Pascal’s Triangle (mod 8)

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Lucas’ theorem gives a congruence for a binomial coefficient modulo a prime. Davis and Webb (Europ. J. Combinatorics, 11 (1990), 229–233) extended Lucas’ theorem to a prime power modulus. Making use of their result, we count the number of times each residue class occurs in the nth row of Pascal’s triangle (mod 8). Our results correct and extend those of Granville (Amer. Math. Monthly, 99 (1992), 318–331).

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1. INTRODUCTION

Let n denote a nonnegative integer. The nth row of Pascal’s triangle consists of the n + 1 binomial coefficients \( \binom{n}{r} \) \( (r = 0, 1, \ldots, n) \). For integers t and m with \( 0 \leq t < m \), we denote by \( N_n(t, m) \) the number of integers in the nth row of Pascal’s triangle which are congruent to t modulo m. Clearly if d is a positive integer dividing m then

\[
\sum_{j=0}^{(m/d)-1} N_n(jd + t, m) = N_n(t, d), \quad 0 \leq t < d. \tag{1.1}
\]

When \( d = 1 \) the right-hand side of (1.1) is \( N_n(0, 1) = n + 1 \). In 1899 Glaisher [3] showed that \( N_n(1, 2) \) is always a power of 2, see Theorem A. In 1991 Davis and Webb [2] determined \( N_n(t, 4) \) for \( t = 0, 1, 2, 3 \), see Theorem B. Their results show that \( N_n(1, 4) \) is always a power of 2 and that \( N_n(3, 4) \) is either 0 or a power of 2. These facts were also observed by Granville [4] in 1992. In addition Granville found for odd \( t \) that \( N_n(t, 8) \) is either 0 or a power of 2. Unfortunately some of Granville’s results on the distribution of the odd binomial coefficients modulo 8 in the nth row of Pascal’s triangle are incorrect. We label Granville’s five assertions following Figure 12 on p. 324 of [4] as (α), (β), (γ), (δ), and (ε). (Note that in the wording describing Figure 12 the assertion ‘each \( u_j \geq 2 \) is not correct as the block of 0’s in \( (n)_{12} \) furthest to the right may contain just one zero, for example, \( n = 78 \) has \( (n)_{12} = 10011100 \).) We comment on each of (α), (γ), (δ), and (ε).

(α) This assertion is false. Take \( n = 3 \) so that \( (n)_{12} = 11 \). Thus \( t_1 = 2 \) and there are no other \( t_j \)’s. Hence \( n = 3 \) falls under (α). However, the third row of Pascal’s triangle is 1, 3, 3, 1 contradicting the assertion of (α).

(β) This assertion is false. Take \( n = 19 \) so that \( (n)_{12} = 100111 \). Thus \( t_1 = 1, u_1 = 2, \) and \( t_2 = 2 \). Hence \( n = 19 \) falls under (β). However, the first half of the 19th row of Pascal’s triangle modulo 8 is 1, 3, 3, 1, 4, 4, 4, 4, 6, 2 so that \( N_{19}(1, 8) = 4, N_{19}(7, 8) = 0 \), contradicting Granville’s claim that \( N_{19}(1, 8) = N_{19}(7, 8) \).

(δ) This assertion does not tell the full story. Take \( n = 39 \) so that \( (n)_{12} = 10011111 \). Thus \( t_1 = 1, u_1 = 2, \) and \( n = 39 \) falls under (δ). Here \( N_{39}(t, 8) = 4 \) \( (t = 1, 3, 5, 7) \). On the other hand if \( n = 156 \) then \( (n)_{12} = 10011100 \) so that \( t_1 = 1, u_1 = 2, t_2 = 3, u_2 = 2, \) and \( n = 156 \) falls under (δ). Here \( N_{156}(1, 8) = N_{156}(3, 8) = 8, N_{156}(5, 8) = N_{156}(7, 8) = 0 \).

(ε) This assertion is false. If \( n = 3699 \) then \( (n)_{12} = 111001110011 \) so that \( t_1 = 3, u_1 = 2, \) \( t_2 = 3, u_2 = 2, \) \( t_3 = 2, \) and \( n = 3699 \) falls under (ε). However \( N_{3699}(1, 8) = N_{3699}(3, 8) = 128, N_{3699}(5, 8) = N_{3699}(7, 8) = 0, \) contradicting Granville’s claim that \( N_{3699}(1, 8) = N_{3699}(3, 8) = N_{3699}(5, 8) = N_{3699}(7, 8) \).
This article was motivated by the desire to find the correct evaluation of \( N_n(t, 8) \) when \( t \) is odd (see Theorem C (third part)). In addition our method enables us to determine the value of \( N_n(t, 8) \) when \( t \) is even, a problem not considered by Granville, see Theorem C (first and second parts). Our evaluation of \( N_n(t, 8) \) involves the binary representation of \( n \), namely

\[
n = a_0 + a_12 + a_22^2 + \cdots + a_\ell2^\ell,
\]

where \( \ell \geq 0 \), each \( a_\ell \) is either 0 or 1, and \( a_\ell = 1 \) unless \( n = 0 \) in which case \( \ell = 0 \) and \( a_0 = 0 \).

For brevity we write \( a_0a_1\ldots a_\ell \) for the binary representation of \( n \). Note that our notation is the reverse of Granville’s notation [4]. On occasion it is more convenient to consider \( a_0a_1\ldots a_\ell \) as a string of 0’s and 1’s. The context will make it clear which interpretation is being used. The length of the \( i \)th block of 0’s (respectively 1’s) in \( a_0a_1\ldots a_\ell \) is denoted by \( v_i \) (respectively \( s_j \)). We consider \( n \) to begin with a block of 0’s and to finish with a block of 1’s. Thus, the binary representation of \( n = 389743 \) is 1111011001001111101 and \( v_1 = 0, s_1 = 4, v_2 = 1, s_2 = 2, v_3 = 2, s_3 = 1, v_4 = 2, s_4 = 5, v_5 = 5, s_5 = 1 \).

Throughout this article \( r \) denotes an arbitrary integer between 0 and \( n \) inclusive. The binary representation of \( r \) is (with additional zeros at the right-hand end if necessary) \( r = b_0b_1\ldots b_\ell \). The exact power of 2 dividing the binomial coefficient \( \binom{n}{r} \) is given by a special case of Kummer’s theorem [5].

**Proposition 1 (Kummer [5]).** Let \( c(n, r) \) denote the number of carries when adding the binary representations of \( r \) and \( n - r \). Then

\[
2^{c(n, r)} \| \binom{n}{r}.
\]

Consider now the addition of the binary representation \( b_0b_1\ldots b_\ell \) of \( r \) to that of \( n - r \) to obtain the binary representation \( a_0a_1\ldots a_\ell \) of \( n \). If no carry occurs in the \( (i - 1) \)th position then there is no carry in the \( i \)th position if \( b_i \leq a_i \), whereas there is a carry in the \( i \)th position if \( b_i > a_i \). This simple observation enables us to say when \( c(n, r) = 0, 1 \) or 2.

**Proposition 2.**

(a) \( c(n, r) = 0 \Leftrightarrow b_i \leq a_i \) (\( i = 0, 1, \ldots, \ell \)).

(b) \( c(n, r) = 1 \) and the carry occurs in the \( f \)th position (\( 0 \leq f \leq \ell - 1 \)) \( \Leftrightarrow a_\ell a_{f+1} = 01, b_f b_{f+1} = 10, \) and \( b_i \leq a_i \) (\( i \neq f, f + 1 \)).

(c) \( c(n, r) = 2 \) and the carries occur in the \( f \)th and \( g \)th positions (\( 0 \leq f < g \leq \ell - 1 \))

\[
\Leftrightarrow a_\ell a_{f+1} = 01, b_f b_{f+1} = 10, a_g a_{g+1} = 01, b_g b_{g+1} = 10, \quad \text{if } g \neq f + 1,
\]

\[
a_\ell a_{f+1}a_{f+2} = 011, \quad b_f b_{f+1}b_{f+2} = 110, \quad \text{or}
\]

\[
a_\ell a_{f+1}a_{f+2} = 001, \quad b_f b_{f+1}b_{f+2} = 100, \quad \text{if } g = f + 1,
\]

and

\[
b_i \leq a_i \quad (i \neq f, f + 1, g, g + 1).
\]

(* denotes 0 or 1.)

