INTEGRAL BASES FOR AN INFINITE FAMILY OF CYCLIC QUINTIC FIELDS

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Abstract. An explicit integral basis is given for infinitely many cyclic quintic fields.

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1. Introduction. We denote the set of integers by \( \mathbb{Z} \) and the set of positive integers by \( \mathbb{N} \). Let \( n \in \mathbb{Z} \). The Lehmer quintic \( f_n(x) \in \mathbb{Z}[x] \) is defined by

\[
f_n(x) = x^5 + n^2x^4 - (2n^3 + 6n^2 + 10n + 10)x^3 \\
+ (n^4 + 5n^3 + 11n^2 + 15n + 5)x^2 + (n^3 + 4n^2 + 10n + 10)x + 1,
\]

see [5, p. 539]. Schoof and Washington [6, p. 548] have shown that \( f_n(x) \) is irreducible for all \( n \in \mathbb{Z} \). Let \( \theta \in \mathbb{C} \) be a root of \( f_n(x) = 0 \). Set \( K = \mathbb{Q}(\theta) \) so that \( [K : \mathbb{Q}] = 5 \). It is known that \( K \) is a cyclic field [6, p. 548]. We denote the ring of integers of \( K \) by \( \mathcal{O}_K \). The discriminant \( d(K) \) of \( K \) has been determined by Jeannin [4, p. 76], see also Spearman and Williams [7, p. 215], namely \( d(K) = f(K)^4 \), where the conductor \( f(K) \) of \( K \) is given by

\[
f(K) = 5^b \prod_{p \equiv 1 \pmod{5}} p, \quad v_p(n^4 + 5n^3 + 15n^2 + 25n + 25) \neq 0 \pmod{5},
\]

where \( v_p(k) \) denotes the exponent of the largest power of the prime \( p \) dividing the nonzero integer \( k \) and

\[
b = \begin{cases} 
0, & \text{if } 5 \nmid n, \\
2, & \text{if } 5 \mid n.
\end{cases}
\]

Set

\[
m = n^4 + 5n^3 + 15n^2 + 25n + 25 \in \mathbb{Z},
\]

\[
d = n^3 + 5n^2 + 10n + 7 \in \mathbb{Z},
\]

\[
a = m^3 - 10m^2 + 5m \in \mathbb{Z}.
\]

From (1.3) we have

\[
m = (n + 2)(n + 1)(n + 1)^2 + 6 + 11
\]

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and, as \((n + 2)(n + 1) \geq 0\) for all \(n \in \mathbb{Z}\), we deduce that \(m \geq 11\) so that
\[(1.6) \quad m \in \mathbb{N}.
\]
Then, from (1.5), we obtain \(a = m^2(m - 10) + 5m \geq 176\) so that
\[(1.7) \quad a \in \mathbb{N}.
\]
As \(x^3 + 5x^2 + 10x + 7\) is irreducible in \(\mathbb{Z}[x]\), we deduce from (1.4) that
\[(1.8) \quad d \neq 0.
\]
A MAPLE calculation gives
\[(1.9) \quad a = (n^3 + 5n^2 + 10n + 7)(n^9 + 10n^8 + 60n^7 + 243n^6 + 730n^5 + 1650n^4 + 2824n^3 + 3520n^2 + 2990n + 1357) + 1.
\]
From (1.2) and (1.3) we observe that
\[(1.10) \quad 5^b \mid m.
\]
From (1.4) and (1.9) we see that
\[(1.11) \quad a = 1 + dk,
\]
where
\[(1.12) \quad k = n^9 + 10n^8 + 60n^7 + 243n^6 + 730n^5 + 1650n^4 + 2824n^3 + 3520n^2 + 2990n + 1357 \in \mathbb{Z} \setminus \{0\}.
\]
Gaál and Pohst [2, p. 1690] have shown that under the condition
\[(1.13) \quad p^2 \nmid m \text{ for any prime } p \neq 5
\]
an integral basis for \(K\) is given by
\[(1.14) \quad \{1, \theta, \theta^2, \theta^3, \omega_3\},
\]
where
\[(1.15) \quad \omega_3 = \frac{1}{d} \left((n + 2) + (2n^2 + 9n + 9)\theta + (2n^2 + 4n - 1)\theta^2 + (-3n - 4)\theta^3 + \theta^4\right).
\]
Although it is very likely that there are infinitely many \(n \in \mathbb{Z}\) such that (1.13) holds this has not yet been proved. Gaál and Pohst used their integral basis in a search for cyclic quintic fields with a power basis. They proved under the condition that \(m\) is squarefree that the field \(K\) admits a power basis if and only if \(n = -1\) or \(n = -2\) [2, Theorem, p. 1695], and noted that these values of \(n\) give the same field \(K\) [2, p. 1689]. They also observed [2, Remark, p. 1695] that their result is a special case of a theorem of Gras [3], which asserts that there is only one cyclic quintic field with a power basis, namely, the maximal real subfield of the cyclotomic field of 11-th roots of unity.

