A cubic transformation formula for $2F_1\left(\frac{1}{3}, \frac{2}{3}; 1; z\right)$ and some arithmetic convolution formulae

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(Received 16 April 2003; revised 21 July 2003)

Abstract

A cubic transformation formula for the hypergeometric function $2F_1\left(\frac{1}{3}, \frac{2}{3}; 1; z\right)$ is proved. As an application of this formula a number of arithmetic convolution sums are evaluated. For example, Melfi’s formula,

$$\sum_{k=1}^{n-1} \sigma(k)\sigma(n-k) = \frac{1}{9}\sigma_3(n), \quad n \equiv 2 \pmod{3},$$

is proved without the use of modular forms.

1. Introduction: preliminaries and statement of main results

As usual we let $\mathbb{C}$ denote the complex plane and $\mathbb{R}$ the real number line in $\mathbb{C}$. We recall that the function $z^\frac{1}{3}$ is defined for all $z \in \mathbb{C}$ by

$$z^\frac{1}{3} = \begin{cases} |z|^\frac{1}{3}e^{\frac{1}{3}\text{arg}z}, & \text{if } z \neq 0, \\ 0, & \text{if } z = 0, \end{cases}$$

where the argument $\text{arg} z$ of the complex number $z$ is chosen to satisfy

$$-\pi < \text{arg} z \leq \pi, \quad z \in \mathbb{C}\setminus\{0\}.$$

In particular if $a \in \mathbb{R}$ is such that $a < 0$ then

$$a^\frac{1}{3} = -|a|^\frac{1}{3}\omega^2,$$

where $\omega$ is the complex cube root of unity given by

$$\omega = e^{\frac{2i\pi}{3}} = \frac{1}{2}(-1 + i\sqrt{3}),$$

† Research supported by Natural Sciences and Engineering Research Council of Canada, grant A-7233.
so that for example we have
\[ (-\frac{1}{8})^{\frac{1}{3}} = -\frac{1}{2}\omega^2. \] (1.1)

The function \( z^{\frac{1}{3}} \) is regular \([8, p. 36]\) in \( \mathbb{C}\backslash(-\infty, 0] \), see \([8, p. 49]\). Let \( D \) denote the domain \([8, p. 15]\) in the complex plane given by
\[ D = \mathbb{C}\backslash[1, \infty). \] (1.2)

Hence, the function \((1 - z)^{\frac{1}{3}}\) is defined for all \( z \in \mathbb{C} \) and is regular in \( D \). (1.3)

We define the function \( h \) by
\[ h(z) = 9z(1 - z)^{\frac{1}{3}} + ((1 - z)^{\frac{1}{3}} - \omega)^4, \quad z \in \mathbb{C}. \] (1.4)

We show that
\[ h(z) \neq 0 \text{ for all } z \in \mathbb{C}. \] (1.5)

To see this we suppose that \( z \in \mathbb{C} \) is such that \( h(z) = 0 \). From (1.1) and (1.4) we have \( z \neq 0, \frac{9}{8} \). Set
\[ z_1 = \omega^2(1 - z)^{\frac{1}{3}} \in \mathbb{C} \]
so that \( z_1 \neq 1, -\frac{1}{2} \). Then \( h(z) = 0 \) gives
\[ 9(1 - z_1^3)\omega z_1 + (\omega z_1 - \omega)^4 = 0 \]
so that
\[ (1 - z_1)(1 + 2z_1)^3 = 0, \]
a contradiction.

In view of (1.3), (1.4) and (1.5), we can define functions \( f \) and \( g \) by
\[ f(z) = \frac{9z(1 - z)^{\frac{1}{3}}}{9z(1 - z)^{\frac{1}{3}} + ((1 - z)^{\frac{1}{3}} - \omega)^4}, \quad z \in \mathbb{C}, \] (1.6)
and
\[ g(z) = \frac{1 + 2\omega^2(1 - z)^{\frac{1}{3}}}{1 + 2\omega^2}, \quad z \in \mathbb{C}. \] (1.7)

From (1.3), (1.4), (1.5), (1.6) and (1.7), we see that
\[ f(z) \text{ and } g(z) \text{ are defined for all } z \in \mathbb{C} \text{ and are regular in } D. \] (1.8)

For \( r \in \mathbb{R}^+ \) we set \( C(r) = \{ z \in \mathbb{C} \mid |z| < r \} \). We note that
\[ f(0) = 0, \quad f(1) = 0, \quad f\left(\frac{9}{8}\right) = \frac{9}{8}, \] (1.9)
\[ g(0) = 1, \quad g(1) = \frac{1}{3}(1 + 2\omega), \quad g\left(\frac{9}{8}\right) = 1 + \omega, \] (1.10)
\[ f(z) = 0 \iff z = 0, 1, \] (1.11)
\[ g(z) \neq 0 \text{ for } z \in \mathbb{C}, \] (1.12)
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\[ f(z) \in C\left(\frac{1}{3}\right) \quad \text{for} \quad z \in C\left(\frac{1}{8}\right), \]  
(1·13)

\[ f(x) \notin \mathbb{R} \quad \text{for} \quad x \in (0, \frac{1}{2}), \]  
(1·14)

\[ f'(0) = \lim_{z \to 0} \frac{f(z)}{z} = \frac{9}{(1 - \omega)^4} = \omega, \]  
(1·15)

\[ g'(0) = \frac{-2\omega^2}{3(1 + 2\omega^2)} = -\frac{2}{3}(1 - \omega). \]  
(1·16)

We just prove (1·13) and (1·14) as the others are clear.

**Proof of (1·13).** For \( z \in C\left(\frac{1}{8}\right) \) we have

\[ |1 - z|^{\frac{1}{3}} < 2, \quad 9|z||1 - z|^{\frac{1}{3}} < \frac{9}{4}. \]

By the generalized binomial theorem, we see that for \( a = \frac{1}{3}, \frac{2}{3}, \frac{4}{3} \) and \( z \in C\left(\frac{1}{8}\right) \) we have

\[ |(1 - z)^a - 1| \leq \sum_{n=1}^{\infty} \binom{a}{n} |z|^n \leq \sum_{n=1}^{\infty} a|z|^n = \frac{a|z|}{1 - |z|} < \frac{a}{7}. \]

Thus

\[ |(1 - z)^{\frac{1}{3}} - \omega|^4 - (1 - \omega)^4| \]

\[ = |(1 - z)^{\frac{1}{3}} - 1 + 4\omega z + 6\omega^2((1 - z)^{\frac{1}{3}} - 1) - 4((1 - z)^{\frac{1}{3}} - 1)| \]

\[ \leq |(1 - z)^{\frac{1}{3}} - 1| + 4|z| + 6|((1 - z)^{\frac{1}{3}} - 1) + 4|(1 - z)^{\frac{1}{3}} - 1| \]

\[ < \frac{4}{21} + \frac{1}{2} + \frac{4}{3} + \frac{4}{21} \]

\[ = \frac{64}{21} \]

\[ = \frac{3}{2}. \]

Hence

\[ |9z(1 - z)^{\frac{1}{3}} + ((1 - z)^{\frac{1}{3}} - \omega)| \]

\[ = |(1 - \omega)^4 + 9z(1 - z)^{\frac{1}{3}} + (((1 - z)^{\frac{1}{3}} - \omega)^4 - (1 - \omega)^4)| \]

\[ \geq |(1 - \omega)^4| - 9|z||1 - z|^{\frac{1}{3}} - |((1 - z)^{\frac{1}{3}} - \omega)^4 - (1 - \omega)^4| \]

\[ > 9 - \frac{9}{4} - \frac{3}{2} \]

\[ = \frac{21}{4}. \]

