# 1 Proof of the Lagrange Inversion Formula

**Theorem 1** *Lagrange Inversion Formula: Suppose $u = u(x)$ is a power series in $x$ satisfying $x = u/\phi(u)$ where $\phi(u)$ is a power series in $u$ with a nonzero constant term. Then for any power series $F(u)$ of $u$, we have*

$$[x^n]F(u(x)) = \frac{1}{n}[u^{n-1}]\left(F'(u)\phi^n(u)\right). \tag{1}$$

**Proof.** Let $\phi(u) = \sum_{k\geq 0} \phi_k u^k$. We first note that $u(x) = \phi_0 x + O(x^2)$ and hence $\sum_{k\geq n} \phi_k u^k$ does not contribute to either side of (1).

Hence we may assume that $\phi(u)$ is a polynomial of $u$, which is analytic everywhere. Similarly we may also assume $F(u)$ is a polynomial in $u$. Since $\phi(0) \neq 0$, $u/\phi(u)$ is analytic at the origin with a nonzero first derivative at 0, by the implicit function theorem, $u(x)$ is also analytic in a small region enclosing the origin. Let $C$ be a small circle enclosing the origin in the $x$-plane and $C'$ be the image of $C$ under the transformation $x = u/\phi(u)$. Using the Cauchy formula, we obtain

$$\begin{aligned}
[x^n]F(u(x)) &= \frac{1}{2\pi i}\int_C \frac{F(u(x))}{x^{n+1}}dx \\
&= \frac{1}{2\pi i}\int_{C'} \frac{F(u)\phi^{n+1}(u)}{u^{n+1}}\left(\phi^{-1}(u) - u\phi^{-2}(u)\right)du \\
&= \frac{1}{2\pi i}\int_{C'} \frac{F(u)\phi^{n}(u)}{u^{n+1}}du - \frac{1}{2\pi i}\int_{C'} \frac{F(u)\phi^{n-1}(u)}{u^{n}}du
\end{aligned}$$

Applying the Cauchy formula again w.r.t. the variable $u$, we obtain

$$[x^n]F(u(x)) = [u^n]F(u)\phi^n(u) - [u^{n-1}]F(u)\phi^{n-1}(u)$$

For any power series $f(u)$, we have

$$[u^n]f(u) = \frac{1}{n}[u^{n-1}]f'(u),$$

and hence

$$\begin{aligned}
[x^n]F(u(x)) &= \frac{1}{n}[u^{n-1}]\left((F'(u)\phi^n(u) + nF(u)\phi^{n-1}(u)\right) - [u^{n-1}]F(u)\phi^{n-1}(u) \\
&= \frac{1}{n}[u^{n-1}]\left((F'(u)\phi^n(u)\right)
\end{aligned}$$

This completes the proof.

# 2 Hayman's Method

**Example 1** *Stirling's formula*

$$\frac{1}{n!} \sim \frac{1}{\sqrt{2\pi n}}\left(\frac{e}{n}\right)^n.$$

*Using Cauchy's formula with the contour $|z| = r$, we have*

$$\frac{1}{n!} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{re^{i\theta}}}{r^n e^{in\theta}} d\theta.$$

*Since*

$$\left| e^{re^{i\theta}} \right| \le e^r, \quad |\theta| \le \pi,$$

*we have*

$$\frac{1}{n!} \le \frac{e^r}{r^n}, \text{ for any } r > 0.$$

*To find a least upper bound, we choose $r$ such that*

$$\frac{d}{dr} \left( \ln(e^r/r^n) \right) = 0, \quad i.e., \ r = n.$$

*Hence*

$$\frac{1}{n!} \le \left( \frac{e}{n} \right)^n.$$

*To obtain a more precise estimate for $1/n!$, we need to estimate the above integral more carefully. To do that we note that $|\exp(ne^{i\theta})| = \exp(n \cos\theta)$ becomes exponentially small when $\theta$ is away from 0. Hence we only need to estimate the integral in a small interval $|\theta| \le \delta$. Expanding $e^{i\theta}$, we obtain*

$$e^{n(e^{i\theta} - i\theta)} = e^{n(1-\theta^2/2+O(\theta^3))} \sim e^{n(1-\theta^2/2)},$$

*provided that $n\delta^3 \to 0$, or $\delta = o(n^{-1/3})$.*

*Hence*

$$\begin{aligned} \frac{1}{n!} &\sim \frac{1}{2\pi} (e/n)^n \int_{-\delta}^{\delta} e^{-n\theta^2/2} d\theta \\ &\sim \frac{1}{2\pi\sqrt{n}} (e/n)^n \int_{-\sqrt{n}\delta}^{\sqrt{n}\delta} e^{-t^2/2} dt \qquad (\theta = t/\sqrt{n}) \\ &\sim \frac{1}{\sqrt{2\pi n}} (e/n)^n, \end{aligned}$$

*provided that $n\delta^2 \to \infty$. We may choose $\delta = n^{-5/12}$.*

Hayman (1954) generalized the above estimate to a class of *Hayman-admissible* functions.

