The Primes for Which an Abelian Cubic Polynomial Splits

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Abstract. Let $X^{3}+AX+B$ be an irreducible abelian cubic polynomial in $Z[X]$. We determine explicitly integers $a_{1}, \cdots, a_{t}, F$ such that, except for finitely many primes $p$,

$$x^{3}+Ax+B\equiv 0 \pmod{p}\text{ has three solutions }\iff p\equiv a_{1}, \cdots, a_{t} \pmod{F}.$$

Let $X^{3}+AX+B$ be an irreducible abelian cubic polynomial in $Z[X]$. We are interested in determining those primes $p$ for which the congruence

$$x^{3}+Ax+B\equiv 0 \pmod{p}$$

has exactly three solutions, that is, those primes $p$ for which $X^{3}+AX+B$ splits completely into distinct linear factors modulo $p$. As $X^{3}+AX+B$ is abelian, it is known from class field theory (see for example [6]) that, apart from a finite number of exceptions, the primes $p$ which split $X^{3}+AX+B$ modulo $p$ lie in certain congruence classes modulo the conductor of $X^{3}+AX+B$. In this note we determine these congruence classes explicitly as well as the exceptional primes.

Let $N_{p}(A, B)$ denote the number of solutions $x \pmod{p}$ of the congruence $x^{3}+Ax+B\equiv 0 \pmod{p}$ and let $K=K(A, B)$ denote the largest positive integer such that $K^{2}|A$ and $K^{3}|B$. Since

$$N_{p}(A, B)=\begin{cases}N_{p}(A/K^{2}, B/K^{3}), & \text{if } p \nmid K, \\1, & \text{if } p|K,
\end{cases}$$

it suffices to determine the primes $p$ for which $N_{p}(A, B)=3$ under the simplifying assumption

(1) $K(A, B)=1$.

The irreducible polynomial $X^{3}+AX+B$ is abelian if and only if its discriminant is a perfect square, that is, if and only if

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(2) \[ -4A^3 - 27B^2 = C^2 \]

for some positive integer \( C \) (see [4: Example 2, p. 308]). We see from (2) that \( A < 0 \), \( B \equiv C \pmod{2} \) and \( A \equiv 0 \) or \( 2 \pmod{3} \). Clearly \( B \neq 0 \) as \( X^3 + AX + B \) is irreducible. From (1) and (2) it is easy to show that exactly one of the following occurs:

(3)(i) \[ 3 \nmid A, \]

(3)(ii) \[ 3 \mid A, 3 \nmid B, \]

(3)(iii) \[ 3^2 \mid A, 3^2 \mid B. \]

If (3) (i) holds then \( 3 \mid C \). If (3) (ii) holds then \( 3^2 \mid C \) and, if \( 3^2 \mid C, \) 3 divides exactly one of \( B \pm (C/9) \). If (3) (iii) holds then \( 3^3 \mid C \) and 3 divides exactly one of \( (B/9) \pm (C/27) \). It is convenient to define the integer \( b = b(A, B) = 0, 1, 2 \) by

\[
\begin{align*}
 b = 0, \quad & \text{if } 3 \nmid A \text{ or } 3 \mid A, 3 \nmid B, 3^3 \mid C, \\
 b = 1, \quad & \text{if } 3 \mid A, 3 \mid B, 3^2 \mid C, 3 \mid B - (C/9) \text{ or } 3^2 \mid A, 3^2 \mid B, 3 \mid (B/9) + (C/27), \\
 b = 2, \quad & \text{if } 3 \mid A, 3 \mid B, 3^2 \mid C, 3 \mid B + (C/9) \text{ or } 3^2 \mid A, 3^2 \mid B, 3 \mid (B/9) - (C/27). 
\end{align*}
\]

We note that

(5) \[ b \neq 0 \iff 3 \mid A, 3 \nmid B, 3^2 \mid C \text{ or } 3^2 \mid A, 3^2 \mid B. \]

