ON A DOUBLE SERIES OF CHAN AND ONG

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Abstract. An arithmetic identity is used to prove a relation satisfied by the double series \( \sum_{m,n=-\infty}^{\infty} q^{mn+2n^2} \). As an application an explicit formula is given for the number of representations of the positive integer \( n \) by the form \( x_1^2 + x_1x_2 + 2x_3^2 + x_4^2 + x_5x_6 + 2x_7^2 + x_8^2 + x_9x_8 + 2x_8^2 \).

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1. Introduction. Let \( \mathbb{N}, \mathbb{N}_0, \mathbb{Z}, \mathbb{R}, \mathbb{C} \) denote the sets of positive integers, nonnegative integers, integers, real numbers, complex numbers, respectively. For \( m \in \mathbb{Z} \) and \( n \in \mathbb{N} \) we define
\[
\sigma_m(n) := \sum_{d \in \mathbb{N}} d^m,
\]
where \( d \) runs through the positive integers dividing \( n \). We also set \( \sigma(n) = \sigma_1(n) = \sum_{d \mid n} d \) and \( d(n) = \sigma_0(n) = \sum_{d \mid n} 1 \). If \( n \not\in \mathbb{N} \), we set \( \sigma_m(n) = 0 \). The Bernoulli numbers \( B_0 = 1, B_1 = -\frac{1}{2}, B_2 = \frac{1}{6}, B_3 = 0, B_4 = -\frac{1}{30}, \ldots \) are defined by
\[
\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} B_n \frac{x^n}{n!}, \quad x \in \mathbb{R}, \quad |x| < 2\pi.
\]
The Eisenstein series \( E_k(q) \ (k \in \mathbb{N}) \) is defined by
\[
E_k(q) := 1 - \frac{4k}{B_{2k}} \sum_{n=1}^{\infty} \sigma_{2k-1}(n)q^n, \quad q \in \mathbb{C}, \quad |q| < 1.
\]
We set
\[
L(q) := E_1(q) = 1 - 24 \sum_{n=1}^{\infty} \sigma(n)q^n.
\]

In this paper we use a recent elementary arithmetic identity due to Huard, Ou, Spearman and Williams [3] to prove in Section 5 the following result, after some preliminary results are proved in Sections 2, 3 and 4.

Theorem 1.1. Let \( n \in \mathbb{N} \). Set \( n = 7^\alpha \beta \), where \( \alpha \in \mathbb{N}_0, \beta \in \mathbb{N} \) and \( \gcd(\beta, 7) = 1 \). Then the number of \( (x, y, z, t) \in \mathbb{Z}^4 \) such that
\[
n = x^2 + xy + 2y^2 + z^2 + zt + 2t^2
\]
is
\[
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In 1999 H. H. Chan and Y. L. Ong [2] introduced the two-dimensional theta series

\[ S(q) := \sum_{m,n=-\infty}^{\infty} q^{m^2 + mn + 2n^2}, \quad q \in \mathbb{C}, \quad |q| < 1. \]  

They proved a result equivalent to the following identity [2, Remark 3, p. 1742].

**Theorem 1.2.** \( S^2(q) = \frac{7}{6} L(q^7) - \frac{1}{6} L(q). \)

This identity is also equivalent to the one stated by Ramanujan as entry 5 of his second notebook [10] and first proved by Berndt [1, p. 467, entry 5(i)]. Both Berndt and Chan and Ong used modular equations of degree 7 in their proofs of Theorem 1.2. We show in Section 6 that Theorem 1.2 is a simple consequence of Theorem 1.1 and thus can be viewed as an elementary identity.

Klein and Fricke in their book [6, p. 400] gave an analytic proof of the following theorem.

