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SOME PRODUCT-TO-SUM IDENTITIES

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Abstract

The infinite products

$$\prod_{n=1}^{\infty}\left(1+\sqrt{2}q^n+q^{2n}\right)^2, \quad \prod_{n=1}^{\infty}\left(1+\sqrt{3}q^n+q^{2n}\right)^3$$

and

$$\prod_{n=1}^{\infty}\left(1+\frac{(1+\sqrt{5})}{2}q^n+q^{2n}\right)^5$$

are determined.

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1. Introduction

Let $\mathbb{N}=\{1,2,3,\ldots\}$ and $\mathbb{Z}=\{0,\pm1,\pm2,\ldots\}$. Throughout this paper q denotes a complex number satisfying $|q|<1$. For $k\in\mathbb{N}$ we define

$$E_k=E_k(q):=\prod_{n=1}^{\infty}(1-q^{kn}). \tag{1.1}$$

Ramanujan's theta function φ is defined by

$$\varphi(q)=\sum_{n=-\infty}^{\infty}q^{n^2}, \tag{1.2}$$

see for example [2, p. 6]. The following infinite product representation of φ follows from Jacobi's triple product identity [2, p. 10], namely,

$$\varphi(q)=\frac{E_2^5}{E_1^2E_4^2}, \tag{1.3}$$

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see for example [2, p. 11]. As

$$E_1(-q) = \frac{E_2^3}{E_1E_4},$$

changing q to $-q$ in (1.3), we obtain

$$\varphi(-q) = \frac{E_1^2}{E_2}. \tag{1.4}$$

The two-dimensional theta function $a(q)$ of the Borweins [3] is defined by

$$a(q) = \sum_{x,y=-\infty}^{\infty} q^{x^2+xy+y^2}. \tag{1.5}$$

For $a \in \mathbb{Z},\ m \in \mathbb{N}$ and $n \in \mathbb{N}$, we define

$$d_{a,m}(n) = \sum_{\substack{d \mid n \\ d \equiv a \pmod m}} 1. \tag{1.6}$$

The following expansions are well-known:

$$\varphi^2(q) = 1 + 4\sum_{n=1}^{\infty}(d_{1,4}(n) - d_{3,4}(n))q^n, \qquad \text{[2, p. 58]} \tag{1.7}$$

$$\varphi(q)\varphi(q^2) = 1 + 2\sum_{n=1}^{\infty}\big(d_{1,8}(n) + d_{3,8}(n) - d_{5,8}(n) - d_{7,8}(n)\big)q^n, \quad \text{[2, p. 74]} \tag{1.8}$$

$$\varphi(q)\varphi(q^3) = 1 + 2\sum_{n=1}^{\infty}(d_{1,3}(n) - d_{2,3}(n)q^n + 4\sum_{n=1}^{\infty}(d_{1,3}(n) - d_{2,3}(n))q^{4n}, \quad \text{[2, p. 75]} \tag{1.9}$$

$$a(q) = 1 + 6\sum_{n=1}^{\infty}(d_{1,3}(n) - d_{2,3}(n)q^n. \qquad \text{[4, p. 43]} \tag{1.10}$$

From (1.7) we obtain

$$\begin{aligned} \varphi^2(q) &= 1 + 4\sum_{n=1}^{\infty}(d_{1,12}(n) + d_{5,12}(n) + d_{9,12}(n) - d_{3,12}(n) - d_{7,12}(n) - d_{11,12}(n))q^n \\ &= 1 + 4\sum_{n=1}^{\infty}(d_{1,12}(n) + d_{5,12}(n) - d_{7,12}(n) - d_{11,12}(n))q^n \\ &\quad -4\sum_{n=1}^{\infty}(d_{1,4}(n) - d_{3,4}(n))q^{3n} \end{aligned}$$

so that

$$\varphi^2(q) + \varphi^2(q^3) = 2 + 4\sum_{n=1}^{\infty}(d_{1,12}(n) + d_{5,12}(n) - d_{7,12}(n) - d_{11,12}(n))q^n. \tag{1.11}$$

From (1.10) we deduce

$$\begin{aligned} a(q) &= 1+6\sum_{n=1}^{\infty}(d_{1,6}(n)+d_{4,6}(n)-d_{2,6}(n)-d_{5,6}(n))q^n \\ &= 1+6\sum_{n=1}^{\infty}(d_{1,6}(n)-d_{5,6}(n))q^n-6\sum_{n=1}^{\infty}(d_{1,3}(n)-d_{2,3}(n))q^{2n} \end{aligned}$$

so that

$$a(q)+a(q^2)=2+6\sum_{n=1}^{\infty}(d_{1,6}(n)-d_{5,6}(n))q^n. \tag{1.12}$$

Recently Alaca, Alaca, Uygul and Williams [1] determined the number of representations of a positive integer by certain diagonal, sextenary, quadratic forms whose coefficients are 1, 2 and 4. In the course of the proof of an identity needed in the proof of their results, they established the new identity

$$(\sqrt{2}-1)\prod_{n=1}^{\infty}\left(1-\sqrt{2}q^n+q^{2n}\right)^2+(\sqrt{2}+1)\prod_{n=1}^{\infty}\left(1+\sqrt{2}q^n+q^{2n}\right)^2=2\sqrt{2}\frac{E_4^4}{E_1^2E_2E_8}, \tag{1.13}$$

see [1, eq. (3.9), p. 299]. They deduced this result from a theorem about Weierstrass sigma functions, see [9, Example 3, p. 451]. Moreover they proved the identity (1.13) without determining $\prod_{n=1}^{\infty}\left(1+\sqrt{2}q^n+q^{2n}\right)^2$ and $\prod_{n=1}^{\infty}\left(1-\sqrt{2}q^n+q^{2n}\right)^2$ individually. In this paper we evaluate each of the two infinite products $\prod_{n=1}^{\infty}\left(1+\sqrt{2}q^n+q^{2n}\right)^2$ and $\prod_{n=1}^{\infty}\left(1-\sqrt{2}q^n+q^{2n}\right)^2$ in terms of φ, E_1, E_2, E_4 and E_8. We prove

Theorem 1.1. *For $q\in\mathbb{C}$ with $|q|<1$ we have*

$$\prod_{n=1}^{\infty}\left(1+\sqrt{2}q^n+q^{2n}\right)^2=(\sqrt{2}-1)\frac{E_2E_8}{E_1^2E_4}\left(\varphi(q)+\sqrt{2}\varphi(q^2)\right)$$

and

$$\prod_{n=1}^{\infty}\left(1-\sqrt{2}q^n+q^{2n}\right)^2=(-\sqrt{2}-1)\frac{E_2E_8}{E_1^2E_4}\left(\varphi(q)-\sqrt{2}\varphi(q^2)\right).$$

It is easy to check that the evaluations given in Theorem 1.1 satisfy (1.13) as $\varphi(q^2)=E_4^5/(E_2^2E_8^2)$ by (1.3). Moreover, as

$$\prod_{n=1}^{\infty}\left(1+\sqrt{2}q^n+q^{2n}\right)\left(1-\sqrt{2}q^n+q^{2n}\right)=\prod_{n=1}^{\infty}(1+q^{4n})=\frac{E_8}{E_4},$$

multipying the two evaluations in Theorem 1.1 together, we recover the well-known theta function identity [2, p. 72]

$$\varphi^2(q)-2\varphi^2(q^2)=-\frac{E_1^4}{E_2^2}=-\varphi^2(-q).$$

We deduce Theorem 1.1 as a special case of a general product-to-sum identity, which we prove in Section 2, see Proposition 2.1. We also deduce from Proposition 2.1 the analogous results to Theorem 1.1 when $\sqrt{2}$ is replaced by $\sqrt{3}$ and $(1+\sqrt{5})/2$.