If \( S \) denotes a nonempty string of 0’s and 1’s, we denote by \( n_S \) the number of occurrences of \( S \) in the string \( a_0a_1\ldots a_\ell \). For example, if \( n = 1496 = 00011011101 \) then \( n_0 = 5, n_1 = 6, n_{00} = 2, n_{01} = 3, n_{10} = 2, n_{11} = 3, n_{000} = 1, n_{001} = 1 \).
Pascal’s Triangle \((\text{mod } 8)\)

Propositions 1 and 2(a) enable us to prove Glaisher’s formulae \([3]\).

**THEOREM A (GLAISHER [3]).** \(N_n(0, 2) = n + 1 - 2^n, \ N_n(1, 2) = 2^n.\)

**PROOF.** We have

\[
N_n(1, 2) = \sum_{r=0}^{n} \binom{n}{r} \equiv 1 \pmod{2}
\]

The formula for \(N_n(0, 2)\) now follows from (1.1) with \(d = 1\) and \(m = 2.\)

Similarly we can prove Davis and Webb’s formulae \([2]\).

**THEOREM B (FIRST PART, DAVIS AND WEBB [2]).**

\[
N_n(0, 4) = n + 1 - 2^n - n_{01}2^{n-1}, \quad N_n(2, 4) = n_{01}2^{n-1}.
\]

**PROOF.** Appealing to Propositions 1 and 2(b), we have

\[
N_n(2, 4) = \sum_{f=0}^{n} \prod_{j=0, a_f + 1 = 01, j \neq f+1} (a_j + 1) = \sum_{f=0}^{n-1} 2^{n-1} = n_{01}2^{n-1}.
\]

From (1.1) with \(m = 4, \ d = 2, \) and \(t = 0,\) we have \(N_n(0, 4) + N_n(2, 4) = N_n(0, 2),\) from which the value of \(N_n(0, 4)\) follows.

Likewise we can use Propositions 1 and 2(c) to determine \(N_n(0, 8)\) and \(N_n(4, 8).\)

**THEOREM C (FIRST PART).**

\[
N_n(0, 8) = n + 1 - (n_{001} + 1)2^n - n_{011}2^{n-2} - n_{01}(n_{01} + 3)2^{n-3},
\]
\[
N_n(4, 8) = n_{001}2^n + n_{011}2^{n-2} + n_{01}(n_{01} - 1)2^{n-3}.
\]
Proof. We have

\[ N_n(4, 8) = \sum_{r=0}^{n} \frac{1}{c(n,r) = 2} \quad (\ell \equiv 4 \pmod{8}) \]

\[ = \sum_{f=0}^{\ell-2} \frac{a_0, \ldots, a_{f-1}, a_{f+3}, \ldots, a_{\ell}}{b_0, \ldots, b_{f-1}, b_{f+3}, \ldots, b_{\ell} = 0} 1 + \sum_{f=0}^{\ell-2} \frac{a_0, \ldots, a_{f-1}, a_{f+3}, \ldots, a_{\ell}}{b_0, \ldots, b_{f-1}, b_{f+3}, \ldots, b_{\ell} = 0} \]

\[ + \sum_{f=0}^{\ell-3} \frac{a_0, \ldots, a_{f-1}, a_{f+2}, \ldots, a_{\ell}}{b_0, \ldots, b_{f-1}, b_{f+2}, \ldots, b_{\ell} = 0} 1 \]

\[ = \sum_{f=0}^{\ell-2} \frac{\prod_{j=0}^{\ell-1} (a_j + 1)}{a_{f+1} = 011} + \sum_{j=0}^{\ell-2} \frac{\prod_{j=0}^{\ell-1} (a_j + 1)}{a_{f+1} = 001} 2^{n-2} + \sum_{f=0}^{\ell-2} \frac{\prod_{j=0}^{\ell-1} (a_j + 1)}{a_{f+1} = 001} 2^{n-1} \]

\[ = n_{011}2^{n-1} + n_{001}2^{n-1} + n_{001}(n_{001} - 1)2^{n-2} \]

The value of \( N_n(0, 8) \) follows from (1.1) with \( m = 8, d = 4, \) and \( t = 0 \).

Although Kummer’s result (Proposition 1) enabled us to determine \( N_n(1,2), N_n(2,4), \) and \( N_n(4,8) \), it is clear that we need a more precise congruence for \( \binom{\ell}{2} \) to be able to determine \( N_n(t,4) \) for \( t = 1,3 \) and \( N_n(t,8) \) for \( t = 1,2,3,5,6,7 \). The required congruences for \( \binom{\ell}{2} \) modulo 4 and modulo 8 are provided by the Davis–Webb congruence, which is the subject of the next section.

It is understood throughout that an empty sum has the value 0, an empty product the value 1, and

\[ 0^n = \begin{cases} 1, & \text{if } n = 0, \\ 0, & \text{if } n \geq 1. \end{cases} \]

2. The Davis–Webb Congruence

In order to state the Davis–Webb congruence for \( \binom{\ell}{2} \) modulo \( 2^h \), we need the binary version of the symbol \( \binom{\ell}{2} \) defined by Davis and Webb [1] for arbitrary nonnegative integers \( c = c_0c_1 \ldots c_s \) and \( d = d_0d_1 \ldots d_t \) (where additional zeros have been included at the right-hand of either \( c \) or \( d \), if necessary, to make their binary representations the same length). If \( c_0c_1 \ldots c_i < d_0d_1 \ldots d_i \) for \( i = 0,1,\ldots,s \), we set

\[ \left\lfloor \frac{c}{d} \right\rfloor = 2^x + 1. \]

Otherwise we let \( u \) denote the largest integer between 0 and \( s \) inclusive for which \( c_0c_1 \ldots c_u \geq d_0d_1 \ldots d_u \), and set

\[ \left\lfloor \frac{c}{d} \right\rfloor = 2^{s-u} \left( \frac{c_0c_1 \ldots c_u}{d_0d_1 \ldots d_u} \right). \]
### Table 1.

<table>
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<th>( d_0 )</th>
<th>( c_0 )</th>
<th>( d_1 )</th>
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### Table 2.

<table>
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<th>( d_0d_1 )</th>
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</thead>
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<td>01</td>
</tr>
<tr>
<td>01</td>
<td>11</td>
</tr>
<tr>
<td>11</td>
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</tbody>
</table>

Thus, for example, if \( c = 26 = 010110 \) and \( d = 39 = 111001 \), we have \( s = 5 \) and \( u = 4 \) so that

\[
\binom{26}{39} = 2^{5-4} \binom{01011}{11100} = 2 \binom{26}{7}.
\]

The symbol \( \binom{c}{d} \) is an extension of the ordinary binomial coefficient since for \( 0 \leq d \leq c \) we have \( u = s \) and so \( \binom{c}{d} = \binom{c}{s} \). The odd part of \( \binom{c}{d} \) is denoted by \( \left[ \binom{c}{d} \right] \). The values of \( \left[ \binom{c}{d} \right] \) for \( s = 0, 1, \) and 2 are given in Tables 1–3 respectively.

For our purposes it is also convenient to set for \( s \geq 1 \)

\[
\left[ \binom{c}{d} \right] = \text{odd part of } \binom{c_0c_1 \ldots c_s}{d_0d_1 \ldots d_s}.
\]

The values of \( \left[ \binom{c}{d} \right] \) for \( s = 1 \) and 2 are given in Tables 4 and 5 respectively. From Tables 1–5 we obtain the assertions of Lemma 1.

