firstly, the associativity of matrix multiplications cannot be taken for granted, secondly, the existence of a nontrivial solution to a linear system of infinitely many equations cannot be taken for granted, and thirdly, the method of dealing with a recurrent matrix and a transient matrix should be distinguished. When the Markov chain is positive recurrent, these issues have been successfully addressed in the literature, for example, see Heyman (1995). Ramaswami (1988) presented a stable recursion, equivalent to the factorization of Heyman, for the steady state vector for Markov chains of $M/G/1$ type. Also, Meini (1997) studied the matrix of $M/G/1$ type in terms of a method of factorization. For quasistationary distributions, the method employed by Bean, Pollett and Taylor (2000) to the quasi-birth-and-death process is essentially equivalent to the factorization method used in this paper. However, they did not indicate how the expressions for the β-invariant measure are constructed.

It is our belief that the idea presented here can also be used to study other types of block-structured matrices, for example, matrices of $GI/M/1$ type and, more generally, $GI/G/1$ type.

The rest of the paper is organized as follows.

In Section 2, some basic properties about the matrix βP are provided, including properties on the existence of an inverse of $I - \beta P$, the minimal nonnegative inverse and the fundamental matrix. These properties are needed in later sections.

When P is transient, the states of P can be further classified as α-recurrent or α-transient according to $\widehat{\alpha P} = \sum_{k=0}^{\infty} \alpha^k P^k = \infty$ or $< \infty$, respectively. The matrix $\widehat{\alpha P}$ is referred as the fundamental matrix of αP. If P is α-recurrent, either $\lim_{n\to\infty} \alpha^n p^{(n)}_{(i,r),(j,s)} > 0$ for all states (i, r) and (j, s), or $\lim_{n\to\infty} \alpha^n p^{(n)}_{(i,r),(j,s)} = 0$ for all states (i, r) and (j, s). In the former case, P is called α-positive and in the latter case, α-null. In Section 3, we determine the radius α of convergence and the α-classification of the transient states, based on the combination of the result on determining the radius $\bar{\alpha}$ of convergence for the matrix of $M/G/1$ type without boundaries and a new treatment for the boundary.

In Section 4, the RG-factorization for matrix $I - \beta P$ is proved. We show that

$$I - \beta P = [I - R_U(\beta)][I - U_D(\beta)][I - G_L(\beta)],$$

where $R_U(\beta)$ is a block-form upper triangular matrix involving only the R-measure, $G_L(\beta)$ is a block-form lower triangular matrix involving only the G-measure, and $U_D(\beta)$ is a block-form diagonal matrix. The R-measure is a sequence of matrices defined by (15) and (16) and the G-measure for the matrix $M/G/1$ type consists of two matrices defined by (7) and (17). Probabilistic interpretations for both R- and G-measuers are provided after the definition formulas. In this section, we also show that the RG-factorization exists for the matrix of level-dependent $M/G/1$ type.

In Section 5, based on the RG-factorization, expressions for the β-invariant measure are obtained. There are two different sets of expressions: One is for the α-invariant measure when P is α-recurrent. In this case, the α-invariant measure is unique up to a multiple of a positive constant. For all other cases, we provide a common expression for the β-invariant measure. When the β-invariant measure cannot be summed, this uniqueness is no longer guaranteed.

The Final section, Section 6, consists of concluding remarks.

2 Preliminaries

In this section, we provide some properties of the discounted matrix βP, which will be used in later sections. Most of these results can be viewed as generalizations of the counterparts for a stochastic or strictly substochastic matrix. Proofs of these properties may not be obvious. However, since they can be proved either in the same way as that for a stochastic or strictly substochastic matrix or in a similar fashion, we omit most of the proofs. Relevant references are Seneta (1981), Kemeny, Snell and Knapp (1976), Çinlar (1975) among possible others.

A general statement on the existence and uniqueness of an α-invariant measure can be found in the literature, for example Seneta (1981) which is stated in the following lemma. In order to do so, we need the concept of subinvariant measure (or superregular measure). A row vector x is called a subinvariant measure of P if $x \geq xP$. A row vector x is called a β-subinvariant measure of P if $x \geq \beta xP$. A 1-subinvariant measure is simply subinvariant.

Lemma 1 *For irreducible aperiodic matrix P, there always exists a positive α-subinvariant measure x. If P is α-recurrent, then the unique α-subinvariant measure x, up to a multiple of a positive constant, of P is α-invariant and positive.*

The following are some basic properties about the existence of an inverse, minimal nonnegative inverse and the fundamental matrix.

Lemma 2 *(i) For $0 < \beta < \alpha$ if P is α-recurrent, or for $0 < \beta \leq \alpha$ if P is α-transient, $(I - \beta P)$ is invertible. (ii) If $(I - \beta P)$ is invertible, then*

$$\widehat{\beta P} = \sum_{k=0}^{\infty} \beta^k P^k \tag{2}$$

is the minimal nonnegative inverse of $(I - \beta P)$, which is often referred to as the fundamental matrix of βP. (iii) Let P be partitioned into

$$P = \begin{bmatrix} T & H \\ L & Q \end{bmatrix}. \tag{3}$$

Then, both $(I - \beta T)$ and $(I - \beta Q)$ are invertible for $0 < \beta \leq \alpha$.

The following lemma plays an important role in later sections, which will be used to establish a relationship between block-entries of the fundamental matrix $\widehat{\beta P}$.

Lemma 3 *Let P be partitioned as in (3) and let βP be partitioned accordingly* *as*

$$\beta P = \begin{bmatrix} \beta T & \beta H \\ \beta L & \beta Q \end{bmatrix}, \quad 0 < \beta \le \alpha. \tag{4}$$

Assume that $I - \beta P$ is invertible. Then, the minimal nonnegative inverse $\widehat{\beta P}$ of $(I - \beta P)$ is given by

$$\widehat{\beta P} = \begin{bmatrix} (I - \beta T - \beta H \widehat{\beta Q} \beta L)^{-1}_{\min} & (I - \beta T - \beta H \widehat{\beta Q} \beta L)^{-1}_{\min} \beta H \widehat{\beta Q} \\ \widehat{\beta Q} \beta L (I - \beta T - \beta H \widehat{\beta Q} \beta L)^{-1}_{\min} & \widehat{\beta Q} + \widehat{\beta Q} \beta L (I - \beta T - \beta H \widehat{\beta Q} \beta L)^{-1}_{\min} \beta H \widehat{\beta Q} \end{bmatrix} \tag{5}$$

or equivalently,

$$\widehat{\beta P} = \begin{bmatrix} \widehat{\beta T} + \widehat{\beta T} \beta H (I - \beta Q - \beta L \widehat{\beta T} \beta H)^{-1}_{\min} \beta L \widehat{\beta T} & \widehat{\beta T} \beta H (I - \beta Q - \beta L \widehat{\beta T} \beta H)^{-1}_{\min} \\ (I - \beta Q - \beta L \widehat{\beta T} \beta H)^{-1}_{\min} \beta L \widehat{\beta T} & (I - \beta Q - \beta L \widehat{\beta T} \beta H)^{-1}_{\min} \end{bmatrix}, \tag{6}$$

where $(I - X)^{-1}_{\min} = \sum_{i=0}^{\infty} X^i$ is the minimal nonnegative inverse of $I - X$.