Theorem 1.2. *For $q \in \mathbb{C}$ with $|q| < 1$ we have*

$$\prod_{n=1}^{\infty}\left(1+\sqrt{3}q^n+q^{2n}\right)^3 = \frac{(3\sqrt{3}-5)}{4}\frac{E_2^5E_3^3E_{12}^3}{E_1^5E_4^2E_6^6}\left(\varphi^2(q)+9\varphi^2(q^3)+6\sqrt{3}\frac{\varphi^3(-q^6)}{\varphi(-q^2)}\right)$$

and

$$\prod_{n=1}^{\infty}\left(1-\sqrt{3}q^n+q^{2n}\right)^3 = \frac{(-3\sqrt{3}-5)}{4}\frac{E_2^5E_3^3E_{12}^3}{E_1^5E_4^2E_6^6}\left(\varphi^2(q)+9\varphi^2(q^3)-6\sqrt{3}\frac{\varphi^3(-q^6)}{\varphi(-q^2)}\right).$$

Theorem 1.3. *For $q \in \mathbb{C}$ with $|q| < 1$ we have*

$$\prod_{n=1}^{\infty}\left(1+\frac{(1+\sqrt{5})}{2}q^n+q^{2n}\right)^5 = \frac{E_5^3}{E_1^5}\left(1+\frac{(5\sqrt{5}-5)}{2}\sum_{n=1}^{\infty}(d_{1,5}(n)-d_{4,5}(n))q^n + \frac{(35-15\sqrt{5})}{2}\sum_{n=1}^{\infty}(d_{2,5}(n)-d_{3,5}(n))q^n\right)$$

and

$$\prod_{n=1}^{\infty}\left(1+\frac{(1-\sqrt{5})}{2}q^n+q^{2n}\right)^5 = \frac{E_5^3}{E_1^5}\left(1+\frac{(-5\sqrt{5}-5)}{2}\sum_{n=1}^{\infty}(d_{1,5}(n)-d_{4,5}(n))q^n + \frac{(35+15\sqrt{5})}{2}\sum_{n=1}^{\infty}(d_{2,5}(n)-d_{3,5}(n))q^n\right).$$

2. A Product-to-Sum Identity

In this section we prove the product-to-sum formula given in Proposition 2.1. In Section 3 we deduce Theorem 1.1 from Proposition 2.1 by taking $m = 8$. In Section 4 we take $m = 12$ to obtain Theorem 1.2 and in Section 5 we take $m = 5$ to obtain Theorem 1.3.

Proposition 2.1. *Let m be a positive integer with $m \geq 4$. Define*

$$\omega_m := e^{2\pi i/m}, \quad \lambda_m := \omega_m + \omega_m^{-1},$$

$$\Delta(k,m) := 3\omega_m^k - \omega_m^{3k} + \omega_m^{(m-3)k} - 3\omega_m^{(m-1)k}, \quad k \in \mathbb{Z},$$

and

$$c_m := \begin{cases} -\omega_m^3(1-\omega_m^3)\left(1+\omega_m^2+\omega_m^4+\cdots+\omega_m^{m-3}\right)^3 & \text{if } m \text{ is odd}, \\ \dfrac{8}{m^3}\left(1-\omega_m^3\right)\left(1-\omega_m\right)^3 \displaystyle\prod_{r=2}^{m/2-1}\left(1-\omega_m^{2r}\right)^3 & \text{if } m \text{ is even}. \end{cases}$$

Then

$$\prod_{n=1}^{\infty}\frac{\left(1-(\lambda_m^2-2)q^n+q^{2n}\right)^3\left(1-q^n\right)^2}{\left(1-\lambda_m q^n+q^{2n}\right)^3\left(1-(\lambda_m^3-3\lambda_m)q^n+q^{2n}\right)} = 1+c_m\sum_{k=1}^{m-1}\Delta(k,m)\sum_{n=1}^{\infty}d_{k,m}(n)q^n.$$

Proof. Let q be a complex number satisfying $|q|<1$. Let a be a complex variable with $a\neq 0$. We consider the function

$$f(a) := \prod_{n=1}^{\infty} \frac{\left(1-a^2q^n\right)^3\left(1-a^{-2}q^n\right)^3\left(1-q^n\right)^2}{\left(1-aq^n\right)^3\left(1-a^{-1}q^n\right)^3\left(1-a^3q^n\right)\left(1-a^{-3}q^n\right)}. \tag{2.1}$$

As $|q|<1$ and $a\neq 0$ the infinite products

$$\prod_{n=1}^{\infty}(1-bq^n), \quad b=1,a,a^{-1},a^2,a^{-2},a^3,a^{-3},$$

converge (absolutely) as $|b|\sum_{n=1}^{\infty}|q|^n$ converges. Provided $a^3\neq q^n$ for any $n\in\mathbb{Z}\setminus\{0\}$ the infinite products

$$\prod_{n=1}^{\infty}(1-bq^n), \quad b=a,a^{-1},a^3,a^{-3},$$

do not converge to 0. Thus $f(a)$ is an analytic function of $a\in\mathbb{C}\setminus\{0\}$ except for poles at points a where $a^3=q^n$ for some $n\in\mathbb{Z}\setminus\{0\}$.

