Censoring Technique in Studying Block-Structured Markov Chains

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Abstract: Markov chains with block-structured transition matrices find many applications in various areas. Such Markov chains are characterized by partitioning the state space into subsets called levels, each level consisting of a number of stages. Examples include Markov chains of $GI/M/1$ type and $M/G/1$ type, and, more generally, Markov chains of Toeplitz type, or $GI/G/1$ type. In the analysis of such Markov chains, a number of properties and measures which relate to transitions among levels play a dominant role, while transitions between stages within the same level are less important. The censoring technique has been frequently used in the literature in studying these measures and properties. In this paper, we use this same technique to study block-structured Markov chains. New results and new proofs on factorizations and convergence of algorithms will be provided.
Keywords: Censored Markov chains, block-structured transition matrices, factorizations, convergence of algorithms.

1 Introduction

In this paper, we consider Markov chains of Toeplitz type or $GI/G/1$ type, whose transition matrix is given by

$$P = \begin{bmatrix} D_0 & D_1 & D_2 & D_3 & \cdots & \cdots \\ D_{-1} & C_0 & C_1 & C_2 & \cdots & \cdots \\ D_{-2} & C_{-1} & C_0 & C_1 & \cdots & \cdots \\ D_{-3} & C_{-2} & C_{-1} & C_0 & \cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}, \tag{1}$$

where all entries are blocks (submatrices) of size $m \times m$. Entries C_i are called repeating blocks and D_i the boundary blocks. Examples include: Quasi-

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birth-and-death (QBD) processes if $D_i = D_{-i} = C_i = C_{-i} = 0$ for $i \geq 2$. Markov chains of $GI/M/1$ type if $D_i = C_i = 0$ for $i \geq 2$. Markov chains of $M/G/1$ type if $D_{-i} = C_{-i} = 0$ for $i \geq 2$. Markov chains of non-skip-free $GI/M/1$ type if $D_i = C_i = 0$ for $i > N$ with N a positive integer. Markov chains of non-skip-free $M/G/1$ type if $D_{-i} = C_{-i} = 0$ for $i > N$ with N a positive integer.

For such a Markov chain, a number of measures and properties which relate to transitions among levels play a dominant role, while transitions between stages within the same level are less important. The censoring technique has been found useful and often used in the literature in studying this type of Markov chains.

There exists a large volume of references on Markov chains with repeating transition blocks. For example, see Neuts (1981, 1989), Latouche and Ramaswami (1999) and the references therein. References on the censoring technique are also plenty. Among them are Kemeny, Snell and Knapp (1966), Freedman (1983), Hajek (1982), Grassmann and Heyman (1990, 1993), Latouche (1993), Zhao, Li and Braun (1998a, 1998b), Zhao, Li and Alfa (1999), and Latouche and Ramaswami (1999). The censoring technique applies not only for the case where all transition blocks have the same size and also for the case where the boundary blocks have different sizes. Moreover, this technique can be used to study Markov chains with other types of block structures, for example, the level-dependent quasi-birth-and-death (LDQBD) processes or Markov chains with repeating block-entries of countable size.

Definition 1 *Consider a discrete-time irreducible Markov chain $\{X_n; n = 1, 2, \ldots\}$ with state space S. Let E be a non-empty subset of S. Suppose that the successive visits of X_n to E take place at time epochs $0 < n_1 < n_2 < \cdots$. Then the process $\{X_t^E = X_{n_t}; t = 1, 2, \ldots\}$ is called the censored process with censoring set E.*

Censored Markov chains are also called restricted, watched or embedded Markov chains.

In the following, we define four probabilistic measures, called the R, G, A and B-measures, that play a dominant role in studying Markov chains of Toeplitz type.

Consider an arbitrary discrete-time Markov chain $\{X_n; n = 1, 2, \ldots\}$ whose state space S is partitioned as $S = \cup_{i=0}^{\infty} L_i$, where $L_i = \{(i, j); j = 1, 2, \ldots, m_i\}$. In a state (i, j), i is called a *level* and j a *phase*. We also write $L_{\leq i} = \cup_{k=0}^{i} L_k$ and $L_{\geq i}$ for the complement of $L_{\leq (i-1)}$. Partition the transition matrix of the Markov chain according to levels as: $P = (P_{i,j})_{i \geq 0, 1 \leq j \leq m_i}$, where $P_{i,j}$ is a matrix of $m_i \times m_j$.

