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On the Representations of a Positive Integer by the Forms $x^2 + y^2 + z^2 + 2t^2$ and $x^2 + 2y^2 + 2z^2 + 2t^2$

Kenneth S. Williams

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Abstract

In this paper we present elementary arithmetic proofs of Liouville's formulae for the number of representations of a positive integer by the forms $x^2 + y^2 + z^2 + 2t^2$ and $x^2 + 2y^2 + 2z^2 + 2t^2$.

Keywords: Liouville's formulae, quaternary quadratic forms, representations.

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1 Introduction

Let $\mathbb{N}$, $\mathbb{N}_0$ and $\mathbb{Z}$ denote the sets of positive integers, nonnegative integers and integers respectively. Let $n \in \mathbb{N}$. Set $n = 2^\alpha N$, where $\alpha \in \mathbb{N}_0, N \in \mathbb{N}$ and $\gcd(N, 2) = 1$. In 1861 Liouville [11] asserted without proof that the number $A(n)$ of solutions $(x, y, z, t) \in \mathbb{Z}^4$ of

$$n = x^2 + y^2 + z^2 + 2t^2 \tag{1.1}$$

is given by

$$A(n) = 2\left(2^{\alpha+2}\left(\frac{8}{N}\right) - 1\right)\sum_{d|N} d\left(\frac{8}{d}\right) \tag{1.2}$$

and the number $B(n)$ of solutions $(x, y, z, t) \in \mathbb{Z}^4$ of

$$n = x^2 + 2y^2 + 2z^2 + 2t^2 \tag{1.3}$$

is given by

$$B(n) = 2\left(2^{\alpha+1}\left(\frac{8}{N}\right) - 1\right)\sum_{d|N} d\left(\frac{8}{d}\right), \tag{1.4}$$

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where d runs through the positive integers dividing N and $\left(\frac{8}{d}\right)$ $(d \in \mathbb{N})$ is the Legendre-Jacobi-Kronecker symbol for discriminant 8, namely,

$$\left(\frac{8}{d}\right) = \begin{cases} 0, & \text{if } d \equiv 0 \pmod 2, \\ 1, & \text{if } d \equiv 1, 7 \pmod 8, \\ -1, & \text{if } d \equiv 3, 5 \pmod 8. \end{cases}$$

A search of the literature by the author found five proofs of (1.2) and two of (1.4). In 1884 Pepin [12, pp. 189-196] gave long proofs of (1.2) and (1.4) using Liouville's elementary methods and recurrence relations between $A(n)$ and $B(n)$ as well as between $A(2^\alpha N)$ and $A(N)$. In 1901 Petr [13, p. 8] gave some theta function identities from which a proof of (1.2) can be deduced. In 1964 Benz [5, pp. 168-175] gave proofs of (1.2) and (1.4) using theta functions and recurrence relations such as the easily proved relation $B(2n) = A(n)$ $(n \in \mathbb{N})$. In 1968 Demuth [6, pp. 241-243] used Siegel's mass formula to prove (1.2). In 1974 Wild [14] used modular forms to prove (1.2).

Recently Alaca, Alaca, Lemire and Williams [3] have used theta functions to give new analytic proofs of (1.2) and (1.4). They have treated the representation of integers by related quaternary quadratic forms in [1], [2] and [4].

In 2000 Huard, Ou, Spearman and Williams [8] gave an elementary identity, the proof of which requires only the manipulation of finite sums. We show that Liouville's formulae for $A(n)$ and $B(n)$ are simple consequences of this identity, thereby providing direct elementary arithmetic proofs of (1.2) and (1.4). No recurrence relations involving $A(n)$ and $B(n)$ are needed.