Further, we define for $a\neq 0$ (and $|q|<1$)

$$F(a) := \frac{(1+a)^3}{1-a^3}\prod_{n=1}^{\infty} \frac{\left(1-a^2q^n\right)^3\left(1-a^{-2}q^n\right)^3\left(1-q^n\right)^2}{\left(1-aq^n\right)^3\left(1-a^{-1}q^n\right)^3\left(1-a^3q^n\right)\left(1-a^{-3}q^n\right)} \tag{2.2}$$

so that

$$F(a) = \frac{(1+a)^3}{1-a^3}f(a). \tag{2.3}$$

A straightforward calculation shows that

$$F(aq) = F(a) \tag{2.4}$$

from which it follows that

$$F(aq^n) = F(a),\ n\in\mathbb{Z}. \tag{2.5}$$

Let ω denote a complex cube root of unity. Simple calculations show that

$$f(1)=f(\omega)=f(\omega^2)=1,\ f(-1)=\prod_{n=1}^{\infty}\frac{\left(1-q^n\right)^8}{\left(1+q^n\right)^8}\neq 0,\ F(-1)=0. \tag{2.6}$$

Fix a cube root $q^{1/3}$ of q. It is easy to deduce from (2.1) that for $r\in\{0,1,2\}$

$$\begin{cases} \lim\limits_{b\to\omega^r q^{1/3}}(1-b^{-3}q)f(b) = \dfrac{1-q}{\left(1+\omega^r q^{1/3}\right)^3}, \\ \lim\limits_{b\to\omega^r q^{-1/3}}(1-b^{3}q)f(b) = \dfrac{1-q}{\left(1+\omega^{-r} q^{1/3}\right)^3}. \end{cases} \tag{2.7}$$

The poles of $F(a)$ arise from the factors $1-a^3$, $1-a^3q^n$ $(n\in\mathbb{N})$ and $1-a^{-3}q^n$ $(n\in\mathbb{N})$. No poles arise from $1-aq^n$ as the factor $\left(1-aq^n\right)^3$ in the denominator of $F(a)$ cancels into the factor $\left(1-a^2q^{2n}\right)^3$ in the numerator. Similarly no poles arise from $1-a^{-1}q^n$. Thus all the poles of $F(a)$ are given by $a^3=q^n$ $(n\in\mathbb{Z})$, that is, by

$$a=\omega^r q^{n/3},\ r\in\{0,1,2\},\ n\in\mathbb{Z}. \tag{2.8}$$

All the poles are simple. Before determining the residue of $F(a)$ at the simple pole $a=\omega^r q^{n/3}$, we note the following limit. For $r\in\{0,1,2\}$ we have

$$\lim_{b\to\omega^r}\frac{b-\omega^r}{1-b^{-3}}=\lim_{b\to\omega^r}\frac{b^3}{\dfrac{b^3-1}{b-\omega^r}}=\frac{\omega^{3r}}{3\omega^{2r}}=\frac{1}{3}\omega^r. \tag{2.9}$$

From (2.9) we deduce

$$\lim_{b\to\omega^r q^{1/3}}\frac{b-\omega^r q^{1/3}}{1-b^{-3}q}=\frac{1}{3}\omega^r q^{1/3} \tag{2.10}$$

and

$$\lim_{b\to\omega^r q^{-1/3}}\frac{b-\omega^r q^{-1/3}}{1-b^3q}=-\frac{1}{3}\omega^r q^{-1/3}. \tag{2.11}$$

For $n\in\mathbb{Z}$ set $n=3k+s$, where $k\in\mathbb{Z}$ and $s\in\{-1,0,1\}$. The residue of $F(a)$ at the simple pole $a=\omega^r q^{n/3}=\omega^r q^{s/3}q^k$ is

$$\begin{aligned}\operatorname{Res}_{a=\omega^r q^{s/3}q^k}F(a) &= \lim_{a\to\omega^r q^{s/3}q^k}(a-\omega^r q^{s/3}q^k)F(a)=\lim_{b\to\omega^r q^{s/3}}(bq^k-\omega^r q^{s/3}q^k)F(bq^k)\\ &= q^k\lim_{b\to\omega^r q^{s/3}}(b-\omega^r q^{s/3})F(b) \qquad \text{(by (2.5))}\\ &= q^k\lim_{b\to\omega^r q^{s/3}}(b-\omega^r q^{s/3})\frac{(1+b)^3}{1-b^3}f(b) \qquad \text{(by (2.3))}.\end{aligned}$$

If $s=0$ the residue is

$$\begin{aligned}q^k\lim_{b\to\omega^r}(b-\omega^r)\frac{(1+b)^3}{1-b^3}f(b) &= q^k(1+\omega^r)^3 f(\omega^r)(-\omega^r)^{-3}\lim_{b\to\omega^r}\frac{b-\omega^r}{1-b^{-3}}\\ &= \frac{-1}{3}q^k(1+\omega^r)^3\omega^r \qquad \text{(by (2.6) and (2.9))}\\ &= \begin{cases}-\dfrac{8}{3}q^k & \text{if } r=0,\\ \dfrac{1}{3}q^k\omega^r & \text{if } r=1,2.\end{cases}\end{aligned}$$

If $s=1$ the residue is

$$q^k \lim_{b\to\omega^r q^{1/3}} (b-\omega^r q^{1/3}) \frac{(1+b)^3}{1-b^3} f(b) = q^k \frac{(1+\omega^r q^{1/3})^3}{1-q} \lim_{b\to\omega^r q^{1/3}} \frac{b-\omega^r q^{1/3}}{1-b^{-3}q}(1-b^{-3}q) f(b)$$

$$= q^k \frac{(1+\omega^r q^{1/3})^3}{1-q} \frac{1}{3}\omega^r q^{1/3} \frac{1-q}{(1+\omega^r q^{1/3})^3} \qquad \text{(by (2.7) and (2.10))}$$

$$= \frac{1}{3} q^{k+1/3}\omega^r.$$

Similarly, if $s=-1$ the residue is $(1/3)q^{k-1/3}\omega^r$ by (2.7) and (2.11). Hence in all three cases we have

$$\text{Res}_{a=\omega^r q^{n/3}} F(a) = \lambda(r,n)\omega^r q^{n/3}, \tag{2.12}$$

where for $r\in\{0,1,2\}$ and $n\in\mathbb{Z}$

$$\lambda(r,n) := \begin{cases} -\dfrac{8}{3} & \text{if } r=0 \text{ and } n\equiv 0 \pmod 3, \\ \dfrac{1}{3} & \text{otherwise.} \end{cases} \tag{2.13}$$

Our next step is to construct a function $G(a)$ from the principal parts of $F(a)$ at the poles $a=\omega^r q^{n/3}$, $r\in\{0,1,2\}$, $n\in\mathbb{Z}$, in such a way that $F(a)-G(a)$ is analytic in $\mathbb{C}$ except possibly at $a=0$. We make use of the simple identities

$$\sum_{r=0}^{2} \frac{\omega^r y}{x-\omega^r y} = \frac{3y^3}{x^3-y^3} \tag{2.14}$$

and

$$\sum_{r=0}^{2} \frac{x}{x-\omega^r} = \frac{3x^3}{x^3-1}. \tag{2.15}$$

We begin by showing that the three infinite series

$$\sum_{n=1}^{\infty} \frac{\lambda(r,n)\omega^r q^{n/3}}{a-\omega^r q^{n/3}},\ r\in\{0,1,2\}, \quad |q|<1,$$

converge absolutely for $a\neq 0$ and $a\neq \omega^r q^{n/3}$ for any $r\in\{0,1,2\}$ and any $n\in\mathbb{N}$. As $a\neq 0$ we have $|a|>0$. Thus as $|q|<1$ there exists a positive integer $N=N(a,q)$ such that