For $0 \le i < j$ or $1 \le i \le j$, $R_{i,j}$ is defined as a matrix of size $m_i \times m_j$ whose (r,s)th entry is the expected number of visits to state (j,s) before hitting any state in $L_{\le(j-1)}$, given that the process starts in state (i,r), or

$$R_{i,j}(r,s) = E[\text{number of visits to } (j,s) \text{ before hitting } L_{\le(j-1)} | X_0 = (i,r)].$$

For $i > j \ge 0$, $G_{i,j}$ is defined as a matrix of size $m_i \times m_j$ whose (r,s)th entry is the probability of hitting state (j,s) when the process enters $L_{\le(i-1)}$ for the first time, given that the process starts in state (i,r), or

$$G_{i,j}(r,s) = P[\text{ hitting } (j,s) \text{ upon entering } L_{\le(i-1)} \text{ for first time } | X_0 = (i,r)].$$

For $i \ge 0$ and $j \ge 0$ with $i \ne j$, $A_{i,j}$ is defined as a matrix of size $m_i \times m_j$ whose (r,s)th entry is the expected number of visits to state (j,s) before hitting any state in level i, given that the process starts in state (i,r), or

$$A_{i,j}(r,s) = E[\text{ number of visits to } (j,s) \text{ before hitting } L_i | X_0 = (i,r)].$$

For $i \ge 0$ and $j \ge 0$, $B_{i,j}$ is defined as a matrix of size $m_i \times m_j$ whose (r,s)th entry is the probability of visiting (or returning if $i = j$) level j for the first time by hitting state (j,s), given that the process starts in state (i,r), or

$$B_{i,j}(r,s) = P[\text{ hitting } (j,s) \text{ upon entering } L_j \text{ for first time } | X_0 = (i,r)].$$

In Section 2, we provide some preliminary results on censoring, including the relationship between these four probabilistic measures. Results in Section 2 will be used to prove factorizations in Section 3 and the convergence of computational algorithms in Section 4. Our main focus is on discussing new results or new proofs.

Throughout the paper, we will only consider discrete-time Markov chains. Parallel results for continuous-time Markov chains can be similarly obtained. In many situations, we will not distinguish a Markov chain and its transition matrix. We will allow a transition matrix to be only sub-stochastic. The transpose of a matrix M is denoted by M^t and the complement of a set E by E^c. For a sequence M_n of matrices, we say $\lim_n M_n = M$ if for every entry there is a limit.

2 The censored Markov chain

The most important property about the censored process is that the censored process is also a Markov chain. For a non-negative matrix M, let $\widehat{M} = \sum_{k=0}^{\infty} M^k$, which is often called the fundamental matrix for M. For an arbitrary non-negative matrix M, entries in the fundamental matrix $\widehat{M}$ might be infinite. However, if M is the transition matrix, either stochastic or sub-stochastic, of a transient Markov chain, then the fundamental matrix is finite.

Theorem 2 *Let P be the transition probability matrix of the Markov chain $\{X_n; n = 1, 2, \ldots\}$ partitioned according to subsets E and E^c:*

$$P = \begin{array}{c c} & \begin{array}{cc} E & E^c \end{array} \\ \begin{array}{c} E \\ E^c \end{array} & \left[\begin{array}{cc} T & U \\ D & Q \end{array} \right] \end{array} . \tag{2}$$

Then, the censored process is a Markov chain and its transition probability matrix is given by

$$P^E = T + U\widehat{Q}D \tag{3}$$

with $\widehat{Q} = \sum_{k=0}^{\infty} Q^k$. If P has a unique stationary distribution $\{x_k\}$, then the stationary distribution $\{x_k^E\}$ of the censored Markov chain is given by

$$x_k^E = \frac{x_k}{\sum_{i \in E} x_i}, \quad k \in E. \tag{4}$$

One may refer to Kemeny, Snell and Knapp (1966) for a proof. In general, entries of $\widehat{Q}$ might be infinite. In that case, it should be understood that $0 \cdot \infty = 0$ in the above theorem.

A probabilistic interpretation for each of the components in the expression for P^E is provided below. Here, $C_{i,j}$ stands for the (i, j)th entry in matrix C.

1. $(\widehat{Q})_{i,j}$ is the expected number of visits to state $j \in E^c$ before entering E given that the process started in state $i \in E^c$.

2. $(U\widehat{Q})_{i,j}$ is the expected number of visits to state $j \in E^c$ before returning to E given that the process started in state $i \in E$.

3. $(\widehat{Q}D)_{i,j}$ is the probability that the process leaves E and upon entering E the first state visited is $j \in E$, given that the process started in state $i \in E^c$.

4. $(U\widehat{Q}D)_{i,j}$ is the probability that upon returning to E the first state visited is $j \in E$, given that the process started in state $i \in E$.

5. $(T + U\widehat{Q}D)_{i,j}$ is the probability that the next state visited in E is j given that the process started in state $i \in E$.