Remark 1 *By a sample path argument or the above lemma, we can show that the fundamental matrix is invariant under censoring. Let E be a subset of the state space. Let βP be partitioned according to E and its complement E^c as in (4). And let the fundamental matrix $\widehat{\beta P}$ of βP be expressed as in (3). Then, the fundamental matrix of the censored matrix $(\beta P)^E$ is equal to the block-entry corresponding to the states in E in the fundamental matrix $\widehat{\beta P}$.*

3 Radius of convergence and classification of states

Let α be the radius of convergence for P. If $\alpha = 1$, the classification of states is conventional. So, we are only interested in the classification of states when $\alpha > 1$. This corresponds to a further classification of the transient states. The main purpose of this section is to determine the radius of convergence α and to provide conditions on classification of the states. To pursue that, we first define the matrix $G(\beta)$ which, together with matrix $G_{1,0}(\beta)$ defined in Section 4, is referred to as the G-measure for the transition matrix P of $M/G/1$ type. The main results in this section will be expressed in terms of the G-measure through the analysis of the fundamental matrix and censored matrices $N(\beta)$ and $N_0(\beta)$. By introducing the G-measure, not only can the theoretical analysis be carried out, but it is also computable.

Partition the discounted transition matrix βP of $M/G/1$ type as in (4) with $\beta T = \beta D_1$, and βH, βL and βQ being determined accordingly. Notice that, in the partition, Q is the transition matrix of $M/G/1$ type without boundaries. Let $\widehat{\beta Q} = (\widehat{Q}_{i,j}(\beta))_{i,j=1,2,\ldots}$ be the fundamental matrix for βQ partitioned in blocks, where $\widehat{Q}_{i,j}(\beta)$ is the (i,j)th block. Write $N(\beta) = \widehat{Q}_{1,1}(\beta)$.

The matrix $G(\beta)$ is defined by

$$G(\beta) = N(\beta)\beta C_0. \tag{7}$$

$G(\beta)$ is a matrix of size m. The (r,s)th entry of $G(\beta)$ can be interpreted as the total expected discounted reward with rate β induced by hitting state (i,s) upon the process entering $L_{\leq i}$ for the first time, given that the process starts in state $(i+1,r)$.

Remark 2 *Though the matrix $G(\beta)$ is defined as the product of $N(\beta)$ and βC_0, we usually first compute $G(\beta)$ and then determine $N(\beta)$ in terms of $G(\beta)$. To do so, we need the following lemma, that says that all the other block-entries in the first block-column in $\widehat{\beta Q}$ can be explicitly expressed in terms of $N(\beta)$, the $(1,1)$st block-entry in $\widehat{\beta Q}$.*

Lemma 4 *For the fundamental matrix $\widehat{\beta Q} = (\widehat{Q}_{i,j}(\beta))_{i,j=1,2,\ldots}$,*

$$\widehat{Q}_{j,1}(\beta) = G(\beta)^{j-1}N(\beta), \quad j \geq 1. \tag{8}$$

Proof: It follows from (3) in Lemma 3 that

$$(\widehat{Q}_{2,1}(\beta)^T, \widehat{Q}_{3,1}(\beta)^T, \ldots)^T = \widehat{\beta Q}\beta L N(\beta).$$

The repeating structure and the property of skip-free-to-left of the transition matrix βQ leads to

$$(\widehat{Q}_{2,1}(\beta)^T, \widehat{Q}_{3,1}(\beta)^T, \ldots)^T = (N(\beta)^T, \widehat{Q}_{2,1}(\beta)^T, \ldots)^T \beta C_0 N(\beta).$$

The proof is completed by the above recursive expression and repeatedly using $N(\beta)\beta C_0 = G(\beta)$. ∎

For the discounted transition matrix βP of $M/G/1$ type, we partition the fundamental matrix $\widehat{\beta P}$ of βP according to levels. The block-entries of $\widehat{\beta P}$ are denoted by $\widehat{P}_{i,j}(\beta)$. It is clear that to study the radius of convergence and to classify the states, it is sufficient to only consider an arbitrary block-entry in $\widehat{\beta P}$. For the block-structured transition matrix P in (1), partition P according to (3) with $T = D_1$. It suffices to consider the $(1,1)$st block-entry, denoted by $N_0(\beta)$, in $\widehat{\beta P}$. We express $N(\beta)$ in terms of $G(\beta)$ and $N_0(\beta)$ in terms of $N(\beta)$. This will enable us to determine the radius of convergence α and provide conditions for classification of the states.

Theorem 1 *For the transition matrix of $M/G/1$ type, the $(1,1)$st block-entry $N(\beta)$ in $\widehat{\beta Q}$ can be expressed as*

$$N(\beta) = [I - \sum_{k=1}^{\infty} \beta C_k G(\beta)^{k-1}]^{-1}, \tag{9}$$

or $N(\beta)$ is the fundamental matrix for $U(\beta) = \beta \sum_{k=1}^{\infty} C_k G(\beta)^{k-1}$. The $(1,1)$st block-entry $N_0(\beta)$ in $\widehat{\beta P}$ can be expressed as

$$N_0(\beta) = [I - U_0(\beta)]^{-1}, \tag{10}$$

where

$$U_0(\beta) = \beta D_1 + \sum_{k=1}^{\infty} \beta D_{k+1} G(\beta)^{k-1} N(\beta) \beta D_0, \tag{11}$$

or $N_0(\beta)$ is the fundamental matrix for $U_0(\beta)$.

Proof: Apply Lemma 3 to the discounted transition matrix βQ. It follows from (3) that $N(\beta) = \widehat{Q}_{1,1}(\beta)$ is the fundamental matrix for $\beta T + \beta H \widehat{\beta Q} \beta L$. Then,

$$\begin{aligned} \beta T + \beta H \widehat{\beta Q} \beta L &= \beta C_1 + \beta H (\widehat{Q}_{1,1}(\beta), \widehat{Q}_{2,1}(\beta), \dots)^T \beta C_0 \\ &= \beta C_1 + \sum_{k=2}^{\infty} \beta C_k \widehat{Q}_{k-1,1}(\beta) \beta C_0. \end{aligned}$$

Noticing that $N(\beta)\beta C_0 = G(\beta)$ and using Lemma 4 will complete the proof to the first assertion.