$$0<|q^{1/3}|^n<|a| \quad \text{for all } n\geq N.$$

Hence for $n\geq N$ we have

$$\begin{aligned} \left|\frac{\lambda(r,n)\omega^r q^{n/3}}{a-\omega^r q^{n/3}}\right| &\leq \frac{8}{3}\frac{|q^{1/3}|^n}{|a-\omega^r q^{n/3}|} \qquad \text{(by (2.13))} \\ &\leq \frac{8}{3}\frac{|q^{1/3}|^n}{|a|-|q^{1/3}|^n} \qquad \text{(by the triangle inequality)} \\ &\leq \frac{8}{3}\frac{1}{|a|-|q^{1/3}|^N}|q^{1/3}|^n \end{aligned}$$

and the three series $\sum_{n=N}^{\infty} \lambda(r,n)\omega^r q^{n/3}/(a-\omega^r q^{n/3})$ $(r \in \{0,1,2\})$ converge absolutely by comparison with the series $\sum_{n=N}^{\infty} |q^{1/3}|^n$. Hence, as $a \neq \omega^r q^{n/3}$ for any $r \in \{0,1,2\}$ and any $n \in \mathbb{N}$, the three series

$$\sum_{n=1}^{\infty} \frac{\lambda(r,n)\omega^r q^{n/3}}{a-\omega^r q^{n/3}}, \quad r \in \{0,1,2\},$$

converge absolutely. Thus we can form the sum $G_+(a)$ of the principal parts of $F(a)$ at the poles $a = \omega^r q^{n/3}$, $r \in \{0,1,2\}$, $n \in \mathbb{N}$, namely

$$G_+(a) := \sum_{r=0}^{2} \sum_{n=1}^{\infty} \frac{\lambda(r,n)\omega^r q^{n/3}}{a-\omega^r q^{n/3}}. \tag{2.16}$$

By (2.13) and (2.16) we have

$$G_+(a) = \frac{1}{3} \sum_{n=1}^{\infty} \sum_{r=0}^{2} \frac{\omega^r q^{n/3}}{a-\omega^r q^{n/3}} - 3 \sum_{\substack{n=1 \\ n \equiv 0 \pmod 3}}^{\infty} \frac{q^{n/3}}{a-q^{n/3}}. \tag{2.17}$$

Appealing to (2.14), (2.17) becomes

$$G_+(a) = \sum_{n=1}^{\infty} \frac{q^n}{a^3-q^n} - 3 \sum_{n=1}^{\infty} \frac{q^n}{a-q^n}.$$

Hence

$$G_+(a) = \sum_{n=1}^{\infty} \frac{a^{-3}q^n}{1-a^{-3}q^n} - 3 \sum_{n=1}^{\infty} \frac{a^{-1}q^n}{1-a^{-1}q^n}. \tag{2.18}$$

Next we form the sum of the principal parts of $F(a)$ at the poles $a = \omega^r$, $r \in \{0,1,2\}$, namely

$$\frac{-8/3}{a-1} + \frac{\omega/3}{a-\omega} + \frac{\omega^2/3}{a-\omega^2} = \frac{3}{1-a} - \frac{1}{1-a^3}$$

and define

$$G_0(a) := \frac{3}{1-a} - \frac{1}{1-a^3} - 2$$

so that

$$G_0(a) = \frac{3a}{1-a} - \frac{a^3}{1-a^3}. \tag{2.19}$$

Finally we treat the principal parts of $F(a)$ at the poles $a = \omega^r q^{n/3}$, $r \in \{0,1,2\}$, $n \in -\mathbb{N}$. As each of the three infinite series

$$\sum_{n=-\infty}^{-1} \frac{\lambda(r,n)\omega^r q^{n/3}}{a-\omega^r q^{n/3}}, \ r \in \{0,1,2\}, \ |q| < 1,$$

diverges (since

$$\lim_{\substack{n\to-\infty \\ n\not\equiv 0 \pmod 3}} \frac{\lambda(r,n)\omega^r q^{n/3}}{a-\omega^r q^{n/3}} = \frac{\omega^r}{3} \lim_{\substack{n\to\infty \\ n\not\equiv 0 \pmod 3}} \frac{q^{-n/3}}{a-\omega^r q^{-n/3}}$$
$$= \frac{\omega^r}{3} \lim_{\substack{n\to\infty \\ n\not\equiv 0 \pmod 3}} \frac{1}{aq^{n/3}-\omega^r} = \frac{\omega^r}{3}\frac{1}{0-\omega^r} = -\frac{1}{3} \neq 0\ ),$$

we cannot just take the sum of the principal parts, instead we must modify the sum appropriately. We let

$$G_-(a) := \sum_{r=0}^{2}\sum_{n=-\infty}^{-1}\left(\frac{\lambda(r,n)\omega^r q^{n/3}}{a-\omega^r q^{n/3}} + \lambda(r,n)\right)$$
$$= \sum_{r=0}^{2}\sum_{n=-\infty}^{-1} \frac{\lambda(r,n)a}{a-\omega^r q^{n/3}} = \sum_{r=0}^{2}\sum_{n=1}^{\infty} \frac{\lambda(r,-n)a}{a-\omega^r q^{-n/3}},$$

that is

$$G_-(a) = \sum_{r=0}^{2}\sum_{n=1}^{\infty} \frac{\lambda(r,n)aq^{n/3}}{aq^{n/3}-\omega^r}. \tag{2.20}$$

We show that the three infinite series

$$\sum_{n=1}^{\infty} \frac{\lambda(r,n)aq^{n/3}}{aq^{n/3}-\omega^r},\ r\in\{0,1,2\},\ \ |q|<1,$$

converge absolutely for $a\neq 0$ and $a \neq \omega^r q^{-n/3}$ for any $r\in\{0,1,2\}$ and any $n\in\mathbb{N}$. As $a\neq 0$ we have $1/|a|>0$. Since $|q|<1$ there exists a positive integer $N=N(a,q)$ such that

$$0<|q^{1/3}|^n < \frac{1}{|a|} \ \text{ for all } n\geq N.$$

Hence, for all $n \geq N$, we have

$$\begin{aligned}
\left|\frac{\lambda(r,n)aq^{n/3}}{aq^{n/3}-\omega^r}\right| &\leq \frac{8}{3}\frac{|a|\ |q^{1/3}|^n}{|aq^{n/3}-\omega^r|} && \text{(by (2.13))}\\
&\leq \frac{8}{3}\frac{|a|\ |q^{1/3}|^n}{1-|a|\ |q^{1/3}|^n} && \text{(by the triangle inequality)}\\
&\leq \frac{8}{3}\frac{1}{\left(\frac{1}{|a|}-|q^{1/3}|^N\right)}|q^{1/3}|^n
\end{aligned}$$

and the three series $\sum_{n=N}^{\infty}\lambda(r,n)aq^{n/3}/(aq^{n/3}-\omega^r)$, $r\in\{0,1,2\}$ converge absolutely by comparison with the series $\sum_{n=N}^{\infty}|q^{1/3}|^n$. Hence, as $a\neq\omega^r q^{-n/3}$ for any $r\in\{0,1,2\}$ and any $n\in\mathbb{N}$, the three series

$$\sum_{n=1}^{\infty} \frac{\lambda(r,n)aq^{n/3}}{aq^{n/3}-\omega^r},\quad r\in\{0,1,2\},$$

converge absolutely. By (2.13) and (2.20), we obtain

$$G_-(a) = \sum_{n=1}^{\infty}\sum_{r=0}^{2} \frac{\lambda(r,n)aq^{n/3}}{aq^{n/3}-\omega^r} = \frac{1}{3}\sum_{n=1}^{\infty}\sum_{r=0}^{2}\frac{aq^{n/3}}{aq^{n/3}-\omega^r} - 3\sum_{\substack{n=1 \\ n\equiv 0 \pmod 3}}^{\infty} \frac{aq^{n/3}}{aq^{n/3}-1}.$$