Freedman (1983) studied the convergence of censored Markov chains to the original Markov chain. One of the consequences of his study is

Lemma 3 *Let P be a transition matrix with state space S and let E_n for $n = 1, 2, \ldots$ be a sequence of subsets of S such that $E_n \subseteq E_{n+1}$ and $lim_{n\to\infty} E_n = S$. Then, for any $i, j \in E_n$, $\lim_{n\to\infty} P_{i,j}^{E_n} = P_{i,j}$.*

It follows from the definition that the censored Markov chain has a clear sample path structure. One consequence from this sample path structure is that the R, G, A and B-measures defined above are all invariant under censoring. More specifically, for an arbitrary block-partitioned transition matrix P, let $R_{i,j}^{(n)}$ and $G_{i,j}^{(n)}$ be the R and G-measures, respectively, defined for the censored Markov chain with censoring set $L_{\leq n}$. Then, given $0 = i < j$ or $0 \leq i \leq j$, $R_{i,j}^{(n)} = R_{i,j}$ for all $n \geq j$, and given $0 \leq j < i$, $G_{i,j}^{(n)} = G_{i,j}$ for all $n \geq i$. Similarly, let $A_{i,j}^{(n)}$ and $B_{i,j}^{(n)}$ be the A and B-measures, respectively, defined for the censored Markov chain with censoring set $L_{\leq n}$. Then, given $j > 0$, $A_{0,j}^{(n)} = A_{0,j}$ for all $n \geq j$, and given $i > 0$, $B_{i,0}^{(n)} = B_{i,0}$ for all $n \geq i$. One may refer to Zhao, Li and Braun (1998b) for a proof based on censoring.

We mentioned in the Introduction that the R, G, A and B-measures are important in the analysis of block-structured Markov chains. For example, they can be used to express the stationary probability distribution and other interesting measures and to characterize conditions of classification of the states. Numerically, these measures can be efficiently computed. The invariant property under censoring allows us to study the R, G, A and B-measures for a censored Markov chain (usually finite) instead of studying these measures for the original Markov chain (usually infinite).

For an arbitrary block-partitioned matrix P, let Q_n be the southeast corner of P from level n, that is, $Q_n = (P_{i,j})_{i \geq n, 1 \leq j \leq m_i}$. For convenience, the ith block-row and the jth block-column of $\widehat{Q_n}$ are denoted by $\widehat{Q_n}(i, \cdot)$ and $\widehat{Q_n}(\cdot, j)$, respectively. Notice that i and j here do not correspond to level i and level j. We also define $P_{[i]}$ to be the matrix obtained by deleting the ith block-row and ith block-column in matrix P.

The following expressions are useful and can be verified based on probabilistic interpretations. They also apply to finite Markov chains.

Lemma 4 **(a)** *For $0 = i < j$ or $1 \leq i \leq j$,*

$$R_{i,j} = (P_{i,j}, P_{i,j+1}, P_{i,j+2}, \ldots)\widehat{Q_j}(\cdot, 1);$$

(b) *for $i > j \geq 0$,*

$$G_{i,j} = \widehat{Q_i}(1, \cdot)(P_{i,j}, P_{i+1,j}, P_{i+2,j}, \ldots)^t;$$

(c) *for* $i \geq 0$ *and* $j \geq 0$ *with* $i \neq j$,

$$(A_{i,0}, \ldots, A_{i,i-1}, A_{i,i+1}, A_{i,i+2}, \ldots) = (P_{i,0}, \ldots, P_{i,i-1}, P_{i,i+1}, P_{i,i+2}, \ldots)\widehat{P_{[i]}};$$

(d) *for* $i \geq 0$ *and* $j \geq 0$ *with* $i \neq j$,

$$(B_{0,j}, \ldots, B_{j-1,j}, B_{j+1,j}, B_{j+2,j}, \ldots)^t = \widehat{P_{[j]}}(P_{0,j}, \ldots, P_{j-1,j}, P_{j+1,j}, P_{j+2,j}, \ldots)^t;$$

(e) *for* $i = j \geq 0$,

$$\begin{aligned} B_{j,j} &= P_{j,j} + (P_{j,0}, P_{j,1}, \ldots, P_{j,j-1}, P_{j,j+1}, P_{j,j+2}, \ldots) \\ &\quad \widehat{P_{[j]}}(P_{0,j}, P_{1,j}, \ldots, P_{i-1,j}, P_{i+1,j}, P_{i+2,j}, \ldots)^t. \end{aligned}$$

These expressions can be significantly simplified for the transition matrix of Toeplitz type. Details are left for readers.

The following equations provide a relationship between probabilistic measures defined above. These equations are a key to develop computational algorithms for the stationary probabilities and to study properties of these measures and the Markov chain itself.

Theorem 5 *For an arbitrary block-partitioned transition matrix* P, *let* $A_{0,0} = P^{(0)}$, *the censored chain with censoring set* L_0. *Then*

$$A_{0,j} = P_{0,j} + \sum_{i=1}^{\infty} A_{0,i} P_{i,j}, \quad j \geq 0, \tag{5}$$

and

$$B_{j,0} = P_{j,0} + \sum_{i=1}^{\infty} P_{j,i} B_{i,0}, \quad j \geq 0. \tag{6}$$

Proof: We only prove the first result; the second one can be proved similarly.

