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EVALUATION OF TWO CONVOLUTION SUMS INVOLVING THE SUM OF DIVISORS FUNCTION

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The convolution sums $\sum_{m<n/5} \sigma(m)\sigma(n-5m)$ and $\sum_{m<n/7} \sigma(m)\sigma(n-7m)$ are evaluated for all $n \in \mathbb{N}$.

1. INTRODUCTION.

Let $\mathbb{N}$ denote the set of natural numbers. We set

$$\sigma_k(n) = \sum_{d|n} d^k, \quad n, k \in \mathbb{N},$$

where d runs through the positive integers dividing n. If $n \notin \mathbb{N}$ we set $\sigma_k(n) = 0$. We write $\sigma(n)$ for $\sigma_1(n)$. We define the convolution sum $W_k(n)$ by

$$W_k(n) := \sum_{m<n/k} \sigma(m)\sigma(n-km), \tag{1.1}$$

where m runs through the positive integers $< n/k$. The sum $W_k(n)$ has been evaluated for $k = 1$, 2, 3, 4, 6, 8, 9 and 16 for all $n \in \mathbb{N}$, see [**4, 7**] ($k = 1$), [**7, 8, 9**] ($k = 2$, 3, 4), [**2**] ($k = 6$), [**11**] ($k = 8$), [**8, 9, 10, 12**] ($k = 9$), [**1**] ($k = 16$). For $k = 5$ Melfi [**8, 9**] has shown that for $n \equiv 8 \pmod{16}, n \not\equiv 0 \pmod 5$

$$W_5(n) = \frac{5}{312}\sigma_3(n) + \left(\frac{1}{24} - \frac{1}{20}n\right)\sigma(n). \tag{1.2}$$

In this paper we make use of some recent results of Berndt, Chan, Sohn and Son [**3**] to evaluate $W_5(n)$ and $W_7(n)$ for all $n \in \mathbb{N}$. We prove

THEOREM 1. *For all $n \in \mathbb{N}$*

$$W_5(n) = \frac{5}{312}\sigma_3(n) + \frac{125}{312}\sigma_3\left(\frac{n}{5}\right) + \left(\frac{1}{24} - \frac{1}{20}n\right)\sigma(n) + \left(\frac{1}{24} - \frac{1}{4}n\right)\sigma\left(\frac{n}{5}\right) - \frac{1}{130}c_5(n),$$

where the $c_5(n)$ ($n \in \mathbb{N}$) are integers defined by

$$q\prod_{n=1}^{\infty}(1-q^n)^4(1-q^{5n})^4 = \sum_{n=1}^{\infty} c_5(n)q^n. \tag{1.3}$$

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The first twenty values of $c_5(n)$ are $c_5(1) = 1$, $c_5(2) = -4$, $c_5(3) = 2$, $c_5(4) = 8$, $c_5(5) = -5$, $c_5(6) = -8$, $c_5(7) = 6$, $c_5(8) = 0$, $c_5(9) = -23$, $c_5(10) = 20$, $c_5(11) = 32$, $c_5(12) = 16$, $c_5(13) = -38$, $c_5(14) = -24$, $c_5(15) = -10$, $c_5(16) = -64$, $c_5(17) = 26$, $c_5(18) = 92$, $c_5(19) = 100$, $c_5(20) = -40$.

Theorem 2. *For all $n \in \mathbb{N}$*

$$W_7(n) = \frac{1}{120}\sigma_3(n) + \frac{49}{120}\sigma_3\Big(\frac{n}{7}\Big) + \Big(\frac{1}{24} - \frac{1}{28}n\Big)\sigma(n) + \Big(\frac{1}{24} - \frac{1}{4}n\Big)\sigma\Big(\frac{n}{7}\Big) - \frac{1}{70}c_7(n),$$

where the $c_7(n)$ ($n \in \mathbb{N}$) are integers defined by

$$(1.4)\quad q\Big(\prod_{n=1}^{\infty}(1-q^n)^{16}(1-q^{7n})^8 + 13q\prod_{n=1}^{\infty}(1-q^n)^{12}(1-q^{7n})^{12} + 49q^2\prod_{n=1}^{\infty}(1-q^n)^8(1-q^{7n})^{16}\Big)^{1/3} = \sum_{n=1}^{\infty} c_7(n)q^n.$$

The first twenty values of $c_7(n)$ are $c_7(1) = 1$, $c_7(2) = -1$, $c_7(3) = -2$, $c_7(4) = -7$, $c_7(5) = 16$, $c_7(6) = 2$, $c_7(7) = -7$, $c_7(8) = 15$, $c_7(9) = -23$, $c_7(10) = -16$, $c_7(11) = -8$, $c_7(12) = 14$, $c_7(13) = 28$, $c_7(14) = 7$, $c_7(15) = -32$, $c_7(16) = 41$, $c_7(17) = 54$, $c_7(18) = 23$, $c_7(19) = -110$, $c_7(20) = -112$.