Appealing to (2.15), we deduce

$$G_-(a) = \sum_{n=1}^{\infty}\frac{a^3q^n}{a^3q^n-1} - 3\sum_{n=1}^{\infty}\frac{aq^n}{aq^n-1}$$

so that

$$G_-(a) = 3\sum_{n=1}^{\infty}\frac{aq^n}{1-aq^n} - \sum_{n=1}^{\infty}\frac{a^3q^n}{1-a^3q^n}. \tag{2.21}$$

For $a \neq 0$ we define

$$G(a) := G_+(a) + G_0(a) + G_-(a). \tag{2.22}$$

By (2.18), (2.19) and (2.21), we deduce that

$$\begin{aligned} G(a) \quad &= \frac{3a}{1-a} - \frac{a^3}{1-a^3} + 3\sum_{n=1}^{\infty}\frac{aq^n}{1-aq^n} - 3\sum_{n=1}^{\infty}\frac{a^{-1}q^n}{1-a^{-1}q^n} \\ &\quad - \sum_{n=1}^{\infty}\frac{a^3q^n}{1-a^3q^n} + \sum_{n=1}^{\infty}\frac{a^{-3}q^n}{1-a^{-3}q^n}. \end{aligned} \tag{2.23}$$

From (2.23) we deduce

$$G(-1) = -1 \tag{2.24}$$

and

$$G(aq) = G(a). \tag{2.25}$$

Finally we define

$$H(a) := F(a) - G(a). \tag{2.26}$$

By (2.6), (2.24) and (2.26), we have

$$H(-1) = 1. \tag{2.27}$$

By (2.4), (2.25) and (2.26), we have

$$H(aq) = H(a). \tag{2.28}$$

By the definition of $G(a)$ in terms of sums of the principal parts of $F(a)$ at its poles, it is clear that $H(a)$ is analytic in $\mathbb{C}$ except possibly at $a = 0$. Hence $H(a)$ has a Laurent expansion

$$H(a) = \sum_{n=-\infty}^{\infty} h_n a^n, \tag{2.29}$$

where each h_n depends (at most) upon q but not on a. By (2.28) and (2.29) we have

$$\sum_{n=-\infty}^{\infty} h_n q^n a^n = H(aq) = H(a) = \sum_{n=-\infty}^{\infty} h_n a^n.$$

By the uniqueness of the Laurent expansion, we deduce

$$h_n q^n = h_n, \ \ n \in \mathbb{Z}.$$

Thus, as $|q| < 1$, we deduce

$$h_n = 0, \ \ n \in \mathbb{Z} \setminus \{0\}. \tag{2.30}$$

Hence, by (2.29), (2.30) and (2.27), we have

$$H(a) = h_0 = H(-1) = 1 \tag{2.31}$$

and so, by (2.26) and (2.31), we deduce

$$F(a) = 1 + G(a). \tag{2.32}$$

Appealing to (2.2), (2.23) and (2.32), we obtain for $|q| < 1$, $a \neq 0$ and $a^3 \neq q^n$ for any $n \in \mathbb{Z}$

$$\begin{aligned}
&\frac{(1+a)^3}{1-a^3} \prod_{n=1}^{\infty} \frac{\left(1-a^2q^n\right)^3\left(1-a^{-2}q^n\right)^3\left(1-q^n\right)^2}{\left(1-aq^n\right)^3\left(1-a^{-1}q^n\right)^3\left(1-a^3q^n\right)\left(1-a^{-3}q^n\right)} \\
&\quad = 1 + \frac{3a}{1-a} - \frac{a^3}{1-a^3} + \sum_{n=1}^{\infty}\left(\frac{3aq^n}{1-aq^n} - \frac{3a^{-1}q^n}{1-a^{-1}q^n} - \frac{a^3q^n}{1-a^3q^n} + \frac{a^{-3}q^n}{1-a^{-3}q^n}\right) \\
&\quad = \frac{(1+a)^3}{1-a^3} + \sum_{n=1}^{\infty}\left(\frac{3aq^n}{1-aq^n} - \frac{3a^{-1}q^n}{1-a^{-1}q^n} - \frac{a^3q^n}{1-a^3q^n} + \frac{a^{-3}q^n}{1-a^{-3}q^n}\right).
\end{aligned}$$

Multiplying both sides by

$$\frac{1-a^3}{(1+a)^3} = \frac{(1-a)^3(1-a^3)}{(1-a^2)^3}$$

we obtain

$$\begin{aligned}
&\prod_{n=1}^{\infty} \frac{\left(1-a^2q^n\right)^3\left(1-a^{-2}q^n\right)^3\left(1-q^n\right)^2}{\left(1-aq^n\right)^3\left(1-a^{-1}q^n\right)^3\left(1-a^3q^n\right)\left(1-a^{-3}q^n\right)} \\
&\quad = 1 + \frac{(1-a)^3(1-a^3)}{(1-a^2)^3} \sum_{n=1}^{\infty}\left(\frac{3aq^n}{1-aq^n} - \frac{3a^{-1}q^n}{1-a^{-1}q^n} - \frac{a^3q^n}{1-a^3q^n} + \frac{a^{-3}q^n}{1-a^{-3}q^n}\right),
\end{aligned} \tag{2.33}$$

which is valid for any $q \in \mathbb{C}$ with $|q| < 1$ and any $a \in \mathbb{C}$ with $a \neq 0$, $a \neq \pm 1$ and $a^3 \neq q^n$ for any $n \in \mathbb{Z}$. We now choose $a = \omega_m = e^{2\pi i/m}$, where $m \in \mathbb{N}$ satisfies $m \geq 4$. Clearly $a \neq 0$. The conditions $m \geq 4$ and $|q| < 1$ ensure that $a \neq \pm 1$ and $a^3 \neq q^n$ for any $n \in \mathbb{Z}$. Hence this choice of a satisfies the conditions for the validity of (2.33). Now

$$a + a^{-1} = \omega_m + \omega_m^{-1} = \lambda_m,$$

$$a^2+a^{-2}=(a+a^{-1})^2-2=\lambda_m^2-2,$$
$$a^3+a^{-3}=(a+a^{-1})^3-3(a+a^{-1})=\lambda_m^3-3\lambda_m,$$

so

$$\prod_{n=1}^{\infty}\frac{\left(1-a^2q^n\right)^3\left(1-a^{-2}q^n\right)^3\left(1-q^n\right)^2}{\left(1-aq^n\right)^3\left(1-a^{-1}q^n\right)^3\left(1-a^3q^n\right)\left(1-a^{-3}q^n\right)}$$
$$=\prod_{n=1}^{\infty}\frac{\left(1-(\lambda_m^2-2)q^n+q^{2n}\right)^3\left(1-q^n\right)^2}{\left(1-\lambda_m q^n+q^{2n}\right)^3\left(1-(\lambda_m^3-3\lambda_m)q^n+q^{2n}\right)}.$$

