International Journal of Number Theory
Vol. 4, No. 4 (2008) 677–689

BERNDT'S CURIOUS FORMULA

AYŞE ALACA*, ŞABAN ALACA† and KENNETH S. WILLIAMS‡

Centre for Research in Algebra and Number Theory
School of Mathematics and Statistics, Carleton University
Ottawa, Ontario, Canada K1S 5B6
*aalaca@math.carleton.ca
†salaca@math.carleton.ca
‡kwilliam@connect.carleton.ca

Received 4 April 2007
Accepted 2 May 2007

Dedicated to Bruce Berndt on the Occasion of his 68th Birthday

A curious arithmetic formula deduced by Berndt from an analytic formula of Ramanujan is proved arithmetically and used to prove the formulae given by Liouville for the number of representations of a positive integer by the forms $x^2 + y^2 + z^2 + t^2 + 2u^2 + 2v^2$ and $x^2 + y^2 + 2z^2 + 2t^2 + 2u^2 + 2v^2$.

Keywords: Berndt's curious formula; two quadratic forms in six variables.

Mathematics Subject Classification 2000: 11A25, 11E20, 11E25

1. Introduction

In [1] and [2, p. 64] Berndt deduced a curious arithmetic formula from an analytic formula of Ramanujan. In this paper we prove Berndt's curious formula arithmetically and then use it to establish results of Liouville [4, 5] on the number of representations of a positive integer by the forms $x^2 + y^2 + z^2 + t^2 + 2u^2 + 2v^2$ and $x^2 + y^2 + 2z^2 + 2t^2 + 2u^2 + 2v^2$.

2. Notation

Following Berndt [2, p. 64] we set

$$\sigma_\nu^*(n) := \sum_{2k-1|n} (-1)^{k-1}(2k-1)^\nu, \quad n \in \mathbb{N}, \quad \nu \in \mathbb{N} \cup \{0\}, \tag{2.1}$$

$$\tilde{\sigma}_\nu(n) := \sum_{k|n} (-1)^{k+\frac{n}{k}} k^\nu, \quad n \in \mathbb{N}, \quad \nu \in \mathbb{N} \cup \{0\}. \tag{2.2}$$

Berndt's curious formula is

$$\sigma_2^*(n) = 8\sum_{k=0}^{n} \sigma_0^*(k)\tilde{\sigma}_1(n-k), \quad n \in \mathbb{N} \cup \{0\}, \tag{2.3}$$

where

$$\sigma_0^*(0) = \frac{1}{4}, \quad \tilde{\sigma}_1(0) = -\frac{1}{8}, \quad \sigma_2^*(0) = -\frac{1}{4}. \tag{2.4}$$

In preparation for proving Berndt's formula (2.3), we define (following McAfee [6, p. 156])

$$J_i(n) := \sum_{d \mid n} \left(\frac{-4}{d}\right) d^i, \quad n \in \mathbb{N}, \quad i \in \mathbb{Z}, \tag{2.5}$$

and

$$K_i(n) := \sum_{d \mid n} \left(\frac{-4}{n/d}\right) d^i, \quad n \in \mathbb{N}, \quad i \in \mathbb{Z}, \tag{2.6}$$

where for $k \in \mathbb{N}$

$$\left(\frac{-4}{k}\right) = \begin{cases} 1, & \text{if } k \equiv 1 \pmod 4, \\ -1, & \text{if } k \equiv 3 \pmod 4, \\ 0, & \text{if } k \equiv 0 \pmod 2. \end{cases} \tag{2.7}$$

McAfee [6, Theorem 4.7.1, p. 165] has used an arithmetic identity of Huard, Ou, Spearman and Williams [3] to prove the three formulae:

$$\sum_{\substack{(a,b,x,y)\in\mathbb{N}^4 \\ ax+by=n}} \left(\frac{-4}{a}\right) b = \frac{1}{24}(J_0(n) + J_2(n) + 16K_2(n) - 12nJ_0(n) - 6\sigma(n)), \tag{2.8}$$

$$\sum_{\substack{(a,b,x,y)\in\mathbb{N}^4 \\ ax+2by=n}} \left(\frac{-4}{a}\right) b = \frac{1}{24}(J_0(n) + J_2(n) + 4K_2(n) - 6nJ_0(n) - 6\sigma(n/2)), \tag{2.9}$$

$$\sum_{\substack{(a,b,x,y)\in\mathbb{N}^4 \\ ax+4by=n}} \left(\frac{-4}{a}\right) b = \frac{1}{24}(J_0(n) + J_2(n) + K_2(n) - 3nJ_0(n) - 6\sigma(n/4)), \tag{2.10}$$

where

$$\sigma(n) = \begin{cases} \displaystyle\sum_{d \mid n} d, & \text{if } n \in \mathbb{N}, \\ 0, & \text{if } n \notin \mathbb{N}. \end{cases} \tag{2.11}$$

We need the following consequence of formulae (2.8)–(2.10).