Then, for \( z \in C\left(\frac{1}{8}\right) \), we have

\[ |f(z)| = \frac{9|z||1 - z|^{\frac{1}{3}}}{|9z(1 - z)^{\frac{1}{3}} + ((1 - z)^{\frac{1}{3}} - \omega)|} < \frac{9}{\frac{21}{4}} = \frac{9}{\frac{21}{4}} < \frac{1}{2}. \]

**Proof of (1·14).** Suppose that \( f(x) \in \mathbb{R} \) for \( x \in (0, \frac{1}{8}) \). Then \( f(x) = \overline{f(x)} \). As \( 1 - x > 0 \) we have \((1 - x)^{\frac{1}{3}} \in \mathbb{R} \) so that by (1·6)

\[ ((1 - x)^{\frac{1}{3}} - \omega)^4 = ((1 - x)^{\frac{1}{3}} - \omega^2)^4. \]
Expanding the fourth powers we obtain, after cancelling the factor $\omega - \omega^2$,

$$6(1 - x)^\frac{5}{3} = -3 + 4x.$$  

Cubing both sides, we deduce that

$$64x^3 - 360x^2 + 540x - 243 = 0.$$  

Hence

$$(8x - 9)(8x^2 - 36x + 27) = 0$$  

so that

$$x = \frac{9}{8}(\approx 1.1), \quad \frac{9-3\sqrt{3}}{8}(\approx 0.9) \text{ or } \frac{9+3\sqrt{3}}{8}(\approx 3.5).$$

This is a contradiction as none of these values lies in the interval $(0, \frac{1}{8})$.

The integral

$$\int_0^1 t^{-\frac{1}{3}}(1 - t)^{-\frac{2}{3}}(1 - zt)^{-\frac{1}{2}}dt$$

is uniformly convergent in any closed domain of $D$ and so represents an analytic function of $z$, which is regular in $D$ [8, p. 249]. We set

$$w(z) := \frac{\sqrt{3}}{2\pi} \int_0^1 t^{-\frac{1}{3}}(1 - t)^{-\frac{2}{3}}(1 - zt)^{-\frac{1}{2}}dt, \quad z \in D. \quad (1.17)$$

Hence

$$w(z) \text{ is defined and regular in } D. \quad (1.18)$$

For $z \in C(1)$ it is known that [8, p. 249]

$$\frac{\sqrt{3}}{2\pi} \int_0^1 t^{-\frac{1}{3}}(1 - t)^{-\frac{2}{3}}(1 - zt)^{-\frac{1}{2}}dt = \sum_{n=0}^{\infty} \frac{\left(\frac{1}{3}\right)_n \left(\frac{2}{3}\right)_n}{(1)_n} \frac{z^n}{n!},$$

where for $a \in \mathbb{R}^+$ and $n \in \{0, 1, 2, \ldots\}$ the Pochhammer symbol $(a)_n$ is defined by

$$(a)_n = a(a + 1) \cdots (a + n - 1), \quad n \in \mathbb{N}; \quad (a)_0 = 1. \quad (1.19)$$

Thus $w(z)$ is the analytic continuation of the hypergeometric function $\mathbf{2}F_1\left(\frac{1}{3}, \frac{2}{3}; 1; z\right)$ from $C(1)$ to $D$, so that

$$w(z) = \mathbf{2}F_1\left(\frac{1}{3}, \frac{2}{3}; 1; z\right), \quad z \in C(1).$$

The function $w(z)$ is the unique solution of the hypergeometric differential equation

$$z(1 - z)w'' + (1 - 2z)w' - \frac{2}{9}w = 0, \quad (1.20)$$

which is regular in $D$ and satisfies

$$w(0) = 1, \quad w'(0) = \frac{2}{9}. \quad (1.21)$$
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see [8, pp. 246–248]. We now show that

$$w(z) \neq 0 \text{ for } z \in C\left(\frac{1}{2}\right).$$  \hspace{1cm} (1.22)

As

$$\left(\frac{1}{3}\right)_n = \frac{1 \cdot 4 \cdot 7 \cdots (3n - 2)}{3^n}, \quad \left(\frac{2}{3}\right)_n = \frac{2 \cdot 5 \cdot 8 \cdots (3n - 1)}{3^n}, \quad (1)_n = n!,$$

by (1.19), we have

$$\frac{\left(\frac{1}{3}\right)_n \left(\frac{2}{3}\right)_n}{(1)_n} = \frac{3n!}{n!^3 3^{3n}}$$

so that

$$|w(z) - 1| = \left| \sum_{n=1}^{\infty} \frac{3n!}{n!^3 3^{3n}} z^n \right| \leq \sum_{n=1}^{\infty} \frac{3n!}{n!^3 3^{3n}} |z|^n, \quad |z| < 1.$$

The multinomial coefficient

$$\binom{3n}{n, n, n} = \frac{3n!}{n!^3}$$

is one of the terms in the multinomial expansion of $(1 + 1 + 1)^{3n}$ so that

$$\frac{3n!}{n!^3} \leq 3^{3n}.$$

Hence

$$|w(z) - 1| \leq \sum_{n=1}^{\infty} |z|^n = \frac{|z|}{1 - |z|}.$$

If $z \in C\left(\frac{1}{2}\right)$ then $\frac{|z|}{1 - |z|} < 1$, so that $|w(z) - 1| < 1$, proving (1.22).

For $x \in \mathbb{R}$ with $0 \leq x < 1$ we have

$$w(x) = 1 + \sum_{n=1}^{\infty} \frac{3n!}{n!^3 3^{3n}} x^n$$

so that

$$w(x) \geq 1, \quad 0 \leq x < 1,$$

and thus in particular

$$w(x) \neq 0, \quad 0 \leq x < 1. \hspace{1cm} (1.23)$$

It is convenient to define the subset $D^*$ of $D$ by

$$D^* = \{z \in D \mid f(z) \in D\}. \hspace{1cm} (1.24)$$

By (1.8), (1.18) and (1.24) we see that

$$w(f(z)) \text{ is defined and regular in } D^* \hspace{1cm} (1.25)$$
and

\[ g(z)w(z) \text{ is defined and regular in } D. \] (1.26)

Let \( z \in C\left(\frac{1}{3}\right) \). By (1.2) we have \( z \in D \). By (1.13) we have \( f(z) \in C\left(\frac{1}{2}\right) \) so that \( f(z) \in D \). Hence, by (1.24), we have

\[ C\left(\frac{1}{4}\right) \subseteq D^* \subseteq D. \] (1.27)

The main result of this paper is the following cubic transformation formula for \( w(z) \).

**Theorem 1.1.** For \( z \in D^* \) we have

\[ w(f(z)) = g(z)w(z). \] (1.28)

The proof of Theorem 1.1 is given in Section 2. Theorem 1.1 is similar to the cubic transformation formula of Ramanujan [13, second notebook, p. 258]

\[ w\left( 1 - \left( \frac{1 - x}{1 + 2x} \right)^3 \right) = (1 + 2x)w(x^3), \] (1.29)

which is valid for \( x \in \mathbb{R} \) with \( |x| \) sufficiently small, see [4, corollary 2.4, p. 97]. Proofs of (1.29) have been given by Berndt, Bhargava and Garvan [5, corollary 2.4, p. 4170], Borwein and Borwein [6, p. 694] and Chan [7, sections 5 and 6, pp. 201–203]. We have not been able to deduce Theorem 1.1 from (1.29).