Next, as $|\omega_m^k q^n|=|q|^n<1$ for all $k,n\in\mathbb{N}$, we have

$$\begin{aligned}
&\sum_{n=1}^{\infty}\Big(\frac{3aq^n}{1-aq^n}-\frac{3a^{-1}q^n}{1-a^{-1}q^n}-\frac{a^3q^n}{1-a^3q^n}+\frac{a^{-3}q^n}{1-a^{-3}q^n}\Big)\\
&=\sum_{n=1}^{\infty}\Big(\frac{3\omega_m q^n}{1-\omega_m q^n}-\frac{3\omega_m^{m-1}q^n}{1-\omega_m^{m-1}q^n}-\frac{\omega_m^3q^n}{1-\omega_m^3q^n}+\frac{\omega_m^{m-3}q^n}{1-\omega_m^{m-3}q^n}\Big)\\
&=\sum_{n=1}^{\infty}\sum_{r=1}^{\infty}(3\omega_m^r-3\omega_m^{(m-1)r}-\omega_m^{3r}+\omega_m^{(m-3)r})q^{nr}\\
&=\sum_{N=1}^{\infty}q^N\sum_{r|N}(3\omega_m^r-3\omega_m^{(m-1)r}-\omega_m^{3r}+\omega_m^{(m-3)r})\\
&=\sum_{N=1}^{\infty}q^N\sum_{k=0}^{m-1}\sum_{\substack{r\mid N\\ r\equiv k \text{ (mod m)}}}(3\omega_m^r-3\omega_m^{(m-1)r}-\omega_m^{3r}+\omega_m^{(m-3)r})\\
&=\sum_{k=0}^{m-1}(3\omega_m^k-\omega_m^{3k}+\omega_m^{(m-3)k}-3\omega_m^{(m-1)k})\sum_{N=1}^{\infty}q^N\sum_{\substack{r\mid N\\ r\equiv k \text{ (mod m)}}}1=\sum_{k=1}^{m-1}\Delta(k,m)\sum_{N=1}^{\infty}d_{k,m}(N)q^N.
\end{aligned}$$

Finally, we examine

$$\frac{(1-a)^3(1-a^3)}{(1-a^2)^3}=\frac{(1-\omega_m)^3(1-\omega_m^3)}{(1-\omega_m^2)^3}.$$

If $m\equiv 1 \pmod 2$ then $m\geq 5$ and $m-1$ is an even positive integer greater than or equal to 4, and we have

$$\frac{(1-\omega_m)^3(1-\omega_m^3)}{(1-\omega_m^2)^3}=\frac{(\omega_m^m-\omega_m)^3(1-\omega_m^3)}{(1-\omega_m^2)^3}=-\omega_m^3\frac{(1-\omega_m^{m-1})^3(1-\omega_m^3)}{(1-\omega_m^2)^3}$$
$$=-\omega_m^3(1+\omega_m^2+\omega_m^4+\cdots+\omega_m^{m-3})^3(1-\omega_m^3)=c_m.$$

If $m\equiv 0 \pmod 2$ then $m/2$ is a positive integer greater than or equal to 2, and we have

$$\frac{(1-\omega_m)^3(1-\omega_m^3)}{(1-\omega_m^2)^3}=(1-\omega_m)^3(1-\omega_m^3)\frac{\prod_{r=2}^{m/2-1}(1-\omega_m^{2r})^3}{\prod_{r=1}^{m/2-1}(1-\omega_m^{2r})^3}.$$

Now

$$\prod_{r=1}^{m/2-1}(1-\omega_m^{2r}) = \lim_{x\to 1}\prod_{r=1}^{m/2-1}(x-\omega_m^{2r}) = \lim_{x\to 1}\frac{x^{m/2}-1}{x-1} = \frac{m}{2}$$

so

$$\frac{(1-\omega_m)^3(1-\omega_m^3)}{(1-\omega_m^2)^3} = (1-\omega_m)^3(1-\omega_m^3)\frac{\prod_{r=2}^{m/2-1}(1-\omega_m^{2r})^3}{(m/2)^3} = c_m.$$

The proposition now follows from (2.33). ∎

Formula (2.33) has its origins in the identity relating the Weierstrass sigma and zeta functions given in [7, p. 187]. Formulae for these functions can be found in [7] and [9]. Our proof of Proposition 2.1 is based on the ideas in Dobbie [6].

3. Proof of Theorem 1.1

We choose $m = 8$ in Proposition 2.1. Here

$$\omega_8 = \frac{1+i}{\sqrt{2}},\ \ \omega_8^{-1} = \frac{1-i}{\sqrt{2}},\ \ \lambda_8 = \sqrt{2},\ \ \lambda_8^2-2 = 0,\ \ \lambda_8^3-3\lambda_8 = -\sqrt{2},$$

so

$$\prod_{n=1}^{\infty}\frac{\left(1-(\lambda_8^2-2)q^n+q^{2n}\right)^3\left(1-q^n\right)^2}{\left(1-\lambda_8 q^n+q^{2n}\right)^3\left(1-(\lambda_8^3-3\lambda_8)q^n+q^{2n}\right)}$$

$$= \prod_{n=1}^{\infty}\frac{\left(1+q^{2n}\right)^3\left(1-q^n\right)^2}{\left(1-\sqrt{2}q^n+q^{2n}\right)^3\left(1+\sqrt{2}q^n+q^{2n}\right)} = \frac{E_1^2E_4^6}{E_2^3E_8^3}\prod_{n=1}^{\infty}\left(1+\sqrt{2}q^n+q^{2n}\right)^2$$

as

$$(1-\sqrt{2}q^n+q^{2n})(1+\sqrt{2}q^n+q^{2n}) = \left(1+q^{2n}\right)^2-2q^{2n} = 1+q^{4n} = \frac{1-q^{8n}}{1-q^{4n}}.$$