#### Lemma 1.

\[(a)\] \[ \begin{bmatrix} c_0 \\ d_0 \end{bmatrix} = 1. \]

\[(b)\] \[ \begin{bmatrix} c_0 & c_1 \\ d_0 & d_1 \end{bmatrix} = \begin{cases} 1 \pmod{4}, & \text{if } c_0c_1 \neq 11, \\ (-1)^{d_0+d_1} \pmod{4}, & \text{if } c_0c_1 = 11. \end{cases} \]

\[(c)\] \[ \begin{bmatrix} c_0 & c_1 \\ d_0 & d_1 \end{bmatrix} = \begin{cases} 1 \pmod{8}, & \text{if } c_0c_1 \neq 11, \\ (-1)^{d_0+d_1} 5^{d_0+d_1} \pmod{8}, & \text{if } c_0c_1 = 11. \end{cases} \]

\[(d)\] \[ \begin{bmatrix} c_0 & c_1 \\ d_0 & d_1 \end{bmatrix} \equiv \begin{cases} 1 \pmod{4}, & \text{if } c_0c_1 \neq 11, \\ (-1)^{d_0+d_1} \pmod{4}, & \text{if } c_0c_1 = 11. \end{cases} \]

\[(e)\] \[ \begin{bmatrix} c_0 & c_1 & 0 \\ d_0 & d_1 & d_2 \end{bmatrix} = 1. \]
TABLE 3.
Values of $[c_0, c_1, c_2]$.  

<table>
<thead>
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<td>3</td>
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TABLE 4.
Values of $[c_0, c_1]'$.

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</tr>
<tr>
<td>11</td>
<td>1</td>
<td>1</td>
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</tr>
</tbody>
</table>

\[
\begin{align*}
(f) & \begin{bmatrix} c_0 & 0 & 1 \\ d_0 & 1 & 0 \end{bmatrix}' \equiv (-1)^{1+c_0} \pmod{4}, \text{ if } c_0 \geq d_0. \\
(g) & \begin{bmatrix} 0 & 1 & c_2 \\ 1 & 0 & d_2 \end{bmatrix}' \equiv (-1)^{c_2} \pmod{4}. \\
(h) & \begin{bmatrix} c_0 & 1 & 1 \\ d_0 & d_1 & d_2 \end{bmatrix}' \equiv (-1)^{d_1+d_2} \pmod{4}, \text{ if } c_0 \geq d_0. \\
(i) & \begin{bmatrix} c_0 & 1 & c_2 \\ d_0 & d_1 & d_2 \end{bmatrix}' \equiv 1 \pmod{4}, \text{ if } c_1c_2 \neq 11 \text{ and } c_1 \geq d_1. \\
(j) & \begin{bmatrix} 1 & 0 & 1 \\ d_0 & 0 & d_2 \end{bmatrix}' \equiv 5^{d_0+d_2} \pmod{8}. \\
(k) & \begin{bmatrix} 0 & 1 & 1 \\ 0 & d_1 & d_2 \end{bmatrix}' \equiv (-1)^{d_1+d_2} \pmod{8}. \\
(l) & \begin{bmatrix} 1 & 1 & 1 \\ d_0 & d_1 & d_2 \end{bmatrix}' \equiv (-1)^{d_1+d_2}5^{d_0+d_2} \pmod{8}. \\
(m) & \begin{bmatrix} c_0 & c_1 & c_2 \\ d_0 & d_1 & d_2 \end{bmatrix}' \equiv 1 \pmod{8}, \text{ if } c_0c_1c_2 \neq 101, 011, 111, \text{ and } c_i \geq d_i (i = 0, 1, 2).
\end{align*}
\]

Let $h$ be an integer with $h \geq 2$. When $\ell \geq h - 1$, Davis and Webb [1] have given a congruence for \( \binom{n}{r} \pmod{p^h} \) for any prime $p$. (Granville’s Proposition 2 in [4] is the special case of Davis and Webb’s congruence when $p \nmid \binom{n}{r}$.) When $p = 2$ their congruence can be expressed using Proposition 1 in the form:

**Davis–Webb Congruence (mod $2^h$).** For $2 \leq h \leq \ell + 1$

\[
\binom{n}{r} \equiv 2^{\nu(n,r)} \frac{d_0d_1 \ldots d_{h-2}}{b_0b_1 \ldots b_{h-2}} \prod_{i=0}^{\ell-h+1} \left[ a_i \prod_{j=1}^{h+i-1} b_{j}b_{j+1} \ldots b_{j+h-1} \right] \pmod{2^h}. \tag{2.1}
\]
Our next task is to make the Davis–Webb congruence (mod \(2^h\)) explicit in certain cases when \(h = 2\) and \(h = 3\) by means of Lemma 1. It is convenient to set

\[
E_1 = \sum_{i=0}^{\ell-1} (b_i + b_{i+1}), \quad E_2 = \sum_{i=0}^{\ell-2} (b_i + b_{i+2}).
\]

For an integer \(f\) with \(0 \leq f \leq \ell - 1\) we also set

\[
H_f = \sum_{i \neq f-1, f, f+1, a_i a_{i+1} = 11} (b_i + b_{i+1}).
\]

**Davis–Webb Congruence (mod 4).** For \(\ell \geq 1\) and \(c(n, r) = 0\), we have

\[
\binom{n}{r} \equiv (-1)^{E_1} \pmod{4}.
\]

**Proof.** Taking \(h = 2\) and \(c(n, r) = 0\) in (2.1), we obtain for \(\ell \geq 1\)

\[
\binom{n}{r} \equiv \left[ \frac{d_0}{b_0} \prod_{i=0}^{\ell-1} \frac{a_i}{b_i} \frac{a_{i+1}}{b_{i+1}} \right]' \pmod{4}.
\]

Appealing to Lemma 1(a)(d), we obtain

\[
\binom{n}{r} \equiv \prod_{i=0}^{\ell-1} (-1)^{b_i + b_{i+1}} \equiv (-1)^{c(a_{i+1} = 11)} \equiv (-1)^{E_1} \pmod{4}.
\]

**Davis–Webb Congruence (mod 8).** (a) For \(\ell \geq 2\) and \(c(n, r) = 0\), we have

\[
\binom{n}{r} \equiv (-1)^{E_1} 5^{E_2} \pmod{8}, \quad \text{if } a_0 a_1 \neq 11,
\]

\[
\binom{n}{r} \equiv (-1)^{E_1} 5^{b_0 + b_1 + E_2} \pmod{8}, \quad \text{if } a_0 a_1 = 11.
\]
(b) For \( \ell \geq 2 \) and \( c(n, r) = 1 \), we have

\[
\binom{n}{r} = 2(-1)^{1+a_f-1+a_{f+2}+H_f} \pmod{8},
\]

where \( f \) (\( 0 \leq f \leq \ell - 1 \)) is the position of the carry when adding the binary representations of \( r \) and \( n-r \), and

\[
a_{-1} = -1, \quad a_{\ell+1} = 0.
\]

**Proof.** (a) Taking \( h = 3 \) and \( c(n, r) = 0 \) in (2.1), we obtain for \( \ell \geq 2 \)

\[
\binom{n}{r} = \left[ \begin{array}{ccc} a_0 & a_1 & a_{\ell+1} \\ b_0 & b_1 & b_{\ell+1} \end{array} \right] \prod_{i=0}^{\ell-2} \left( \begin{array}{c} a_i \\ b_i \end{array} \right) \cdot \prod_{i=0}^{\ell-2} \left( (-1)^{b_{i+1}+b_{i+2}} \right) \pmod{8}.
\]

By Proposition 2(a) we have \( b_i \leq a_i \) for \( i = 0, \ldots, \ell \). Appealing to Lemma 1(j)(k)(l)(m), we obtain mod 8:

\[
\binom{n}{r} = \left[ \begin{array}{ccc} a_0 & a_1 & a_{\ell+1} \\ b_0 & b_1 & b_{\ell+1} \end{array} \right] \prod_{i=0}^{\ell-2} \left( \begin{array}{c} a_i \\ b_i \end{array} \right) \cdot \prod_{i=0}^{\ell-2} \left( (-1)^{b_{i+1}+b_{i+2}} \right) \pmod{8}.
\]