**Theorem 1.3.** Let \( n \in \mathbb{N}. \) Then the number of \((x, y, z, t) \in \mathbb{Z}^4\) such that

\[ 4n = x^2 + y^2 + 7z^2 + 7t^2, \quad x \equiv z \pmod{2} \]

is

\[ 4 \sum_{d|n \atop \gcd(d,n) = 1} d. \]

We show in Section 7 that Theorem 1.3 is also an elementary consequence of Theorem 1.1, thus providing an elementary proof of Theorem 1.3. The elementary proof of Theorem 1.3 given by Humbert [4] is restricted to odd \( n. \)

Next, making use of a result, which was proved recently by Lemire and Williams [8, Lemma 4.6, p. 113] in order to evaluate the convolution sum

\[ \sum_{m \in \mathbb{N} \atop 1 \leq m < \frac{n}{7}} \sigma(m) \sigma(n - 7m), \]

in conjunction with Theorem 1.2, we prove in Section 8 the following result.

**Theorem 1.4.** Let \( n \in \mathbb{N}. \) Then the number of \((x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8) \in \mathbb{Z}^8\) such that

\[ n = x_1^2 + x_1 x_2 + 2x_2^2 + x_3^2 + x_3 x_4 + 2x_4^2 + x_5^2 + x_5 x_6 + 2x_6^2 + x_7^2 + x_7 x_8 + 2x_8^2 \]

is given by

\[ \frac{24}{5} \sigma_3(n) + \frac{1176}{5} \sigma_3 \left( \frac{n}{7} \right) + \frac{16}{5} c_7(n), \]
where the \( c_7(n) \) \((n \in \mathbb{N})\) are integers defined by

\[
\sum_{n=1}^{\infty} c_7(n)q^n = q \left( \prod_{n=1}^{\infty} (1-q^n)^{16}(1-q^{7n})^8 + 13q \prod_{n=1}^{\infty} (1-q^n)^{12}(1-q^{7n})^{12} + 49q^2 \prod_{n=1}^{\infty} (1-q^n)^{8}(1-q^{7n})^{16} \right)^{\frac{1}{2}}. \tag{1.6}
\]

This result should be compared with that of Kachakhidze [5].

Finally, we make use of a classical identity of Jacobi, which is given for example in [7, Corollary 6, p. 37], to prove the following formula for \( c_7(n) \) \((n \in \mathbb{N})\) in Section 9.

**Theorem 1.5.** For \( n \in \mathbb{N} \) we have

\[
c_7(n) = -1 \sum_{(r,s) \in \mathbb{N}_0^2} \frac{r(r+1)}{2} + r s, \quad r(s+1), \quad s(n-1) + 1 = n-1
\]

\[
+ 2 \sum_{(r,s) \in \mathbb{N}_0^2 \times \mathbb{N}} (-1)^{r+s}(2r+1)(2s+1) \sum_{d \in \mathbb{N}} \left( \frac{-7}{d} \right) .
\]

Here

\[
\left( \frac{-7}{d} \right) = \begin{cases} 
1, & \text{if } d \equiv 1, 2, 4 \pmod{7}, \\
-1, & \text{if } d \equiv 3, 5, 6 \pmod{7}, \\
0, & \text{if } d \equiv 0 \pmod{7}, 
\end{cases}
\]

is the Legendre–Jacobi–Kronecker symbol for discriminant \(-7\).

2. **Some properties of** \( F_k(n) \). For \( k \in \mathbb{N} \) and \( n \in \mathbb{Z} \) we define

\[
F_k(n) := \begin{cases} 
1, & \text{if } k \mid n, \\
0, & \text{if } k \nmid n. 
\end{cases}
\tag{2.1}
\]

Let \( a \in \mathbb{Z} \). Denote the gcd of \( k \) and \( a \) by \( (k,a) \). Clearly

\[
F_k(an) = F_{k,(k,a)}(n). \tag{2.2}
\]

For \( x \in \mathbb{R} \) we denote the greatest integer less than or equal to \( x \) by \( \lfloor x \rfloor \). The following results are easily proved:

\[
\sum_{d \mid n} F_k(d) = d \left( \frac{n}{k} \right); \tag{2.3}
\]

\[
\sum_{d \mid n} dF_k(d) = k \sigma \left( \frac{n}{k} \right); \tag{2.4}
\]

\[
\sum_{d \mid n} \frac{n}{d} F_k(d) = \sigma \left( \frac{n}{k} \right); \tag{2.5}
\]
Adding (2.9) and (2.12) we obtain

\[ \sum_{d|n} F_k(l) = \sum_{d|n} \left\lfloor \frac{d}{2k} \right\rfloor - \sum_{d|n} \left\lfloor \frac{d}{2k} \right\rfloor, \quad \text{if } 2|k; \]

\[ \sum_{d|n} F_k(l) = \left\lfloor \frac{2d}{k} \right\rfloor, \quad \text{if } 2|k. \]