We define the subset \( D^{**} \) of \( D^* \) by

\[ D^{**} = \{ z \in D^* \mid w(z) \neq 0, w(f(z)) \neq 0, 1 - z \in D, 1 - f(z) \in D \}. \] (1.30)

It is clear from (1.30) that

\[ \frac{2\pi}{\sqrt{3}} \frac{w(1 - z)}{w(z)} - \frac{2\pi}{\sqrt{3}} \frac{w(1 - f(z))}{w(f(z))} \text{ is defined and regular in } D^{**}. \] (1.31)

For \( x \in \left( 0, \frac{1}{3} \right) \) we have

\[ x \in D^*, \text{ by (1.27)}, \] (1.32)

\[ w(x) \neq 0, \text{ by (1.22)}, \] (1.33)

\[ w(f(x)) \neq 0, \text{ by (1.13) and (1.22)}, \] (1.34)

\[ 1 - x \in D, \text{ by (1.2)}, \] (1.35)

\[ 1 - f(x) \in D, \text{ by (1.2) and (1.14)}. \] (1.36)

From (1.30) and (1.32) – (1.36) we see that

\[ \left( 0, \frac{1}{3} \right) \subseteq D^{**}. \] (1.37)

We use Theorem 1.1 to prove the following important identity satisfied by \( w(z) \).

**Theorem 1.2.** For \( z \in D^{**} \) we have

\[ e^{-\frac{2\pi}{\sqrt{3}} w(f(z))} = \omega e^{-\frac{2\pi}{\sqrt{3}} w(z)}. \] (1.38)
The proof of Theorem 1·2 is given in Section 3. Theorem 1·2 is the analogue of Jacobi’s “change of sign” formula [3, p. 126] for a “change of cube root of unity.”

We now recall the Eisenstein series

\[ L(q) = 1 - 24 \sum_{n=1}^{\infty} \sigma(n)q^n, \quad q \in C(1), \quad (1·39) \]

\[ M(q) = 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n)q^n, \quad q \in C(1), \quad (1·40) \]

\[ N(q) = 1 - 504 \sum_{n=1}^{\infty} \sigma_5(n)q^n, \quad q \in C(1), \quad (1·41) \]

treated by Ramanujan in [12], [14, pp. 136–162]. (Ramanujan actually used \( P, Q, R \) in place of \( L, M, N \), see [12, equation (25), p. 140].) Now let \( q \in \mathbb{R} \) be such that \( 0 < q < 1 \) so that \( 0 < -\log q < +\infty \). The derivative of the function

\[ y(x) := \frac{2\pi}{\sqrt{3}} \frac{2F_1 \left( \frac{1}{3}, \frac{2}{3}; 1; 1 - x \right)}{2F_1 \left( \frac{1}{3}, \frac{2}{3}; 1; x \right)} = \frac{2\pi}{\sqrt{3}} \frac{w(1 - x)}{w(x)}, \quad 0 < x < 1, \]

is, [2, p. 87],

\[ y'(x) = \frac{-1}{x(1 - x)\{w(x)\}^2}, \quad 0 < x < 1. \quad (1·42) \]

By (1·23), we have \( w(x) \neq 0 \) for \( 0 < x < 1 \), so that

\[ y'(x) < 0, \quad 0 < x < 1. \]

Hence, as \( x \) increases from \( 0 \) to \( 1 \), \( y(x) \) decreases from \( y(0) = +\infty \) to \( y(1) = 0 \), so that there is a unique real number \( x \) with \( 0 < x < 1 \) such that

\[ \frac{2\pi}{\sqrt{3}} \frac{w(1 - x)}{w(x)} = -\log q, \]

equivalently

\[ q = \exp \left( -\frac{2\pi}{\sqrt{3}} \frac{w(1 - x)}{w(x)} \right) = e^{-y(x)}, \quad (1·43) \]

see [4, equation (1·7), p. 91]. It is also convenient to set

\[ w := w(x) = 2F_1 \left( \frac{1}{3}, \frac{2}{3}; 1; x \right). \]

It is known that

\[ L(q) = (1 - 4x)w^2 + 12x(1 - x)w \frac{dw}{dx}, \quad (1·44) \]

\[ M(q) = (1 + 8x)w^4, \quad (1·45) \]

\[ N(q) = (1 - 20x - 8x^2)w^6, \quad (1·46) \]

see [4, lemma 4·1, theorem 4·2, theorem 4·3, pp. 105–106]. In Section 4, we apply Theorem 1·2 to (1·44), (1·45) and (1·46) to obtain \( L(\omega q) \), \( M(\omega q) \) and \( N(\omega q) \).
Theorem 1.3.

\[ L(\omega q) = \left( 11 - 12x + 8\omega^2(1 - x)^{\frac{1}{3}} + 8\omega(1 - x)^{\frac{2}{3}} \right) \frac{w^2}{3} + 12x(1 - x)w \frac{dw}{dx}, \]  

\[ M(\omega q) = \left( 249 - 248x - 120(1 - x)^{\frac{1}{3}} - 120(1 - x)^{\frac{2}{3}} + 80x(1 - x)^{\frac{1}{3}} \right) \frac{w^4}{9} + (\omega - \omega^2) \left( 80x(1 - x)^{\frac{1}{3}} - 120(1 - x)^{\frac{2}{3}} + 120(1 - x)^{\frac{1}{3}} \right) \frac{w^6}{27}. \]  

\[ N(\omega q) = \left( 6579 - 8604x + 2024x^2 - 3276(1 - x)^{\frac{1}{3}} - 3276(1 - x)^{\frac{2}{3}} \right) \frac{w^6}{27}. \]  

For \( j = 1, 2, 3 \) we introduce the series

\[ L_j(q) = \sum_{n=1}^{\infty} \sigma(n)q^n; \]  

\[ M_j(q) = \sum_{n=1}^{\infty} \sigma_3(n)q^n; \]  

\[ N_j(q) = \sum_{n=1}^{\infty} \sigma_5(n)q^n. \]  

In Section 5 we use Theorem 1.3 and the method of triplication [4, pp. 101–102] to evaluate \( L_1(q), L_2(q), \ldots, N_3(q) \).

Theorem 1.4.

\[ L_1(q) = \left( 1 + (1 - x)^{\frac{1}{3}} - 2(1 - x)^{\frac{2}{3}} \right) \frac{w^2}{27}, \]  

\[ L_2(q) = \left( 1 - 2(1 - x)^{\frac{1}{3}} + (1 - x)^{\frac{2}{3}} \right) \frac{w^2}{27}, \]  

\[ L_3(q) = \frac{1}{24} + \left( -25 + 36x + 8(1 - x)^{\frac{1}{3}} + 8(1 - x)^{\frac{2}{3}} \right) \frac{w^2}{216} - \frac{1}{2}x(1 - x)w \frac{dw}{dx}, \]  

\[ M_1(q) = \left( -3 + 4x - 3(1 - x)^{\frac{1}{3}} + 6(1 - x)^{\frac{2}{3}} + 2x(1 - x)^{\frac{1}{3}} \right) \frac{w^4}{81}, \]  

\[ M_2(q) = \left( -3 + 4x + 6(1 - x)^{\frac{1}{3}} - 3(1 - x)^{\frac{2}{3}} - 4x(1 - x)^{\frac{1}{3}} \right) \frac{w^4}{81}, \]  

\[ M_3(q) = -\frac{1}{240} + \left( 507 - 424x - 240(1 - x)^{\frac{1}{3}} - 240(1 - x)^{\frac{2}{3}} + 160x(1 - x)^{\frac{1}{3}} \right) \frac{w^4}{6480}. \]
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\[ N_1(q) = \left( 117 - 144x + 40x^2 + 117(1 - x)^{\frac{1}{3}} - 234(1 - x)^{\frac{2}{3}} \right) \frac{w^6}{729}, \]  
\[ N_2(q) = \left( 117 - 144x + 40x^2 - 234(1 - x)^{\frac{1}{3}} + 117(1 - x)^{\frac{2}{3}} \right) \frac{w^6}{729}, \]  
\[ N_3(q) = \frac{1}{504} + \left( -355995 + 479196x - 103464x^2 + 176904(1 - x)^{\frac{1}{3}} \right) \frac{w^6}{1102248}. \]

The relationships in Theorems 1·5, 1·6 and 1·7 below follow easily from Theorem 1·4, for proofs see Sections 6, 7 and 8.