In this work we give an integral basis for \(K\) under the weaker condition
\[(1.16) \quad m \text{ is cubefree}.
\]
From now on we assume that (1.16) holds except in Lemma 2.2. In view of (1.6), (1.10) and (1.16), we have

\begin{equation}
m = 5^b PQ^2,
\end{equation}

where \( b \) is given by (1.2) and \( P, Q \in \mathbb{N} \) are such that

\begin{equation}
5 \mid P, \quad 5 \mid Q, \quad (P, Q) = 1, \quad P, Q \text{ squarefree}.
\end{equation}

By [4, Lemme 2.1.1] every prime factor \((\neq 5)\) of \( m \) is \( \equiv 1 \pmod{5} \). Hence, by (1.1), we have

\begin{equation}
f(K) = 5^b PQ
\end{equation}

and

\begin{equation}
p \text{ (prime)} \mid PQ \implies p \equiv 1 \pmod{5}.
\end{equation}

By (1.17) we have \( Q \mid m \). By (1.5) we have \( m \mid a \). Hence \( Q \mid a \). Then, by (1.11), we have \( m \mid 1 + dk \) from which we deduce

\begin{equation}
(d, Q) = 1.
\end{equation}

We define

\begin{equation}
v_4 = \frac{1}{Q} \left( \theta - \frac{n^2}{5}(Q - 1) \right)^3 \in K
\end{equation}

and

\begin{equation}
v_5 = \frac{ad\omega_5 + (1 - a)Qv_4\theta}{dQ} \in K.
\end{equation}

We note that (1.8) ensures that \( v_5 \) is well-defined. We prove

**Theorem.** Under the assumption (1.16)

\[ \{1, \theta, \theta^2, v_4, v_5\} \]

is an integral basis for \( K \).

We note that if (1.13) holds then

\[ Q = 1, \quad v_4 = \theta^3, \quad v_5 = \frac{ad\omega_5 + (1 - a)\theta^4}{d}. \]

Appealing to (1.11) we deduce

\[ v_5 = \omega_5 + k(d\omega_5 - \theta^4). \]

As \( d\omega_5 - \theta^4 \) is a cubic polynomial in \( \theta \) with coefficients in \( \mathbb{Z} \), we deduce from the theorem that \( \{1, \theta, \theta^2, \theta^3, \omega_5\} \) is an integral basis for \( K \) showing that our theorem includes that of Gaál and Pohst [2, p. 1690].

By a theorem of Erdős [1] there exists an infinite set \( S \) of integers \( n \) such that

\[ m = n^4 + 5n^3 + 15n^2 + 25n + 25 \]

is cubefree. For \( n \in S \) the integer \( m \) has the form (1.17). Clearly \( S \) contains an infinite subset \( S_1 \) such that the values of \( 5^b PQ \) are distinct for \( n \in S_1 \). Thus, by (1.19), the conductors \( f(K) \) are distinct for \( n \in S_1 \) thus ensuring that the cyclic quintic fields \( K \) are distinct for \( n \in S_1 \). Thus our theorem gives an integral basis for infinitely many cyclic quintic fields.
2. Proof of Theorem. We require a number of lemmas.

Lemma 2.1. Under the assumption (1.16), we have \( v_4 \in O_K \).