In order to state our main result we need the notion of a cubic residue symbol. An Eisenstein integer \( \theta \) is a complex number of the form \( \theta = x + y\omega \), where \( x \) and \( y \) are rational integers and \( \omega = (-1 + \sqrt{-3})/2 \) is a complex cube root of unity. Equivalently \( \theta \) is of the form \( (a_1 + a_2\sqrt{-3})/2 \), where \( a_1 \) and \( a_2 \) are rational integers with \( a_1 \equiv a_2 \pmod{2} \). The complex conjugate of \( \theta \) is denoted by \( \bar{\theta} \). The norm \( N(\theta) \) of \( \theta \) is the rational integer \( \theta \bar{\theta} \). The Eisenstein integer \( \theta \) is called a unit if \( N(\theta) = 1 \). The only units are \( \pm 1, \pm \omega, \pm \omega^2 \). An Eisenstein integer \( \theta \) is said to be primary if \( \theta \equiv -1 \pmod{3} \). For each Eisenstein integer \( \theta \) not divisible by \( \sqrt{-3} \) there is a unique unit \( \eta = \eta(\theta) \) such that \( \eta \theta \) is primary. The Eisenstein primes (up to multiplication by a unit) are \( \sqrt{-3} \), rational primes of the form \( 3n + 2 \), and Eisenstein integers with norm equal to a rational prime of the form \( 3n + 1 \). Each nonzero Eisenstein integer can be written uniquely as a product of a unit, a nonnegative integral power of the Eisenstein prime \( \sqrt{-3} \), and nonnegative integral powers of primary Eisenstein primes. If \( \pi \) is an Eisenstein prime with \( N(\pi) \neq 3 \), and \( \theta \) is an Eisenstein integer not divisible by \( \pi \), then the cubic residue symbol \( [\theta/\pi]_3 \) is defined to be the unique cuberoot of unity such that

\[ \theta^{(N(\pi) - 1)/3} \equiv [\theta/\pi]_3 \pmod{\pi}. \]

The basic properties of the cubic residue symbol, extended multiplicatively to denominators not divisible by \( \sqrt{-3} \), are given in [3].

Before stating and proving our main result, we introduce some notation. If \( a \) is a
rational integer, the integers $a'$ and $a''$ are given uniquely by

$$a = 3a' + a'', \quad a'' = -1, 0, 1.$$ 

As usual $\phi$ denotes Euler's phi function.

We prove the following theorem.

**THEOREM.** Let $X^3 + AX + B \in \mathbb{Z}[X]$ be an irreducible abelian cubic polynomial in $\mathbb{Z}[X]$ satisfying (1). Let $C$ be the positive integer given by (2). Let $\lambda$ denote the Eisenstein integer

$$\lambda = (3B + C) + 3B\omega = \frac{1}{2}(C + 3B\sqrt{-3})$$

of norm $N(\lambda) = -A^3$.

(i) We have

$$N(\rho) = \prod_{q | A, q | B} q$$

where $3^c \nmid A^3$.

Let $\tau$ be the (possibly empty) product of primary Eisenstein primes such that $\lambda/(\sqrt{-3})^c \tau^3$ is cubefree. Then there is a unique product $\rho$ of primary Eisenstein primes such that

$$N(\rho) = \prod_{q \equiv 1 (mod 3)} q$$

and

$$\rho N(\rho) \mid \lambda/(\sqrt{-3})^c \tau^3.$$

(ii) With $b$ as defined in (4), we set

$$F = 3^e N(\rho),$$

where

$$e = \begin{cases} 0, & \text{if } b = 0, \\ 2, & \text{if } b \neq 0. \end{cases}$$

Then $F \neq 1$ and there are $\phi(F)/3$ integers $a$ satisfying

$$1 \leq a < F, \quad \gcd(a, F) = 1, \quad [a/\rho]_3 = \omega^{ba'a''}.$$ 

(iii) Let $a_1, \cdots, a_{\phi(F)/3}$ be the $\phi(F)/3$ integers satisfying (11). Then, except for finitely many primes $p$, we have

$$x^3 + Ax + B \equiv 0 (mod p) \text{ has 3 solutions} \iff p \equiv a_1, \cdots, a_{\phi(F)/3} (mod F).$$

The exceptional primes are those primes $p$ ($\neq 3$) such that $p | C$, $p | F$ together with the prime 3 if $3^4 | C$.