3. An identity of Huard, Ou, Spearman and Williams. Using nothing more than the rearrangement of terms in finite sums, Huard, Ou, Spearman and Williams [3] proved the following elementary arithmetic formula.
Theorem 3.1. Let $F : \mathbb{Z}^4 \to \mathbb{C}$ be such that
\[
F(a, b, x, y) - F(x, y, a, b) = F(-a, -b, x, y) - F(x, y, -a, -b)
\]
for all $(a, b, x, y) \in \mathbb{Z}^4$. Then, for $n \in \mathbb{N}$, we have
\[
\sum_{(a, b, x, y) \in \mathbb{N}^4 \atop az + by = n} \left( F(a, b, x, -y) - F(a, -b, x, y) + F(a, a - b, x + y, y) - F(a + b, b, x, x + y) \right)
\]
\[
= \sum_{d \in \mathbb{N}} \left( F(0, n/d, x, d) + F(n/d, 0, d, x) + F(n/d, n/d, d - x, -x) - F(x, d - n/d, n/d, n/d) - F(d, x, n/d, 0) \right).
\]

Taking $F(a, b, x, y) = f(b)$ in Theorem 3.1, where $f : \mathbb{Z} \to \mathbb{C}$ is an even function, we obtain

Corollary 3.1. Let $f : \mathbb{Z} \to \mathbb{C}$ be an even function. Then for $n \in \mathbb{N}$ we have
\[
\sum_{(a, b, x, y) \in \mathbb{N}^4 \atop az + by = n} \left( f(a - b) - f(a + b) \right)
\]
\[
= f(0)(\sigma(n) - d(n)) + \sum_{d \mid n} f(d) - \sum_{d \mid n} df(d) + 2 \sum_{d \mid n} \frac{n}{d} f(d) - 2 \sum_{d \mid n} \sum_{1 \leq t \leq d} f(l).
\]

Corollary 3.1 was stated but not proved by Liouville in [9]. Replacing $n$ by $2n$ in Theorem 3.1, and choosing $F(a, b, x, y) = F_2(a)f(b)F_2(y)$, where $f : \mathbb{Z}^4 \to \mathbb{C}$ is even, we obtain

Corollary 3.2. Let $f : \mathbb{Z} \to \mathbb{C}$ be an even function. Then for $n \in \mathbb{N}$ we have
\[
\sum_{(a, b, x, y) \in \mathbb{N}^4 \atop az + by = n} \left( f(2a - b) - f(2a + b) \right)
\]
\[
= f(0) \left( \frac{1}{2} \sigma(n) - \frac{1}{2} d(n) - \frac{1}{2} d \left( \frac{n}{2} \right) \right)
\]
\[
+ \frac{1}{2} \sum_{d \mid n} f(d) - \frac{1}{2} \sum_{d \mid n} df(d) + 2 \sum_{d \mid n} \frac{n}{d} f(d)
\]
\[
+ \frac{1}{2} \sum_{d \mid n} f(2d) + \frac{1}{2} \sum_{d \mid n} \frac{n}{d} f(2d) - \sum_{d \mid n} \sum_{1 \leq l \leq 2d} f(l).
\]
Let $k \in \mathbb{N}$. Taking $f(x) = F_k(x)$ ($x \in \mathbb{Z}$) in Corollary 3.1 and appealing to (2.3), (2.4), (2.5) and (2.6), we obtain

**Theorem 3.2.** Let $k, n \in \mathbb{N}$. Then

$$
\sum_{(a,b,x,y) \in \mathbb{N}^4 \atop ax + by = n} \left( F_k(a - b) - F_k(a + b) \right)
= \sigma(n) - (k - 2)\sigma\left(\frac{n}{k}\right) - d(n) + d\left(\frac{n}{k}\right) - 2 \sum_{d|n} \left\lfloor \frac{d}{k} \right\rfloor.
$$