Further, using MAPLE, we find

$$c_8 = \frac{8}{8^3}\left(1-\omega_8^3\right)\left(1-\omega_8\right)^3\left(1-\omega_8^4\right)^3\left(1-\omega_8^6\right)^3 = \frac{i(1-\sqrt{2})}{\sqrt{2}}$$

and

$$\Delta(k,8) = 3\omega_8^k-\omega_8^{3k}+\omega_8^{5k}-3\omega_8^{7k} = \begin{cases} 0 & \text{if } k=0,4,\\ 2i\sqrt{2} & \text{if } k=1,3,\\ 8i & \text{if } k=2,\\ -2i\sqrt{2} & \text{if } k=5,7,\\ -8i & \text{if } k=6. \end{cases}$$

Then Proposition 2.1 with $m=8$ gives (appealing to (1.7) and (1.8))

$$\begin{aligned}
&\frac{E_1^2E_4^6}{E_2^3E_8^3}\prod_{n=1}^{\infty}\left(1+\sqrt{2}q^n+q^{2n}\right)^2\\
&\quad=1+\frac{i(1-\sqrt{2})}{\sqrt{2}}\Big(2i\sqrt{2}\sum_{n=1}^{\infty}\left(d_{1,8}(n)+d_{3,8}(n)-d_{5,8}(n)-d_{7,8}(n)\right)q^n\\
&\qquad\qquad+8i\sum_{n=1}^{\infty}\left(d_{2,8}(n)-d_{6,8}(n)\right)q^n\Big)\\
&\quad=1+2(\sqrt{2}-1)\sum_{n=1}^{\infty}\left(d_{1,8}(n)+d_{3,8}(n)-d_{5,8}(n)-d_{7,8}(n)\right)q^n\\
&\qquad+4\sqrt{2}(\sqrt{2}-1)\sum_{n=1}^{\infty}\left(d_{1,4}(n)-d_{3,4}(n)\right)q^{2n}\\
&\quad=1+(\sqrt{2}-1)\left(\varphi(q)\varphi(q^2)-1\right)+\sqrt{2}(\sqrt{2}-1)\left(\varphi^2(q^2)-1\right)\\
&\quad=(\sqrt{2}-1)\varphi(q^2)\left(\varphi(q)+\sqrt{2}\varphi(q^2)\right)
\end{aligned}$$

so that

$$\prod_{n=1}^{\infty}\left(1+\sqrt{2}q^n+q^{2n}\right)^2=(\sqrt{2}-1)\frac{E_2^3E_8^3}{E_1^2E_4^6}\varphi(q^2)\left(\varphi(q)+\sqrt{2}\varphi(q^2)\right).$$

By (1.3) we have $\varphi(q^2)=E_4^5/(E_2^2E_8^2)$ and the first formula of Theorem 1.1 follows.

The second formula follows by conjugation. ∎

4. Proof of Theorem 1.2

Here we apply Proposition 2.1 with $m=12$. We have

$$\omega_{12}=\frac{\sqrt{3}+i}{2},\ \ \omega_{12}^{-1}=\frac{\sqrt{3}-i}{2},\ \ \lambda_{12}=\sqrt{3},\ \ \lambda_{12}^2-2=1,\ \ \lambda_{12}^3-3\lambda_{12}=0,$$

so

$$\begin{aligned}
&\prod_{n=1}^{\infty}\frac{\left(1-(\lambda_{12}^2-2)q^n+q^{2n}\right)^3\left(1-q^n\right)^2}{\left(1-\lambda_{12}q^n+q^{2n}\right)^3\left(1-(\lambda_{12}^3-3\lambda_{12})q^n+q^{2n}\right)}\\
&\quad=\prod_{n=1}^{\infty}\frac{\left(1-q^n+q^{2n}\right)^3\left(1-q^n\right)^2}{\left(1-\sqrt{3}q^n+q^{2n}\right)^3\left(1+q^{2n}\right)}=\frac{E_1^5E_4^2E_6^6}{E_2^5E_3^3E_{12}^3}\prod_{n=1}^{\infty}\left(1+\sqrt{3}q^n+q^{2n}\right)^3,
\end{aligned}$$

as

$$\prod_{n=1}^{\infty}(1+q^{2n})=\prod_{n=1}^{\infty}\frac{1-q^{4n}}{1-q^{2n}}=\frac{E_4}{E_2},$$

$$\prod_{n=1}^{\infty}(1-q^n+q^{2n})=\prod_{n=1}^{\infty}\frac{1+q^{3n}}{1+q^n}=\prod_{n=1}^{\infty}\frac{1-q^{6n}}{1-q^{3n}}\frac{1-q^n}{1-q^{2n}}=\frac{E_1E_6}{E_2E_3},$$

and

$$\prod_{n=1}^{\infty}(1-\sqrt{3}q^n+q^{2n})(1+\sqrt{3}q^n+q^{2n})$$
$$=\prod_{n=1}^{\infty}\left((1+q^{2n})^2-3q^{2n}\right)=\prod_{n=1}^{\infty}(1-q^{2n}+q^{4n})=\frac{E_2E_{12}}{E_4E_6}.$$

Further, using MAPLE, we find that

$$c_{12}=\frac{8}{12^3}(1-\omega_{12}^3)(1-\omega_{12})^3(1-\omega_{12}^4)^3(1-\omega_{12}^6)^3(1-\omega_{12}^8)^3(1-\omega_{12}^{10})^3=(5-3\sqrt{3})i$$

and

$$\Delta(k,12)=3\omega_{12}^k-\omega_{12}^{3k}+\omega_{12}^{9k}-3\omega_{12}^{11k}=\begin{cases}0 & \text{if } k=0,6,\\ i & \text{if } k=1,5,\\ -i & \text{if } k=7,11,\\ 3i\sqrt{3} & \text{if } k=2,4,\\ -3i\sqrt{3} & \text{if } k=8,10,\\ 8i & \text{if } k=3,\\ -8i & \text{if } k=9.\end{cases}$$