To prove the second, apply Lemma 3 to the discounted transition matrix βP. Then, $N_0(\beta)$ is the fundamental matrix for $\beta T + \beta H \widehat{\beta Q} \beta L$, where $\beta T = \beta D_1$, $\beta H = \beta(D_2, D_3, \dots)$, $\widehat{\beta Q}$ is the fundamental matrix of βQ and $\beta L = \beta(D_0, 0, \dots)$. Therefore,

$$U_0(\beta) = \beta D_1 + \sum_{k=2}^{\infty} \beta D_k \widehat{Q}_{k-1,1}(\beta) \beta D_0.$$

The proof is complete by using Lemma 4. ■

Remark 3 *It follows from the definition equation (7) and equation (9) that* $G(\beta)$ *satisfies the following equation:*

$$G(\beta) = \sum_{k=0}^{\infty} \beta C_k G(\beta)^k. \tag{12}$$

We can further prove that $G(\beta)$ *is the minimal nonnegative solution to equation (12).*

The determination of the radius of convergence α and the conditions on classification of the states given below are based on the combination of the classification result for the matrix without boundaries given by Kijima (1993) and the treatment of the boundary. For convenience, we state two results by Kijima here.

For the transition matrix P of $M/G/1$ type in (1) without boundaries, or all $D_k = C_k$ for $k = 0, 1, \ldots,$ Kijima (1993) provided a method for determining the radius of convergence $\bar{\alpha}$ and showed that P is always α-transient.

Lemma 5 (Kijima) *Let* $C^*(z)$ *be defined by*

$$C^*(z) = \sum_{k=0}^{\infty} C_k z^k, \quad 0 \le z < z_0. \tag{13}$$

Let $\chi(z)$ *be the Perron-Frobenius eigenvalue of* $C^*(z)$*. If* $z_0 > 1$*, then there always exists a unique* γ *such that* $\chi(z) \ge \gamma z$ *for all* $0 < z < z_0$*, and there exists some* θ *with* $0 < \theta \le z_0$ *such that* $\chi(\theta) = \theta\gamma$*. If* $\theta = z_0$*, then* $\gamma = \chi(z_0)/z_0$*. Otherwise,* γ *and* θ *can be determined by solving the simultaneous equations*

$$\chi(\theta) = \gamma\theta \quad \text{and} \quad \chi'(\theta) = \gamma. \tag{14}$$

By using this lemma, Kijima was able to show the following result.

Theorem 2 (Kijima) *For the transition matrix* P *of* $M/G/1$ *type without boundaries (*$D_k = C_k$ *for all* $k \ge 0$*), if* γ *is the quantity determined in Lemma 5, then the radius of convergence* $\bar{\alpha}$ *of* P *satisfies* $\bar{\alpha} = 1/\gamma$ *and* P *is* $\bar{\alpha}$*-transient.*

Remark 4 *In fact,* θ *given in the above lemma is the maximal eigenvalue of the* $G(\bar{\alpha})$*. Makimoto (1993) obtained two types of expressions for the quasistationary distributions of the* $PH/PH/c$ *queue in terms of* θ *and* γ*, Li (1997) and Kijima and Makimoto (1999) generalized those results to the matrix of* $GI/M/1$ *type without boundaries.*

Remark 5 *Kijima (1993) also related θ and γ to the mean drift. The fact that the matrix of $M/G/1$ type without boundaries is always $\bar{\alpha}$-transient is independent of the mean drift. However, the matrix with boundaries can be α-transient, α-positive recurrent or α-null recurrent.*

For P of $M/G/1$ type in (1) with boundaries, we can perform the spectral analysis on the censored matrix to level 0 to obtain conditions on classifications of the transient states and a determination of the radius of convergence. The censored matrix can be calculated according to Lemma 3 and Remark 1 as $U_0(\beta)$. However, it seems more convenient to reach this goal by considering the relationship between the censored matrix $U_0(\beta)$ and its fundamental matrix $N_0(\beta)$.

Let $u_0(\beta)$ and $n_0(\beta)$ be the maximal eigenvalues of the censored matrix $U_0(\beta)$ and its fundamental matrix $N_0(\beta)$, respectively. Then $n_0(\beta) = \frac{1}{1-u_0(\beta)}$. It follows from results of linear algebra that the first two statements of the following lemma are true, for example, Seneta (1981), and the other two follow from the definitions of the radius of convergence and $N_0(\beta)$.

Lemma 6 *Let $\bar{\alpha}$ and α be the radii of convergence of Q and P respectively. In i) and ii), assume $0 < \beta \le \bar{\alpha}$.*

i) *Both $u_0(\beta)$ and $n_0(\beta)$ are strictly increasing in β, and*

ii) *$u_0(\beta) < 1$ if and only if $N_0(\beta) < \infty$.*

iii) *$N_0(\beta) < \infty$ if $\beta < \alpha$ and $N_0(\beta) = \infty$ if $\beta > \alpha$.*

iv) *$\alpha \le \bar{\alpha}$.*

The classification of the states is characterized by the following conditions.

Theorem 3

i) *If for all $0 < \beta \le \bar{\alpha}$, $u_0(\beta) < 1$, then $N_0(\bar{\alpha}) < \infty$ and $\alpha = \bar{\alpha}$. Therefore, P is α-transient;*

ii) *If there exists a β^* with $0 < \beta^* \le \bar{\alpha}$ such that $u_0(\beta^*) = 1$, then $\alpha = \beta^*$ and $N_0(\alpha) = \infty$. Therefore, P is α-recurrent.*

Proof: Based on the facts: $n_0(\beta) = \frac{1}{1-u_0(\beta)}$ and $n_0(\beta) < \infty$ if and only if $N_0(\beta) < \infty$, we discuss the following two cases: i) there exists no solution to $1 - u_0(\beta) = 0$ for $0 < \beta \le \bar{\alpha}$, and ii) there exists a solution β^* to $1 - u_0(\beta) = 0$ for $0 < \beta^* \le \bar{\alpha}$. In the first case, $n_0(\bar{\alpha}) < \infty$. Hence $N_0(\bar{\alpha}) < \infty$. Therefore, $\alpha \ge \bar{\alpha}$. This, together with iv) of Lemma 6, implies $\alpha = \bar{\alpha}$. Hence, P is α-transient. In the second case, $n_0(\beta^*) = \infty$, hence there exists at least one infinite entry of $N_0(\beta)$. This leads to $\alpha = \beta^* \leq \bar{\alpha}$. Therefore, P is α-recurrent. This completes the proof. ∎

Remark 6 *Theorem 3 is also a generalization of classifying an irreducible stochastic matrix into either a recurrent or transient matrix based on censoring. For example, P is recurrent if and only if every censored matrix of P is stochastic. Therefore, the maximal eigenvalue of the censored matrix is one, or $u_0(1) = 1$. P is transient if and only if every censored finite matrix of P is strictly substochastic. Therefore, $u_0(1) < 1$ and $\alpha \geq 1$. If we replace $u_0(1)$ mentioned above by $u_0(\alpha)$, we then have the conditions for α-recurrence and α-transience.*

The above result provides a way to classify the states into either α-transient or α-recurrent and to determine the radius of convergence of P. For an α-recurrent P, the following theorem further provides conditions to determine when it is α-positive or α-null.