Finally, taking $f(x) = F_k(x)$ ($x \in \mathbb{Z}$) in Corollary 3.2, and appealing to (2.2), (2.3), (2.4), (2.5), (2.6) and (2.13), we obtain

**Theorem 3.3.** Let $k, n \in \mathbb{N}$. Then if $k$ is odd we have

$$
\sum_{(a,b,x,y) \in \mathbb{N}^4 \atop ax + by = n} (F_k(2a - b) - F_k(2a + b))
= \frac{1}{2} \sigma(n) + \frac{(5-k)}{2} \sigma\left(\frac{n}{k}\right) - \frac{1}{2} d(n) - \frac{1}{2} d\left(\frac{n}{2}\right) + d\left(\frac{n}{k}\right)

- \sum_{d|n} \left\lfloor \frac{2d}{k} \right\rfloor - \sum_{d|n/2} \left\lfloor \frac{d}{k} \right\rfloor
- \sum_{d|n} \left\lfloor \frac{d + k}{2k} \right\rfloor + \sum_{d|n/2} \left\lfloor \frac{2d + k}{2k} \right\rfloor
$$

and if $k$ is even

$$
\sum_{(a,b,x,y) \in \mathbb{N}^4 \atop ax + by = n} (F_k(2a - b) - F_k(2a + b))
= \frac{1}{2} \sigma(n) + \frac{(4-k)}{2} \sigma\left(\frac{n}{k}\right) + \frac{1}{2} \sigma\left(\frac{n}{k/2}\right) - \frac{1}{2} d(n) - \frac{1}{2} d\left(\frac{n}{2}\right) + \frac{1}{2} d\left(\frac{n}{k}\right)

+ \frac{1}{2} d\left(\frac{n}{k/2}\right) - \sum_{d|n} \left\lfloor \frac{2d}{k} \right\rfloor - \sum_{d|n/2} \left\lfloor \frac{d}{k} \right\rfloor
- \sum_{d|n} \left\lfloor \frac{d + k}{2k} \right\rfloor
\sum_{d|n/2} \left\lfloor \frac{2d + k}{2k} \right\rfloor
$$

4. **Evaluation of some finite sums.** Our task in this section is to give the values of the sums $\sum_{d|n} \left\lfloor \frac{d}{k} \right\rfloor$, $\sum_{d|n} \left\lfloor \frac{d}{k} \right\rfloor$. $\sum_{d|n} \left\lfloor \frac{2d}{k} \right\rfloor$, $\sum_{d|n/2} \left\lfloor \frac{d}{k} \right\rfloor$, $\sum_{d|n/2} \left\lfloor \frac{d + k}{2k} \right\rfloor$ and $\sum_{d|n/2} \left\lfloor \frac{2d + k}{2k} \right\rfloor$ occurring in Theorems 3.2 and 3.3 in the special case where $k = 7$. 
For $a \in \mathbb{Z}$ and $m, n \in \mathbb{N}$ we define
\[
d_{a,m}(n) := \sum_{d|n, \quad d \equiv a \pmod{m}} 1,
\]
so that
\[
\sum_{a=0}^{m-1} d_{a,m}(n) = d(n). \tag{4.2}
\]
In particular we set
\[
d_i := d_{i,7}(n), \quad i = 0, 1, 2, 3, 4, 5, 6, \tag{4.3}
\]
and
\[
e_i := d_{i,14}(n), \quad i = 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13. \tag{4.4}
\]
Clearly,
\[
d_i = e_i + e_{i+7}, \quad i = 0, 1, 2, 3, 4, 5, 6. \tag{4.5}
\]
Also,
\[
d_0 = d_{0,7}(n) = \sum_{d|n, \quad d \equiv 0 \pmod{7}} 1 = \sum_{d|n/7} 1 = d\left(\frac{n}{7}\right) \tag{4.6}
\]
and, similarly,
\[
e_0 = d\left(\frac{n}{14}\right). \tag{4.7}
\]
Thus
\[
e_7 = d_0 - e_0 = d\left(\frac{n}{7}\right) - d\left(\frac{n}{14}\right). \tag{4.8}
\]
We need the following results, all of which are simple to prove.
\[
d(n) = e_0 + e_1 + e_2 + \cdots + e_{13}. \tag{4.9}
\]
\[
d\left(\frac{n}{2}\right) = e_0 + e_2 + e_4 + \cdots + e_{12}. \tag{4.10}
\]
\[
d\left(\frac{n}{7}\right) = e_0 + e_7. \tag{4.11}
\]
\[
d_{i,7}\left(\frac{n}{2}\right) = e_{2i}, \quad i = 0, 1, 2, \ldots, 6. \tag{4.12}
\]
\[
\sum_{d|n, \quad \left[d\right] \equiv \left[\frac{n}{7}\right]} d \cdot \sigma(n) - \frac{1}{7} \sum_{i=1}^{7} (e_1 + 2e_2 + 3e_3 + 4e_4 + 5e_5 + 6e_6 + e_8 + 2e_9 + 3e_{10} + 4e_{11} + 5e_{12} + 6e_{13}). \tag{4.13}
\]
\[
\sum_{d|n, \quad \left[2d\right] \equiv \left[\frac{n}{7}\right]} \frac{2}{7} \sigma(n) - \frac{1}{7} (2e_1 + 4e_2 + 6e_3 + e_4 + 3e_5 + 5e_6 + 2e_8 + 4e_9 + 6e_{10} + e_{11} + 3e_{12} + 5e_{13}). \tag{4.14}
\]
We are now in a position to prove the three theorems that we will need in the proof of Theorem 1.1 in Section 5.