Lemma 2.1. *For $n \in \mathbb{N}$*

$$\sum_{\substack{(a,b,x,y)\in\mathbb{N}^4\\ ax+by=n}} (-1)^{by}\left(\frac{-4}{a}\right)b$$
$$= \frac{1}{24}\big(J_0(n) + J_2(n) + 4K_2(n) - 12nJ_0(n) + 6\sigma(n) - 36\sigma(n/2) + 24\sigma(n/4)\big).$$

Proof. We have

$$\sum_{\substack{(a,b,x,y)\in\mathbb{N}^4\\ ax+by=n}} (1+(-1)^{by})\left(\frac{-4}{a}\right)b$$
$$= 2\sum_{\substack{(a,b,x,y)\in\mathbb{N}^4\\ ax+by=n\\ 2|by}} \left(\frac{-4}{a}\right)b$$
$$= 2\sum_{\substack{(a,b,x,y)\in\mathbb{N}^4\\ ax+by=n\\ 2|b}} \left(\frac{-4}{a}\right)b + 2\sum_{\substack{(a,b,x,y)\in\mathbb{N}^4\\ ax+by=n\\ 2|y}} \left(\frac{-4}{a}\right)b - 2\sum_{\substack{(a,b,x,y)\in\mathbb{N}^4\\ ax+by=n\\ 2|b,\ 2|y}} \left(\frac{-4}{a}\right)b$$
$$= 6\sum_{\substack{(a,b,x,y)\in\mathbb{N}^4\\ ax+2by=n}} \left(\frac{-4}{a}\right)b - 4\sum_{\substack{(a,b,x,y)\in\mathbb{N}^4\\ ax+4by=n}} \left(\frac{-4}{a}\right)b$$
$$= \frac{1}{4}\big(J_0(n) + J_2(n) + 4K_2(n) - 6nJ_0(n) - 6\sigma(n/2)\big)$$
$$- \frac{1}{6}\big(J_0(n) + J_2(n) + K_2(n) - 3nJ_0(n) - 6\sigma(n/4)\big)$$
$$= \frac{1}{24}\big(2J_0(n) + 2J_2(n) + 20K_2(n) - 24nJ_0(n) - 36\sigma(n/2) + 24\sigma(n/4)\big),$$

by (2.9) and (2.10). Then, by (2.8), we obtain

$$\sum_{\substack{(a,b,x,y)\in\mathbb{N}^4\\ ax+by=n}} (-1)^{by}\left(\frac{-4}{a}\right)b$$
$$= \frac{1}{24}\big(2J_0(n) + 2J_2(n) + 20K_2(n) - 24nJ_0(n) - 36\sigma(n/2) + 24\sigma(n/4)\big)$$
$$- \frac{1}{24}\big(J_0(n) + J_2(n) + 16K_2(n) - 12nJ_0(n) - 6\sigma(n)\big)$$
$$= \frac{1}{24}\big(J_0(n) + J_2(n) + 4K_2(n) - 12nJ_0(n) + 6\sigma(n) - 36\sigma(n/2) + 24\sigma(n/4)\big),$$

as claimed. □

We denote the number of representations of $n \in \mathbb{N} \cup \{0\}$ as the sum of $k \in \mathbb{N}$ squares by $r_k(n)$. Clearly $r_k(0) = 1$. Jacobi showed that

$$r_2(n) = 4\sum_{d|n}\left(\frac{-4}{d}\right) = 4J_0(n), \quad n \in \mathbb{N}, \tag{2.12}$$

$$r_4(n) = 8\sum_{\substack{d|n \\ 4\nmid d}} d, \quad n \in \mathbb{N}, \tag{2.13}$$

and

$$\begin{aligned} r_6(n) &= 16\sum_{d|n}\left(\frac{-4}{n/d}\right)d^2 - 4\sum_{d|n}\left(\frac{-4}{d}\right)d^2 \\ &= 16K_2(n) - 4J_2(n), \quad n \in \mathbb{N}. \end{aligned} \tag{2.14}$$