If \( a_0 a_1 \neq 11 \), then, by Lemma 1(c), we have \( \binom{n}{r} \equiv (-1)^{E_1 E_2} \pmod{8} \). If \( a_0 a_1 = 11 \) then, by Lemma 1(c), we have

\[
\binom{n}{r} \equiv (-1)^{b_0+b_1} 5^{b_0+b_1} 5^{E_2} (-1)^{E_1} \pmod{8}.
\]

(b) Taking \( h = 3 \) and \( c(n, r) = 1 \) in (2.1), we obtain for \( \ell \geq 2 \)

\[
\binom{n}{r} = 2 \left[ \begin{array}{ccc} a_0 & a_1 & a_{\ell+1} \\ b_0 & b_1 & b_{\ell+1} \end{array} \right] \prod_{i=0}^{\ell-2} \left( \begin{array}{c} a_i \\ b_i \end{array} \right) \cdot \prod_{i=0}^{\ell-2} \left( (-1)^{b_{i+1}+b_{i+2}} \right) \pmod{8}.
\]

We let \( f \) (\( 0 \leq f \leq \ell - 1 \)) be the position of the carry so that by Proposition 2(b)
\[ a_f b_{f+1} = 01, b_f b_{f+1} = 10, \text{ and } a_k \geq b_k \text{ for } k \neq f, f + 1. \] 

We have, by Lemma 1(h)(i)(l),

\[
\begin{bmatrix}
    a_0 & a_1 \\
    b_0 & b_1
\end{bmatrix}
= \prod_{i=0}^{f-2} \begin{bmatrix}
    a_i & a_{i+1} & a_{i+2} \\
    b_i & b_{i+1} & b_{i+2}
\end{bmatrix}
\equiv \begin{bmatrix}
    a_0 & a_1 \\
    b_0 & b_1
\end{bmatrix}
= \prod_{i=0}^{f-1} \begin{bmatrix}
    a_i & a_{i+1} \\
    b_i & b_{i+1}
\end{bmatrix}
\equiv \begin{bmatrix}
    a_0 & a_1 \\
    b_0 & b_1
\end{bmatrix}
\equiv (-1)^{\sum_{i=0}^{f-1} (b_i + b_{i+1})}
= (-1)^{\sum_{i=0}^{f-1} n_{a_i+1=11}}
\equiv (-1)^{H_f} \quad (\text{mod } 4).
\]

We now consider four cases: (i) \( f = 0 \); (ii) \( f = 1 \); (iii) \( 2 \leq f \leq \ell - 2 \); and (iv) \( f = \ell - 1 \).

In each case we must determine

\[
P = \prod_{i=f-2}^{f-1} \begin{bmatrix}
    a_i & a_{i+1} & a_{i+2} \\
    b_i & b_{i+1} & b_{i+2}
\end{bmatrix}
\equiv (-1)^{a_2} \quad (\text{mod } 4).
\]

**Case (i).** \( f = 0 \). In this case we have

\[
P = \begin{bmatrix}
    0 & 1 & a_2 \\
    1 & 0 & b_2
\end{bmatrix}
\equiv (-1)^{a_2} \quad (\text{mod } 4),
\]

by Lemma 1(g), so that

\[
\binom{n}{r} \equiv 2(-1)^{H_f} (-1)^{a_2} \equiv 2(-1)^{1+a_{f-1}+a_{f+2}+H_f} \quad (\text{mod } 8).
\]

**Case (ii).** \( f = 1 \). Here

\[
P = \begin{bmatrix}
    a_0 & 0 & 1 \\
    b_0 & 1 & 0
\end{bmatrix}
\equiv \begin{bmatrix}
    0 & 1 & a_3 \\
    1 & 0 & b_3
\end{bmatrix}
\equiv (-1)^{1+a_0} (-1)^{a_3} \equiv (-1)^{1+a_0+a_3} \quad (\text{mod } 4),
\]

by Lemma 1(f)(g), so that

\[
\binom{n}{r} \equiv 2(-1)^{H_f} (-1)^{1+a_0+a_3} \equiv 2(-1)^{1+a_{f-1}+a_{f+2}+H_f} \quad (\text{mod } 8).
\]

**Case (iii).** \( 2 \leq f \leq \ell - 2 \). Here

\[
P = \begin{bmatrix}
    a_{f-2} & a_{f-1} & 0 \\
    b_{f-2} & b_{f-1} & 1
\end{bmatrix}
\begin{bmatrix}
    a_{f-1} & 0 & 1 \\
    b_{f-1} & 1 & 0
\end{bmatrix}
\begin{bmatrix}
    0 & 1 & a_{f+2} \\
    1 & 0 & b_{f+2}
\end{bmatrix}
\equiv (-1)^{1+a_{f-1}+a_{f+2}} \quad (\text{mod } 4),
\]

by Lemma 1(e)(f)(g), so that

\[
\binom{n}{r} \equiv 2(-1)^{H_f} (-1)^{1+a_{f-1}+a_{f+2}} \equiv 2(-1)^{1+a_{f-1}+a_{f+2}+H_f} \quad (\text{mod } 8).
\]
Case (iv). \( f = \ell - 1 \). Here

\[
P = \begin{bmatrix}
  a_{\ell - 3} & a_{\ell - 2} & 0 \\
  b_{\ell - 3} & b_{\ell - 2} & 1
\end{bmatrix} = \begin{bmatrix}
  a_{\ell - 2} & 0 & 1 \\
  b_{\ell - 2} & 1 & 0
\end{bmatrix} \equiv (-1)^{1+a_{\ell-2}} \pmod{4},
\]

by Lemma 1(e)(f), so that

\[
\binom{n}{r} \equiv 2(-1)^{H_f}(-1)^{1+a_{\ell-2}} = 2(-1)^{1+a_{f-1}+a_{f+2}+H_f} \pmod{8}.
\]

\[\square\]

Our final task in this section is to give the mechanism whereby we can count the number of integers \( r \) (\( 0 \leq r \leq n \)) for which \( \binom{n}{r} \) is in a particular residue class \( \pmod{4} \) or \( \pmod{8} \). This mechanism is provided by the next lemma.

**Lemma 2.** Let \( c_0, \ldots, c_\ell \) be integers. Then

\[
a_{i_0, \ldots, i_\ell} \sum_{b_{i_0}, \ldots, b_{i_\ell} = 0} (-1)^{\sum_{i=0}^{\ell} c_i b_i} = \begin{cases} 
2^{n_1}, & \text{if } c_i \equiv 0 \pmod{2} \text{ for each } i = 0, 1, \ldots, \ell \text{ with } a_i = 1, \\
0, & \text{if } c_i \equiv 1 \pmod{2} \text{ for some } i \text{ (} 0 \leq i \leq \ell \text{) with } a_i = 1.
\end{cases}
\]

**Proof.** We have

\[
a_{i_0, \ldots, i_\ell} \sum_{b_{i_0}, \ldots, b_{i_\ell} = 0} (-1)^{\sum_{i=0}^{\ell} c_i b_i} = \prod_{i=0}^{\ell} \left( \sum_{b_i = 0}^{a_i} (-1)^{c_i b_i} \right) = \prod_{i=0}^{\ell} \left( \sum_{a_i = 1}^{1} (-1)^{c_i} \right) = \prod_{i=0}^{\ell} (1 + (-1)^{c_i})
\]

\[
= \begin{cases} 
\prod_{i=0}^{\ell} 2, & \text{if } c_i \equiv 0 \pmod{2} \text{ for each } i \text{ with } a_i = 1, \\
0, & \text{if } c_i \equiv 1 \pmod{2} \text{ for some } i \text{ with } a_i = 1.
\end{cases}
\]

\[\square\]

In applying Lemma 2 in the evaluation of \( N_n(t, 4) \) (\( t = 1, 3 \)) and \( N_n(t, 8) \) (\( t = 1, 2, 3, 5, 6, 7 \)) a number of finite sums involving \( E_1 \) and \( E_2 \) arise. These sums are evaluated in Lemmas 3–7.