Theorem 4 *If $\sum_{k=1}^{\infty} kD_k G(\alpha)^{k-1} < \infty$, $\sum_{k=1}^{\infty} kC_k G(\alpha)^{k-1} < \infty$ and $\alpha < \overline{\alpha}$, then the α-recurrent Markov chain is α-positive; otherwise, it is α-null.*

Proof: This proof is long and needs results in Section 5. Therefore, it is given as an Appendix. ∎

Remark 7 *If $\alpha = 1$ and $\overline{\alpha} > 1$, then, the three conditions in Theorem 4 are the same conditions as that in Remark b of Neuts (1989) (pp. 140-141). This is because in this situation, $G(1)$ is stochastic. Therefore, i) $\sum_{k=1}^{\infty} kD_k G(1)^{k-1} < \infty$ if and only if $\sum_{k=1}^{\infty} kD_k < \infty$; ii) $\sum_{k=1}^{\infty} kC_k G(1)^{k-1} < \infty$ if and only if $\sum_{k=1}^{\infty} kC_k < \infty$; and iii) for the recurrent matrix, $\alpha < \overline{\alpha}$ if and only if $I - R^*(1)$ is invertible, which is equivalent to $\rho < 1$.*

Remark 8 *If $D_k = 0$ and $C_k = 0$, $k \geq 3$, then the transition matrix in (1) is a level-independent QBD process with boundary. In this case, Theorem 4 illustrates that the α-recurrent QBD process is α-positive if and only if $\alpha < \overline{\alpha}$. A similar analysis as in Appendix will show that an α-recurrent level-dependent QBD process is α-positive if and only if $\alpha < \overline{\alpha}$. Now, we compare this result with Lemma 15 in Bean, Pollett and Taylor (2000). If $\alpha < \overline{\alpha}$, then since $\alpha < \overline{\alpha} \leq \alpha_2$, where $\overline{\alpha} = \alpha_1$, $N^{(2)}(z)$ is analytic at $z = \alpha$.Thus, $\frac{d}{d\alpha} N^{(2)}(\alpha) = \frac{d}{dz} N^{(2)}(z)_{|z=\alpha} < \infty$. On the contrary, if $\frac{d}{d\alpha} N^{(2)}(\alpha) < \infty$, then $\alpha < \overline{\alpha}$, since $N^{(2)}(\overline{\alpha}) = \infty$ and $N^{(2)}(z)$ is increasing for $1 \leq z < \alpha_2$.*

4 *RG*-factorization

The RG-factorization of $(I-P)$, where P is stochastic or strictly substochastic, is a version of LU-factorization having probabilistic interpretations. This factorization was discussed by Heyman (1995), Zhao, Li and Braun (1997, 2000), and Zhao (2000). Heyman showed how to use this factorization to determine the stationary probability vector of a positive recurrent Markov chain. When studying the quasistationary behavior of transition matrix P of $M/G/1$ type without boundaries, Li (1997) obtained an LU-factorization for $(I-\beta P)$ without using the R-measure defined in this paper.

The RG-factorization of $(I-\beta P)$ can be proved for an arbitrary transition matrix P, with or without a block-structure. However, in this paper, we only concentrate on the transition matrix of $M/G/1$ type defined in (1). We first need to define the R-measure and the matrix $G_{1,0}(\beta)$.

Consider the fundamental matrix $\widehat{\beta Q}$ of βQ. Let the first block-column of $\widehat{\beta Q}$ be $(\widehat{Q}_{1,1}(\beta)^T, \widehat{Q}_{2,1}(\beta)^T, \dots)^T$. The R-measure for the matrix βP in (1) consists of two sequences of matrices $R_{0,k}(\beta)$ and $R_k(\beta)$, $k=1,2,\dots$, defined by

$$R_{0,k}(\beta) = \sum_{l=1}^{\infty} \beta D_{k+l} \widehat{Q}_{l,1}(\beta) \tag{15}$$

and

$$R_k(\beta) = \sum_{l=1}^{\infty} \beta C_{k+l} \widehat{Q}_{l,1}(\beta). \tag{16}$$

The (r,s)th entry of $R_{0,k}(\beta)$ can be interpreted as the total expected discounted reward with rate β induced by all visits to state (k,s) before hitting any state in $L_{\le k-1}$, given that the process starts in state $(0,r)$. Similarly, the (r,s)th entry of $R_k(\beta)$ can be interpreted as the total expected discounted reward with rate β induced by all visits to state $(i+k,s)$ before hitting any state in $L_{\le i+k-1}$, given that the process starts in state (i,r), where $i \ge 1$.

The G-measure for βP of $M/G/1$ type consists of two matrices, $G(\beta)$ as defined in (7) and $G_{1,0}(\beta)$ defined by

$$G_{1,0}(\beta) = \widehat{Q}_{1,1}(\beta)\beta D_0 = N(\beta)\beta D_0. \tag{17}$$

The (r,s)th entry of $G_{1,0}(\beta)$ can be interpreted as the total expected discounted reward with rate β induced by hitting state $(0,s)$ upon the process entering level 0 for the first time, given that the process starts in state $(1,r)$.

Applying Lemma 4 to (15) and (16), the R-measure can then be expressed as

$$R_{0,k}(\beta) = \sum_{i=1}^{\infty} \beta D_{k+i} G(\beta)^{i-1} N(\beta) \tag{18}$$

and

$$R_k(\beta) = \sum_{i=1}^{\infty} \beta C_{k+i} G(\beta)^{i-1} N(\beta) \tag{19}$$

for $k = 1, 2, \ldots$.

Remark 9 *Up to now, we have obtained all components needed in the factorization equation and expressed then in terms of $G(\beta)$ only.*

For the matrix P of $M/G/1$ type with boundaries, the RG-factorization can be stated in the following theorem.