3. Arithmetic Proof of Berndt's Curious Formula

The formula (2.3) is trivially true for $n = 0$ in view of (2.4) so we may suppose that $n \in \mathbb{N}$. Taking $\nu = 0$ in (2.1) we obtain by (2.5), (2.7), and (2.12)

$$\begin{aligned} \sigma_0^*(n) &= \sum_{2k-1|n} (-1)^{k-1} = \sum_{\substack{d|n \\ d \text{ odd}}} (-1)^{\frac{d-1}{2}} \\ &= \sum_{d|n}\left(\frac{-4}{d}\right) = J_0(n) = \frac{1}{4}r_2(n), \quad n \in \mathbb{N}. \end{aligned} \tag{3.1}$$

Taking $\nu = 1$ in (2.2) we have by (2.13)

$$\begin{aligned} \tilde{\sigma}_1(n) &= \sum_{k|n}(-1)^{k+\frac{n}{k}}k = \sigma(n) - 6\sigma(n/2) + 8\sigma(n/4) \\ &= (-1)^{n-1}\big(\sigma(n) - 4\sigma(n/4)\big) \\ &= (-1)^{n-1}\sum_{\substack{d|n \\ 4\nmid d}} d = \frac{(-1)^{n-1}}{8}r_4(n), \quad n \in \mathbb{N}, \end{aligned} \tag{3.2}$$

where we used the simple fact that

$$\sigma(n) - 3\sigma(n/2) + 2\sigma(n/4) = 0 \tag{3.3}$$

when n is even.

Taking $\nu = 2$ in (2.1), we deduce by (2.5) and (2.7)

$$\begin{aligned} \sigma_2^*(n) &= \sum_{2k-1|n} (-1)^{k-1}(2k-1)^2 = \sum_{\substack{d|n \\ d \text{ odd}}} (-1)^{\frac{d-1}{2}} d^2 \\ &= \sum_{d|n}\left(\frac{-4}{d}\right)d^2 = J_2(n), \quad n \in \mathbb{N}. \end{aligned} \tag{3.4}$$

Thus for $n \in \mathbb{N}$ we have

$$
\begin{aligned}
&\sum_{k=0}^{n} \sigma_0^*(k)\tilde{\sigma}_1(n-k) \\
&\quad = \sigma_0^*(0)\tilde{\sigma}_1(n) + \sum_{k=1}^{n-1} \sigma_0^*(k)\tilde{\sigma}_1(n-k) + \sigma_0^*(n)\tilde{\sigma}_1(0) \\
&\quad = \frac{1}{4}\tilde{\sigma}_1(n) + \sum_{\substack{(k,l)\in\mathbb{N}^2 \\ k+l=n}} \sigma_0^*(k)\tilde{\sigma}_1(l) - \frac{1}{8}\sigma_0^*(n) \quad \text{(by (2.4))} \\
&\quad = \frac{1}{4}(\sigma(n) - 6\sigma(n/2) + 8\sigma(n/4)) - \frac{1}{8}J_0(n) \\
&\qquad + \sum_{\substack{(k,l)\in\mathbb{N}^2 \\ k+l=n}} \left(\sum_{a|k} \left(\frac{-4}{a}\right) \right) \left((-1)^{l-1} \sum_{\substack{b|l \\ 4\nmid b}} b \right) \quad \text{(by (3.1) and (3.2))} \\
&\quad = \frac{1}{4}\sigma(n) - \frac{3}{2}\sigma(n/2) + 2\sigma(n/4) - \frac{1}{8}J_0(n) \\
&\qquad + \sum_{\substack{(a,b,x,y)\in\mathbb{N}^4 \\ ax+by=n \\ 4\nmid b}} (-1)^{by-1} \left(\frac{-4}{a}\right) b \\
&\quad = \frac{1}{4}\sigma(n) - \frac{3}{2}\sigma(n/2) + 2\sigma(n/4) - \frac{1}{8}J_0(n) \\
&\qquad - \sum_{\substack{(a,b,x,y)\in\mathbb{N}^4 \\ ax+by=n}} (-1)^{by} \left(\frac{-4}{a}\right) b + 4 \sum_{\substack{(a,b,x,y)\in\mathbb{N}^4 \\ ax+4by=n}} \left(\frac{-4}{a}\right) b \\
&\quad = \frac{1}{4}\sigma(n) - \frac{3}{2}\sigma(n/2) + 2\sigma(n/4) - \frac{1}{8}J_0(n) \\
&\qquad - \frac{1}{24}(J_0(n) + J_2(n) + 4K_2(n) - 12nJ_0(n) + 6\sigma(n) - 36\sigma(n/2) + 24\sigma(n/4)) \\
&\qquad + \frac{1}{6}(J_0(n) + J_2(n) + K_2(n) - 3nJ_0(n) - 6\sigma(n/4)) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(by Lemma 2.1 and (2.10))} \\
&\quad = \frac{1}{8}J_2(n) \\
&\quad = \frac{1}{8}\sigma_2^*(n) \quad \text{(by (3.4))}.
\end{aligned}
$$