Suppose $f(z)$ is analytic in $|z| < R$, and let

$$a(r) = r \frac{f'(r)}{f(r)}, \quad b(r) = ra'(r).$$

It is called Hayman-admissible if there is a positive number $R_0$ and a function $0 < \delta(r) < \pi$ defined in $R_0 < r < R$ such that

(a) $f(r) > 0$ for $R_0 < r < R$;
(b) $f(re^{i\theta}) \sim f(r)e^{ia(r)\theta - b(r)\theta^2/2}$ as $r \to R$, uniformly for $|\theta| \le \delta(r)$;
(c) $f(re^{i\theta}) = o(f(r)/\sqrt{b(r)})$ as $r \to R$, uniformly for $\delta(r) \le |\theta| \le \pi$.
(d) $b(r) \to \infty$ as $r \to R$.

**Theorem 2 (Hayman)** *Suppose $f(z)$ is Hayman-admissible, $r_n$ satisfies $a(r_n) = n$. Then*

$$[z^n]f(z) \sim \frac{f(r_n)}{\sqrt{2\pi b(r_n)}} r_n^{-n}, \text{ as } n \to \infty.$$

Hayman provided the following large families of admissible functions.

(A) Suppose $P(z)$ is a polynomial with real coefficients, and $[z^n]e^{P(z)} > 0$ for all sufficiently large $n$. Then $e^{P(z)}$ is Hayman admissible in $|z| < \infty$.
(B) Suppose $P(z)$ is a polynomial with real coefficients, and $f(z)$ is Hayman-admissible in $|z| < R$. Then $P(f(z))$ is also Hayman-admissible in $|z| < R$.
(C) If $f(z)$ and $g(z)$ are both Hayman-admissible in $|z| < R$, then $f(z)g(z)$ and $e^{f(z)}$ are also Hayman-admissible in $|z| < R$.

**Example 2** *By (A), $e^z$ is Hayman-admissible with $R = \infty$, and we have $a(r) = r$, $b(r) = r$. Hence $r_n = n$.*

**Example 3** *The EGF for permutations with cycle length at most 2 is $f(z) = e^{z+z^2/2}$. By (A), $f(z)$ is Hayman-admissible with $R = \infty$, and we have $a(r) = r(1+r)$, $b(r) = r(1+2r)$. Hence*

$$\begin{aligned} r_n &= \sqrt{n+1/4} - 1/2 = \sqrt{n}\left(1 + \frac{1}{4n}\right)^{1/2} - 1/2 \\ &= \sqrt{n}\left(1 - \frac{1}{2\sqrt{n}} + \frac{1}{8n} + O\left(n^{-3/2}\right)\right). \end{aligned}$$

*To obtain the asymptotic expression for $[z^n]f(z)$, we need to estimate $r_n$, $\exp(r_n)$, and $r_n^{-n} = \exp(n\ln(1/r_n))$.*

$$\begin{aligned} f(r_n) &\sim e^{n/2+\sqrt{n}/2-1/4}, \\ b(r_n) &\sim 2n, \\ r_n^{-n} &= n^{-n/2}\left(1 - \left(\frac{1}{2\sqrt{n}} - \frac{1}{8n} + O(n^{-3/2})\right)\right)^{-n} \\ &= n^{-n/2}\exp\left(n\ln\left(1 - \left(\frac{1}{2\sqrt{n}} - \frac{1}{8n} + O(n^{-3/2}\right)^{-1}\right)\right) \\ &\sim n^{-n/2}e^{\sqrt{n}/2}. \end{aligned}$$

*Hence*

$$f_n/n! \sim \frac{e^{n/2+\sqrt{n}-1/4}}{2\sqrt{n\pi}} n^{-n/2},$$

*and*

$$f_n \sim \frac{1}{\sqrt{2}} n^{n/2} e^{-n/2+\sqrt{n}-1/4}.$$

*We can also solve $r(1+r) = n$ using Lagrange inversion formula. Since $r \to \infty$ as $n \to \infty$, it is clear that $n \sim r^2$, or $1/r \sim n^{-1/2}$. We can rewrite $r(1+r) = n$ as*

$$\frac{1}{r}(1+1/r)^{-1/2} = n^{-1/2}.$$

*Setting $u = 1/r$ and $t = n^{-1/2}$, we obtain*

$$\frac{u}{(1+u)^{1/2}} = t,$$

*and hence*

$$u = \sum_{k \geq 1} t^k \frac{1}{k}[u^{k-1}](1+u)^{k/2} = \sum_{k \geq 1} \frac{1}{k}\binom{k/2}{k-1} t^k,$$

*or*

$$\frac{1}{r} = n^{-1/2} + \frac{1}{2}n^{-1} + \frac{1}{8}n^{-3/2} + O\left(n^{-2}\right).$$

*This gives*

$$r_n = n^{1/2}\left(1 + (1/2)n^{-1/2} + (1/8)n^{-1} + O(n^{-3/2})\right)^{-1} = n^{1/2}\left(1 - (1/2)n^{-1/2} + (1/8)n^{-1} + O(n^{-3/2})\right),$$

$$\ln\frac{1}{r_n} = \ln n^{-1/2} + \ln\left(1 + (1/2)n^{-1/2} + (1/8)n^{-1} + O(n^{-3/2})\right) = -(1/2)\ln n + (1/2)n^{-1/2} + O\left(n^{-3/2}\right)$$