**Theorem 1·5.**

\[ L_1(q)^{\frac{2}{3}} = \frac{1}{9} M_2(q), \]
\[ L_1(q)L_2(q) = \frac{1}{80} M(q^3) - \frac{1}{80} M(q^9), \]
\[ L_2(q)L_3(q) = \frac{1}{24} L_2(q) + \frac{11}{72} M_2(q) - \frac{1}{4q} \frac{dL_2(q)}{dq}, \]
\[ L_3(q)^{2} = -\frac{1}{40} M(q^3) + \frac{1}{40} M(q^9) + \frac{1}{12} L_3(q) + \frac{5}{12} M_3(q) - \frac{1}{2q} \frac{dL_3(q)}{dq}. \]

**Theorem 1·6.**

\[ L_1(q)L(q^3) = -M_1(q) + 2q \frac{dL_1(q)}{dq}, \]
\[ L_2(q)L(q^3) = -M_2(q) + 2q \frac{dL_2(q)}{dq}, \]
\[ L_3(q)L(q^3) = \frac{1}{24} L(q^3) - \frac{3}{80} M(q^3) - M_3(q) - \frac{3}{16q} \frac{dL(q^9)}{dq} + \frac{7}{2q} \frac{dL_3(q)}{dq} - \frac{1}{240}, \]
\[ L(q)L(q^3) = \frac{1}{10} M(q) + \frac{9}{10} M(q^3) + 2q \frac{dL(q)}{dq} + \frac{9}{2q} \frac{dL(q^9)}{dq} - 36q \frac{dL_3(q)}{dq}. \]

**Theorem 1·7.**

\[ L_2(q)L(q^9) = -\frac{1}{9} M_2(q) + \frac{2}{3q} \frac{dL_2(q)}{dq}, \]
\[ L_3(q)L(q^9) = \frac{1}{24} L(q^9) - \frac{23}{560} M(q^9) - \frac{1}{7} M_3(q) - \frac{1}{4q} \frac{dL(q^9)}{dq} + \frac{2}{3q} \frac{dL_3(q)}{dq} - \frac{1}{1680}. \]

We set (as in [9, p. 255])

\[ S(a, 3) = \sum_{\substack{m=1 \atop m \equiv a \pmod{3}}}^{n-1} \sigma(m)\sigma(n-m), \quad a = 0, 1, 2. \]
By equating coefficients of $q^n$ on both sides of the four identities in Theorem 1.5, we obtain the following arithmetic convolution identities, see [9, theorem 8, p. 256].

**Theorem 1.8.**

$$S(1, 3) = \frac{1}{9}\sigma_3(n), \quad \text{if } n \equiv 2 \mod 3,$$

$$S(1, 3) = S(2, 3) = \frac{1}{9}(\sigma_3(n) - \sigma_3(n/3)), \quad \text{if } n \equiv 0 \mod 3,$$

$$S(0, 3) = S(2, 3) = \frac{1}{72}(11\sigma_3(n) + (3 - 18n)\sigma(n)), \quad \text{if } n \equiv 2 \mod 3,$$

$$S(0, 3) = \frac{1}{36}(7\sigma_3(n) + (3 - 18n)\sigma(n) + 8\sigma_3(n/3)), \quad \text{if } n \equiv 0 \mod 3.$$

The first identity in Theorem 1.8 is due to Melfi [10, 11], who proved it using modular forms. Our proof is the first proof without the use of modular forms, see Section 9.

Similarly, equating coefficients of $q^n$ in the four identities in Theorem 1.6, we obtain the following result.

**Theorem 1.9.**

$$\sum_{m<n/3} \sigma(m)\sigma(n-3m) = \frac{1}{24}(\sigma_3(n) + (1 - 2n)\sigma(n) + 9\sigma_3(n/3) + (1 - 6n)\sigma(n/3)).$$

The first identity in Theorem 1.6 gives the case $n \equiv 1 \mod 3$ of Theorem 1.9, the second the case $n \equiv 2 \mod 3$, and the third $n \equiv 0 \mod 3$. The fourth identity gives the result for all $n$. For $n \not\equiv 0 \mod 3$ Theorem 1.9 is due to Melfi [10, 11]. His proof used modular forms. For general $n$ the theorem is due to Huard, Ou, Spearman and Williams [9]. Their proof is completely elementary. The proof of Theorem 1.9 is given in Section 10.

Finally, by equating coefficients of $q^n$ in the two identities in Theorem 1.7, we obtain the following result, see Section 11.

**Theorem 1.10.**

$$\sum_{k<n/9} \sigma(k)\sigma(n-9k) = \frac{1}{216}(\sigma_3(n) + (9 - 6n)\sigma(n)), \quad \text{if } n \equiv 2 \mod 3,$$

$$\sum_{k<n/9} \sigma(k)\sigma(n-9k) = \frac{1}{36}(6\sigma_3(n/3) + (6 - 4n)\sigma(n/3)$$

$$+ 9\sigma_3(n/9) - (3 + 6n)\sigma(n/9)), \quad \text{if } n \equiv 0 \mod 3.$$
give one example. Differentiating the first identity in Theorem 1.5, we obtain

\[ L_1(q) \frac{dL_1(q)}{dq} = \frac{1}{18} \frac{dM_2(q)}{dq}. \]  

(1.62)

Equating coefficients of \( q^n \ (n \equiv 2 \pmod{3}) \) in (1.62), we obtain the following identity, see Section 12.

**Theorem 1.11.**

\[
\sum_{k=1 \ (k \equiv 1 \pmod{3})}^{n-1} k\sigma(k)\sigma(n-k) = \frac{1}{18} n\sigma_3(n), \text{ if } n \equiv 2 \pmod{3}.
\]

This identity can also be proved by changing the variable \( k \) to \( n-k \) in the sum and appealing to the first identity in Theorem 1.8.