Write P as

$$P = \begin{bmatrix} P_{0,0} & U \\ D & Q \end{bmatrix}.$$

By using (c) of Lemma 4 and the definition of the fundamental matrix $\widehat{Q}$, we have

$$(A_{0,1}, A_{0,2}, \ldots) = U\widehat{Q} = U(I + \widehat{Q}Q) = U + U\widehat{Q}Q = U + (A_{0,1}, A_{0,2}, \ldots)Q$$

which gives (5) for $j \geq 1$. When $j = 0$,

$$P_{0,0}^{(0)} = P_{0,0} + U\widehat{Q}D = P_{0,0} + (A_{0,1}, A_{0,2}, \ldots)D.$$

Therefore, the result is also true when $j = 0$. ∎

The stationary equations in block-form for an arbitrary transition matrix P can be written as

$$x_j = x_0 P_{0,j} + \sum_{i=1}^{\infty} x_i P_{i,j}, \quad j \geq 0, \tag{7}$$

where x_i is a row vector of size m_i. It is clear that if we let $x_i = x_0 A_{0,i}$, then equations in (5) are equivalent to stationary equations. Therefore, $A_{0,i}$ lead to a determination of the stationary probability vectors except x_0.

The following result was obtained by Zhao, Li and Braun (1998b). Here, we provide a new proof.

Theorem 6 *Let the R, G, A and B-measures be defined for an arbitrary transition matrix P. Then, matrices $A_{0,n}$ and $R_{k,n}$ satisfy*

$$A_{0,n} = \begin{cases} R_{0,1}, & \text{if } n = 1, \\ R_{0,n} + \sum_{k=1}^{n-1} A_{0,k} R_{k,n}, & \text{if } n \geq 2, \end{cases} \tag{8}$$

and matrices $B_{n,0}$ and $G_{n,k}$ satisfy

$$B_{n,0} = \begin{cases} G_{1,0}, & \text{if } n = 1, \\ G_{n,0} + \sum_{k=1}^{n-1} G_{n,k} B_{k,0}, & \text{if } n \geq 2. \end{cases} \tag{9}$$

Proof: We only prove the first result; the second one can be proved similarly.

For the transition matrix P, consider the censored Markov chain $P^{(n)} = (P^{(n)}_{i,j})$ with censoring set $L_{\leq n}$. Applying Theorem 5 to $P^{(n)}$ gives us

$$A^{(n)}_{0,n} = P^{(n)}_{0,n} + \sum_{i=1}^{n} A^{(n)}_{0,i} P^{(n)}_{i,n},$$

or

$$A^{(n)}_{0,n}(I - P^{(n)}_{n,n}) = P^{(n)}_{0,n} + \sum_{i=1}^{n-1} A^{(n)}_{0,i} P^{(n)}_{i,n}.$$

Since $P^{(n)}$ is irreducible due to the irreducibility of P, the inverse of $(I - P^{(n)}_{n,n})$ exists. Therefore,

$$A^{(n)}_{0,n} = P^{(n)}_{0,n}(I - P^{(n)}_{n,n})^{-1} + \sum_{i=1}^{n-1} A^{(n)}_{0,i} P^{(n)}_{i,n}(I - P^{(n)}_{n,n})^{-1} = R^{(n)}_{0,n} + \sum_{i=1}^{n-1} A^{(n)}_{0,i} R^{(n)}_{i,n}.$$

Since both A and R-measures are invariant under censoring, the proof is complete now. $\blacksquare$

Another basic property about censoring is the following limit theorem, which was proved recently in Grassmann and Stanford (1999). This theorem is very helpful in dealing with the convergence of computational algorithms. In the literature, the convergence was often treated case by case, though some efforts have been made to unify the treatment. We will discuss the usage of this limiting theorem in a later section.

Theorem 7 *Let $P = (P_{i,j})_{i,j=0,1,\ldots}$ be the transition matrix of a recurrent Markov chain on the non-negative integers. For an integer $\omega \geq 0$, let $P(\omega) = (p_{i,j}(\omega))_{i,j=0,1,\ldots}$ be a matrix such that*

$$p_{i,j}(\omega) = p_{i,j}, \quad \text{for } i, j \leq \omega$$

and $P(\omega)$ is either stochastic or substochastic matrix. For any fixed $n \geq 0$, let $E_n = \{0, 1, \ldots, n\}$ be the censoring set. Then,

$$\lim_{\omega \to \infty} P^{E_n}(\omega) = P^{E_n}. \tag{10}$$

3 Factorizations

In this section, we consider the transition matrix of Toeplitz type defined in (1). By using the repeating property, the expressions in Lemma 4 can be simplified except for the case involving boundary transition blocks. These expressions, together with the fact of $\widehat{Q_1} = \widehat{Q_2} = \widehat{Q_3} = \cdots$, prove that $R_{i,j}$ and $G_{i,j}$ only depend on the difference between i and j, except for $R_{0,j}$ with $j = 1, 2, \ldots$, and $G_{i,0}$ with $i = 1, 2, \ldots$. Thus, we can define

$$R_k = R_{i,j}, \quad \text{for } k = 0, 1, \ldots, \text{ with } k = j - i \text{ and } j \geq i > 0 \tag{11}$$

and

$$G_k = G_{i,j}, \quad \text{for } k = 1, 2, \ldots, \text{ with } k = i - j \text{ and } i > j > 0. \tag{12}$$