Proof. The asserted result is immediate if \( Q = 1 \). Hence we may assume that \( Q > 1 \). By (1.19) we see that \( Q \mid f(K) \). Hence all the prime divisors \( q \) of \( Q \) ramify in \( O_K \). Moreover, as \( K \) is a cyclic quintic field, each prime factor \( q \) ramifies totally. Hence there is a prime ideal \( \mathfrak{p} \) of \( O_K \) such that \( < q > = \mathfrak{p}^5 \) and \( N(\mathfrak{p}) = q \). Let \( g_n(x) \in \mathbb{Z}[x] \) be the minimal polynomial of \( 5\theta + n^2 \). Using MAPLE we find

\[
(2.1) \quad g_n(0) = m(4n^6 + 30n^5 + 65n^4 - 200n^2 - 125n + 125).
\]

From (1.17) and (2.1) we deduce that

\[
(2.2) \quad Q^2 \mid g_n(0) = \pm N(5\theta + n^2).
\]

Let

\[
(2.3) \quad < 5\theta + n^2 > = P_1^{a_1} \cdots P_r^{a_r}
\]

be the prime ideal decomposition of \( < 5\theta + n^2 > \) into distinct prime ideals of \( O_K \) so

\[
(2.4) \quad |N(5\theta + n^2)| = N(< 5\theta + n^2 >) = N(P_1)^{a_1} \cdots N(P_r)^{a_r}.
\]

From (2.2) and (2.4) we see that

\[
(2.5) \quad q^2 \mid N(P_1)^{a_1} \cdots N(P_r)^{a_r}.
\]

Thus \( P_i = q \) and \( a_i \geq 2 \) for some \( i \in \{1, 2, \ldots, r\} \). Hence by (2.3) we have

\[
(2.6) \quad \mathfrak{p}^2 \mid < 5\theta + n^2 >.
\]

Since \( \mathfrak{p}^5 \mid Q \) we deduce from (2.6) that

\[
(2.7) \quad \mathfrak{p}^2 \mid < 5\theta + n^2 - n^2Q >.
\]

As \( 5 \nmid Q \) we have \( \mathfrak{p} \nmid < 5 > \). Also by (1.20) we have \( Q \equiv 1 \pmod{5} \). Thus

\[
\mathfrak{p}^2 \mid < \theta - n^2 \left( \frac{Q-1}{5} \right) >.
\]

Hence

\[
(2.8) \quad \mathfrak{p}^5 \mid < \theta - n^2 \left( \frac{Q-1}{5} \right) >^3.
\]

As (2.8) is true for each prime divisor \( q \) of \( Q \) we have

\[
Q \mid < \theta - n^2 \left( \frac{Q-1}{5} \right) >^3.
\]

This proves that

\[
v_4 = \frac{1}{Q} \left( \theta - \frac{n^2}{5}(Q-1) \right)^3 \in O_K
\]
as asserted. □

**Lemma 2.2.** For all $n \in \mathbb{Z}$ we have $\omega_5 \in O_K$.

*Proof.* The proof is given in [2, pp. 1690-1691], where the case $n = -2$ should be dealt with separately. □

**Lemma 2.3.** Under the assumption (1.16), we have $v_5 \in O_K$.

*Proof.* Let

\[(2.9) \quad \alpha = ad\omega_5 + (1 - a)Qv_4\theta.\]

By Lemmas 2.1 and 2.2 we have $v_4 \in O_K$ and $\omega_5 \in O_K$ so

\[\alpha \in O_K.\]

From (1.5) and (1.17) we have $Q \mid a$. Hence

\[\alpha \equiv 0 \pmod{Q}\]

in $O_K$. From (1.11) we have $d \mid 1 - a$. Hence

\[\alpha \equiv 0 \pmod{d}\]

in $O_K$. Then, by (1.21), we deduce that

\[\alpha \equiv 0 \pmod{dQ}\]

in $O_K$ so that by (1.23) and (2.9)

\[v_5 = \frac{\alpha}{dQ} \in O_K\]

as claimed. □

*Proof of Theorem.* We have

\[\alpha = dQv_5 = ad\omega_5 + (1 - a)Qv_4\theta\]

\[= a \left( \theta^4 + c(\theta) \right) + (1 - a)\theta \left( \theta - \frac{n^2}{5}(Q - 1) \right)^3,\]

where

\[c(\theta) \in \mathbb{Z}[\theta], \quad \deg c(\theta) = 3.\]