2. Notation.

Let $z = x + iy \in \mathbb{C}$ be such that $y > 0$. Set $q = e^{2\pi iz}$ so that $|q| = e^{-2\pi y} < 1$. The Dedekind eta function $\eta(z)$ is defined by

$$(2.1)\qquad \eta(z) := e^{2\pi iz/24}\prod_{n=1}^{\infty}(1 - e^{2\pi inz}).$$

Following Ramanujan (see [**3**, p. 82]) we set

$$(2.2)\qquad f(-q) := e^{-2\pi iz/24}\eta(z) = \prod_{n=1}^{\infty}(1 - e^{2\pi inz}) = \prod_{n=1}^{\infty}(1 - q^n).$$

It is convenient to set

$$(2.3)\qquad A := f(-q),\ B := f(-q^5),\ C := f(-q^7).$$

We define $c_5(n)$ and $c_7(n)$ by (1.3) and (1.4) respectively. It is clear that $c_5(n) \in \mathbb{Z}$. We shall show that $c_7(n) \in \mathbb{Z}$ in the proof of Theorem 2. From (1.3), (1.4), (2.2) and (2.3), we obtain

$$(2.4)\qquad qA^4B^4 = \sum_{n=1}^{\infty} c_5(n)q^n.$$

$$(2.5)\qquad qA^3B^3\Big(\frac{A^7}{C} + 13qA^3C^3 + 49q^2\frac{C^7}{A}\Big)^{1/3} = \sum_{n=1}^{\infty} c_7(n)q^n.$$

The Eisenstein series $P(q)$ and $Q(q)$ (see [3, p. 81]) are defined by

$$P(q) := 1 - 24\sum_{n=1}^{\infty}\sigma(n)q^n \tag{2.6}$$

and

$$Q(q) := 1 + 240\sum_{n=1}^{\infty}\sigma_3(n)q^n. \tag{2.7}$$

3. Proof of Theorem 1.

We begin with a classical result.

Lemma 3.1.

$$P(q)^2 = 1 + \sum_{n=1}^{\infty}\big(240\sigma_3(n) - 288n\sigma(n)\big)q^n.$$

Proof: See for example Glaisher [5, 6]. ∎

Lemma 3.2.

$$P(q^5)^2 = 1 + \sum_{n=1}^{\infty}\left(240\sigma_3\Big(\frac{n}{5}\Big) - \frac{288}{5}n\sigma\Big(\frac{n}{5}\Big)\right)q^n.$$

Proof: Letting $q \to q^5$ in Lemma 3.1 we obtain the result. ∎

Lemma 3.3.

$$\big(P(q) - 5P(q^5)\big)^2 = 16\frac{A^{10}}{B^2} + 352qA^4B^4 + 2000q^2\frac{B^{10}}{A^2}.$$

Proof: This follows from [3, (3.21), (3.34)] and (2.3). ∎

Lemma 3.4.

(a) $$Q(q) = \frac{A^{10}}{B^2} + 250qA^4B^4 + 3125q^2\frac{B^{10}}{A^2}.$$

(b) $$Q(q^5) = \frac{A^{10}}{B^2} + 10qA^4B^4 + 5q^2\frac{B^{10}}{A^2}.$$

Proof: This follows from [3, Theorem 3.1, p. 85] and (2.3). ∎

Lemma 3.5.