**Lemma 3.**

\[
a_{i_0, \ldots, i_\ell} \sum_{b_{i_0}, \ldots, b_{i_\ell} = 0} (-1)^{E_1} = 0^{n_1} 2^{n_1}.
\]

**Proof.** Set \( S = \sum_{b_{i_0}, \ldots, b_{i_\ell} = 0} (-1)^{E_1} \). For \( j = 0, 1, \ldots, \ell \) let \( c_j \) denote the number of occurrences of \( b_j \) in

\[
E_1 = \sum_{a_i, a_{i+1} = 11}^{\ell-1} (b_i + b_{i+1}),
\]

so that \( S = \sum_{b_{i_0}, \ldots, b_{i_\ell} = 0} (-1)^{\sum_{i=0}^{\ell} c_i b_i} \). If \( n_{11} = 0 \) then \( c_j = 0 \) (\( 0 \leq j \leq \ell \)) so \( S = 2^{n_1} \) by Lemma 2. If \( n_{11} > 0 \) let \( u \) be the least integer (\( 0 \leq u \leq \ell - 1 \)) such that \( a_u a_{u+1} = 11 \). Then \( c_u = 1 \) and \( S = 0 \) by Lemma 2.

\[\square\]

**Lemma 4.**

\[
a_{i_0, \ldots, i_\ell} \sum_{b_{i_0}, \ldots, b_{i_\ell} = 0} (-1)^{E_2} = \begin{cases} 
2^{n_1}, & \text{if } n_{101} = n_{111} = 0, \\
0, & \text{if } n_{101} > 0 \text{ or } n_{111} > 0.
\end{cases}
\]
Proof. Set $S = \sum_{b_0, \ldots, b_\ell=0}^{a_0, \ldots, a_\ell} (-1)^{E_2}$. For $j = 0, 1, \ldots, \ell$ let $c_j$ denote the number of occurrences of $b_j$ in

$$E_2 = \sum_{a_i, a_{i+1}=1}^{\ell-2} (b_i + b_{i+2}).$$

If $n_{101} = n_{111} = 0$ then $c_j = 0$ ($0 \leq j \leq \ell$) so, by Lemma 2, we have $S = 2^{n_1}$. If $n_{101} > 0$ or $n_{111} > 0$ let $s$ be the least integer such that $a_s a_{s+2} = 11$ ($0 \leq s \leq \ell - 2$). Then $c_s = 1$ and $S = 0$ by Lemma 2.

Before stating the next lemma, we remind the reader that the length of the $i$th block of 0’s in $a_0 a_1 \ldots a_\ell$ is denoted by $v_i$ and the length of the $i$th block of 1’s by $s_i$. We consider $a_0 a_1 \ldots a_\ell$ to start with a block of 0’s and finish with a block of 1’s.

**Lemma 5.** For $n \geq 1$

$$\sum_{b_0, \ldots, b_\ell=0}^{a_0, \ldots, a_\ell} (-1)^{E_1+E_2} = \begin{cases} 0, & \text{if } n_{101} > 0 \text{ or } n_{1111} > 0, \\ 0, & \text{if } n_{101} = n_{1111} = 0 \text{ and some } s_i = 2, \\ 2^{v_1}, & \text{if } n_{101} = n_{1111} = 0 \text{ and each } s_i = 1 \text{ or } 3. \end{cases}$$

Proof. The lemma is easily checked for $\ell = 0, 1, 2$ so we may suppose that $\ell \geq 3$. Let $S = \sum_{b_0, \ldots, b_\ell=0}^{a_0, \ldots, a_\ell} (-1)^{E_1+E_2}$. For $j = 0, 1, \ldots, \ell$ let $c_j$ denote the number of occurrences of $b_j$ in

$$E_1 + E_2 = \sum_{a_i, a_{i+1}=11}^{\ell-1} (b_i + b_{i+1}) + \sum_{a_i, a_{i+1}=101}^{\ell-2} (b_i + b_{i+2}) + \sum_{a_i, a_{i+1}=111}^{\ell-2} (b_i + b_{i+2}).$$

Suppose first that $n_{101} = n_{1111} = 0$ and some $s_i = 2$, where $i \geq 1$. Hence there exists an integer $u$ ($0 \leq u \leq \ell - 1$) such that $a_u a_{u+1} = 11$, $a_{u-1} = 0$ if $u \geq 1$, and $a_{u+2} = 0$ if $u \leq \ell - 2$. Let $u$ be the least such integer. Then $c_u = 1$ and $S = 0$ by Lemma 2.

Suppose next that $n_{101} = n_{1111} = 0$ and each $s_i = 1$ or 3. Let $j$ ($0 \leq j \leq \ell$) be an integer such that $a_j = 1$. If $j = 0$ and $a_1 = 0$, then $a_2 = 0$ and $c_j = 0$. If $j = 0$ and $a_1 = 1$, then $a_2 = 1$ and $c_j = 2$. If $j = \ell$ and $a_{\ell-2} = 0$, then $a_{\ell-2} = 0$ and $c_j = 0$. If $j = \ell$ and $a_{\ell-2} = 1$, then $a_{\ell-2} = 1$ and $c_j = 2$. Now suppose $1 \leq j \leq \ell - 1$. If $a_{j-1} a_j a_{j+1} = 010$ then $c_j = 0$. If $a_{j-1} a_j a_{j+1} = 110$ then $j \geq 2$ and $a_{j-2} a_{j-1} a_{j+1} = 110$ so that $c_j = 2$. If $a_{j-1} a_j a_{j+1} = 011$ then $j \leq \ell - 2$ and $a_{j-2} a_{j-1} a_{j+1} = 011$ so that $c_j = 2$. If $a_{j-1} a_j a_{j+1} = 111$ then $c_j = 2$. Hence $c_j$ is even for every $j$ with $a_j = 1$. Thus, by Lemma 2, we have $S = 2^{v_1}$.

Now suppose that $n_{101} > 0$. Let $s$ be the least integer such that $a_s a_{s+1} a_{s+2} = 101$. If $s = 0$ then $c_s = 1$. If $s = 1$ and $a_{s-1} = 0$ then $c_s = 1$. If $s = 1$ and $a_0 = 1$ then $c_0 = 1$. If $s \geq 2$, $a_{s-1} = 1$, and $a_{s-2} = 0$ then $c_{s-1} = 1$. If $s \geq 2$, $a_{s-1} = 1$, and $a_{s-2} = 1$ then $c_s = 3$. Hence, by Lemma 2, we have $S = 0$.

Finally suppose that $n_{1111} > 0$. Let $w$ be the least integer such that $a_w a_{w+1} a_{w+2} a_{w+3} = 1111$. Then $c_{w+1} = 3$ and, by Lemma 2, we have $S = 0$.

**Lemma 6.** If $a_0 a_1 = 1$ then

$$\sum_{b_0, \ldots, b_\ell=0}^{a_0, \ldots, a_\ell} (-1)^{b_0+b_1+E_2} = 0.$$
In Sections 4 and 5 we employ the Davis–Webb congruence \((mod \, 8)\) to determine thereby reproving the formulae due to Davis and Webb \([2]\) (see Theorem B (second part)).