We also define $R = \sum_{i=1}^{\infty} R_i$ and $G = \sum_{i=1}^{\infty} G_i$.

In the following, we write $\widehat{Q} = \widehat{Q_i}$. There are two matrix equations, which bridge the R and G-measures, and the repeating transition probability blocks C_i. They can be considered as Wiener-Hopf-type equations. These equations are useful, especially in developing computational algorithms. These two equations were first mentioned in Grassmann and Heyman (1990). We provide a rigorous proof based on censoring. First, we prove a lemma.

Lemma 8 *Let P be the transition matrix of Toeplitz type given in (1) and let $P^{(n)} = (P^{(n)}_{i,j})$ be the block-partitioned transition matrix of the censored Markov chain with censoring set $L_{\leq n}$ with $n \geq 0$. Then,*

$$P^{(n)}_{i,j} = P^{(n+1)}_{i+1,j+1} = \cdots, \quad \textit{for all } i,j = 1,2,\ldots,n.$$

Proof: It follows from the repeating structure of transition blocks in (1) and the expression for the censored Markov chain (3) that

$$P^{(n)}_{n-i,n-j} = C_{i-j} + (C_{i+1}, C_{i+2}, \ldots)\widehat{Q} \begin{pmatrix} C_{-(j+1)} \\ C_{-(j+2)} \\ \vdots \end{pmatrix},$$

which equals $P^{(n+1)}_{(n+1)-i,(n+1)-j}$ for $i, j = 0, 1, \ldots, n-1$. ■

This lemma says that for any two levels $n_1 < n_2$, the southeast corner of the transition matrix $P^{(n_1)}$ corresponding to levels from 1 to n_1 are the same as the southeast corner of the transition matrix $P^{(n_2)}$ of the same size, which corresponds to levels from $n_2 - n_1 + 1$ to n_2. We also notice that for a fixed n, blocks in the transition matrix $P^{(n)}$ for the censored chain are generally no longer repeating. As a special case of the above lemma, for $i = 0, 1, 2, \ldots$, we can define

$$\Phi_i = P^{(n)}_{n-i,n} \quad \text{and} \quad \Phi_{-i} = P^{(n)}_{n,n-i}, \tag{13}$$

where $n > i$.

Corollary 9 *For $i = 0, 1, 2, \ldots$, $R_i = \Phi_i(I - \Phi_0)^{-1}$, and for $j = 0, 1, 2, \ldots$, $G_j = (I - \Phi_0)^{-1}\Phi_{-j}$.*

Proof: Consider the censored Markov chain $P^{(n)}$ with $n > i$. Define the R-measure for $P^{(n)}$ and denote it by $R^{(n)}_{i,j}$, $i = 0, 1, \ldots, n$. For $i = 0, 1, \ldots, n-1$, $R^{(n)}_{n-i,n}$ can be expressed as

$$R^{(n)}_{n-i,n} = P^{(n)}_{n-i,n}(I - P^{(n)}_{n,n})^{-1} = \Phi_i(I - \Phi_0)^{-1}$$

according to the above definition and Lemma 4. It follows from the invariance of the R-measure under censoring that $R_i = R^{(n)}_{n-i,n}$. The other half of the corollary can be proved similarly. ■

Theorem 10 *For the transition matrix of Toeplitz type given in (1),*

$$R_i(I - \Phi_0) = C_i + \sum_{k=1}^{\infty} R_{i+k}(I - \Phi_0)G_k, \quad i \geq 0, \tag{14}$$

$$(I-\Phi_0)G_j = C_{-j} + \sum_{k=1}^{\infty} R_k (I-\Phi_0) G_{j+k}, \quad j \geq 0, \tag{15}$$

and

$$\Phi_0 = C_0 + \sum_{i=1}^{\infty} R_i (I-\Phi_0) G_i. \tag{16}$$

Proof: We only prove the first equation and the other two can be proved similarly.