**Theorem 4.1.** Let \( n \in \mathbb{N} \). Then

\[
\sum_{d|n} \left\lfloor \frac{d+7}{14} \right\rfloor = \frac{1}{14} \sigma(n) - \frac{1}{14}(e_1 + 2e_2 + 3e_3 + 4e_4 + 5e_5 + 6e_6 - 7e_7 - 6e_8 - 5e_9 - 4e_{10} - 3e_{11} - 2e_{12} - e_{13}).
\]

**Proof.** This result follows by taking \( k = 7 \) in Theorem 3.2 and appealing to (4.5), (4.9), (4.11) and (4.13).

**Theorem 4.2.** Let \( n \in \mathbb{N} \). Then

\[
\sum_{d|n/2} \left\lfloor \frac{d}{7} \right\rfloor = \frac{1}{7} \sigma(\frac{n}{2}) - \frac{1}{7}(e_2 + 2e_4 + 3e_6 + 4e_8 + 5e_{10} + 6e_{12}).
\]

**Theorem 4.3.**

For all \( a, b \in \mathbb{N} \)

\[
\sum_{x+y=n} \left( F_7(a-b) - F_7(a+b) \right)
\]

\[
= \frac{5}{7} \sigma(n) - 5\sigma\left(\frac{n}{7}\right) - \frac{5}{7}d_1 - \frac{3}{7}d_2 - \frac{1}{7}d_3 + \frac{3}{7}d_4 + \frac{3}{7}d_5 + \frac{5}{7}d_6.
\]

**Proof.** This result follows by taking \( k = 7 \) in Theorem 3.3 and appealing to (4.5), (4.9), (4.10), (4.11), (4.14), (4.15), (4.16) and (4.17).

**Theorem 4.4.**

For all \( a, b \in \mathbb{N} \)

\[
\sum_{x+y=n} \left( F_7(2a-b) - F_7(2a+b) \right)
\]

\[
= \frac{1}{7} \sigma(n) - \sigma\left(\frac{n}{7}\right) - \frac{2}{7}d_1 - \frac{4}{7}d_2 - \frac{4}{7}d_3 - \frac{2}{7}d_4 + \frac{1}{7}d_5 + \frac{1}{7}d_6.
\]

**Proof.** This result follows by taking \( k = 7 \) in Theorem 3.3 and appealing to (4.5), (4.9), (4.10), (4.11), (4.14), (4.15), (4.16) and (4.17).