This completes our arithmetic proof of Berndt's curious formula (2.3). □

4. A Companion to Berndt's Curious Formula

From Sec. 3 we see that Berndt's curious formula can be stated in the form

$$8\sum_{k=0}^{n}\sigma_0^*(k)\hat{\sigma}_1(n-k)=J_2(n),\quad n\in\mathbb{N}. \tag{4.1}$$

We next give an analogue of this formula for $K_2(n)$. We set

$$\hat{\sigma}_\nu(n):=\sum_{\substack{2k-1\mid n\\ n/(2k-1)\ \text{odd}}}(2k-1)^\nu,\quad n\in\mathbb{N},\quad \nu\in\mathbb{N}\cup\{0\}. \tag{4.2}$$

Clearly for $n\in\mathbb{N}$ we have

$$\begin{aligned}\hat{\sigma}_1(n)&=\sum_{\substack{2k-1\mid n\\ n/(2k-1)\ \text{odd}}}(2k-1)\\ &=\sum_{\substack{d\mid n\\ d\ \text{odd}\\ n/d\ \text{odd}}}d\\ &=\frac{1}{2}(1-(-1)^n)\sigma(n)\end{aligned}$$

so that by (3.3) we have

$$\hat{\sigma}_1(n)=\sigma(n)-3\sigma(n/2)+2\sigma(n/4),\quad n\in\mathbb{N}. \tag{4.3}$$

We set

$$\hat{\sigma}_1(0)=0. \tag{4.4}$$

We prove:

Theorem 4.1. *For $n\in\mathbb{N}$*

$$4\sum_{k=0}^{n}\sigma_0^*(k)\hat{\sigma}_1(n-k)=K_2(n).$$

Proof. For $n\in\mathbb{N}$ we have by (2.4), (4.4), (4.3), (3.1), (2.8) and Lemma 2.1,

$$\begin{aligned}\sum_{k=0}^{n}\sigma_0^*(k)\hat{\sigma}_1(n-k)&=\sigma_0^*(0)\hat{\sigma}_1(n)+\sum_{k=1}^{n-1}\sigma_0^*(k)\hat{\sigma}_1(n-k)+\sigma_0^*(n)\hat{\sigma}_1(0)\\ &=\frac{1}{4}\hat{\sigma}_1(n)+\sum_{k=1}^{n-1}\sigma_0^*(n-k)\hat{\sigma}_1(k)\\ &=\frac{1}{4}\sigma(n)-\frac{3}{4}\sigma(n/2)+\frac{1}{2}\sigma(n/4)\\ &\quad+\frac{1}{2}\sum_{k=1}^{n-1}(1-(-1)^k)\sum_{a\mid n-k}\left(\frac{-4}{a}\right)\sum_{b\mid k}b\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{4}\sigma(n) - \frac{3}{4}\sigma(n/2) + \frac{1}{2}\sigma(n/4) \\
&\quad + \frac{1}{2} \sum_{\substack{(a,b,x,y)\in\mathbb{N}^4 \\ ax+by=n}} \left(\frac{-4}{a}\right) b \\
&\quad - \frac{1}{2} \sum_{\substack{(a,b,x,y)\in\mathbb{N}^4 \\ ax+by=n}} (-1)^{by} \left(\frac{-4}{a}\right) b \\
&= \frac{1}{4}K_2(n),
\end{aligned}
$$

as asserted. □

5. Application of Berndt's Curious Formula to Liouville's Formulae

In 1864 Liouville [4, 5] stated without proof the two formulae in Theorem 5.1. No proof of these claims seems to appear in the literature. We show that they are simple consequences of Berndt's curious formula.