2. **Proof of Theorem 1.1**

We set

\[ a(z) = w(f(z)), \quad z \in D^*, \quad (2.1) \]
\[ b(z) = g(z)w(z), \quad z \in D^*. \]

(2.2)

Let \( z \in C(\frac{1}{z}) \subseteq D^* \). By (1.24) – (1.26), we know that \( a(z) \) and \( b(z) \) are defined and regular in \( D^* \). Differentiating (2.1) twice, and appealing to (1.20), we find that \( a(z) \) satisfies the differential equation

\[ a'' + p(z)a' + q(z)a = 0, \]

where

\[ p(z) = \frac{(1 - 2f(z))f'(z)}{f(z)(1 - f(z))} - \frac{f''(z)}{f'(z)}, \quad q(z) = \frac{-2f'(z)^2}{9f(z)(1 - f(z))}. \]

(2.3)

Similarly, differentiating (2.2) twice, and appealing to (1.20), we find that \( b(z) \) satisfies the differential equation

\[ b'' + r(z)b' + s(z)b = 0, \]

where

\[ r(z) = \frac{1 - 2z}{z(1 - z)} - \frac{2g'(z)}{g(z)}, \]
\[ s(z) = \frac{2g'(z)^2}{g(z)^2} - \frac{g''(z)}{g(z)} - \frac{2}{9z(1 - z)} - \frac{(1 - 2z)g'(z)}{z(1 - z)g(z)}. \]

(2.4)

(2.5)

A simple MAPLE calculation shows that

\[ p(z) = r(z), \quad q(z) = s(z), \]

(2.6)

so that \( a(z) \) and \( b(z) \) satisfy the same second order homogeneous linear differential equation

\[ y'' + r(z)y' + q(z)y = 0. \]

(2.7)

From (1.10), (1.16) and (2.4), we deduce that

\[ \lim_{z \to 0} zr(z) = \lim_{z \to 0} \left( \frac{1 - 2z}{1 - z} - \frac{2zg'(z)}{g(z)} \right) = 1. \]

(2.8)
and, from (1·9), (1·15) and (2·3), that
\[ \lim_{z \to 0} \frac{z^2 q(z)}{z} = \lim_{z \to 0} \left( \frac{-2zf'(z)^2}{f(z)(1 - f(z))} \right) = \frac{-2 \cdot 0 \cdot 9 \cdot \omega^2}{9 \cdot \omega \cdot 1} = 0. \]

We deduce from (2·8) and (2·9) that \( z = 0 \) is a regular singular point of (2·7) [8, p. 237]. The indicial equation is \( \alpha(\alpha - 1) + 1 \alpha + 0 = 0 \) [8, p. 238], which has the double root \( \alpha = 0 \). Thus the general solution of (2·7) in a neighbourhood of \( z = 0 \) is of the form
\[ y = Ay_1(z) + By_2(z), \quad A, B \in \mathbb{C}, \]
where \( y_1(z) \) is a solution of (2·7), which is regular in a neighbourhood of \( z = 0 \), and \( y_2(z) \) is a solution of (2·7) possessing a logarithmic branch point at \( z = 0 \) [8, p. 242]. Hence (2·7) has a unique solution \( y(z) \) regular in a neighbourhood of \( z = 0 \) with a prescribed value for \( y(0) \). Since (by (2·1), (1·9), (1·10) and (2·2))
\[ a(0) = w(f(0)) = w(0) = g(0)w(0) = b(0), \]
this proves that
\[ w(f(z)) = a(z) = b(z) = g(z)w(z) \text{ for } z \in C\left(\frac{1}{\delta}\right). \]

Thus as \( w(f(z)) \) and \( g(z)w(z) \) are defined and regular in \( D^* \), and equal in the subset \( C\left(\frac{1}{\delta}\right) \) of \( D^* \), we have
\[ w(f(z)) = g(z)w(z) \text{ for } z \in D^*, \]
which is (1·28). This completes the proof of Theorem 1·1.

3. Proof of Theorem 1·2

In this section we use Theorem 1·1 to prove Theorem 1·2. By (1·31) and (1·37) the function
\[ \frac{2\pi}{\sqrt{3}} \frac{w(1 - z)}{w(z)} = \frac{2\pi}{\sqrt{3}} \frac{w(1 - f(z))}{w(f(z))} \]
is defined and regular in \( D^{**} \supseteq \left(0, \frac{1}{\delta}\right) \). For \( x \in \left(0, \frac{1}{\delta}\right) \) we have [2, p. 87]
\[ \frac{d}{dx} \left( \frac{2\pi}{\sqrt{3}} \frac{w(1 - x)}{w(x)} \right) = -\frac{1}{x(1 - x)w(x)^2}. \]

Thus
\[ \frac{d}{dx} \left( \frac{2\pi}{\sqrt{3}} \frac{w(1 - f(x))}{w(f(x))} \right) = -\frac{f'(x)}{f(x)(1 - f(x))w(f(x))^2}. \]

By Theorem 1·1 we have \( w(f(x)) = g(x)w(x) \) so that
\[
\frac{d}{dx} \left( \frac{2\pi}{\sqrt{3}} \frac{w(1 - x)}{w(x)} - \frac{2\pi}{\sqrt{3}} \frac{w(1 - f(x))}{w(f(x))} \right) = \left( \frac{f'(x)}{f(x)(1 - f(x))g(x)^2} - \frac{1}{x(1 - x)} \right) \frac{1}{w(x)^2}. 
\]
A MAPLE calculation shows that
\[
\frac{f'(x)}{f(x)(1 - f(x))g(x)^2} = \frac{1}{x(1 - x)}
\]
so that there is a constant \( K \in \mathbb{C} \) such that
\[
\frac{2\pi}{\sqrt{3}} \frac{w(1 - x)}{w(x)} - \frac{2\pi}{\sqrt{3}} \frac{w(1 - f(x))}{w(f(x))} = K, \quad \text{for } 0 < x < \frac{1}{8}.
\]
Thus
\[
\lim_{x \to 0^+} (g(x)w(1 - x) - w(1 - f(x)))
\]
\[
= \lim_{x \to 0^+} \left( g(x)w(x) \left( \frac{w(1 - x)}{w(x)} - \frac{w(1 - f(x))}{w(f(x))} \right) \right)
\]
\[
= g(0)w(0) \frac{\sqrt{3}}{2\pi} K
\]
\[
= \frac{\sqrt{3}}{2\pi} K,
\]
by (1·10) and (1·21). On the other hand by [1, formula (15.3.10), p. 559], we have

\[
w(1 - x) = -\frac{\sqrt{3}}{2\pi} w(x) \log x + A(x), \quad |x| < 1,
\]
where

\[
A(x) = \frac{\sqrt{3}}{2\pi} \sum_{n=0}^{\infty} \frac{\left( \frac{1}{3} \right)_n}{(n!)^2} \left( 2\psi(n + 1) - \psi(n + \frac{1}{3}) - \psi(n + \frac{2}{3}) \right) x^n,
\]
and

\[
\psi(x) := \frac{\Gamma'(x)}{\Gamma(x)}, \quad x > 0,
\]
is the digamma function. Hence
\[
\lim_{x \to 0^+} (g(x)w(1 - x) - w(1 - f(x)))
\]
\[
= \lim_{x \to 0^+} (g(x)A(x) - A(f(x)) + \frac{\sqrt{3}}{2\pi} \log f(x) - \log x)w(f(x))
\]
\[
= g(0)A(0) - A(0) + \frac{\sqrt{3}}{2\pi} \lim_{x \to 0^+} \log \frac{f(x)}{x}
\]
\[
= \frac{\sqrt{3}}{2\pi} \log \omega \quad \text{(by (1·15))}
\]
\[
= \frac{i}{\sqrt{3}}.
\]
Hence
\[
K = \frac{2\pi i}{3}.
\]
Thus
\[
\frac{2\pi}{\sqrt{3}} \frac{w(1 - x)}{w(x)} - \frac{2\pi}{\sqrt{3}} \frac{w(1 - f(x))}{w(f(x))} - \frac{2\pi i}{3} = 0 \text{ for } x \in \left( 0, \frac{1}{8} \right).
\]
As the left-hand side of (3·1) is defined and regular in $D^{**}$ and $D^{**} \supseteq (0, \frac{1}{8})$, we have
$$\frac{2\pi}{\sqrt{3}} \frac{w(1-z)}{w(z)} - \frac{2\pi}{\sqrt{3}} \frac{w(1-f(z))}{w(f(z))} - \frac{2\pi i}{3} = 0 \text{ for } z \in D^{**},$$
which is (1·38). This completes the proof of Theorem 1·2.