Let \(k(1 \leq k \leq \ell)\) be the largest integer such that \(a_0 a_1 \ldots a_k = 11 \ldots 1\). Then \(c_{k-1} = 1\). Hence, by Lemma 2, \(S = 0\). □

**Lemma 7.** If \(a_0 a_1 = 11\) then

\[
\sum_{b_0, \ldots, b_{\ell-2}=0}^{a_0, \ldots, a_{\ell}} (-1)^{b_0+b_1+E_1+E_2} = \begin{cases} 
0, & \text{if } n_{101} > 0 \text{ or } n_{1111} > 0, \\
0, & \text{if } n_{101} = n_{1111} = 0 \text{ and some } s_i = 2 \text{ with } i \geq 2, \\
0, & \text{if } n_{101} = n_{1111} = 0 \text{ and each } s_i = 1 \text{ or } 3, \\
2^{n_1}, & \text{if } n_{101} = n_{1111} = 0, s_1 = 2, \\
& \text{and each } s_i = 1 \text{ or } 3 \text{ with } i \geq 2.
\end{cases}
\]

**Proof.** Set \(S = \sum_{b_0, \ldots, b_{\ell-2}=0}^{a_0, \ldots, a_{\ell}} (-1)^{b_0+b_1+E_1+E_2}\). For \(j = 0, 1, \ldots, \ell\), let \(c_j\) denote the number of occurrences of \(b_j\) in

\[
b_0 + b_1 + \sum_{i=0}^{\ell-1} (b_i + b_{i+1}) + \sum_{i=0}^{\ell-2} (b_i + b_{i+2}).
\]

Suppose first that \(n_{101} > 0\). Let \(s(0 \leq s \leq \ell - 2)\) be the least integer such that \(a_s a_{s+1} a_{s+2} = 101\). As \(a_0 a_1 = 11\) we have \(s \geq 1\). If \(s = 1\) then \(a_s = a_0 = 1\) and \(c_1 = 3\). If \(s \geq 2\) and \(a_{s-1} = 0\), then \(a_s = 0\) and \(s \geq 4\), so that \(c_s = 1\). If \(a_s = 2\) and \(a_{s-2} = a_{s-1} = 1\) then \(c_s = 3\). If \(s = 2\) and \(a_s = 0\), then \(a_{s-1} = 1\), then \(s \geq 3\) and \(a_{s-3} = 0\), so that \(c_{s-1} = 1\). Hence, by Lemma 2, \(S = 0\).

Suppose next that \(n_{1111} > 0\). Let \(s(0 \leq s \leq \ell - 3)\) be the least integer such that \(a_s a_{s+1} a_{s+2} a_{s+3} = 1111\). If \(s = 0\) then \(c_0 = 3\). If \(s \geq 1\) then \(a_{s-1} = 0\) and \(c_{s-1} = 3\). Hence, by Lemma 2, \(S = 0\).

Now suppose that \(n_{101} = n_{1111} = 0\) and some \(s_i = 2\) with \(i \geq 2\), say \(a_s a_{s+1} = 11\). As \(a_0 a_1 = 11\) and \(n_{101} = 0\) we have \(s \geq 4\). Clearly \(a_{s-2} a_{s-3} = 00\). Hence \(c_s = 1\), and, by Lemma 2, we have \(S = 0\).

Next suppose that \(n_{101} = n_{1111} = 0\) and each \(s_i = 1\) or \(3\). Then, as \(a_0 a_1 = 11\), we must have \(a_2 = 1\). Thus \(c_0 = 3\) and, by Lemma 2, we have \(S = 0\).

Finally suppose that \(n_{101} = n_{1111} = 0\), \(s_1 = 2\), and each \(s_i(i \geq 2) = 1\) or \(3\). Clearly \(c_0 = c_1 = 2\), \(c_i = 0\) if \(a_{i-1} a_{i+1} = 010\) (\(4 \leq i \leq \ell - 1\)), \(c_i = 0\) if \(a_{i-1} a_i = 01\), and \(c_{i-1} = c_i = c_{i+1} = 2\) if \(a_{i-1} a_i a_{i+1} = 111\) (\(5 \leq i \leq \ell - 1\)). Thus \(c_i\) is even for all \(i\) with \(a_i = 1\) so that, by Lemma 2, we have \(S = 2^{n_1}\). □

In Section 3 we use the Davis–Webb congruence \((mod \, 4)\) to determine \(N_n(1, 4)\) and \(N_n(3, 4)\), thereby reproving the formulae due to Davis and Webb \([2]\) (see Theorem B (second part)).

In Sections 4 and 5 we employ the Davis–Webb congruence \((mod \, 8)\) to determine \(N_n(t, 8)\) \((t = 2, 6)\) (see Theorem C (second part) in Section 4) and \(N_n(t, 8)\) \((t = 1, 3, 5, 7)\) (see Theorem C (third part) in Section 5).

### 3. Evaluation of \(N_n(1, 4)\) and \(N_n(3, 4)\)

In this section we illustrate our methods by re-establishing the formulæ for \(N_n(1, 4)\) and \(N_n(3, 4)\) due to Davis and Webb \([2]\).
**Theorem B (Second Part, Davis and Webb [2]).**

\[ N_n(1, 4) = \begin{cases} 2^{n_1}, & \text{if } n_{11} = 0, \\ 2^{n_1-1}, & \text{if } n_{11} > 0, \end{cases} \quad N_n(3, 4) = \begin{cases} 0, & \text{if } n_{11} = 0, \\ 2^{n_1-1}, & \text{if } n_{11} > 0. \end{cases} \]

**Proof.** It is easily checked that the formulae hold for \( n = 0, 1 \) so that we may take \( n \geq 2 \). Thus \( \ell \geq 1 \). For \( t = 1 \) and 3, we have

\[ N_n(t, 4) = \sum_{r=0}^{n} \frac{1}{2^{\ell}} \sum_{b_0, \ldots, b_\ell = 0} \left( 1 + (-1)^{\frac{1}{2}(t-1)+E_1} \right) \]

by Proposition 1, Proposition 2(a), and the Davis–Webb congruence (mod 4). Hence

\[ N_n(t, 4) = 2^{n_1-1} + \frac{1}{2} (-1)^{\frac{1}{2}(t-1)} \sum_{b_0, \ldots, b_\ell = 0} (-1)^{E_1}, \]

by Lemma 3.

\[ 4. \text{ Evaluation of } N_n(2, 8) \text{ and } N_n(6, 8) \]

In this section we evaluate \( N_n(2, 8) \) and \( N_n(6, 8) \).

**Theorem C (Second Part).**

\[ N_n(2, 8) = \begin{cases} n_0^2 2^{n_1-1} - n_0 2^{n_1-1}, & \text{if } n_{11} = 0, \\ n_0 2^{n_1-1} - n_{11} 2^{n_1-2} + n_0 2^{n_1-1}, & \text{if } n_{11} = 1, \\ n_0 2^{n_1-1}, & \text{if } n_{11} \geq 2. \end{cases} \]

\[ N_n(6, 8) = \begin{cases} n_0 2^{n_1-1}, & \text{if } n_{11} = 0, \\ n_0 2^{n_1-2} - n_{11} 2^{n_1-2} - n_0 2^{n_1-1}, & \text{if } n_{11} = 1, \\ n_0 2^{n_1-2}, & \text{if } n_{11} \geq 2. \end{cases} \]

**Proof.** It is easily checked that the theorem holds for \( n = 0, 1, 2, 3 \) (equivalently \( \ell = 0, 1 \)). Hence we may assume that \( \ell \geq 2 \). For \( t = 2 \) and 6 we have, by Proposition 1,

\[ N_n(t, 8) = \sum_{r=0}^{\ell} 1 = \sum_{r=0}^{n} 1 = \ell - 1 \sum_{r=0}^{\ell} 1. \]

Before continuing it is convenient to introduce some notation. Let \( S \) be a string of 0’s and 1’s of length \( k \). For \( 0 \leq i \leq i + k - 1 \leq \ell \) we set

\[ \left( \begin{array}{c} a_i a_{i+1} \ldots a_{i+k-1} \\ S \end{array} \right) = \begin{cases} 1, & \text{if } a_i a_{i+1} \ldots a_{i+k-1} = S, \\ 0, & \text{if } a_i a_{i+1} \ldots a_{i+k-1} \neq S. \end{cases} \]
Now, by the Davis–Webb congruence (mod 8), we have
\[
\left( \frac{n}{r} \right) \equiv t \pmod{8} \iff \frac{1}{4} (t + 2) + a_{f-1} + a_{f+2} + H_f \equiv 0 \pmod{2}.
\]

Next, let
\[
E_1' = \sum_{i=0}^{f-2} \left( b_i + b_{i+1} \right), \quad E_1'' = \sum_{i=f+2}^{\ell-1} \left( b_i + b_{i+1} \right),
\]
so that \( E_1' + E_1'' = H_f \). Hence the Davis–Webb congruence (mod 8) becomes
\[
\left( \frac{n}{r} \right) \equiv t \pmod{8} \iff \frac{1}{4} (t + 2) + a_{f-1} + a_{f+2} + E_1' + E_1'' \equiv 0 \pmod{2}.
\]