Theorem 5.1. *Let $n \in \mathbb{N}$. Set $n = 2^\alpha N$, where $\alpha \in \mathbb{N} \cup \{0\}$, $N \in \mathbb{N}$ and $N \equiv 1 \pmod 2$.*

(i) *The number $R(n)$ of representations of n by the form $x^2 + y^2 + z^2 + t^2 + 2u^2 + 2v^2$ is given by*

$$R(n) = \left(2^{2\alpha+3} - 2(1 + (-1)^{2^\alpha})\left(\frac{-4}{N}\right)\right) \sum_{d|N} \left(\frac{-4}{N/d}\right) d^2. \tag{5.1}$$

(ii) *The number $S(n)$ of representations of n by the form $x^2 + y^2 + 2z^2 + 2t^2 + 2u^2 + 2v^2$ is given by*

$$S(n) = \left(2^{2\alpha+2} - 2\big(1 + (-1)^{2^\alpha}\big)\left(\frac{-4}{N}\right)\right) \sum_{d|N} \left(\frac{-4}{N/d}\right) d^2. \tag{5.2}$$

Proof. (i) By (2.4) and (3.1) we have

$$\sigma_0^*(k) = \frac{1}{4} r_2(k), \quad k \in \mathbb{N} \cup \{0\}. \tag{5.3}$$

By (2.4) and (3.2) we have

$$\tilde{\sigma}_1(k) = \frac{(-1)^{k-1}}{8} r_4(k), \quad k \in \mathbb{N} \cup \{0\}. \tag{5.4}$$

Using (5.3) and (5.4) in Berndt's formula (2.3), we obtain by (3.4)

$$\sum_{k=0}^{n} (-1)^k r_2(k) r_4(n-k) = 4(-1)^{n-1} J_2(n). \tag{5.5}$$

Appealing to (2.14) we have

$$\sum_{k=0}^{n} r_2(k)r_4(n-k) = r_6(n) = 16K_2(n) - 4J_2(n). \tag{5.6}$$

Adding (5.5) and (5.6), we deduce

$$2\sum_{\substack{k=0\\ k \text{ even}}}^{n} r_2(k)r_4(n-k) = 16K_2(n) - 4(1+(-1)^n)J_2(n). \tag{5.7}$$

As $r_2(2m) = r_2(m)$ $(m \in \mathbb{N} \cup \{0\})$ we deduce from (5.7)

$$R(n) = \sum_{0 \le m \le \frac{n}{2}} r_2(m)r_4(n-2m) = 8K_2(n) - 2(1+(-1)^n)J_2(n). \tag{5.8}$$

An easy calculation shows that

$$J_2(n) = J_2(N), \quad K_2(n) = 2^{2\alpha}K_2(N), \quad J_2(N) = \left(\frac{-4}{N}\right)K_2(N). \tag{5.9}$$

Hence, from (5.8) and (5.9), we obtain

$$R(n) = 8 \cdot 2^{2\alpha}K_2(N) - 2(1+(-1)^n)\left(\frac{-4}{N}\right)K_2(N),$$

which gives Liouville's formula (5.1).

(ii) We let

$$A_n := \{(x,y,z,t,u,v) \in \mathbb{Z}^6 \mid n = x^2 + y^2 + 2z^2 + 2t^2 + 2u^2 + 2v^2\}.$$

Suppose first that n is even so that $n/2 \in \mathbb{N}$ and $\alpha \ge 1$. Let

$$B_n := \{(x,y,z,t,u,v) \in \mathbb{Z}^6 \mid n/2 = x^2 + y^2 + z^2 + t^2 + u^2 + v^2\}.$$

For $(x,y,z,t,u,v) \in A_n$ define

$$f((x,y,z,t,u,v)) = ((x-y)/2, (x+y)/2, z, t, u, v).$$

Then $f : A_n \to B_n$ is a bijection. Thus, by (2.14) and (5.9), we have

$$\begin{aligned}
S(n) &= \text{card } A_n \\
&= \text{card } B_n \\
&= r_6(n/2) \\
&= 16K_2(n/2) - 4J_2(n/2) \\
&= 16 \cdot 2^{2(\alpha-1)}K_2(N) - 4\left(\frac{-4}{N}\right)K_2(N) \\
&= \left(2^{2\alpha+2} - 2(1+(-1)^{2^\alpha})\left(\frac{-4}{N}\right)\right)K_2(N).
\end{aligned}$$