4. Proof of Theorem 1·3

We just give the details for (1·48) as (1·47) and (1·49) can be proved in a similar manner. The function $M(q)$ is analytic in $C(1)$. For $0 < x < 1$ we have by (1·43) and (1·45)
$$M(e^{-\frac{2\pi}{\sqrt{3}} \frac{w(1-x)}{w(x)}}) = (1 + 8x)w(x)^4.$$ By the principle of analytic continuation, $M$ has an analytic continuation $M^*$ such that
$$M^*(e^{-\frac{2\pi}{\sqrt{3}} \frac{w(1-z)}{w(z)}}) = (1 + 8z)w(z)^4, \quad z \in D, \quad w(z) \neq 0,$$ and
$$M^*(q) = M(q), \quad q \in C(1).$$
For $z \in D^{**}$, we have $f(z) \in D$, $1 - f(z) \in D$ and $w(f(z)) \neq 0$, so that
$$M^*(e^{-\frac{2\pi}{\sqrt{3}} \frac{w(1-f(z))}{w(f(z))}}) = (1 + 8f(z))w(f(z))^4.$$ Then, by Theorems 1·1 and 1·2, we obtain
$$M^*(\omega e^{-\frac{2\pi}{\sqrt{3}} \frac{w(1-z)}{w(z)}}) = (1 + 8f(z))(g(z)w(z))^4.$$ Now take $z = x \in (0, \frac{1}{8}) \subseteq D^{**}$. Then
$$M^*(\omega q) = (1 + 8f(x))(g(x)w(x))^4.$$ As $|\omega q| = |q| < 1$ we have $M^*(\omega q) = M(\omega q)$ so that
$$M(\omega q) = (1 + 8f(x))(g(x)w(x))^4. \quad (4·1)$$ Replacing $f(x)$ and $g(x)$ in (4·1) by the expressions in (1·6) and (1·7) respectively, we obtain (1·48).

5. Proof of Theorem 1·4

We begin by applying the principle of triplication to (1·44), (1·45) and (1·46). As triplication sends
$$q \longrightarrow q^3, \quad x \longrightarrow \left(\frac{1 - (1 - x)^{\frac{1}{3}}}{1 + 2(1 - x)^{\frac{1}{3}}} \right)^3, \quad w \longrightarrow \frac{1}{3} \left(1 + 2(1 - x)^{\frac{1}{3}}\right) w,$$
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see [4, theorem 3.1, p. 101], we obtain

\[ L(q^3) = \left(1 - \frac{4}{3}x\right)w^2 + 4x(1-x)w \frac{dw}{dx}, \] (5.1)

\[ M(q^3) = \left(1 - \frac{8}{9}x\right)w^4, \] (5.2)

\[ N(q^3) = \left(1 - \frac{4}{3}x + \frac{8}{27}x^2\right)w^6, \] (5.3)

see [4, equation (13.17), p. 178, theorems 4.4 and 4.5, p. 107]. Applying the principle of triplication to (5.1), (5.2) and (5.3), we obtain

\[ L(q^9) = \left(11 - 12x + 8(1-x)^{\frac{1}{3}} + 8(1-x)^{\frac{2}{3}}\right)\frac{w^2}{27} + \frac{4}{3}x(1-x)w \frac{dw}{dx}, \] (5.4)

\[ M(q^9) = \left(249 - 248x + 240(1-x)^{\frac{1}{3}} + 240(1-x)^{\frac{2}{3}} - 160x(1-x)^{\frac{1}{3}}\right)\frac{w^4}{729}, \] (5.5)

\[ N(q^9) = \left(6579 - 8604x + 2024x^2 + 6552(1-x)^{\frac{1}{3}} + 6552(1-x)^{\frac{2}{3}} - 6384(1-x)^{\frac{1}{3}} - 4704x(1-x)^{\frac{2}{3}}\right)\frac{w^6}{19683}. \] (5.6)

Clearly, from (1.50) - (1.52) and (1.39) - (1.41), we obtain

\[ L_1(q) + L_2(q) + L_3(q) = \sum_{n=1}^{\infty} \sigma(n)q^n = \frac{1 - L(q)}{24}, \] (5.7)

\[ M_1(q) + M_2(q) + M_3(q) = \sum_{n=1}^{\infty} \sigma_3(n)q^n = \frac{M(q) - 1}{240}, \] (5.8)

\[ N_1(q) + N_2(q) + N_3(q) = \sum_{n=1}^{\infty} \sigma_5(n)q^n = \frac{1 - N(q)}{504}. \] (5.9)

Now for all \( k \in \mathbb{N} \) we have

\[ \sigma(3k) = 4\sigma(k) - 3\sigma(k/3) \]

so that

\[ L_3(q) = \sum_{k=1}^{\infty} \sigma(3k)q^{3k} \]
\[ = 4\sum_{k=1}^{\infty} \sigma(k)q^{3k} - 3\sum_{k=1}^{\infty} \sigma(k/3)q^{3k} \]
\[ = 4\sum_{k=1}^{\infty} \sigma(k)q^{3k} - 3\sum_{k=1}^{\infty} \sigma(k)q^{9k} \]
\[ = 4\left(\frac{1 - L(q^3)}{24}\right) - 3\left(\frac{1 - L(q^9)}{24}\right), \]

that is

\[ L_3(q) = \frac{1 - 4L(q^3) + 3L(q^9)}{24}. \] (5.10)
Then, from (5·7) and (5·10), we deduce
\[ L_1(q) + L_2(q) = \frac{-L(q) + 4L(q^3) - 3L(q^9)}{24}. \] (5·11)

Similarly, using
\[ \sigma_3(3k) = 28\sigma_3(k) - 27\sigma_3(k/3), \]
\[ \sigma_5(3k) = 244\sigma_5(k) - 243\sigma_5(k/3), \]
we obtain
\[ M_3(q) = \frac{-1 + 28M(q^3) - 27M(q^9)}{240}; \]
\[ M_1(q) + M_2(q) = \frac{M(q) - 28M(q^3) + 27M(q^9)}{240}; \]
\[ N_3(q) = \frac{1 - 244N(q^3) + 243N(q^9)}{504}; \]
\[ N_1(q) + N_2(q) = \frac{-N(q) + 244N(q^3) - 243N(q^9)}{504}. \] (5·15)

From (5·1)–(5·6), (5·10), (5·12) and (5·14), we obtain
\[ L_3(q) = \frac{1}{24} + \left( -25 + 36x + 8(1 - x)^{\frac{1}{3}} + 8(1 - x)^{\frac{2}{3}} \right) \frac{w^2}{216} \]
\[ \quad - \frac{1}{2}x(1 - x)w \frac{dw}{dx}, \] (5·16)
\[ M_3(q) = \frac{-1}{240} + \left( 507 - 424x - 240(1 - x)^{\frac{1}{3}} - 240(1 - x)^{\frac{2}{3}} + 160x(1 - x)^{\frac{1}{3}} \right) \frac{w^4}{6480}; \] (5·17)
\[ N_3(q) = \frac{1}{504} + \left( -355995 + 479196x - 103464x^2 + 176904(1 - x)^{\frac{1}{3}} + 176904(1 - x)^{\frac{2}{3}} - 127008x(1 - x)^{\frac{1}{3}} \right) \frac{w^6}{1102248}. \] (5·18)