Theorem 5 *For the matrix P of $M/G/1$ type in (1), $I-\beta P$ can be factorized as*

$$I - \beta P = [I - R_U(\beta)][I - U_D(\beta)][I - G_L(\beta)], \tag{20}$$

where

$$[I - R_U(\beta)] = \begin{bmatrix} I & -R_{0,1}(\beta) & -R_{0,2}(\beta) & -R_{0,3}(\beta) & \cdots \\ & I & -R_1(\beta) & -R_2(\beta) & \cdots \\ & & I & -R_1(\beta) & \cdots \\ & & & I & \cdots \\ & & & & \ddots \end{bmatrix}, \tag{21}$$

$U_D(\beta)$ is the diagonal matrix in block-form with the first block-entry on the diagonal equal to $U_0(\beta)$ and all the other diagonal block-entries equal to $U(\beta)$, or $U_D(\beta) =$ diag$(U_0(\beta, U(\beta), U(\beta), \ldots)$, and

$$[I - G_L(\beta)] = \begin{bmatrix} I & & & & \\ -G_{1,0}(\beta) & I & & & \\ & -G(\beta) & I & & \\ & & -G(\beta) & I & \\ & & & \ddots & \ddots \end{bmatrix}. \tag{22}$$

Proof: We only prove the factorization equation for the first block row and first block-column entries. The remaining can be similarly proved.

The entry $(1, 0)$ on the right-hand side is $-[I - U(\beta)]G_{1,0}(\beta)$, which is equal to $-\beta D_0$ from the definition of $G_{1,0}(\beta)$.

The entry $(0,k)$ with $k \geq 1$ on the right-hand side is

$$\begin{aligned}
& - R_{0,k}(\beta)[I - U(\beta)] + R_{0,k+1}(\beta)[I - U(\beta)]G(\beta) \\
=& - \sum_{i=1}^{\infty} \beta D_{i+k} G(\beta)^{i-1} + \sum_{i=1}^{\infty} \beta D_{i+k+1} G(\beta)^{i-1} G(\beta) \\
=& - \beta D_{k+1},
\end{aligned}$$

where the first equality is due to Lemma 4.

Finally, to see that the entry $(0,0)$ on the right-hand side is equal to the corresponding entry on the left-hand side, we have

$$\begin{aligned}
& [I - U_0(\beta)] + R_{0,1}(\beta)[I - U(\beta)]G_{1,0}(\beta) \\
=& [I - U_0(\beta)] + \sum_{i=1}^{\infty} \beta D_{i+1} G(\beta)^{i-1} N(\beta) \beta D_0 \\
=& [I - \beta D_1 - \sum_{k=1}^{\infty} \beta D_{k+1} G(\beta)^{k-1} N(\beta) \beta D_0] + \sum_{i=1}^{\infty} \beta D_{i+1} G(\beta)^{i-1} N(\beta) \beta D_0 \\
=& I - \beta D_1.
\end{aligned}$$

where the first equality is due to Lemma 4 and the second one due to (11). ■

Remark 10 *As we mentioned earlier, the RG-factorization can be obtained for an arbitrary transition matrix P. Therefore, the approach of this paper is still valid for using the RG-factorization to obtain expressions for the β-invariant measure of a level-dependent transition matrix of M/G/1 type.*

5 β-invariant measures

In this section, we use the RG-factorization to obtain β-invariant measures for the transition matrix P of $M/G/1$ type with boundaries, where $0 < \beta \leq \alpha$. Since the RG-factorization is a version of the LU-factorization for a matrix of infinite size, the procedure of obtaining an expression for the β-invariant measure is similar to the Gaussian elimination for solving a finite linear system. We present two sets of expressions, one for an α-recurrent matrix with $\beta = \alpha$ and the other for all the other cases. Since for an α-recurrent matrix, its α-invariant measure is unique up to a multiple of a positive constant, the solution given here is a unique solution up to a multiple of a positive constant. When P is α-transient, the β-invariant measure may not be unique. Examples and remarks will be given.

In the RG-factorization in (20), the three matrices, $[I - R_U(\beta)]$, $[I - U_D(\beta)]$ and $[I - G_L(\beta)]$, are associative. We can also prove that they are associative with any nonnegative vector π, which will lead to solutions for the β-invariant measure.

Lemma 7 *Let P be the transition matrix of $M/G/1$ type and let π be any nonnegative row vector. Then,*

$$\begin{aligned}\pi[I-\beta P] &= \{\pi[I-R_U(\beta)]\}\{[I-U_D(\beta)][I-G_L(\beta)]\} \\ &= \{\pi[I-R_U(\beta)][I-U_D(\beta)]\}[I-G_L(\beta)].\end{aligned}$$

Proof: This is clear, for example, from the sufficient conditions provided in Corollary 1-9 of Kemeny *et al.*. ■

5.1 α-recurrent with $\beta = \alpha$

In this case, we solve $\pi(I-\alpha P)=0$ by two steps. In the first step, we let

$$x = \pi[I-R_U(\alpha)]. \tag{23}$$

If $x=(x_0,x_1,\dots)$ and $\pi=(\pi_0,\pi_1,\dots)$ are partitioned according to levels, then (23) is equivalent to

$$\begin{aligned} x_0 &= \pi_0, \\ x_k &= -\pi_0 R_{0,k}(\alpha) - \sum_{i=1}^{k-1}\pi_i R_{k-i}(\alpha) + \pi_k, \quad k\ge 1. \end{aligned}$$

Expressing π_k in terms of x_k, we have

$$\pi_0 = x_0, \tag{24}$$

$$\pi_k = \pi_0 R_{0,k}(\alpha) + \sum_{i=1}^{k-1}\pi_i R_{k-i}(\alpha) + x_k, \quad k\ge 1. \tag{25}$$

In the second step, we solve

$$x[I-U_D(\alpha)][I-G_L(\alpha)] = 0 \tag{26}$$

for a nontrivial nonnegative x. If such a solution exists, then π given in (24) and (25) will be nonnegative and nonzero. According to Lemma 7, the above π is an α-invariant measure of P and it is unique up to a multiple of a positive constant.

Equation (26) is equivalent to

$$\begin{aligned} &x_0[I-U_0(\alpha)] - x_1[I-U(\alpha)]G_{1,0}(\alpha) = 0, \\ &x_k[I-U(\alpha)] - x_{k+1}[I-U(\alpha)]G(\alpha) = 0, \quad k\ge 1. \end{aligned}$$

Since P is α-recurrent, it follows from Theorem 3 that the maximal eigenvalue of $U_0(\alpha)$ is $u_0(\alpha) = 1$. Therefore, for nonnegative and irreducible $U_0(\alpha)$, there exists a positive x_0 such that

$$x_0[I - U_0(\alpha)] = 0.$$

Hence, $(x_0, 0, 0, \dots)$ is a solution to (26).

Theorem 6 *If P is α-recurrent, then the unique, up to multiplication by a positive constant, α-invariant measure is given by*

$$\pi_0 = x_0, \tag{27}$$

$$\pi_k = \pi_0 R_{0,k}(\alpha) + \sum_{i=1}^{k-1} \pi_i R_{k-i}(\alpha), \tag{28}$$

where x_0 is the unique, up to a multiple of positive constant, solution to $x_0[I - U_0(\alpha)] = 0$.