**Theorem 4.5.**

For all \( a, b \in \mathbb{N} \)

\[
\left( \begin{array}{c} -7 \\ ab \end{array} \right) = \left( F_7(a-b) - F_7(a+b) \right) + \left( F_7(a-2b) - F_7(a+2b) \right)
\]

\[
+ \left( F_7(2a-b) - F_7(2a+b) \right).
\]

**Proof.** If \( a \equiv 0 \pmod{7} \) or \( b \equiv 0 \pmod{7} \) both the left-hand side and right-hand side of the asserted formula are zero. Thus we may suppose that \( a \not\equiv 0 \pmod{7} \) and \( b \not\equiv 0 \pmod{7} \). Define \( c \not\equiv 0 \pmod{7} \) by \( a \equiv bc \pmod{7} \). Then the assertion of the theorem becomes

\[
\left( \begin{array}{c} -7 \\ c \end{array} \right) = \left( F_7(c-1) - F_7(c+1) \right) + \left( F_7(c-2) - F_7(c+2) \right).
\]
This is easily checked for the six cases \( c \equiv 1, 2, 3, 4, 5, 6 \pmod{7} \).

5. Proof of Theorem 1.1. For \( m \in \mathbb{N}_0 \) we let

\[
 r(m) = \text{number of } (x, y) \in \mathbb{Z}^2 \text{ such that } x^2 + xy + 2y^2 = m. \tag{5.1}
\]

Clearly,

\[
 r(0) = 1. \tag{5.2}
\]

For \( n \in \mathbb{N} \) it is a classical result that

\[
 r(n) = 2 \sum_{d \mid n} \left( \frac{-7}{d} \right). \tag{5.3}
\]

Thus

\[
 r(n) = 2d_1 + 2d_2 - 2d_3 + 2d_4 - 2d_5 - 2d_6. \tag{5.4}
\]

The number of \( (x, y, z, t) \in \mathbb{Z}^4 \) such that

\[
 n = x^2 + xy + 2y^2 + z^2 + zt + 2t^2
\]

is (appealing to (5.2), (5.3), Theorem 4.3, Theorem 4.1, Theorem 4.2 and (5.4))

\[
 \sum_{(k,l) \in \mathbb{N}_0^2} \sum_{k+l=n} r(k)r(l) = 2r(n) + \sum_{k=1}^{n-1} r(k)r(n-k)
\]

\[
 = 2r(n) + \sum_{k=1}^{n-1} \left( 2 \sum_{a \mid k} \left( \frac{-7}{a} \right) \right) \left( 2 \sum_{b \mid n-k} \left( \frac{-7}{b} \right) \right)
\]

\[
 = 2r(n) + 4 \sum_{(a,b,x,y) \in \mathbb{N}^4} \left( \frac{-7}{ab} \right)
\]

\[
 = 2r(n) + 4 \sum_{(a,b,x,y) \in \mathbb{N}^4} \left( F_7(a - b) - F_7(a + b) \right)
\]

\[
 + 4 \sum_{(a,b,x,y) \in \mathbb{N}^4} \left( F_7(a - 2b) - F_7(a + 2b) \right)
\]

\[
 + 4 \sum_{(a,b,x,y) \in \mathbb{N}^4} \left( F_7(2a - b) - F_7(2a + b) \right)
\]

\[
 = 2r(n) + 4 \sum_{(a,b,x,y) \in \mathbb{N}^4} \left( F_7(a - b) - F_7(a + b) \right)
\]
This completes the proof of Theorem 1.1.

6. Proof of Theorem 1.2. We have by (1.5), Theorem 1.1 and (1.4)

\[ S^2(q) = \sum_{x,y,z,t \in \mathbb{Z}} q^{x^2 + xy + 2y^2 + z^2 + zt + 2t^2} \]

\[ = 1 + \sum_{n=1}^{\infty} \left( 4\sigma(n) - 28\sigma\left(\frac{n}{7}\right) \right) q^n \]

\[ = \frac{7}{6} L(q^7) - \frac{1}{6} L(q). \]