2 Notation

The Legendre-Jacobi-Kronecker symbols $\left(\frac{-4}{d}\right)$ and $\left(\frac{-8}{d}\right)$ $(d \in \mathbb{N})$ for discriminants -4 and -8 respectively are

$$\left(\frac{-4}{d}\right) = \begin{cases} 0, & \text{if } d \equiv 0 \pmod 2, \\ 1, & \text{if } d \equiv 1 \pmod 4, \\ -1, & \text{if } d \equiv 3 \pmod 4, \end{cases} \tag{2.1}$$

and

$$\left(\frac{-8}{d}\right) = \begin{cases} 0, & \text{if } d \equiv 0 \pmod 2, \\ 1, & \text{if } d \equiv 1, 3 \pmod 8, \\ -1, & \text{if } d \equiv 5, 7 \pmod 8. \end{cases} \tag{2.2}$$

We extend the definition of $\left(\frac{8}{d}\right)$ to all $d \in \mathbb{Z}$ by

$$\left(\frac{8}{-d}\right) = \left(\frac{8}{d}\right) \quad (d \in \mathbb{N}) \quad \text{and} \quad \left(\frac{8}{0}\right) = 0. \tag{2.3}$$

The following two results can be easily verified: if $a, b \in \mathbb{N}$ then

$$\left(\frac{8}{2a-b}\right) - \left(\frac{8}{2a+b}\right) = 2\left(\frac{-4}{a}\right)\left(\frac{-8}{b}\right), \tag{2.4}$$

and for $m \in \mathbb{N}$ we have

$$\sum_{k=1}^{m}\left(\frac{8}{k}\right) = \begin{cases} \frac{1}{2}\left(\frac{-8}{m}\right) + \frac{1}{2}\left(\frac{8}{m}\right), & \text{if } 2 \nmid m, \\ \left(\frac{-4}{m/2}\right), & \text{if } 2 \mid m \,. \end{cases} \tag{2.5}$$

For $n \in \mathbb{Z}$ we also set

$$\delta(n) = \begin{cases} 1, & \text{if } 2 \mid n, \\ 0, & \text{if } 2 \nmid n \,. \end{cases} \tag{2.6}$$

3 An Elementary Arithmetic Identity

We now state the elementary arithmetic identity due to Huard, Ou, Spearman and Williams [8] that we shall use.

Theorem 3.1. *Let $F : \mathbb{Z}^4 \to \mathbb{C}$ be such that*

$$F(a,b,x,y) - F(x,y,a,b) = F(-a,-b,x,y) - F(x,y,-a,-b)$$

for all $(a,b,x,y) \in \mathbb{Z}^4$. Then, for $n \in \mathbb{N}$, we have

$$\begin{aligned} &\sum_{\substack{(a,b,x,y)\in\mathbb{N}^4 \\ ax+by=n}} \Big(F(a,b,x,-y) - F(a,-b,x,y) + F(a,a-b,x+y,y) \\ &\qquad - F(a,a+b,y-x,y) + F(b-a,b,x,x+y) - F(a+b,b,x,x-y) \Big) \\ &= \sum_{\substack{d\in\mathbb{N} \\ d\mid n}} \sum_{x=1}^{d-1} \Big(F(0,n/d,x,d) + F(n/d,0,d,x) + F(n/d,n/d,d-x,-x) \\ &\qquad - F(x,x-d,n/d,n/d) - F(x,d,0,n/d) - F(d,x,n/d,0) \Big). \end{aligned}$$

4 Proof of the Formula for $A(n)$

Replacing n by $2n$ ($n \in \mathbb{N}$) in Theorem 3.1 and choosing

$$F(a,b,x,y) = \delta(a)\left(\frac{8}{b}\right)\delta(y), \tag{4.1}$$

we obtain after a short calculation using (2.4) and (2.5)

$$\sum_{\substack{(a,b,x,y)\in\mathbb{N}^4\\ ax+by=n}} \left(\frac{-4}{a}\right)\left(\frac{-8}{b}\right) = -\frac{1}{2}\sum_{d\mid n}\left(\frac{-4}{d}\right) - \frac{1}{4}\sum_{d\mid n}\left(\frac{-8}{d}\right) - \frac{1}{4}\sum_{d\mid n} d\left(\frac{8}{d}\right) + \sum_{d\mid n}\frac{n}{d}\left(\frac{8}{d}\right). \tag{4.2}$$

For $n \in \mathbb{N}_0$ we set

$$r(n) = |\{(x,y)\in\mathbb{Z}^2 : n = x^2+y^2\}| \quad \text{and} \quad s(n) = |\{(x,y)\in\mathbb{Z}^2 : n = x^2+2y^2\}|.$$