From (1·44) – (1·46), (5·1) – (5·6), (5·11), (5·13) and (5·15), we obtain
\[ L_1(q) + L_2(q) = \left( 2 - (1 - x)^{\frac{1}{3}} - (1 - x)^{\frac{2}{3}} \right) \frac{w^2}{27}, \] (5·19)
\[ M_1(q) + M_2(q) = \left( -6 + 8x + 3(1 - x)^{\frac{1}{3}} + 3(1 - x)^{\frac{2}{3}} - 2x(1 - x)^{\frac{1}{3}} \right) \frac{w^4}{81}; \] (5·20)
\[ N_1(q) + N_2(q) = \left( 234 - 288x + 80x^2 - 117(1 - x)^{\frac{1}{3}} - 117(1 - x)^{\frac{2}{3}} + 114x(1 - x)^{\frac{1}{3}} + 84x(1 - x)^{\frac{2}{3}} \right) \frac{w^6}{729}. \] (5·21)

The formulae (1·39) – (1·41) are valid for all \( q \in \mathbb{C} \) with \( |q| < 1 \). Replacing \( q \) by \( \omega q \) in each of (1·39), (1·40) and (1·41), we obtain respectively
\[ L(\omega q) = 1 - 24\omega L_1(q) - 24\omega^2 L_2(q) - 24L_3(q), \] (5·22)
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\[ M(\omega q) = 1 + 240\omega M_1(q) + 240\omega^2 M_2(q) + 240 M_3(q), \]
\[ N(\omega q) = 1 - 504\omega N_1(q) - 504\omega^2 N_2(q) - 504 N_3(q). \]

From (1-47), (5-16) and (5-22), we obtain
\[ \omega L_1(q) + \omega^2 L_2(q) = \left(-1 + (\omega - 2\omega^2)(1 - x)^{\frac{1}{3}} + (-2\omega + \omega^2)(1 - x)^{\frac{2}{3}}\right) \frac{w^2}{27}. \]

Solving (5-19) and (5-25) for \( L_1(q) \) and \( L_2(q) \), we obtain
\[ L_1(q) = \left(1 + (1 - x)^{\frac{1}{3}} - 2(1 - x)^{\frac{2}{3}}\right) \frac{w^2}{27}, \]
\[ L_2(q) = \left(1 - 2(1 - x)^{\frac{1}{3}} + (1 - x)^{\frac{2}{3}}\right) \frac{w^2}{27}. \]

From (1-48), (5-17) and (5-23), we obtain
\[ \omega M_1(q) + \omega^2 M_2(q) = \left(6 - 8x - 3(1 - x)^{\frac{1}{3}} - 3(1 - x)^{\frac{2}{3}} + 2x(1 - x)^{\frac{1}{3}}\right)
+ (\omega - \omega^2)(-9(1 - x)^{\frac{1}{3}} + 9(1 - x)^{\frac{2}{3}}
+ 6x(1 - x)^{\frac{1}{3}}) \frac{w^4}{162}. \]

Solving (5-20) and (5-28) for \( M_1(q) \) and \( M_2(q) \), we obtain
\[ M_1(q) = \left(-3 + 4x - 3(1 - x)^{\frac{1}{3}} + 6(1 - x)^{\frac{2}{3}} + 2x(1 - x)^{\frac{1}{3}}\right) \frac{w^4}{81}, \]
\[ M_2(q) = \left(-3 + 4x + 6(1 - x)^{\frac{1}{3}} - 3(1 - x)^{\frac{2}{3}} - 4x(1 - x)^{\frac{1}{3}}\right) \frac{w^4}{81}. \]

From (1-49), (5-18) and (5-24), we obtain
\[ \omega N_1(q) + \omega^2 N_2(q) = \left(A + (\omega - \omega^2)B\right) \frac{x^6}{1458}. \]

where
\[ A = -234 + 288x - 89x^2 + 117(1 - x)^{\frac{1}{3}} + 117(1 - x)^{\frac{2}{3}} - 114x(1 - x)^{\frac{1}{3}} - 84x(1 - x)^{\frac{2}{3}} \]
and
\[ B = 351(1 - x)^{\frac{1}{3}} - 351(1 - x)^{\frac{2}{3}} - 342x(1 - x)^{\frac{1}{3}} + 252x(1 - x)^{\frac{2}{3}}. \]

Solving (5-21) and (5-31) for \( N_1(q) \) and \( N_2(q) \), we obtain
\[ N_1(q) = \left(117 - 144x + 40x^2 + 117(1 - x)^{\frac{1}{3}} - 234(1 - x)^{\frac{1}{3}}
- 114x(1 - x)^{\frac{1}{3}} + 168x(1 - x)^{\frac{2}{3}}\right) \frac{w^6}{729}, \]
\[ N_2(q) = \left(117 - 144x + 40x^2 - 234(1 - x)^{\frac{1}{3}} + 117(1 - x)^{\frac{2}{3}}
+ 228x(1 - x)^{\frac{1}{3}} - 848x(1 - x)^{\frac{2}{3}}\right) \frac{w^6}{729}. \]

This completes the proof of Theorem 1-4.
We just give the proof of the first identity in Theorem 1.5 as the remaining three identities can be proved similarly. From (1.53) and (1.57), we obtain

\[ L_1(q)^2 = \left( 1 + (1 - x)^{\frac{1}{3}} + 4(1 - x)(1 - x)^{\frac{1}{3}} + 2(1 - x)^{\frac{2}{3}} \right) \frac{w^4}{729} \]

\[ = \left( -3 + 4x + 6(1 - x)^{\frac{1}{3}} - 3(1 - x)^{\frac{2}{3}} - 4x(1 - x)^{\frac{1}{3}} \right) \frac{w^4}{729} \]

\[ = \frac{1}{9} M_2(q) \]

as asserted.

7. Proof of Theorem 1.6

We just give the proof of the second identity in Theorem 1.6 as the remaining three identities can be proved similarly. By Theorem 1.4 we have

\[ L_2(q) = \left( 1 - 2(1 - x)^{\frac{1}{3}} + (1 - x)^{\frac{2}{3}} \right) \frac{w^2}{27} \]

so that

\[ \frac{dL_2}{dx} = \left( \frac{2}{3}(1 - x)^{-\frac{2}{3}} - \frac{2}{3}(1 - x)^{-\frac{1}{3}} \right) \frac{w^2}{27} + \left( 1 - 2(1 - x)^{\frac{1}{3}} + (1 - x)^{\frac{2}{3}} \right) 2w \frac{dw}{dx}. \]

By (1.43) and (1.42), we have

\[ \frac{dq}{dx} = -e^{-y} \frac{dy}{dx} = \frac{q}{x(1 - x)w^2} \]

so that

\[ q \frac{dL_2}{dq} = x(1 - x)w^2 \frac{dL_2}{dx} \]

\[ = x\left( (1 - x)^{\frac{1}{3}} - (1 - x)^{\frac{2}{3}} \right) \frac{2}{81} w^4 \]

\[ + x(1 - x) \left( 1 - 2(1 - x)^{\frac{1}{3}} + (1 - x)^{\frac{2}{3}} \right) \frac{2w^3}{27} \frac{dw}{dx}. \]