Now suppose that n is odd so that $\alpha = 0$. In this case the solutions $(x,y,z,t,u,v) \in \mathbb{Z}^6$ of

$$n = x^2 + y^2 + z^2 + t^2 + 2u^2 + 2v^2 \tag{5.10}$$

are such that exactly one of x, y, z, t is odd or exactly one of x, y, z, t is even. Thus

$$R(n) = 4R'(n), \tag{5.11}$$

where $R'(n)$ denotes the number of solutions of (5.10) with $x \equiv 1 \pmod 2$, $y \equiv z \pmod 2$ and $t \equiv 0 \pmod 2$. Also the solutions $(x,y,z,t,u,v) \in \mathbb{Z}^6$ of

$$n = x^2 + y^2 + 2z^2 + 2t^2 + 2u^2 + 2v^2 \tag{5.12}$$

have exactly one of x and y odd. Thus

$$S(n) = 2S'(n), \tag{5.13}$$

where $S'(n)$ denotes the number of solutions of (5.12) with $x \equiv 1 \pmod 2$ and $y \equiv 0 \pmod 2$. Let

$$\begin{aligned} C_n := \{(x,y,z,t,u,v) \in \mathbb{Z}^6 \mid n = x^2 + y^2 + z^2 + t^2 + 2u^2 + 2v^2, \\ x \equiv 1 \,(\text{mod } 2),\ y \equiv z \,(\text{mod } 2),\ t \equiv 0 \,(\text{mod } 2)\} \end{aligned}$$

and

$$\begin{aligned} D_n := \{(x,y,z,t,u,v) \in \mathbb{Z}^6 \mid n = x^2 + y^2 + 2z^2 + 2t^2 + 2u^2 + 2v^2, \\ x \equiv 1 \,(\text{mod } 2),\ y \equiv 0 \,(\text{mod } 2)\}. \end{aligned}$$

For $(x,y,z,t,u,v) \in C_n$ we define

$$g((x,y,z,t,u,v)) = (x, t, (y-z)/2, (y+z)/2, u, v).$$

Then $g : C_n \to D_n$ is a bijection. Thus

$$R'(n) = \text{card } C_n = \text{card } D_n = S'(n), \tag{5.14}$$

Hence

$$R(n) = 4R'(n) = 4S'(n) = 2S(n)$$

so by part (i) as $\alpha = 0$

$$\begin{aligned} S(n) &= \frac{1}{2} R(n) \\ &= 4K_2(N) \\ &= \left(2^{2\alpha+2} - 2(1 + (-1)^{2^\alpha})\left(\frac{-4}{N}\right)\right) K_2(N). \end{aligned}$$

This completes the proof of Theorem 5.1. □

In the proof of part (i) we used $r_2(2m) = r_2(m)$ $(m \in \mathbb{N})$. To adapt the proof of part (i) to prove part (ii) in a similar manner we would expect to use $r_4(2m) = (2 - (-1)^m) r_4(m)$ $(m \in \mathbb{N})$. However the term $(-1)^m r_4(m)$ leads to a sum that has no obvious evaluation. Thus it does not seem possible to prove part (ii) just by tweaking the proof of part (i).

6. A Corollary to Lemma 2.1

In this section we give a formula similar to formulae (2.8)–(2.10), which is a consequence of Lemma 2.1. We have

$$\begin{aligned}
&\sum_{\substack{(a,b,x,y)\in\mathbb{N}^4\\ ax+by=n}} (-1)^{by}\left(\frac{-4}{a}\right)b\\
&\quad= \sum_{\substack{(a,b,x,y)\in\mathbb{N}^4\\ ax+by=n}} (-1)^{n-ax}\left(\frac{-4}{a}\right)b\\
&\quad= (-1)^n \sum_{\substack{(a,b,x,y)\in\mathbb{N}^4\\ ax+by=n}} (-1)^{ax}\left(\frac{-4}{a}\right)b\\
&\quad= (-1)^n \sum_{\substack{(a,b,x,y)\in\mathbb{N}^4\\ ax+by=n\\ 2\nmid a}} (-1)^{ax}\left(\frac{-4}{a}\right)b\\
&\quad= (-1)^n \sum_{\substack{(a,b,x,y)\in\mathbb{N}^4\\ ax+by=n\\ 2\nmid a}} (-1)^{x}\left(\frac{-4}{a}\right)b\\
&\quad= (-1)^n \sum_{\substack{(a,b,x,y)\in\mathbb{N}^4\\ ax+by=n}} (-1)^{x}\left(\frac{-4}{a}\right)b\\
&\quad= (-1)^n \left(2\sum_{\substack{(a,b,x,y)\in\mathbb{N}^4\\ ax+by=n\\ 2\mid x}} \left(\frac{-4}{a}\right)b - \sum_{\substack{(a,b,x,y)\in\mathbb{N}^4\\ ax+by=n}} \left(\frac{-4}{a}\right)b\right)\\
&\quad= 2(-1)^n \sum_{\substack{(a,b,x,y)\in\mathbb{N}^4\\ 2ax+by=n}} \left(\frac{-4}{a}\right)b - (-1)^n \sum_{\substack{(a,b,x,y)\in\mathbb{N}^4\\ ax+by=n}} \left(\frac{-4}{a}\right)b.
\end{aligned}$$