Then Proposition 2.1 with $m=12$ gives

$$\frac{E_1^5E_4^2E_6^6}{E_2^5E_3^3E_{12}^3}\prod_{n=1}^{\infty}\left(1+\sqrt{3}q^n+q^{2n}\right)^3$$
$$=1+(5-3\sqrt{3})i\sum_{n=1}^{\infty}\Big(i\big(d_{1,12}(n)+d_{5,12}(n)-d_{7,12}(n)-d_{11,12}(n)\big)$$
$$+3i\sqrt{3}(d_{2,12}(n)+d_{4,12}(n)-d_{8,12}(n)-d_{10,12}(n))+8i(d_{3,12}(n)-d_{9,12}(n))\Big)q^n$$
$$=1+(3\sqrt{3}-5)\sum_{n=1}^{\infty}\left(d_{1,12}(n)+d_{5,12}(n)-d_{7,12}(n)-d_{11,12}(n)\right)q^n$$
$$+3\sqrt{3}(3\sqrt{3}-5)\sum_{n=1}^{\infty}\left(d_{1,6}(n)-d_{5,6}(n)\right)q^{2n}$$
$$+3\sqrt{3}(3\sqrt{3}-5)\sum_{n=1}^{\infty}\left(d_{1,3}(n)-d_{2,3}(n)\right)q^{4n}$$
$$+8(3\sqrt{3}-5)\sum_{n=1}^{\infty}\left(d_{1,4}(n)-d_{3,4}(n)\right)q^{3n}$$
$$=1+(3\sqrt{3}-5)\left(\frac{\varphi^2(q)+\varphi^2(q^3)-2}{4}\right)\qquad\text{(by (1.11))}$$
$$+3\sqrt{3}(3\sqrt{3}-5)\left(\frac{a(q^2)+a(q^4)-2}{6}\right)\qquad\text{(by (1.12))}$$
$$+3\sqrt{3}(3\sqrt{3}-5)\left(\frac{a(q^4)-1}{6}\right)\qquad\text{(by (1.10))}$$
$$+8(3\sqrt{3}-5)\left(\frac{\varphi^2(q^3)-1}{4}\right)\qquad\text{(by (1.7))}$$
$$=\frac{(3\sqrt{3}-5)}{4}(\varphi^2(q)+9\varphi^2(q^3))+(3\sqrt{3}-5)\frac{\sqrt{3}}{2}(a(q^2)+2a(q^4)).$$

Borwein, Borwein and Garvan [4, eq. (2.27), p. 44] have proved that

$$a(q)+2a(q^2)=3\frac{\varphi^3(-q^3)}{\varphi(-q)}. \tag{4.1}$$

Hence

$$\frac{E_1^5E_4^2E_6^6}{E_2^5E_3^3E_{12}^3}\prod_{n=1}^{\infty}\left(1+\sqrt{3}q^n+q^{2n}\right)^3=\frac{(3\sqrt{3}-5)}{4}\left(\varphi^2(q)+9\varphi^2(q^3)+6\sqrt{3}\frac{\varphi^3(-q^6)}{\varphi(-q^2)}\right),$$

from which the first asserted formula of Theorem 1.2 follows.

The second formula follows by conjugation. ■

5. Proof of Theorem 1.3

Here we apply Proposition 2.1 with $m=5$. We have

$$\omega_5=\frac{\sqrt{5}-1}{4}+i\frac{\sqrt{10+2\sqrt{5}}}{4},\ \ \omega_5^{-1}=\frac{\sqrt{5}-1}{4}-i\frac{\sqrt{10+2\sqrt{5}}}{4},$$

$$\lambda_5=\frac{\sqrt{5}-1}{2},\ \ \lambda_5^2-2=\frac{-1-\sqrt{5}}{2},\ \ \lambda_5^3-3\lambda_5=\frac{-1-\sqrt{5}}{2},$$

so

$$\prod_{n=1}^{\infty}\frac{\left(1-(\lambda_5^2-2)q^n+q^{2n}\right)^3\left(1-q^n\right)^2}{\left(1-\lambda_5q^n+q^{2n}\right)^3\left(1-(\lambda_5^3-3\lambda_5)q^n+q^{2n}\right)}$$

$$=\prod_{n=1}^{\infty}\frac{\left(1+(\frac{1+\sqrt{5}}{2})q^n+q^{2n}\right)^2\left(1-q^n\right)^2}{\left(1+(\frac{1-\sqrt{5}}{2})q^n+q^{2n}\right)^3}=\frac{E_1^5}{E_5^3}\prod_{n=1}^{\infty}\left(1+\left(\frac{1+\sqrt{5}}{2}\right)q^n+q^{2n}\right)^5,$$

as

$$\left(1+\left(\frac{1-\sqrt{5}}{2}\right)q^n+q^{2n}\right)\left(1+\left(\frac{1+\sqrt{5}}{2}\right)q^n+q^{2n}\right)$$

$$=\left(1+\frac{1}{2}q^n+q^{2n}\right)^2-\frac{5}{4}q^{2n}=1+q^n+q^{2n}+q^{3n}+q^{4n}=\frac{1-q^{5n}}{1-q^n}.$$

Next we find

$$\begin{aligned}c_5\ &=-\omega_5^3(1-\omega_5^3)\left(1+\omega_5^2\right)^3=\omega_5-2\omega_5^2+2\omega_5^3-\omega_5^4\\ &=\frac{1}{2}i\left(\sqrt{10+2\sqrt{5}}-2\sqrt{10-2\sqrt{5}}\right)=-\frac{1}{2}i\sqrt{50-22\sqrt{5}}\end{aligned}$$

and

$$\Delta(k,5)=3\omega_5^k+\omega_5^{2k}-\omega_5^{3k}-3\omega_5^{4k}=\begin{cases}0 & \text{if } k=0,\\ i\sqrt{25+10\sqrt{5}} & \text{if } k=1,\\ i\sqrt{25-10\sqrt{5}} & \text{if } k=2,\\ -i\sqrt{25-10\sqrt{5}} & \text{if } k=3,\\ -i\sqrt{25+10\sqrt{5}} & \text{if } k=4.\end{cases}$$

Hence, by Proposition 2.1 with $m = 5$, we obtain

$$\begin{aligned}
\frac{E_1^5}{E_5^3}\prod_{n=1}^{\infty}\Big(1+\frac{1+\sqrt{5}}{2}q^n+q^{2n}\Big)^5
&= 1-\frac{1}{2}i\sqrt{50-22\sqrt{5}}\Big(i\sqrt{25+10\sqrt{5}}\sum_{n=1}^{\infty}\big(d_{1,5}(n)-d_{4,5}(n)\big)q^n \\
&\qquad\qquad +i\sqrt{25-10\sqrt{5}}\sum_{n=1}^{\infty}\big(d_{2,5}(n)-d_{3,5}(n)\big)q^n\Big) \\
&= 1+\frac{1}{2}(5\sqrt{5}-5)\sum_{n=1}^{\infty}\big(d_{1,5}(n)-d_{4,5}(n)\big)q^n \\
&\qquad +\frac{1}{2}(35-15\sqrt{5})\sum_{n=1}^{\infty}\big(d_{2,5}(n)-d_{3,5}(n)\big)q^n,
\end{aligned}$$

which gives the first formula.

The second formula follows by conjugation. ∎

6. Final Remarks

The referee has pointed out that Theorem 1.1 is equivalent to Theorem 3.2 in Yuttanan [10], and that further results involving the products in Theorem 1.3 have been given by Huber [8].

The choice $m = 4$ in Proposition 2.1 yields using (1.3) Jacobi's identity (1.7). Thus Jacobi's two squares theorem is the special case $m = 4$ of Proposition 2.1. The choice $m = 6$ in Proposition 2.1 gives using (1.3) and (1.10) the identity (4.1) of Borwein, Borwein and Garvan. It would be interesting to investigate Proposition 2.1 for other values of m such as $m = 10$. In this connection the referee has suggested consulting Cooper and Hirschhorn [5] for some results along these lines.

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