Hence
\[
N_n(t, 8) = \sum_{a_f = 0}^{\ell-1} \sum_{a_{f-1} = 0}^{a_0 \ldots a_{f-1} = 0} \frac{1}{2} (1 + (-1)^{\frac{1}{2} (t+2) + a_{f-1} + a_{f+2} + E_1' + E_1''})
\]
\[
= n_{01} 2^{n-2} + \frac{1}{2} (-1)^{\frac{1}{2} (t+2)} \sum_{a_f = 0}^{\ell-1} \sum_{a_{f-1} = 0}^{a_0 \ldots a_{f-1} = 0} (-1)^{a_{f-1} + a_{f+2}}
\]
\[
\sum_{b_0, \ldots, b_{f-1} = 0}^{a_f = 0} (-1)^{E_1'} \quad \sum_{b_f, \ldots, b_{\ell-1} = 0}^{a_f = 0} (-1)^{E_1''} = 0^{n_1'} 2^{n_1''}, \quad \sum_{b_f, \ldots, b_{\ell-1} = 0}^{a_f = 0} (-1)^{E_1'} = 0^{n_1'} 2^{n_1''},
\]

where \( n_1' \) is the number of 1’s in \( \alpha_0 \alpha_1 \ldots \alpha_{f-1} \), \( n_1'' \) is the number of 1’s in \( \alpha_f \ldots \alpha_{\ell} \), \( n_1'^1 \) is the number of occurrences of 11 in \( \alpha_0 \alpha_1 \ldots \alpha_{f-1} \), and \( n_1'' \) is the number of occurrences of 11 in \( \alpha_f \ldots \alpha_{\ell} \). Hence
\[
\sum_{b_0, \ldots, b_{f-1} = 0}^{a_f = 0} (-1)^{E_1'} \sum_{b_f, \ldots, b_{\ell-1} = 0}^{a_f = 0} (-1)^{E_1''} = 0^{n_1' + n_1''} 2^{n_1'} + n_2'' = \begin{cases} 0^{n_1' + n_1''} 2^{n_1'} - 1, & \text{if } a_{f+2} = 0, \\ 0^{n_1' + n_1''} 2^{n_1'} - 1, & \text{if } a_{f+2} = 1. \end{cases}
\]

Thus for \( n_1' > 1 \) we have \( N_n(t, 8) = n_01 2^{n-2} \). Next for \( n_1' = 1 \) we have
\[
N_n(t, 8) = n_01 2^{n-2} + \frac{1}{2} (-1)^{\frac{1}{2} (t+2)} \sum_{a_f = 0}^{\ell-1} \sum_{a_{f-1} = 0}^{a_0 \ldots a_{f-1} = 0} (-1)^{a_{f-1} + a_{f+2}} 2^{n_1' - 1}
\]
\[
= n_01 2^{n-2} - (-1)^{\frac{1}{2} (t+2) + a_{f-1}} \sum_{a_f = 0}^{\ell-1} (-1)^{a_{f-1}}
\]
\[
= n_01 2^{n-2} + (-1)^{\frac{1}{2} (t+2) - 2} 2^{n_1 - 2} \left( -\left( \begin{array}{ccc} a_0 & a_1 & a_2 \\ 0 & 1 & 1 \end{array} \right) + n_{011} - n_{1011} \right)
\]
\[
= n_01 2^{n-2} + (-1)^{\frac{1}{2} (t+2) - 2} 2^{n_1 - 2} (2n_{011} - n_{011}),
\]
as
\[
\left( \begin{array}{ccc}
  a_0 & a_1 & a_2 \\
  0 & 1 & 1 \\
\end{array} \right) = n_{011} - n_{001} - n_{1011}.
\]

Hence
\[
N_n(2, 8) = n_{012}2^{n_1-2} + 2^{n_1-1}n_{001} - 2^{n_1-2}n_{011}
\]
and
\[
N_n(6, 8) = n_{012}2^{n_1-2} - 2^{n_1-1}n_{001} + 2^{n_1-2}n_{011}.
\]

Finally for \( n_{11} = 0 \) we have
\[
N_n(t, 8) = n_{012}2^{n_1-2} + \frac{1}{2}(-1)^{\frac{1}{2}(t+2)} \sum_{f=0}^{\ell-1} (-1)^{\alpha_f - 1 + \alpha_{f+2}} 2^{n_1-1}
\]

\[
= n_{012}2^{n_1-2} + (\frac{1}{2}t + 2)2^{n_1-2} \cdot \left[ \left( \begin{array}{ccc}
  a_0 & a_1 & a_2 \\
  0 & 1 & 0 \\
\end{array} \right) + \left( \begin{array}{ccc}
  a_{\ell-2} & a_{\ell-1} & a_\ell \\
  0 & 0 & 1 \\
\end{array} \right) \right] + \sum_{f=1}^{\ell-2} (-1)^{\alpha_f - 1}.\]

Now as \( n_{11} = 0 \) we have
\[
\left( \begin{array}{ccc}
  a_0 & a_1 & a_2 \\
  0 & 1 & 0 \\
\end{array} \right) = \left( \begin{array}{ccc}
  a_0 & a_1 & a_2 \\
  0 & 1 & 0 \\
\end{array} \right) = n_{01} - n_{001} - n_{101},
\]
\[
\left( \begin{array}{ccc}
  a_{\ell-2} & a_{\ell-1} & a_\ell \\
  0 & 0 & 1 \\
\end{array} \right) = n_{001} - n_{0010} - n_{0011} = n_{001} - n_{0010},
\]
\[
\left( \begin{array}{ccc}
  a_{\ell-2} & a_{\ell-1} & a_\ell \\
  1 & 0 & 0 \\
\end{array} \right) = n_{101} - n_{1010} - n_{1011} = n_{101} - n_{1010},
\]
\[
\sum_{f=1}^{\ell-2} (-1)^{\alpha_f - 1} = \sum_{f=1}^{\ell-2} 1 - \sum_{f=1}^{\ell-2} 1 = n_{0010} - n_{1010},
\]
so that
\[
-\left( \begin{array}{ccc}
  a_0 & a_1 & a_2 \\
  0 & 1 & 0 \\
\end{array} \right) + \left( \begin{array}{ccc}
  a_{\ell-2} & a_{\ell-1} & a_\ell \\
  0 & 0 & 1 \\
\end{array} \right) - \left( \begin{array}{ccc}
  a_{\ell-2} & a_{\ell-1} & a_\ell \\
  1 & 0 & 0 \\
\end{array} \right)
\]
\[
+ \sum_{f=1}^{\ell-2} (-1)^{\alpha_f - 1} = -n_{01} + 2n_{001}.
\]

Hence
\[
N_n(t, 8) = n_{012}2^{n_1-2} + (-1)^{\frac{1}{2}(t+2)}(-n_{01} + 2n_{001})2^{n_1-2}
\]
\[
= \begin{cases} 
  n_{012}2^{n_1-1} - n_{001}2^{n_1-1}, & \text{if } t = 2, \\
  n_{001}2^{n_1-1}, & \text{if } t = 6.
\end{cases}
\]
\[\square\]
5. Evaluation of $N_n(t, 8)$, $t = 1, 3, 5, 7$

In this section we carry out the evaluation of $N_n(t, 8)$ for $t = 1, 3, 5, 7$ using the Davis–Webb congruence (mod 8).