(a) $$\frac{A^{10}}{B^2} = 1 + \sum_{n=1}^{\infty}\left(-\frac{5}{13}\sigma_3(n) + \frac{3125}{13}\sigma_3\Big(\frac{n}{5}\Big) - \frac{125}{13}c_5(n)\right)q^n.$$

(b) $$q^2\frac{B^{10}}{A^2} = \sum_{n=1}^{\infty}\left(\frac{1}{13}\sigma_3(n) - \frac{1}{13}\sigma_3\Big(\frac{n}{5}\Big) - \frac{1}{13}c_5(n)\right)q^n.$$

PROOF: From Lemma 3.4 we obtain

$$\frac{A^{10}}{B^2} = -\frac{1}{624}Q(q) + \frac{625}{624}Q(q^5) - \frac{125}{13}qA^4B^4 \tag{3.1}$$

and

$$q^2\frac{B^{10}}{A^2} = \frac{1}{3120}Q(q) - \frac{1}{3120}Q(q^5) - \frac{1}{13}qA^4B^4. \tag{3.2}$$

Letting $q \to q^5$ in (2.7), we obtain

$$Q(q^5) = 1 + 240\sum_{n=1}^{\infty}\sigma_3\left(\frac{n}{5}\right)q^n. \tag{3.3}$$

Using (2.4), (2.7) and (3.3) in (3.1) and (3.2), we obtain the asserted results. ∎

LEMMA 3.6.

$$\left(P(q) - 5P(q^5)\right)^2 = 16 + \sum_{n=1}^{\infty}\left(\frac{1920}{13}\sigma_3(n) + \frac{4800}{13}\sigma_3\left(\frac{n}{5}\right) + \frac{576}{13}c_5(n)\right)q^n.$$

PROOF: This follows from Lemma 3.3, Lemma 3.5 and (2.4). ∎

LEMMA 3.7.

$$P(q)P(q^5) = 1 + \sum_{n=1}^{\infty}\left(\frac{120}{13}\sigma_3(n) + \frac{3000}{13}\sigma_3\left(\frac{n}{5}\right) - \frac{144}{5}n\sigma(n) - 144n\sigma\left(\frac{n}{5}\right) - \frac{288}{65}c_5(n)\right)q^n.$$

PROOF: This follows from Lemmas 3.1, 3.2, 3.6 and the identity

$$P(q)P(q^5) = \frac{1}{10}P(q)^2 + \frac{5}{2}P(q^5)^2 - \frac{1}{10}\left(P(q) - 5P(q^5)\right)^2.$$

∎

PROOF OF THEOREM 1. Letting $q \to q^5$ in (2.6), we obtain

$$P(q^5) = 1 - 24\sum_{n=1}^{\infty}\sigma\left(\frac{n}{5}\right)q^n. \tag{3.4}$$

Next

$$\begin{aligned}
\sum_{n=1}^{\infty}W_5(n)q^n &= \sum_{n=1}^{\infty}\left(\sum_{m<n/5}\sigma(m)\sigma(n-5m)\right)q^n \\
&= \left(\sum_{l=1}^{\infty}\sigma(l)q^l\right)\left(\sum_{m=1}^{\infty}\sigma(m)q^{5m}\right) \\
&= \left(\frac{1-P(q)}{24}\right)\left(\frac{1-P(q^5)}{24}\right) \\
&= \frac{1}{576} - \frac{1}{576}P(q) - \frac{1}{576}P(q^5) + \frac{1}{576}P(q)P(q^5).
\end{aligned}$$

Appealing to (2.6), (3.1) and Lemma 3.7, and equating coefficients of q^n ($n \in \mathbb{N}$), we obtain the required result. $\square$

Since $c_5(mn) = c_5(m)c_5(n)$ for all $m, n \in \mathbb{N}$ with $(m,n) = 1$ and $c_5(8) = 0$ (see [9]), we deduce that $c_5(n) = 0$ for all $n \in \mathbb{N}$ with $n \equiv 8 \pmod{16}$. Hence for $n \equiv 8 \pmod{16}$ Theorem 1 gives

$$W_5(n) = \frac{5}{312}\sigma_3(n) + \frac{125}{312}\sigma_3\left(\frac{n}{5}\right) + \left(\frac{1}{24} - \frac{1}{20}n\right)\sigma(n) + \left(\frac{1}{24} - \frac{1}{4}n\right)\sigma\left(\frac{n}{5}\right).$$

Melfi's result (1.2) is the special case of this result when $n \not\equiv 0 \pmod 5$.