Based on the censored matrix $P^{(n+1)}$, the transition blocks $P^{(n)}_{n-i,n}$ for the censored Markov chain with censoring set $L_{\leq n}$ can be expressed as

$$\begin{aligned}
P^{(n)}_{n-i,n} &= P^{(n+1)}_{n-i,n} + P^{(n+1)}_{n-i,n+1}(I - P^{(n+1)}_{n+1,n+1})^{-1} P^{(n+1)}_{n+1,n} \\
&= P^{(n+1)}_{n-i,n} + R^{(n+1)}_{i+1} P^{(n+1)}_{n+1,n} \\
&= P^{(n+1)}_{n-i,n} + R_{i+1}(I - P^{(n+1)}_{n+1,n+1})(I - P^{(n+1)}_{n+1,n+1})^{-1} P^{(n+1)}_{n+1,n} \\
&= P^{(n+1)}_{n-i,n} + R_{i+1}(I-\Phi_0)G_1 \\
&= P^{(n+2)}_{n-i,n} + R_{i+1}(I-\Phi_0)G_1 + R_{i+1}(I-\Phi_0)G_2 \\
&= P^{(n+K)}_{n-i,n} + \sum_{k=1}^{K} R_{i+k}(I-\Phi_0)G_k.
\end{aligned}$$

The first equality holds due to (3), the second one due to Lemma 4, the fourth one due to Lemma 4 and (13), and the next two hold by repeating the previous steps. The proof is complete by noticing that $\lim_{K\to\infty} P^{(n+K)}_{n-i,n} = P_{n-i,n}$ (Lemma 3), Corollary 9 and (13). ■

The version for the scalar case was obtained in Grassmann (1985).

For the repeating blocks in the transition matrix (1), we defined characteristic functions: $C(z) = -I + \sum_{i=-\infty}^{\infty} C_i z^i$, $R(z) = -I + \sum_{i=1}^{\infty} R_i z^i$ and $G(z) = -I + \sum_{i=1}^{\infty} G_i z^{-i}$.

Theorem 11 *For the characteristic functions $C(z)$, $R(z)$ and $G(z)$ of the transition matrix P given in (1) with both P and $C = \sum_{i=-\infty}^{\infty} C_i$ stochastic and irreducible,*

$$C(z) = -R(z)(I-\Phi_0)G(z). \tag{17}$$

In the above factorization, $C(z)$ is a Laurent series in matrix form. This factorization is equivalent to the Wiener-Hopf-type factorization. Each of the two factorizations has its own advantages and disadvantages. Both of them can be used to develop numerical algorithms for computing R and G-measures. The Laurent series factorization was considered in Zhao, Li and Braun (1998a, 1998b), which can be conveniently used in spectral analysis of the transition matrix of Toeplitz type. An equivalent form of the factorization to (17) was obtained in Gail, Hantler and Taylor (1997) for Markov chains of non-skip-free $GI/M/1$ type and $M/G/1$ type. However, only one of the two factors in the factorization was probabilistically interpreted by them. Notice that the above factorizations provide us with a method to study non-boundary behaviour. For a complete study, the boundary blocks have to be included. One of the methods is to prove a similar Wiener-Hopf-type equations for the boundary blocks, which leads to a factorization of matrix $I - P$. Let $\Psi_0 = P^{(0)}$, the censored Markov chain with censoring set L_0.

Theorem 12 *For the transition matrix of Toeplitz type,*

$$R_{0,k}(I - \Phi_0) = D_k + \sum_{i=1}^{\infty} R_{0,i+k}(I - \Phi_0)G_i, \quad k \geq 1, \tag{18}$$

$$(I - \Phi_0)G_{k,0} = D_{-k} + \sum_{j=1}^{\infty} R_j(I - \Phi_0)G_{j+k,0}, \quad k \geq 1, \tag{19}$$

and

$$\Psi_0 = D_0 + \sum_{i=1}^{\infty} R_{0,i}(I - \Phi_0)G_{i,0}. \tag{20}$$

Combining the both sets of Wiener-Hopf-type equations for the repeating blocks and for the boundary blocks, we can directly verify the following factorization.

Theorem 13 *For the transition matrix P of Toeplitz type, the following factorization holds:*

$$(I - P) = (I - R_U)(I - P_D)(I - G_L), \tag{21}$$

where $P_D = \mathrm{diag}(\Psi_0, \Phi_0, \Phi_0, \ldots)$ is the block diagonal matrix with block entries Ψ_0, Φ_0, Φ_0, $\ldots$,

$$R_U = \begin{bmatrix} 0 & R_{0,1} & R_{0,2} & R_{0,3} & \cdots \\ & 0 & R_1 & R_2 & \cdots \\ & & 0 & R_1 & \cdots \\ & & & 0 & \cdots \\ & & & & \ddots \end{bmatrix} \tag{22}$$

and

$$G_L = \begin{bmatrix} 0 & & & & \\ G_{1,0} & 0 & & & \\ G_{2,0} & G_1 & 0 & & \\ G_{3,0} & G_2 & G_1 & 0 & \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}. \tag{23}$$

The above factorization is called the RG-factorization, which is a version of UL-factorization for an infinite matrix. This type of factorization was proved by Heyman (1985) for the recurrent case. Our presentation of the factorization is slightly different from that in Heyman (1985) and is also valid for the transient case. This factorization can be used to obtain expressions for the stationary probability vectors in terms of the R and G-measures.