**Theorem C (third part).**

<table>
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<th>Case No.</th>
<th>$n_{111}$</th>
<th>$n_{11}$</th>
<th>$n_{101}$</th>
<th>$n_{111}$</th>
<th>$v_1$</th>
<th>$s_1$ ($i \geq 2$)</th>
<th>$N_n(1, 8)$</th>
<th>$N_n(3, 8)$</th>
<th>$N_n(5, 8)$</th>
<th>$N_n(7, 8)$</th>
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<td>2</td>
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<td>0</td>
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<tr>
<td>(iii)</td>
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<td>2</td>
<td>$2^n_{1-2}$</td>
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<td>0</td>
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<tr>
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<td>$2^n_{1-2}$</td>
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<tr>
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<td>$2^n_{1-2}$</td>
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<td>$2^n_{1-2}$</td>
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<tr>
<td>(vii)</td>
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<td>$2^n_{1-2}$</td>
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<td>$2^n_{1-2}$</td>
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<td>$2^n_{1-2}$</td>
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<td>$2^n_{1-2}$</td>
<td>0</td>
<td>0</td>
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</table>

**Proof.** It is easily checked that the theorem holds for $n = 0, 1, 2, 3$ (equivalently $\ell = 0, 1$). Hence we may assume that $\ell \geq 2$. For $t = 1, 3, 5, 7$ we have

$$N_n(t, 8) = \sum_{r=0}^{n \choose t} 1 = \sum_{r=0}^{n \choose t} 1 = \sum_{b_0, \ldots, b_\ell=0 \atop \ell \equiv t (mod 8)} a_0 \cdots a_\ell ,$$

by Propositions 1 and 2(a), and part (a) of the Davis–Webb congruence (mod 8). Set

$$\alpha(t) = (t - 1)/2, \quad \beta(t) = (t^2 - 1)/8,$$

so that $t \equiv (-1)^{\alpha(t)} s^{\beta(t)}$ (mod 8). Hence

$$(-1)^{E_1} s^{(b_0 + b_1)(c_0 c_1)} + E_2 \equiv t \quad (mod 8),$$

$$\Leftrightarrow E_1 \equiv \alpha(t) \quad (mod 2), \quad (b_0 + b_1) \left( \begin{array}{cc} a_0 & a_1 \\ 1 & 1 \end{array} \right) + E_2 \equiv \beta(t) \quad (mod 2).$$

Thus

$$N_n(t, 8) = \frac{1}{4} \sum_{b_0, \ldots, b_\ell=0} a_0 \cdots a_\ell \left( 1 + (-1)^{\alpha(t)} (-1)^{E_1} \right) \left( 1 + (-1)^{\beta(t)} (-1)^{b_0 + b_1}(c_0 c_1) + E_2 \right)$$

$$= 2^n_{1-2} + \frac{(-1)^{\alpha(t)}}{4} \sum_{b_0, \ldots, b_\ell=0} (-1)^{E_1}$$

$$+ \frac{(-1)^{\beta(t)}}{4} \sum_{b_0, \ldots, b_\ell=0} (-1)^{b_0 + b_1}(c_0 c_1) + E_2$$

$$+ \frac{(-1)^{\alpha(t) + \beta(t)}}{4} \sum_{b_0, \ldots, b_\ell=0} (-1)^{b_0 + b_1}(c_0 c_1) + E_1 + E_2.$$
We treat the two cases $a_0a_1 \neq 11$ and $a_0a_1 = 11$ separately.

If $a_0a_1 \neq 11$ then $(\binom{a_0}{a_1}) = 0$ and appealing to Lemmas 3, 4 and 5, we obtain

$$N_n(t, 8) = 2^{n_1-2} + \begin{cases} (-1)^{a(t)} 2^{n_1-2}, & \text{if } n_{11} = 0 \\ 0, & \text{if } n_{11} > 0 \end{cases} + \begin{cases} (-1)^{\beta(t)} 2^{n_1-2}, & \text{if } n_{101} = n_{111} = 0 \\ 0, & \text{if } n_{101} \text{ or } n_{111} > 0 \end{cases} + \begin{cases} (-1)^{a(t)+\beta(t)} 2^{n_1-2}, & \text{if } n_{101} = n_{111} = 0 \text{ and each } s_i = 1 \text{ or } 3 \\ 0, & \text{if } n_{101} = n_{111} > 0 \text{ or } n_{111} > 0. \end{cases}$$

Appealing to the case definitions given in the statement of the theorem we obtain the value of $N_n(t, 8)$.

Cases (i), (iv), (xii). $N_n(t, 8) = 2^{n_1-2} + 0 + 0 + 0 = 2^{n_1-2}$.

Case (ii). Here $n_{11} = 0$ so that each $s_i = 1$.

$$N_n(t, 8) = 2^{n_1-2} + (-1)^{a(t)} 2^{n_1-2} + (-1)^{\beta(t)} 2^{n_1-2} + (-1)^{a(t)+\beta(t)} 2^{n_1-2} = \begin{cases} 2^{n_1}, & \text{if } t = 1, \\ 0, & \text{if } t = 3, 5, 7. \end{cases}$$

Case (iii). $N_n(t, 8) = 2^{n_1-2} + (-1)^{a(t)} 2^{n_1-2} = \begin{cases} 2^{n_1-1}, & \text{if } t = 1, 5, \\ 0, & \text{if } t = 3, 7. \end{cases}$

Cases (v), (vi). Here $n_{11} > 0$, $n_{111} = 0$ implies that some $s_i = 2$.

$$N_n(t, 8) = 2^{n_1-2} + 0 + (-1)^{\beta(t)} 2^{n_1-2} + \begin{cases} 2^{n_1-1}, & \text{if } t = 1, 7, \\ 0, & \text{if } t = 3, 5. \end{cases}$$

Cases (vii), (viii), (ix). Here $v_1 = 0$ and $s_1 = 2$ or 3 so that $a_0a_1 = 11$, contradicting $a_0a_1 \neq 11$. These cases cannot occur.

Case (x). Here $v_1 = 0$ and $s_1 = 1$ or 2. As $a_0a_1 \neq 11$ we have $s_1 = 1$.

$$N_n(t, 8) = 2^{n_1-2} + 0 + 0 + 0 = 2^{n_1-2}.$$

Case (xi) (Here $v_1 = 0$ and $s_1 = 1$ or 2: as $a_0a_1 \neq 11$ we have $s_1 = 1$.) and Case (xiii).

$$N_n(t, 8) = 2^{n_1-2} + 0 + 0 + (-1)^{a(t)+\beta(t)} 2^{n_1-2} = \begin{cases} 2^{n_1-1}, & \text{if } t = 1, 3, \\ 0, & \text{if } t = 5, 7. \end{cases}$$

If $a_0a_1 = 11$, appealing to Lemmas 3, 6, and 7, we obtain

$$N_n(t, 8) = 2^{n_1-2} + \begin{cases} (-1)^{a(t)+\beta(t)} 2^{n_1-2}, & \text{if } n_{101} = n_{111} = 0, s_1 = 2, \\ 0, & \text{each } s_i (i \geq 2) = 1 \text{ or } 3, \\ \text{otherwise.} \end{cases}$$

Cases (i), (iv), (ix), (x). $N_n(t, 8) = 2^{n_1-2} + 0 = 2^{n_1-2}$.

Cases (ii), (iii). Here $a_0a_1 = 11$ implies $n_{11} > 0$ so these cases cannot occur.
Case (v). Here \( a_0a_1 = 11 \) implies \( v_1 = 0 \) so this case cannot occur.

Case (vi). Here \( a_0a_1 = 11 \) implies \( v_1 = 0 \) and \( s_1 \geq 2 \) so this case cannot occur.

Case (vii). \( N_n(t, 8) = 2^{n_1-2} + 0 = 2^{n_1-2} \).

Case (xi) (Here \( a_0a_1 = 11 \) implies \( v_1 = 0 \) and \( s_1 \geq 2 \)) and Case (viii).

\[
N_n(t, 8) = 2^{n_1-2} + (-1)^{\alpha(t)+\beta(t)}2^{n_1-2} = \begin{cases} 
2^{n_1-1}, & \text{if } t = 1, 3, \\
0, & \text{if } t = 5, 7.
\end{cases}
\]

Cases (xii), (xiii). Here \( v_1 > 0 \) contradicting \( a_0a_1 = 11 \). These cases cannot occur. \( \square \)

REFERENCES