We show that the evaluation

$$c_5(5^k) = (-1)^k 5^k, \quad k \in \mathbb{N} \cup \{0\}, \tag{3.5}$$

is a consequence of Theorem 1. Let $n \in \mathbb{N}$. The elementary identities

$$\sigma(5n) = 6\sigma(n) - 5\sigma(n/5) \tag{3.6}$$

and

$$\sigma_3(5n) = 126\sigma_3(n) - 125\sigma(n/5) \tag{3.7}$$

are easily proved. Thus

$$\begin{aligned} W_5(5n) &= \sum_{m<n} \sigma(m)\sigma(5n - 5m) \\ &= 6\sum_{m<n} \sigma(m)\sigma(n - m) - 5\sum_{m<n} \sigma(m)\sigma\big((n - m)/5\big). \end{aligned}$$

It is a classical result that

$$\sum_{m<n} \sigma(m)\sigma(n - m) = \frac{5}{12}\sigma_3(n) + \left(\frac{1}{12} - \frac{n}{2}\right)\sigma(n),$$

see for example [7]. Also

$$\sum_{m<n} \sigma(m)\sigma\big((n - m)/5\big) = \sum_{t<n/5} \sigma(n - 5t)\sigma(t) = W_5(n).$$

Hence

$$W_5(5n) + 5W_5(n) = \frac{5}{2}\sigma_3(n) + \left(\frac{1}{2} - 3n\right)\sigma(n).$$

Appealing to Theorem 1, (3.6) and (3.7), we obtain

$$c_5(5n) = -5c_5(n), \quad n \in \mathbb{N},$$

from which (3.5) follows.

4. Proof of Theorem 2.

It is convenient to set

$$H := \left(\frac{A^7}{C} + 13qA^3C^3 + 49q^2\frac{C^7}{A}\right)^{1/3}, \tag{4.1}$$

where A and C are defined in (2.3).

Lemma 4.1.

$$P(q^7)^2 = 1 + \sum_{n=1}^{\infty}\left(240\sigma_3\left(\frac{n}{7}\right) - \frac{288}{7}n\sigma\left(\frac{n}{7}\right)\right)q^n.$$

Proof: Letting $q \to q^7$ in Lemma 3.1, we obtain the result. ∎

Lemma 4.2.

$$\left(P(q) - 7P(q^7)\right)^2 = 36H^4.$$

Proof: From [3, Lemma 6.2, p. 109] we obtain

$$P(q) - 7P(q^7) = -6\left(\frac{A^7}{C}\right)^{2/3}\left(1 + 13q\frac{C^4}{A^4} + 49q^2\frac{C^8}{A^8}\right)^{2/3}.$$

Squaring both sides we obtain

$$\begin{aligned}\left(P(q) - 7P(q^7)\right)^2 &= 36\left(\frac{A^7}{C}\right)^{4/3}\left(1 + 13q\frac{C^4}{A^4} + 49q^2\frac{C^8}{A^8}\right)^{4/3}\\ &= 36\left(\frac{A^7}{C} + 13qA^3C^3 + 49q^2\frac{C^7}{A}\right)^{43}\\ &= 36H^4.\end{aligned}$$

∎

Lemma 4.3.

(a) $$Q(q) = \left(\frac{A^7}{C} + 245qA^3B^3 + 2401q^2\frac{C^7}{A}\right)H.$$

(b) $$Q(q^7) = \left(\frac{A^7}{C} + 5qA^3B^3 + q^2\frac{C^7}{A}\right)H.$$

Proof: This follows from [3, Theorem 5.1, p. 100]. ∎

Lemma 4.4.

$$qA^3B^3H = \sum_{n=1}^{\infty} c_7(n)q^n.$$

Proof: This follows from (2.5) and (4.1). ∎

Lemma 4.5.

(a) $$\frac{A^7}{C}H = 1 + \sum_{n=1}^{\infty}\left(-\frac{1}{10}\sigma_3(n) + \frac{2401}{10}\sigma_3\left(\frac{n}{7}\right) - \frac{49}{10}c_7(n)\right)q^n.$$

(b) $$q^2\frac{C^7}{A}H = \sum_{n=1}^{\infty}\left(\frac{1}{10}\sigma_3(n) - \frac{1}{10}\sigma_3\left(\frac{n}{7}\right) - \frac{1}{10}c_7(n)\right)q^n.$$

PROOF: From Lemma 4.3 we obtain

$$\frac{A^7}{C} = -\frac{1}{2400}Q(q) + \frac{2401}{2400}Q(q^7) - \frac{49}{10}qA^3C^3H \tag{4.2}$$

and

$$q^2\frac{C^7}{A}H = \frac{1}{2400}Q(q) - \frac{1}{2400}Q(q^7) - \frac{1}{10}qA^3C^3H. \tag{4.3}$$