7. Proof of Theorem 1.3. Let \( n \in \mathbb{N} \). Set

\[ A(n) := \{(x,y,z,t) \in \mathbb{Z}^4 \mid 4n = x^2 + y^2 + 7z^2 + 7t^2, x \equiv z \pmod{2} \} \quad (7.1) \]

and

\[ B(n) := \{(x,y,z,t) \in \mathbb{Z}^4 \mid n = x^2 + xy + 2y^2 + z^2 + zt + 2t^2 \}. \quad (7.2) \]

Let \((x,y,z,t) \in A(n)\). Then \( 4n = x^2 + y^2 + 7z^2 + 7t^2 \) and \( x \equiv z \pmod{2} \) so \( \frac{x-z}{2} \in \mathbb{Z} \) and

\[ y-t \equiv y^2 - t^2 \equiv y^2 + 7t^2 = 4n - x^2 - 7z^2 \equiv x - z \equiv 0 \pmod{2} \]

so \( \frac{y-t}{2} \in \mathbb{Z} \). Further

\[ \left( \frac{x-z}{2} \right)^2 + \left( \frac{x-z}{2} \right) z + 2z^2 + \left( \frac{y-t}{2} \right)^2 + \left( \frac{y-t}{2} \right) t + 2t^2 \]

\[ = \frac{1}{4}(x^2 + 7z^2 + y^2 + 7t^2) = n \]

so \( \left( \frac{x-z}{2}, z, \frac{y-t}{2}, t \right) \in B(n) \). Thus we can define \( \lambda : A(n) \to B(n) \) by

\[ \lambda((x,y,z,t)) = \left( \frac{x-z}{2}, z, \frac{y-t}{2}, t \right). \]
Clearly, $\lambda$ is injective. Let $(x_1, y_1, z_1, t_1) \in B(n)$. Set $x = 2x_1 + y_1 \in \mathbb{Z}$, 
$y = 2z_1 + t_1 \in \mathbb{Z}$, 
$z = y_1 \in \mathbb{Z}$, 
$t = t_1 \in \mathbb{Z}$. Clearly, $x \equiv y \equiv z \pmod{2}$. Also 
\[x^2 + y^2 + 7z^2 + 7t^2 = (2x_1 + y_1)^2 + (2z_1 + t_1)^2 + 7y_1^2 + 7t_1^2 = 4(x_1^2 + x_1y_1 + 2y_1^2 + z_1^2 + z_1t_1 + 2t_1^2) = 4n.\]
Hence $(x, y, z, t) \in A(n)$. Moreover 
\[\lambda((x, y, z, t)) = \left(\frac{x - z}{2}, \frac{z - t}{2}, \frac{y - t}{2}, t\right) = (x_1, y_1, z_1, t_1)\]
so $\lambda$ is surjective. Thus $\lambda$ is a bijection and we have by Theorem 1.1
\[\text{card } A(n) = \text{card } B(n) = 4\sigma(n) - 28\sigma\left(\frac{n}{7}\right) = 4\sum_{d|n} d \quad \text{as asserted.} \]

8. Proof of Theorem 1.4. Let $n \in \mathbb{N}$. Let $N(n)$ denote the number of 
$(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8) \in \mathbb{Z}^8$ such that
\[n = x_1^2 + x_1x_2 + 2x_2^2 + x_3^2 + x_3x_4 + 2x_4^2 + x_5^2 + x_5x_6 + 2x_6^2 + x_7^2 + x_7x_8 + 2x_8^2.\]
Then by Theorem 1.2 we have
\[\sum_{n=0}^{\infty} N(n)q^n = \left(\sum_{(x,y) \in \mathbb{Z}^2} q^{x^2+y+2y^2}\right)^4 = S^4(q) = \frac{1}{36} \left(L(q) - 7L(q^7)\right)^2.\]
Appealing to [8, Lemma 4.6, p. 113] we obtain
\[\sum_{n=0}^{\infty} N(n)q^n = 1 + \sum_{n=1}^{\infty} \left(\frac{24}{5}\sigma_3(n) + \frac{1176}{5}\sigma_3\left(\frac{n}{7}\right) + \frac{16}{5}c_7(n)\right)q^n.\]
Equating coefficients of $q^n$ ($n \in \mathbb{N}$), we obtain the asserted result. \(\square\)