4 Convergence of algorithms

We will only discuss convergence of certain algorithms. It is also possible to deal with the convergence of other algorithms if one can modify the limiting theorem, Theorem 7, mentioned earlier. From the construction of $P(\omega)$ in Theorem 7, we can find that it does not include some of the often used augmentations as its special cases. For example, the main diagonal augmentation and many of the linear augmentations are not covered by this theorem. However, it does include several important approximations, including the northwest corner matrix approximation, Zhao, Braun and Li (1999), which can be used to approximate both recurrent and non-recurrent Markov chains; the last block-column augmentation, Li and Zhao (1998), which is the best in the sense of stochastic block-monotonicity; and of course the censored matrix, Zhao and Liu (1996), which is the best in the sense of l_1 norm. We first start with a discussion on convergence of an algorithm for computing $R = R_1$ matrix and $G = G_1$ matrix for the transition matrix of a QBD process. Then, we discuss the convergence of algorithms for $GI/M/1$ type and $M/G/1$ type.

Recall that for a stochastic or substochastic matrix Q, the fundamental matrix of Q is $\widehat{Q} = \sum_{k=0}^{\infty} Q^k$. It is possible that an entry of $\widehat{Q}$ is infinite. In this case, $0 \times \infty = 0$ as defined earlier. If the size of matrix Q is finite and $Q < \infty$, then $\widehat{Q} = (I - Q)^{-1}$. In practical situations, we often choose an initial matrix such that the fundamental matrix is finite. This is equivalent to the resulting censored Markov chain being irreducible. For more information, one may refer to Remark 8.3.2 of Latouche and Ramaswami (1999).

Theorem 14 *For the transition matrix P of a recurrent QBD process,*

$$Y_k = C_0 + C_1 \widehat{Y}_{k-1} C_{-1}$$

converges to $Y = P_{n,n}^{(n)} = \Phi_0$ as $k \to \infty$ for an initial non-negative matrix Y_0 such that $C_{-1} + Y_0$ is either a stochastic or substochastic matrix. Therefore, $R^{(k)} = C_1 \widehat{Y}_k$ converges to R and $G^{(k)} = \widehat{Y}_k C_{-1}$ to G as $k \to \infty$. Here, $\widehat{Y}_i$ is the fundamental matrix of Y_i.

Proof: In order to use Theorem 7, for $\omega \geq 1$, define block-partitioned transition matrix $P(\omega) = (P_{i,j}(\omega))$ as follows: All $P_{i,j}(\omega) = 0$ if $i > \omega + 1$ or $j > \omega + 1$ and all $P_{i,j}(\omega)$ are the same as that in P if $i \leq \omega + 1$ and $j \leq \omega + 1$, except that $P_{\omega+1,\omega+1}(\omega) = Y_0$. It follows from Theorem 7 that for a fixed n,

$$\lim_{\omega \to \infty} P^{(n)}(\omega) = P^{(n)}.$$

Let $k = \omega - n$. Because of Lemma 8, the only thing we need to show is that the farthest element, denoted as Y_k, in the southeast corner of $P^{(n)}(\omega)$ can be expressed as

$$Y_k = C_0 + C_1 \widehat{Y}_{k-1} C_{-1}.$$

If $k = 1$, it is clear from (3) that

$$Y_1 = C_0 + C_1 \widehat{Y}_0 C_{-1}.$$

For $k > 1$, censor $P(\omega)$ using the censoring set $L_{\leq \omega}$ first and then censor the resulting process level by level until the censoring set equals $L_{\leq n}$. It is clear that $P^{(n)}(\omega) = (\cdots (P^{(\omega)}(\omega))^{(\omega-1)} \cdots)^{(n)}$. The rest of the proof follows from the invariant property under censoring of the R and G-measures and the expressions for the R and G-measures of the censored Markov chain. ■

The algorithm given in the theorem is the same as one of the algorithms discussed in Latouche (1993). This theorem was recently proved by Grassmann and Stanford (1999), also based on censoring and Theorem 7. The censoring technique was also used in Grassmann and Stanford (1999) to discuss the Markov chains of $GI/M/1$ type and $M/G/1$ type. Mathematically, our following discussion is essentially equivalent to theirs.

For the transition matrix P of $GI/M/1$ type, consider the censored Markov chain $P^{(n)}$. It is clear from the expression (3) that only the last block-row in $P^{(n)}$ is different from that in P. Thus, the $R = R_1$ matrix has the same expression as that for the QBD process:

$$R = C_1 \widehat{P}^{(n)}_{n,n},$$

where $\widehat{P}^{(n)}_{n,n}$ is the fundamental matrix of $P^{(n)}_{n,n}$. Therefore, it is enough to compute $\Phi_0 = P^{(n)}_{n,n}$ in order to compute R. The G-measure: G_i for $i = 1, 2, \ldots$, can then be computed also.