Letting $q \to q^7$ in (2.7) we obtain

$$Q(q^7) = 1 + 240\sum_{n=1}^{\infty}\sigma_3\Big(\frac{n}{7}\Big)q^n. \tag{4.4}$$

Using (2.7), (4.4) and Lemma 4.3 in (4.2) and (4.3), we obtain the asserted results. □

LEMMA 4.6.

$$\big(P(q) - 7P(q^7)\big)^2 = 36 + \sum_{n=1}^{\infty}\Big(\frac{864}{5}\sigma_3(n) + \frac{42336}{5}\sigma_3\Big(\frac{n}{7}\Big) + \frac{576}{5}c_7(n)\Big)q^n.$$

PROOF: This follows from Lemmas 4.2, 4.5 and 4.6. □

LEMMA 4.7.

$$P(q)P(q^7) = 1 + \sum_{n=1}^{\infty}\Big(\frac{24}{5}\sigma_3(n) + \frac{1176}{5}\sigma_3\Big(\frac{n}{7}\Big) - \frac{144}{7}n\sigma(n) - 144n\sigma\Big(\frac{n}{5}\Big) - \frac{288}{35}c_7(n)\Big)q^n.$$

PROOF: This follows from Lemmas 3.1, 4.1, 4.7 and the identity

$$P(q)P(q^7) = \frac{1}{14}P(q)^2 + \frac{49}{14}P(q^7)^2 - \frac{1}{14}\big(P(q) - 7P(q^7)\big)^2.$$

□

PROOF OF THEOREM 2. Letting $q \to q^7$ in (2.6), we obtain

$$P(q^7) = 1 - 24\sum_{n=1}^{\infty}\sigma\Big(\frac{n}{7}\Big)q^n. \tag{4.5}$$

Next

$$\begin{aligned}\sum_{n=1}^{\infty}W_7(n)q^n &= \sum_{n=1}^{\infty}\Big(\sum_{m<\frac{n}{7}}\sigma(m)\sigma(n-7m)\Big)q^n\\ &= \Big(\sum_{l=1}^{\infty}\sigma(l)q^l\Big)\Big(\sum_{m=1}^{\infty}\sigma(m)q^{7m}\Big)\\ &= \Big(\frac{1-P(q)}{24}\Big)\Big(\frac{1-P(q^7)}{24}\Big)\\ &= \frac{1}{576} - \frac{1}{576}P(q) - \frac{1}{576}P(q^7) + \frac{1}{576}P(q)P(q^7).\end{aligned}$$

Appealing to (2.6), (4.5) and Lemma 4.7, and equating coefficients of q^n, we obtain the formula for $W_7(n)$ in Theorem 2.

We now show that the $c_7(n)$ are integers. It is easy to check that for all $n \in \mathbb{N}$ we have

$$\sigma_3(n) \equiv \sigma(n) \pmod 3$$

and

$$\sigma_3(n) \equiv (2n-1)\sigma(n) \pmod 4.$$

Hence

$$7\sigma_3(n) + 343\sigma_3(n/7) + (35-30n)\sigma(n) + (35-210n)\sigma(n/7) \\ \equiv \sigma_3(n) + \sigma_3(n/7) - \sigma(n) - \sigma(n/7) \equiv 0 \pmod 3$$

and

$$7\sigma_3(n) + 343\sigma_3(n/7) + (35-30n)\sigma(n) + (35-210n)\sigma(n/7) \\ \equiv -\sigma_3(n) - \sigma_3(n/7) - (1+2n)\sigma(n) - (1+2n)\sigma(n/7) \equiv 0 \pmod 4$$

so

$$7\sigma_3(n) + 343\sigma_3(n/7) + (35-30n)\sigma(n) + (35-210n)\sigma(n/7) \equiv 0 \pmod{12}.$$

Thus

$$e(n) := \frac{7\sigma_3(n) + 343\sigma_3(n/7) + (35-30n)\sigma(n) + (35-210n)\sigma(n/7)}{12} \in \mathbb{Z}.$$

Hence

$$c_7(n) = e(n) - 70W_7(n) \in \mathbb{Z}.$$ □

Similarly to the proof of (3.5) we can use Theorem 2 to show that

(4.6) $$c_7(7^k) = (-1)^k 7^k, \quad k \in \mathbb{N} \cup \{0\}.$$

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