Hence, by Lemma 2.1 and (2.8), we obtain

$$\begin{aligned}
&\sum_{\substack{(a,b,x,y)\in\mathbb{N}^4\\ 2ax+by=n}} \left(\frac{-4}{a}\right)b\\
&\quad= \frac{1}{2}(-1)^n \sum_{\substack{(a,b,x,y)\in\mathbb{N}^4\\ ax+by=n}} (-1)^{by}\left(\frac{-4}{a}\right)b + \frac{1}{2}\sum_{\substack{(a,b,x,y)\in\mathbb{N}^4\\ ax+by=n}} \left(\frac{-4}{a}\right)b\\
&\quad= \frac{(-1)^n}{48}(J_0(n)+J_2(n)+4K_2(n)-12nJ_0(n)+6\sigma(n)-36\sigma(n/2)+24\sigma(n/4))\\
&\qquad+\frac{1}{48}(J_0(n)+J_2(n)+16K_2(n)-12nJ_0(n)-6\sigma(n)).
\end{aligned}$$

Appealing to (3.3) we have the following result.

Corollary 6.1. *Let $n \in \mathbb{N}$. If n is odd then*

$$\sum_{\substack{(a,b,x,y)\in\mathbb{N}^4\\ 2ax+by=n}} \left(\frac{-4}{a}\right) b = \frac{1}{4}\big(K_2(n) - \sigma(n)\big)$$

and if n is even

$$\sum_{\substack{(a,b,x,y)\in\mathbb{N}^4\\ 2ax+by=n}} \left(\frac{-4}{a}\right) b = \frac{1}{24}\big(J_0(n) + J_2(n) + 10K_2(n) - 12nJ_0(n) - 6\sigma(n)\big).$$

7. A Question of McAfee

In [6, p. 169] McAfee asked whether there is an evaluation of the sum

$$\sum_{\substack{(a,b,x,y)\in\mathbb{N}^4\\ ax+8by=n}} \left(\frac{-4}{a}\right) b$$

similar to those in (2.8)–(2.10). We use Theorem 5.1(ii) to show that such an evaluation exists.

Theorem 7.1. *Let $n \in \mathbb{N}$. Then*

$$\sum_{\substack{(a,b,x,y)\in\mathbb{N}^4\\ ax+8by=n}} \left(\frac{-4}{a}\right) b$$
$$= \frac{1}{24}(J_0(n) + \frac{(5+3(-1)^n)}{8} J_2(n) + \frac{1}{4}K_2(n) - \frac{3}{2}nJ_0(n) - 6\sigma(n/8)).$$

Proof. As in [6, p. 155] we denote the sum in (2.9) by $B(n)$. For $n \in \mathbb{N}$ we have by (2.9), (2.12) and (2.13)

$$\begin{aligned}\sum_{\substack{(a,b,x,y)\in\mathbb{N}^4\\ ax+8by=n}} \left(\frac{-4}{a}\right) b &= \sum_{\substack{(a,b,x,y)\in\mathbb{N}^4\\ ax+2by=n\\ 4\mid b}} \left(\frac{-4}{a}\right) \frac{b}{4}\\ &= \frac{1}{4}\sum_{\substack{(a,b,x,y)\in\mathbb{N}^4\\ ax+2by=n}} \left(\frac{-4}{a}\right) b - \frac{1}{4}\sum_{\substack{(a,b,x,y)\in\mathbb{N}^4\\ ax+8by=n\\ 4\nmid b}} \left(\frac{-4}{a}\right) b\\ &= \frac{1}{4}B(n) - \frac{1}{4}\sum_{\substack{l,m\geq 1\\ l+2m=n}} \sum_{a\mid l} \left(\frac{-4}{a}\right) \sum_{\substack{b\mid m\\ 4\nmid b}} b\end{aligned}$$

$$= \frac{1}{4}B(n) - \frac{1}{128}\sum_{\substack{l,m\geq 1\\ l+2m=n}} r_2(l)r_4(m)$$

$$= \frac{1}{4}B(n) - \frac{1}{128}\sum_{\substack{l,m\geq 0\\ l+2m=n}} r_2(l)r_4(m)$$

$$+ \frac{1}{128}r_2(0)r_4(n/2) + \frac{1}{128}r_2(n)r_4(0)$$

$$= \frac{1}{4}B(n) - \frac{1}{128}S(n) + \frac{1}{16}\sigma(n/2) - \frac{1}{4}\sigma(n/8) + \frac{1}{32}J_0(n)$$

from which the asserted result follows on appealing to (2.9) and Theorem 5.1(ii). □

We close this section by giving a companion formula to Lemma 2.1. The proof is similar to that of Lemma 2.1 except that Theorem 7.1 is used in place of (2.8).

