ON LIOUVILLE’S TWELVE SQUARES THEOREM

KENNETH S. WILLIAMS

Centre for Research in Algebra and Number Theory
School of Mathematics and Statistics
Carleton University
Ottawa, Ontario, Canada K1S 5B6
e-mail: kwilliam@connect.carleton.ca

Abstract

A simple proof is given of a formula for the number of representations of a positive integer as the sum of twelve squares.

1. Introduction

Let \( q \) be a complex variable with \( |q| < 1 \). Following [1, p. 6] we set

\[
\varphi(q) := \sum_{n=-\infty}^{\infty} q^{n^2}.
\] (1.1)

Then, as in [1, p. 120], we set

\[
x := 1 - \frac{\varphi^4(-q)}{\varphi^4(q)}, \quad z := \varphi^2(q).
\] (1.2)

Let \( \mathbb{N} \) denote the set of positive integers. For \( k, n \in \mathbb{N} \) we define

\[
\sigma_k(n) = \sum_{d|n} q^k.
\] (1.3)

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If \( n \notin \mathbb{N} \) we set \( \sigma_k(n) = 0 \). The Eisenstein series \( E_{2k}(q) \) is defined by

\[
E_{2k}(q) := 1 + \frac{2}{\zeta(1 - 2k)} \sum_{n=1}^{\infty} \sigma_{2k-1}(n) q^n,
\]

(1.4)

where \( \zeta \) denotes the Riemann zeta function. For brevity we set

\[
R(q) := E_6(q) = 1 - 504 \sum_{n=1}^{\infty} \sigma_5(n) q^n.
\]

(1.5)

It is shown in [1, pp. 127, 128] that

\[
R(q) = (1 - 33x - 33x^2 + x^3)z^6
\]

(1.6)

and

\[
R(q^4) = \left( 1 - \frac{3}{2} x + \frac{15}{32} x^2 + \frac{1}{64} x^3 \right)z^6.
\]

(1.7)

Ramanujan’s discriminant function \( \Delta(q) \) is defined by

\[
\Delta(q) := q \prod_{n=1}^{\infty} (1 - q^n)^{24}.
\]

(1.8)

From [4, eq. (26), p. 392], we have

\[
\Delta(q^2) := \frac{1}{256} x^2(1 - x)^2 z^{12}.
\]

(1.9)

We define integers \( b(n) \) \((n \in \mathbb{N})\) by

\[
\sum_{n=1}^{\infty} b(n) q^n = q \prod_{n=1}^{\infty} (1 - q^{2n})^{12}
\]

(1.10)

so that

\[
\sum_{n=1}^{\infty} b(n) q^n = \Delta(q^{1/2})^{1/2} = \frac{1}{16} x(1 - x) z^6.
\]

(1.11)

We make use of (1.1), (1.2), (1.6), (1.7), (1.10) and (1.11) to determine a formula for the number \( \rho_{12}(n) \) of representations of \( n \) \((n \in \mathbb{N})\) as a sum of twelve squares, that is, for the quantity
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\[ n_{12}(n) := \text{card}(x_1, ..., x_{12}) \in \mathbb{Z}^{12} | n = x_1^2 + \cdots + x_{12}^2, \]

where \( \mathbb{Z} \) denotes the set of all integers. We prove

**Theorem.** Let \( n \in \mathbb{N} \). Then

\[ n_{12}(n) = 8\sigma_5(n) - 512\sigma_5(n/4) + 16b(n). \]

2. Proof of Theorem

We have

\[
\sum_{n=0}^{\infty} n_{12}(n)q^n = \varphi^{12}(q) \\
= z^6 \\
= \frac{1}{63} (1 - 33x - 33x^2 + x^3)z^6 \\
+ \frac{64}{63} \left( 1 - \frac{3}{2}x + \frac{15}{32}x^2 + \frac{1}{64}x^3 \right)z^6 + x(1 - x)z^6 \\
= \frac{1}{63}R(q) + \frac{64}{63}R(q^4) + 16 \sum_{n=1}^{\infty} b(n)q^n \\
= 1 + \sum_{n=1}^{\infty} (8\sigma_5(n) - 512\sigma_5(n/4) + 16b(n))q^n.
\]

Equating coefficients of \( q^n \) (\( n \in \mathbb{N} \)), we obtain the asserted formula for \( n_{12}(n) \).

From (1.10) we see that

\[ b(n) = 0, \text{ if } n = 0 \text{ (mod 2)}. \]

Hence

\[ n_{12}(n) = 8\sigma_5(n) - 512\sigma_5(n/4), \text{ if } n = 0 \text{ (mod 2)}. \]

This result was stated by Liouville [3] in a slightly different form. For other formulae for \( n_{12}(n) \), see [2].
References


[4] K. S. Williams, The convolution sum \( \sum_{m<n/8} \sigma(m)\sigma(n-8m) \), Pacific J. Math. 228 (2006), 387-396.