Thus

\[\alpha = \theta^4 + d(\theta),\]

where

\[d(\theta) \in \mathbb{Z}[\theta], \quad \deg d(\theta) \leq 3.\]
Similarly
\[ Qv_4 = \theta^3 + e(\theta), \]
where
\[ e(\theta) \in \mathbb{Z}[\theta], \quad \deg e(\theta) \leq 2. \]
Thus
\[ \text{disc}(1, \theta, \theta^2, Qv_4, \alpha) = \text{disc}(1, \theta, \theta^2, \theta^3, \alpha) = \text{disc}(1, \theta, \theta^2, \theta^3, \theta^4) = m^4d^2, \]
by [2, p. 1691]. Therefore
\[ \text{disc}(1, \theta, \theta^2, v_4, v_5) = \frac{\text{disc}(1, \theta, \theta^2, Qv_4, \alpha)}{Q^2(dQ)^2} = \frac{m^4}{Q^4} = 5^{4b}P^4Q^4 = f(K)^4 = d(K). \]
As \( v_4 \in O_K \) and \( v_5 \in O_K \) by Lemmas 2.1 and 2.3 respectively, we deduce that \( \{1, \theta, \theta^2, v_4, v_5\} \) is an integral basis for \( K \). \( \Box \)
We conclude with an example.

**Example.** Let \( n = 14 \) so that
\[ K = \mathbb{Q}(\theta), \quad \theta^5 + 196\theta^4 - 6814\theta^3 + 54507\theta^2 + 3678\theta + 1 = 0. \]
We use the theorem to determine an integral basis for \( K \). Here
\[ m = 11 \times 71^2, \quad b = 0, \quad P = 11, \quad Q = 71, \]
\[ d = 7^2 \times 79, \]
\[ a = 2^4 \times 11 \times 71^2 \times 192141181, \]
\[ k = 5 \times 8807580989, \]
\[ v_4 = \frac{1}{71}(\theta - 2744)^3, \quad v_4 \equiv \frac{5 + 29\theta + 4\theta^2 + \theta^3}{71} \quad (\text{mod } 1), \]
\[ \omega_5 = \frac{16 + 527\theta + 447\theta^2 - 46\theta^3 + \theta^4}{3871}, \]
and
\[ v_5 = \frac{r + s\theta - t\theta^2 + u\theta^3 + \theta^4}{274841} \]
with
\[ r = 2727531680673536, \quad s = 352210381854043816557072, \]
\[ t = 3850620295978378636848, \quad u = 1395473396124589624, \]
so that
\[ v_5 \equiv \frac{50339 + 27624\theta + 112706\theta^2 + 220601\theta^3 + \theta^4}{274841} \quad (\text{mod } 1). \]
Thus by the theorem
\[ \left\{1, \theta, \theta^2, \frac{5 + 29\theta + 4\theta^2 + \theta^3}{71}, \frac{50339 + 27624\theta + 112706\theta^2 + 220601\theta^3 + \theta^4}{274841}\right\} \]
is an integral basis for $K$. As
\[
\frac{65823 + 62463\theta + 70125\theta^2 + 3825\theta^3 + \theta^4}{274841}
\]
\[
= \frac{50339 + 27624\theta + 112706\theta^2 + 220601\theta^3 + \theta^4}{274841}
\]
\[-56 \left( \frac{5 + 29\theta + 4\theta^2 + \theta^3}{71} \right) + (4 + 23\theta + 3\theta^2),
\]
we see that
\[
\left\{ 1, \theta, \theta^2, \frac{5 + 29\theta + 4\theta^2 + \theta^3}{71}, \frac{65823 + 62463\theta + 70125\theta^2 + 3825\theta^3 + \theta^4}{274841} \right\}
\]
is also an integral basis for $K$ in agreement with MAPLE.

We close by remarking that when $m$ is not cubefree the cyclic quintic field $K$ may not have an integral basis of the type given in our theorem. To see this take $n = 44$ so that $m = 41^3 \times 61$. In this case $(18 + 20\theta + \theta^2)/41$ is an integer of $K$ and so $\theta^2$ is not a minimal integer of degree 2. Hence $K$ cannot have an integral basis of the type \{1, $\theta$, $\theta^2$, $\ast$, $\ast$\}.

REFERENCES