Corollary 7.1. *Let $n \in \mathbb{N}$. Then*

$$\sum_{\substack{(a,b,x,y)\in\mathbb{N}^4\\ ax+2by=n}} (-1)^{by}\left(\frac{-4}{a}\right) b = \frac{1}{24}\Big(J_0(n) + \frac{(5-3(-1)^n)}{2}J_2(n) + K_2(n) - 6nJ_0(n) + 6\sigma(n/2) - 36\sigma(n/4) + 24\sigma(n/8)\Big).$$

8. Conclusion

The referee pointed out that another analytic proof of (2.3) follows from [2, Chap. 17, Entries 14(i), 17(vi) and 17(vii)]. Putting these three equations together, we obtain

$$\sum_{k=0}^{\infty}\frac{(-1)^k}{e^{(2k+1)y}-1} - 2\sum_{k=1}^{\infty}\frac{(-1)^{k-1}k}{e^{ky}+1} + \sum_{k=0}^{\infty}\frac{(-1)^k(2k+1)^2}{e^{(2k+1)y}-1} = 8\sum_{k=0}^{\infty}\sum_{l=1}^{\infty}\frac{(-1)^{k+l-1}l}{(e^{(2k+1)y}-1)(e^{ly}+1)}. \tag{8.1}$$

As

$$\sum_{k=0}^{\infty}\frac{(-1)^k}{e^{(2k+1)y}-1} = \sum_{n=1}^{\infty}\sigma_0^*(n)e^{-ny},$$

$$\sum_{k=1}^{\infty}\frac{(-1)^{k-1}k}{e^{ky}+1} = \sum_{n=1}^{\infty}\tilde{\sigma}_1(n)e^{-ny},$$

$$\sum_{k=0}^{\infty}\frac{(-1)^k(2k+1)^2}{e^{(2k+1)y}-1} = \sum_{n=1}^{\infty}\sigma_2^*(n)e^{-ny},$$

$$\sum_{k=0}^{\infty}\sum_{l=1}^{\infty}\frac{(-1)^{k+l-1}l}{(e^{(2k+1)y}-1)(e^{ly}+1)} = \sum_{n=1}^{\infty}\left(\sum_{m=1}^{n-1}\sigma_0^*(m)\tilde{\sigma}_1(n-m)\right)e^{-ny},$$

on equating coefficients of e^{-ny} in (8.1), we obtain (2.3). Similarly, if we use [2, Chap. 17, Entries 15(ix), 17(i) and 17(ii)], we obtain Theorem 4.1, which is the companion formula to (2.3). It is clear that by using other similar equations, we can obtain further arithmetic identities.

Acknowledgments

The research of the third author was supported by Natural Sciences and Engineering Research Council of Canada grant A-7233. The authors would like to thank the unknown referee for his/her valuable comments.

References

[1] B. C. Berndt, Remarks on some of Ramanujan's number theoretical discoveries found in his second notebook, in *Number Theory,* ed. K. Alladi, Lecture Notes in Math., No. 1122 (Springer-Verlag, Berlin, 1985), pp. 47–55.

[2] B. C. Berndt, *Ramanujan's Notebooks Part III* (Springer-Verlag, New York, 1991).

[3] J. G. Huard, Z. M. Ou, B. K. Spearman and K. S. Williams, Elementary evaluation of certain convolution sums involving divisor functions, in *Number Theory for the Millennium II*, eds. M. A. Bennett *et al.* (A. K. Peters, Natick, Massachusetts, 2002), pp. 229–274.

[4] J. Liouville, Sur la forme $x^2+y^2+z^2+t^2+2(u^2+v^2)$, *J. Math. Pures Appl.* **9** (1864) 257–272.

[5] J. Liouville, Sur la forme $x^2+y^2+2(z^2+t^2+u^2+v^2)$, *J. Math. Pures Appl.* **9** (1864) 273–280.

[6] E. McAfee, A three term arithmetic formula of Liouville type with application to sums of six squares, M.Sc. Thesis, Carleton University, Ottawa, Canada (2004).