Theorem 15 *For the transition matrix P of a recurrent Markov chain of $GI/M/1$ type, let*

$$C^{(k-i)}_{-i} = C_{-i} + C_1 \widehat{C}^{(k-i-1)}_0 C^{(k-i-1)}_{-(i+1)}, \quad \textit{for } i = k-1, k-2, \ldots, 1, 0. \tag{24}$$

Then, $C_{-i}^{(k-i)}$ converges to $P_{n,n-i}^{(n)} = \Phi_{-i}$ for $i = 0, 1, \ldots, n-1$, as $k \to \infty$ for an initial non-negative matrix $C_0^{(0)}$ such that $\sum_{i=1}^{\infty} C_{-i} + C_0^{(0)}$ is either a stochastic or substochastic matrix. Therefore, $R^{(k)} = C_1 \widehat{C}_0^{(k)}$ converges to R and $G_i^{(k)} = \widehat{C}_0^{(k)} C_{-i}^{(k-i)}$ to G_i for $i = 1, 2, \ldots, n-1$ as $k \to \infty$. Here, $\widehat{C}_i^{(k)}$ is the fundamental matrix of $C_i^{(k)}$.

Proof: For $\omega \geq 1$, define block-partitioned transition matrix $P(\omega) = (P_{i,j}(\omega))$ as follows: All $P_{i,j}(\omega) = 0$ if $i > \omega + 1$ or $j > \omega + 1$ and all $P_{i,j}(\omega)$ are the same as that in P if $i \leq \omega + 1$ and $j \leq \omega + 1$, except that $P_{\omega+1,\omega+1}(\omega) = C_0^{(0)}$. Let $k = \omega - n$. It is clear that the censored Markov chain $P^{(n)}(\omega)$ can be obtained by censoring $P(\omega)$ using censoring set $L_{\leq \omega}$ first and then censoring the resulting process level by level until that the censoring set equals $L_{\leq n}$. Denote the block-entries, except the first one, in the last block-row in $P^{(\omega+1-k)}(\omega)$ by $C_{k-\omega)}^{(k)}, C_{k+1-\omega}^{(k)}, \ldots, C_0^{(k)}$ for $k = 1, 2, \ldots, \omega + 1 - n$. It follows from Theorem 2 that $C_{-i}^{(k-i)}$ can be expressed as in (24). Therefore, Theorem 7 says that for a fixed n,

$$\lim_{\omega \to \infty} P^{(n)}(\omega) = P^{(n)};$$

or

$$\lim_{\omega \to \infty} C_{-(n-i)}^{(\omega+1-n)} = P_{n,n-i}^{(n)} = \Phi_{-i}, \quad \text{for } i = 0, 1, \ldots, n-1.$$

The last equality follows from Theorem 8. Now, the expressions for R and G_i follow from the invariant property under censoring of the R and G-measures and the expressions for the R and G-measures of the censored Markov chain.

■

As the dual, an algorithm for computing the R and G-measures for a Markov chain of $M/G/1$ type can be also developed based on censoring.

Theorem 16 *For the transition matrix P of a recurrent Markov chain of $M/G/1$ type, let*

$$C_i^{(k-i)} = C_i + C_{i+1}^{(k-i-1)} \widehat{C}_0^{(k-i-1)} C_{-1}, \quad \text{for } i = k-1, k-2, \ldots, 1, 0. \qquad (25)$$

Then, $C_i^{(k-i)}$ converges to $P_{n-i,n}^{(n)} = \Phi_i$ for $i = 0, 1, \ldots, n-1$, as $k \to \infty$ for an initial non-negative matrix $C_0^{(0)}$ such that $C_{-1} + C_0^{(0)}$ is either a stochastic or substochastic matrix. Therefore, $G^{(k)} = \widehat{C}_0^{(k)} C_{-1}$ converges to G and $R_i^{(k)} = C_i^{(k-i)} \widehat{C}_0^{(k)}$ to R_i for $i = 1, 2, \ldots, n-1$ as $k \to \infty$. Here, $\widehat{C}_i^{(k)}$ is the fundamental matrix of $C_i^{(k)}$.

Based on the same technique, algorithms for computing the R and G-measures for a transition matrix of non-skip-free $GI/M/1$ type, non-skip-free $M/G/1$ type or Toeplitz type can also be similarly developed.

5 Concluding remarks

In this paper, we addressed the censoring technique in studying Markov chains of block-repeating transition entries. Since the censoring technique is independent of a structure of the transition matrix, it can be also used to deal with other types of transition matrices; for example, transition matrices of level dependent queues and matrices where the size of a block-entry is countable.

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