We may notice that this form of solution is the same as that of the invariant measure for a recurrent Markov chain as obtained using the same procedure in Heyman (1995) or an equivalent method in Ramaswami (1988).

5.2 α-recurrent with $\beta < \alpha$ or α-transient with $\beta \le \alpha$

In this case, we also proceed in two steps, but the matrices are associated differently. In the first step, let

$$y = \pi[I - R_U(\beta)][I - U_D(\beta)]. \tag{29}$$

This is equivalent to

$$y_0 = \pi_0[I - U_0(\beta)],$$
$$y_1 = [-\pi_0 R_{0,1}(\beta) + \pi_1][I - U(\beta)],$$
$$y_k = \left[-\pi_0 R_{0,k}(\beta) - \sum_{i=1}^{k-1} \pi_i R_{k-i}(\beta) + \pi_k\right][I - U(\beta)], \quad k \ge 2.$$

Since both $[I - U_0(\beta)]$ and $[I - U(\beta)]$ are invertible in this case, we can express π_k in terms of y_k:

$$\pi_0 = y_0[I - U_0(\beta)]^{-1}, \tag{30}$$

$$\pi_1 = \pi_0 R_{0,1}(\beta) + y_1[I - U(\beta)]^{-1}, \tag{31}$$

$$\pi_k = \pi_0 R_{0,k}(\beta) + \sum_{i=1}^{k-1} \pi_i R_{k-i}(\beta) + \pi_k[I - U(\beta)]^{-1}, \quad k \ge 2. \tag{32}$$

In the second step, solve

$$y[I - G_L(\beta)] = 0 \tag{33}$$

for nonnegative nonzero y. If such a solution exists, then π calculated by (30), (31) and (32) is nonnegative and nonzero. According to Lemma 7, the above π is a β-invariant measure of P. Though in many cases such a β-invariant measure is unique up to a multiple of a positive constant, in some other cases, it is simply not unique.

Equation (33) is equivalent to

$$\begin{aligned} y_0 - y_1 G_{1,0}(\beta) &= 0, \\ y_k - y_{k+1} G(\beta) &= 0, \quad k \geq 1. \end{aligned}$$

In the following, we construct a nonnegative nonzero solution y to (33). First, we need the following lemma.

Lemma 8 *For every $0 < \beta \leq \alpha$, there exist a $\theta_\beta > 0$ and a nonnegative nonzero vector z such that*

$$\theta_\beta z = z G(\beta). \tag{34}$$

Proof: Since $G(\beta) \geq 0$, the maximal eigenvalue θ_β of $G(\beta)$ is nonnegative. If $\theta_\beta > 0$, then the lemma is proved by choosing z to be the left eigenvector of $G(\beta)$ associated with θ_β.

It follows from Neuts (1989), by using irreducibility of P, that $\theta_1 > 0$. Therefore, $\theta_\beta > 0$ for all $\beta \geq 1$ since $G(\beta)$ is increasing in β.

For $0 < \beta < 1$, the proof also relies on the irreducibility of P. Suppose that there were an s with $0 < s < 1$ such that $\theta_s = 0$. Then, $\theta_\beta = 0$ for all $0 < \beta \leq s$. Therefore, all the eigenvalues of $G(\beta)$, when $0 < \beta \leq s$, would be zero according to the Perron-Frobenius theorem for nonnegative matrices. It follows from the Cayley-Hamilton theorem that

$$G^m(\beta) = 0, \quad \text{for all } 0 < \beta \leq s, \tag{35}$$

where m is the size of matrix $G(\beta)$. On the other hand, according to the probabilistic interpretation of $G^m(\beta)$ and the assumption of irreducibility on P, $G^m(\beta) \neq 0$, which contradicts (35). ■

By using Lemma 8 and letting $y_0 = zG_{1,0}(\beta)$, we can easily check that $y = (y_0, z, z/\theta_\beta, z/\theta_\beta^2, \ldots)$ is a nonnegative nonzero solution to (33). Substituting y into (30), (31) and (32), a β-invariant measure is found.

Theorem 7 *For $\beta < \alpha$ if P is α-recurrent, or for $\beta \leq \alpha$ if P is α-transient, a β-invariant measure of P is given by*

$$\pi_0 = y_0 N_0(\beta), \tag{36}$$

$$\pi_1 = z[N(\beta) + G_{1,0}(\beta) N_0(\beta) R_{0,1}(\beta)], \tag{37}$$

$$\pi_2 = \frac{z}{\theta_\beta} \{N(\beta) + G(\beta)N(\beta)R_1(\beta) \tag{38}$$

$$+G(\beta)G_{1,0}(\beta)N_0(\beta)[R_{0,1}(\beta)R_1(\beta) + R_{0,2}(\beta)]\}, \tag{39}$$

$$\pi_3 = \frac{z}{\theta_\beta^2} \{N(\beta) + G(\beta)N(\beta)R_1(\beta) + G(\beta)^2 N(\beta)[R_1(\beta)^2 + R_2(\beta)] \tag{40}$$

$$+G(\beta)^2 G_{1,0}(\beta)N_0(\beta)[R_{0,1}(\beta)R_2(\beta) + R_{0,1}(\beta)R_1(\beta)^2 \tag{41}$$

$$+R_{0,2}(\beta)R_1(\beta) + R_{0,3}(\beta)]\}$$

$$\cdots\cdots$$

or it can be written as one common expression for $k \geq 1$:

$$\pi_k = \frac{z}{\theta_\beta^{k-1}} \left\{ N(\beta) + \sum_{i=1}^{k-1} G(\beta)^i N(\beta) \sum_{\substack{0 \leq j_1 \leq \cdots \leq j_i \leq i \\ \sum_t j_t = i}} R_{j_1}(\beta) R_{j_2}(\beta) \cdots R_{j_i}(\beta) \right.$$

$$\left. + G(\beta)^{k-1} G_{1,0}(\beta) N_0(\beta) \sum_{i=1}^{k} R_{0,i}(\beta) \sum_{\substack{0 \leq j_1 \leq \cdots \leq j_{k-i} \leq k-i \\ \sum_t j_t = k-i}} R_{j_1}(\beta) R_{j_2}(\beta) \cdots R_{j_{k-i}}(\beta) \right\}, \tag{42}$$

where $R_0(\beta) = I$.