From (1.54) and (5.1), we obtain

\[ L_2(q)L(q^3) = \left( (3 - 4x) - 2(3 - 4x)(1 - x)^{\frac{1}{3}} + (3 - 4x)(1 - x)^{\frac{2}{3}} \right) \frac{w^4}{81} \]

\[ + 4 \left( 1 - 2(1 - x)^{\frac{1}{3}} + (1 - x)^{\frac{2}{3}} \right) x(1 - x)w^3 \frac{dw}{27} \frac{dx}. \]

Hence, appealing to (1.57), we deduce

\[ L_2(q)L(q^3) + M_2(q) = 4x \left( (1 - x)^{\frac{1}{3}} - (1 - x)^{\frac{2}{3}} \right) \frac{w^4}{81} \]

\[ + 4 \left( 1 - 2(1 - x)^{\frac{1}{3}} + (1 - x)^{\frac{2}{3}} \right) x(1 - x)w^3 \frac{dw}{27} \frac{dx} \]

\[ = 2q \frac{dL_2}{dq}, \]

completing the proof of the identity.
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8. Proof of Theorem 1.7

We just give the proof of the first identity in Theorem 1.7. The second identity can be proved similarly. By (1.54) and (5.4), we obtain

\[
9L_2(q)L(q^9) = \left(3 - 4x - 6(1 - x)^{\frac{1}{3}} - 9(1 - x)^{\frac{2}{3}} + 16x(1 - x)^{\frac{2}{3}}\right) \frac{w^4}{81} + \frac{4}{9} \left(1 - 2(1 - x)^{\frac{1}{3}} + (1 - x)^{\frac{2}{3}}\right) x(1 - x)w^3 \frac{dw}{dx}.
\]

Hence, appealing to (1.57), we have

\[
9L_2(q)L(q^9) + M_2(q) = 4x \left((1 - x)^{\frac{1}{3}} - (1 - x)^{\frac{2}{3}}\right) \frac{w^4}{27} + \frac{4}{9} \left(1 - 2(1 - x)^{\frac{1}{3}} + (1 - x)^{\frac{2}{3}}\right) x(1 - x)w^3 \frac{dw}{dx} = 6q \frac{dL_2}{dq},
\]

completing the proof of the identity.

9. Proof of Theorem 1.8

We just give the proof of the first identity. The other identities can be proved similarly. We have

\[
\sum_{n=1}^{\infty} \sum_{n \equiv 2 \pmod{3}} S(1, 3)q^n = \sum_{n=1}^{\infty} \sum_{n \equiv 2 \pmod{3}} \sum_{k=1}^{n-1} \sigma(k)\sigma(n-k)q^n
\]

\[
= \sum_{N=0}^{\infty} \sum_{k=1}^{3N+1} \sum_{k \equiv 1 \pmod{3}} \sigma(k)\sigma(3N + 2 - k)q^{3N+2}
\]

\[
= \sum_{N=0}^{\infty} \sum_{m=0}^{N} \sigma(3m + 1)\sigma(3(N - m) + 1)q^{3N+2}
\]

\[
= \sum_{m, n=0}^{\infty} \sigma(3m + 1)\sigma(3n + 1)q^{3(m+n)+2}
\]

\[
= \left(\sum_{n=0}^{\infty} \sigma(3n + 1)q^{3n+1}\right)^2 = L_1(q)^2 = \frac{1}{9} M_2(q) = \frac{1}{9} \sum_{n=1}^{\infty} \sigma_3(n)q^n,
\]

by the first identity in Theorem 1.5, and the result follows on equating coefficients of \(q^n\) with \(n \equiv 2 \pmod{3}\).
Appealing to the fourth identity in Theorem 1·6, we obtain

\[
\sum_{n=1}^{\infty} \sum_{m<n/3} \sigma(m)\sigma(n-3m)q^n = \sum_{l=1}^{\infty} \sigma(l)q^l \sum_{m=1}^{\infty} \sigma(m)q^{3m} = \left( \frac{1 - L(q)}{24} \right) \left( \frac{1 - L(q^3)}{24} \right) = \frac{1}{576} \left( 1 - L(q) - L(q^3) + L(q)L(q^3) \right) = \frac{1}{576} \left( 1 - L(q) - L(q^3) + \frac{9}{10} M(q) + \frac{9}{10} M(q^3) + 2q \frac{dL(q)}{dq} + \frac{9}{2} q^3 \frac{dL(q^n)}{dq} - 36q \frac{dL_3(q)}{dq} \right)
\]

\[
= \frac{1}{576} \left( 24 \sum_{n=1}^{\infty} \sigma(n)q^n + 24 \sum_{n=1}^{\infty} \sigma(n/3)q^n + 24 \sum_{n=1}^{\infty} \sigma_3(n)q^n + 216 \sum_{n=1}^{\infty} \sigma_3(n/3)q^n - 48 \sum_{n=1}^{\infty} n\sigma(n)q^n - 108 \sum_{n=1}^{\infty} n\sigma(n/9)q^n - 36 \sum_{n \geq 1, 3 \mid n} n\sigma(n)q^n \right).
\]

Equating coefficients of \(q^n\), and appealing to the elementary result

\[
4\sigma(n/3) - 3\sigma(n/9) = \begin{cases} 
\sigma(n), & \text{if } 3 \mid n, \\
0, & \text{if } 3 \nmid n.
\end{cases}
\]

we obtain the assertion of Theorem 1·9.

11. Proof of Theorem 1·10

We just give the proof of the first identity. The second identity can be proved similarly. We have by (1·50) and (1·39)

\[
9L_2(q) L(q^9) = 9 \left( \sum_{m=1}^{\infty} \sigma(m)q^m \right) \left( 1 - 24 \sum_{l=1}^{\infty} \sigma(l)q^l \right)
\]

\[
= 9 \sum_{n=1}^{\infty} \sigma(n)q^n - 216 \sum_{n=1}^{\infty} q^n \sum_{l, m \geq 1, 9l + m = n} \sigma(l)\sigma(m)
\]

\[
= 9 \sum_{n=1}^{\infty} \sigma(n)q^n - 216 \sum_{n=1}^{\infty} q^n \sum_{l \leq n/9} \sigma(l)\sigma(n - 9l).
\]

On the other hand, by the first identity in Theorem 1·7, we have

\[
9L_2(q) L(q^9) = 6q \frac{dL_2}{dq} - M_2(q) = 6 \sum_{n=1}^{\infty} n\sigma(n)q^n - \sum_{n=1}^{\infty} \sigma_3(n)q^n.
\]
A cubic transformation formula

Equating coefficients of $q^n$ ($n \equiv 2 \pmod{3}$), we obtain the asserted identity.

12. Proof of Theorem 1·11

We have by (1·62)

$$
\sum_{n=1}^{\infty} \sum_{k=1}^{n-1} k\sigma(k)\sigma(n-k)q^n = \sum_{l=1}^{\infty} \sigma(l)q^l \sum_{k=1}^{\infty} k\sigma(k)q^k
$$

$$
= L_1(q)q \frac{dL_1(q)}{dq} = \frac{1}{18} q^{12} \frac{dM_2(q)}{dq}
$$

$$
= \frac{1}{18} \sum_{n=1}^{\infty} n\sigma_3(n)q^n.
$$

Equating coefficients of $q^n$ ($n \equiv 2 \pmod{3}$), we obtain the asserted result.

Acknowledgement. The author would like to thank Nathalie Cheng for her assistance in finding the identities in Theorems 1·5, 1·6 and 1·7.

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