This notation is used in Section 5 as well. Clearly $r(0) = s(0) = 1$. It is well-known that

$$r(n) = 4\sum_{d\mid n}\left(\frac{-4}{d}\right) \text{ and } s(n) = 2\sum_{d\mid n}\left(\frac{-8}{d}\right) \text{ for } n\in\mathbb{N},$$

see, e.g., [7, p. 80]. These formulae are implicit in the work of Jacobi [9] on elliptic functions and are special cases of Dirichlet's formula [10] for discriminants -4 and -8. For each $n \in \mathbb{N}$, we have

$$\begin{aligned} A(n) &= \sum_{k=0}^{n} r(k)s(n-k) \\ &= r(n) + s(n) + \sum_{k=1}^{n-1} r(k)s(n-k) \\ &= 4\sum_{d\mid n}\left(\frac{-4}{d}\right) + 2\sum_{d\mid n}\left(\frac{-8}{d}\right) + 8\sum_{\substack{(a,b,x,y)\in\mathbb{N}^4\\ ax+by=n}}\left(\frac{-4}{a}\right)\left(\frac{-8}{b}\right). \end{aligned}$$

This, together with (4.2), yields that

$$A(n) = 8\sum_{d\mid n}\frac{n}{d}\left(\frac{8}{d}\right) - 2\sum_{d\mid n} d\left(\frac{8}{d}\right). \tag{4.3}$$

As

$$\sum_{d\mid n} d\left(\frac{8}{d}\right) = \sum_{d\mid N} d\left(\frac{8}{d}\right) \text{ and } \sum_{d\mid n}\frac{n}{d}\left(\frac{8}{d}\right) = 2^{\alpha}\left(\frac{8}{N}\right)\sum_{d\mid N} d\left(\frac{8}{d}\right), \tag{4.4}$$

we obtain (1.2). □

5 Proof of the Formula for $B(n)$

Let $n \in \mathbb{N}$. Choosing

$$F(a,b,x,y) = \delta(a)\left(\frac{8}{b}\right) \tag{5.1}$$

in Theorem 3.1, we obtain after a short calculation using (2.4) and (2.5)

$$\sum_{\substack{(a,b,x,y)\in\mathbb{N}^4\\ 2ax+by=n}} \left(\frac{-4}{a}\right)\left(\frac{-8}{b}\right) = -\frac{1}{2}\sum_{d|n/2}\left(\frac{-4}{d}\right) - \frac{1}{4}\sum_{d|n}\left(\frac{-8}{d}\right) - \frac{1}{4}\sum_{d|n} d\left(\frac{8}{d}\right) + \frac{1}{2}\sum_{d|n}\frac{n}{d}\left(\frac{8}{d}\right), \tag{5.2}$$

where the first sum on the right hand side of (5.2) is 0 if $n/2 \notin \mathbb{Z}$. Observe that

$$\begin{aligned} B(n) &= \sum_{0\le k\le n/2} r(k)s(n-2k) \\ &= r(n/2) + s(n) + \sum_{1\le k<n/2} r(k)s(n-2k) \\ &= 4\sum_{d|n/2}\left(\frac{-4}{d}\right) + 2\sum_{d|n}\left(\frac{-8}{d}\right) + 8\sum_{1\le k<n/2}\sum_{a|k}\left(\frac{-4}{a}\right)\sum_{b|n-2k}\left(\frac{-8}{b}\right) \\ &= 4\sum_{d|n/2}\left(\frac{-4}{d}\right) + 2\sum_{d|n}\left(\frac{-8}{d}\right) + 8\sum_{\substack{(a,b,x,y)\in\mathbb{N}^4\\ 2ax+by=n}}\left(\frac{-4}{a}\right)\left(\frac{-8}{b}\right). \end{aligned}$$

This, together with (5.2), yields that

$$B(n) = 4\sum_{d|n}\frac{n}{d}\left(\frac{8}{d}\right) - 2\sum_{d|n} d\left(\frac{8}{d}\right). \tag{5.3}$$

Appealing to (4.4) and (5.3), we obtain (1.4). □

Dedication

This paper is dedicated to my friend and coauthor Pierre Kaplan (18 April 1934 – 10 December 2006).

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Kenneth S. Williams: Centre for Research in Algebra and Number Theory, School of Mathematics and Statistics, Carleton University, Ottawa, Ontario, Canada K1S 5B6

E-mail address: kwilliam@connect.carleton.ca