Remark 11 *For a QBD process with $D_i = 0$ and $C_i = 0$ for $i \geq 3$ in (1), $R_i = R_{0,i} = 0$ for $i \geq 2$ and*

$$U_0(\beta) = \beta D_1 + R_{0,1}(\beta)\beta D_0 = \beta D_1 + \beta D_2 G_{1,0}(\beta),$$

where

$$R_{0,1}(\beta) = \beta D_2 N(\beta), \; G_{1,0}(\beta) = N(\beta)\beta D_0.$$

The β-invariant measure is then given as

$$\begin{aligned}\pi_0 =& y_0 N_0,\\ \pi_k =& \frac{z}{\theta_\beta^{k-1}}[N(\beta) + \sum_{i=1}^{k-1} G(\beta)^i N(\beta) R_1(\beta)^i \\ &+ G(\beta)^{k-1} G_{1,0}(\beta) N_0(\beta) R_{0,1}(\beta) R_1(\beta)^{k-1}]. \qquad (43)\end{aligned}$$

The expression (43) is the same as the one provided in Theorem 8 of Bean, Pollett and Taylor (2000). To see this, noting that in Theorem 8 of Bean, Pollett and Taylor (2000) we have the relations: $x_k = z$ *for* $k \geq 1$*,* $\rho_0 = 1$*,* $\rho_n = \theta_\beta$ *for* $n \geq 1$*,* $G^{(1)}(\beta) = G_{1,0}(\beta)$*,* $G^{(l)}(\beta) = G(\beta)$ *for* $l \geq 2$*, and* $R^{(1)}(\beta) = R_{0,1}(\beta)$*,* $R^{(l)}(\beta) = R(\beta)$ *for* $l \geq 2$*. Then, by taking* $l = 0$*,*

$$\begin{aligned} m_k =& x_k \left(\prod_{n=0}^{k-1} \rho_n^{-1}\right) \sum_{v=0}^{k} \left(\left[\prod_{u=0}^{k-v-1} G^{(k-u)}(\beta)\right] N^{(v)}(\beta) \left[\prod_{u=0}^{k-v-1} R^{(v+1+u)}(\beta)\right]\right) \\ =& \frac{1}{\theta_\beta^{k-1}} z \sum_{v=1}^{k} G(\beta)^{k-v} N(\beta) R(\beta)^{k-v} \\ &+ \frac{1}{\theta_\beta^{k-1}} z \left[G(\beta)^{k-1} G_{1,0}(\beta)\right] N_0(\beta) \left[R_{0,1}(\beta) R(\beta)^{k-1}\right] \\ =& \frac{z}{\theta_\beta^{k-1}}[N(\beta) + \sum_{i=1}^{k-1} G(\beta)^i N(\beta) R_1(\beta)^i \\ &+ G(\beta)^{k-1} G_{1,0}(\beta) N_0(\beta) R_{0,1}(\beta) R_1(\beta)^{k-1}]. \end{aligned}$$

We also remark that a QBD process can be treated as a matrix of GI/M/1 type, the same approach used in this paper for a matrix of GI/M/1 type will lead to a different expression of the β*-measure from (43), e.g., see Makimoto (1993), Li (1997) and Bean, Pollett and Taylor (1998). Furthermore, this expression should be equivalent to the expression obtained in (43).*

Remark 12 *For a fixed value of* β*,* $G(\beta)$ *can be effectively computed by a similar computational scheme for the case of* $\beta = 1$*, for example, Ramaswami (1988), Latouche (1994) and Meini (1997). When* $G(\beta)$ *becomes available, other matrices, including* $U(\beta)$*,* $N(\beta)$*,* $U_0(\beta)$*,* $N_0(\beta)$*, and the R-measure can be computed. Finally, the* β*-invariant measure* π_k *can be computed up to a desired index value. A detailed analysis of the computational scheme has been carried out and computational complexity has been analyzed. We omit all the details here. People should notice that significant efforts should be made for the calculation of* $G(\beta)$ *when* $\beta \to \alpha$*. This is because* α

is exactly the point where the underlying series diverges. In other words, it is exactly where the very long sample paths of very low probability get such a large reward that they start to contribute a significant amount. This means that many steps are required and this can involve terms with many exponents multiplied by each other to get terms of reasonable order.

Remark 13 *To see why we need two different sets of expressions for the β-invariant measure, let us consider the scalar case. If P is α-transient, one is not an eigenvalue of $U_0(\alpha)$. Therefore, $x_0[I - U_0(\alpha)] = 0$ only provides the trivial solution. This means that the method used for the case in 5.1 is not valid. If P is α-recurrent, y given in Section 5.2 is zero. In fact, this y cannot satisfy (29) unless $y_0 = 0$. For example, $I - U_0(\alpha) = 1 - 1 = 0$ for the scalar case, which gives $y_0 = 0$.*

While in many cases there exists a unique β-invariant measure up to a multiple of a positive constant, in some other cases, the β-invariant measure is simply not unique. One such example was provided by Gail, Hantler and Taylor (1998).

6 Concluding remarks

In this paper, we considered the matrix of $M/G/1$ type with boundaries. We generalized the censoring technique such that it can be used to deal with the nonnegative matrix βP. Based on the generalized censoring technique, we proposed a method for determining the radius of convergence, we obtained conditions for classifying transient states, and proved a factorization theorem for the matrix $I - \beta P$. This factorization was then used to obtain expressions for the β-invariant measure.

The method developed here can also be used to study the radius of convergence and β-invariant measures for transition matrices with other types of block-structure, such as, for the matrix of $GI/M/1$ type and even for the matrix of $GI/G/1$ type.

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Appendix

In this appendix, we provide a proof to Theorem 4, which follows from the three lemmas provided here. For simplicity, we assume that the matrix $C^*(1)$ is irreducible and stochastic.

This proof is based on a result in Theorem 6.4 in Seneta (1981), which is restated in the following lemma in the block-partitioned form.

Lemma 9 *Suppose $\pi = (\pi_0, \pi_1, \pi_2, \cdots)$ is a β-invariant measure and $v = \left(v_0^T, v_1^T, v_2^T, \cdots\right)^T$ is a β-invariant vector of the transition matrix P, partitioned according to levels. Then, P is α-positive if $\pi v = \sum_{k=0}^{\infty} \pi_i v_i < +\infty$, in which case $\beta = \alpha$, π is (a multiple of) the unique α-invariant measure of P and v is (a multiple of) the unique α-invariant vector of P. Conversely, if P is α-positive, and π and v are respectively an invariant measure and vector, then $\pi v < +\infty$.*

Based on this lemma, besides the α-invariant measure π provided in Theorem 6, we also need to similarly express the α-invariant vector v according to Subsection 5.1. This is given as

$$v_0 = w_0, \quad v_k = G(\alpha)^{k-1} G_{1,0}(\alpha) w_0, \ k \geq 1, \tag{44}$$

where w_0 is the unique, up to a multiplication of a positive constant, solution of $[I - U_0(\alpha)] w_0 = 0$.