9. Proof of Theorem 1.5. As in [8, equation (4.1), p. 112] we define
\[H = \left(\frac{A^7}{C} + 13qA^3C^3 + 49q^2C^7\right)^{\frac{1}{8}}, \quad (9.1)\]
where
\[A := \prod_{n=1}^{\infty}(1 - q^n), \quad C := \prod_{n=1}^{\infty}(1 - q^{7n}). \quad (9.2)\]
From the proof of Theorem 1.4 and [8, Lemma 4.2, p. 112] we have
\[S^4(q) = \frac{1}{36} \left(L(q) - 7L(q^7)\right)^2 = H^4,\]
so that $S(q) = \omega(q)H(q)$, where $\omega(q)^4 = 1$. From (1.5) and (9.1) we find for $|q| < 1$ that $S(q) = 1 + 2q + 4q^2 + O(q^3)$ and $H = 1 + 2q + 4q^2 + O(q^3)$ so that $\omega(q) = 1$ and
\[H = S(q). \quad (9.3)\]
This also follows from [2, Lemma 2.2, p. 1737] (with a typo corrected). Next, by [8, Lemma 4.4, p. 112] (with a typo corrected) and (9.3) we have

\[ \sum_{n=1}^{\infty} c_{7}(n)q^n = qA^3C^3H = qA^3C^3S(q). \]  

(9.4)

Now, by (1.5), (5.1), (5.2) and (5.3), we have

\[ S(q) = \sum_{x,y=-\infty}^{\infty} q^{x^2+xy+y^2} = \sum_{n=0}^{\infty} r(n)q^n = 1 + 2 \sum_{n=1}^{\infty} \sum_{d|n} \left( \frac{-7}{d} \right) q^n. \]  

(9.5)

Hence, from (9.2), (9.4) and (9.5), we deduce

\[ \sum_{n=1}^{\infty} c_{7}(n)q^n = q \prod_{n=1}^{\infty} (1 - q^n)^3 \prod_{n=1}^{\infty} (1 - q^{7n})^3 \times \left( 1 + 2 \sum_{n=1}^{\infty} \sum_{d|n} \left( \frac{-7}{d} \right) q^n \right). \]  

(9.6)

By Jacobi’s identity [7, Corollary 6, p. 37]

\[ \prod_{n=1}^{\infty} (1 - q^n)^3 = \sum_{r=0}^{\infty} (-1)^r(2r + 1)q^{r(r+1)} \],

equation (9.6) becomes

\[ \sum_{n=1}^{\infty} c_{7}(n)q^{n-1} = \left( \sum_{r=0}^{\infty} (-1)^r(2r + 1)q^{r(r+1)} \right) \left( \sum_{s=0}^{\infty} (-1)^s(2s + 1)q^{s(s+1)} \right) \left( \sum_{t=1}^{\infty} \sum_{d|t} \left( \frac{-7}{d} \right) q^t \right) \]

\[ + 2 \left( \sum_{r=0}^{\infty} (-1)^r(2r + 1)q^{r(r+1)} \right) \left( \sum_{s=0}^{\infty} (-1)^s(2s + 1)q^{s(s+1)} \right) \left( \sum_{t=1}^{\infty} \sum_{d|t} \left( \frac{-7}{d} \right) q^t \right) \]

\[ = \sum_{n=1}^{\infty} \left( \sum_{r,s=0}^{r(r+1)+s(s+1)=n-1} (-1)^{r+s}(2r + 1)(2s + 1) \right) q^{n-1} \]

\[ + 2 \sum_{n=1}^{\infty} \left( \sum_{r,s=0}^{r(r+1)+s(s+1)+t=n-1} (-1)^{r+s+t}(2r + 1)(2s + 1) \right) \sum_{d|t} \left( \frac{-7}{d} \right) q^{n-1}. \]

Equating coefficients of \( q^{n-1} \) \( (n \in \mathbb{N}) \) we obtain the asserted formula for \( c_{7}(n) \).

\[ \square \]

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ON A DOUBLE SERIES OF CHAN AND ONG

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