For convenience, we express the α-invariant measure explicitly in terms of the R-measure, instead of an iterative expression given in Theorem 6. To do this, Let $\Pi^*(z) = \sum_{k=1}^{\infty} z^k \pi_k$, $R^*(z) = \sum_{k=1}^{\infty} z^k R_k(\alpha)$ and $R_0^*(z) = \sum_{k=1}^{\infty} z^k R_{0,k}(\alpha)$. We denote by $t_k * s_k$ the convolution of two sequences $\{t_k\}$ and $\{s_k\}$, and $t_k^{*n} = t_k * t_k^{*(n-1)}$, $n \geq 2$. It follows from Theorem 6 that

$$\Pi^*(z) = x_0 R_0^*(z) [I - R^*(z)]^{-1} = x_0 R_0^*(z) \sum_{n=0}^{\infty} [R^*(z)]^n,$$

which gives

$$\pi_k = x_0 R_{0,k}(\alpha) * \sum_{n=0}^{\infty} R_k(\alpha)^{*n}, \quad k \geq 1. \tag{45}$$

It follows from (44) and (45) that

$$\sum_{k=0}^{\infty} \pi_i v_i = x_0 v_0 + x_0 \sum_{k=1}^{\infty} \left[R_{0,k} (\alpha) * \sum_{n=0}^{\infty} R_k (\alpha)^{*n} \right] G (\alpha)^{k-1} G_{1,0} (\alpha) v_0. \quad (46)$$

Clearly, $\sum_{k=0}^{\infty} \pi_i v_i < \infty$ if and only if

$$\sum_{k=1}^{\infty} \left[R_{0,k} (\alpha) * \sum_{n=0}^{\infty} R_k (\alpha)^{*n} \right] G (\alpha)^{k-1} < \infty. \quad (47)$$

Let g_α and $H (\alpha)$ be the maximal eigenvalue and the associated right eigenvector of $G (\alpha)$, respectively. Since $C^* (1)$ is irreducible, we have $H (\alpha) > 0$. It follows from (47) that

$$\sum_{k=1}^{\infty} \left[R_{0,k} (\alpha) * \sum_{n=0}^{\infty} R_k (\alpha)^{*n} \right] G (\alpha)^{k-1} H (\alpha) = \frac{1}{g_\alpha} R_0^* (g_\alpha) \sum_{n=0}^{\infty} \left[R^* (g_\alpha) \right]^n H (\alpha) .$$

Then, (47) is true if and only if, i) $R_0^* (g_\alpha) < \infty$, ii) $R^* (g_\alpha) < \infty$, and iii) the matrix $I - R^* (g_\alpha)$ is invertible.

The following lemma provides the conditions under which, i) $R_0^* (g_\alpha) < \infty$ and ii) $R^* (g_\alpha) < \infty$.

Lemma 10 *i)* $R_0^* (g_\alpha) < \infty$ *if and only if* $\sum_{k=1}^{\infty} k D_k G (\alpha)^{k-1} < \infty$. *ii)* $R^* (g_\alpha) < \infty$ *if and only if* $\sum_{k=1}^{\infty} k C_k G (\alpha)^{k-1} < \infty$.

Proof: We only prove i) and ii) can be similarly proved.

It follows from (20) that

$$R_0^* (g_\alpha) = \sum_{k=1}^{\infty} g_\alpha^k R_{0,k} (\alpha) = \sum_{k=1}^{\infty} g_\alpha^k \sum_{i=1}^{\infty} \alpha D_{k+i} G (\alpha)^{i-1} N (\alpha) .$$

Hence we obtain

$$R_0^* (g_\alpha) N (\alpha)^{-1} H (\alpha) = \sum_{k=1}^{\infty} \sum_{i=1}^{\infty} \alpha g_\alpha^{k+i-1} D_{k+i} H (\alpha) = \alpha \sum_{k=1}^{\infty} k g (\alpha)^{k-1} D_k H (\alpha) ,$$

which illustrates that $R_0^* (g_\alpha) < \infty$ if and only if $\sum_{k=1}^{\infty} k g (\alpha)^{k-1} D_k < \infty$, and if and only if $\sum_{k=1}^{\infty} k D_k G (\alpha)^{k-1} < \infty$. ■

In what follows we provide a condition under which, iii) the matrix $I - R^* (g_\alpha)$ is invertible.

For the discounted matrix αP, a similar analysis to that used by Zhao, Li and Braun (2001) to obtain the RG-factorization in Theorem 14 (also see Theorem 11 in Zhao (2000)) leads to

$$zI - \alpha C^* (z) = [I - R^* (z)] [I - U (\alpha)] [zI - G (\alpha)] . \tag{48}$$

Let $\chi (z)$ be the maximal eigenvalue of the matrix $C^* (z)$ for $z > 0$. It is clear that property 7 about $\chi (z)$ in Bean, Pollett and Taylor (1998) (pp. 393-394) also holds for the transition matrix of $M/G/1$type. Noting that the matrix $C^* (1)$ is irreducible and stochastic, then the equation $z = \alpha\chi (z)$ has two different roots in $(0, z_0)$ if $1 \leq \alpha < \overline{\alpha}$, and one root repeated twice in $(0, z_0)$ if $\alpha = \overline{\alpha}$, where z_0 is the radius of convergence of $C^* (z)$. Furthermore, the equation $\det (zI - \alpha C^* (z)) = 0$ has two different roots in $(0, z_0)$ if $1 \leq \alpha < \overline{\alpha}$, and one root repeated twice in $(0, z_0)$ if $\alpha = \overline{\alpha}$.

Lemma 11 *i) If $\alpha < \overline{\alpha}$, then the matrix $I - R^* (g_\alpha)$ is invertible. ii) If $\alpha = \overline{\alpha}$, then the matrix $I - R^* (g_\alpha)$ is singular.*

Proof: i) If $\alpha < \overline{\alpha}$, then the equation $\det (zI - \alpha C^* (z)) = 0$ has two different roots in $(0, z_0)$. Since

$$\begin{aligned} \{0 < z < z_0 : \det (zI - \alpha C^* (z)) = 0\} &= \{0 < z < z_0 : \det (I - R^* (z)) = 0\} \\ &\cup \{0 < z < z_0 : \det (zI - G (\alpha)) = 0\} \end{aligned}$$

and $z = g_\alpha$ is a positive root to the equation $\det (zI - G (\alpha)) = 0$, it is not a positive root to the equation $\det (I - R^* (z)) = 0$. Thus, $I - R^* (g_\alpha)$ is invertible.

ii) If $\alpha = \overline{\alpha}$, then the equation $\det (zI - \alpha C^* (z)) = 0$ has one root repeated twice in $(0, z_0)$. Since $z = g_\alpha$ is a positive and simple root to the equation $\det (zI - G (\alpha)) = 0$, it must be a positive and simple root to the equation $\det (I - R^* (z)) = 0$. Thus, $I - R^* (g_\alpha)$ is singular. ■

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