We note that as an exceptional prime $p$ divides $C$, it divides the discriminant of the polynomial $X^3 + AX + B$ and so $N_p(A, B) \neq 3$.

Before proving this theorem we give two illustrative examples.
EXAMPLE 1. We consider the irreducible abelian cubic $X^3 - 21X - 17$. Here $A = -21 = -3 \cdot 7$, $B = -17$ and by (2) $C = 171 = 3^2 \cdot 19$. From (4), (8), (10), (9) we see respectively that $b = 1$, $\rho = 1$, $\alpha = 2$, $F = 9$. By (11) the $\phi(F)/3 = 2$ integers $a_1, a_2$ are the solutions $a$ of

$$ 1 \leq a < 9, \quad \gcd(a, 9) = 1, \quad \omega^{a\alpha''} = 1. $$

The following table

<table>
<thead>
<tr>
<th>$a$</th>
<th>1</th>
<th>2</th>
<th>4</th>
<th>5</th>
<th>7</th>
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<tbody>
<tr>
<td>$a'$</td>
<td>0</td>
<td>1</td>
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<td>2</td>
<td>3</td>
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<tr>
<td>$a''$</td>
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<td>$\omega^{a\alpha''}$</td>
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shows that $a_1 = 1$, $a_2 = 8$. By Theorem (iii) the only exceptional prime is $p = 19$, so that for a prime $p \neq 19$ we have

$$ x^3 - 21x - 17 \equiv 0 (\text{mod } p) $$

has 3 solutions $\Leftrightarrow p \equiv 1, 8 \pmod{9}$.

EXAMPLE 2. We consider the irreducible abelian cubic $X^3 - 21X + 35$. Here $A = -21 = -3 \cdot 7$, $B = 35 = 5 \cdot 7$ and by (2) $C = 63 = 3^2 \cdot 7$. Thus from (6) we have $\lambda = \frac{1}{2}(63 + 105\sqrt{-3})$. By (7) we see that $(\sqrt{-3})^3 \| \lambda$. Further, as

$$ \frac{\lambda}{(\sqrt{-3})^3} = \frac{-35 + 7\sqrt{-3}}{2} = \omega^2 \left(\frac{1 + 3\sqrt{-3}}{3}\right) \left(\frac{1 - 3\sqrt{-3}}{2}\right)^2, $$

we see by (8) that $\tau = 1$ and $\rho = \frac{1}{2}(1 - 3\sqrt{-3})$. From (4), (10), (9) we deduce respectively $b = 2$, $\alpha = 2$, $F = 3^2 \cdot 7 = 63$. By (11) the $\phi(F)/3 = 12$ integers $a_1, \ldots, a_{12}$ are the solutions $a$ of

$$ 1 \leq a < 63, \quad \gcd(a, 63) = 1, \quad \left[ \frac{a}{\frac{1}{2}(1 - 3\sqrt{-3})} \right]_3 = \omega^{2a\alpha''}. $$

Clearly we have

$$ \omega^{2a\alpha''} = \begin{cases} 1, & \text{if } a \equiv \pm 1 \pmod{9}, \\ \omega, & \text{if } a \equiv \pm 2 \pmod{9}, \\ \omega^2, & \text{if } a \equiv \pm 4 \pmod{9}, \end{cases} $$

and, as $N(\rho) = 7$ and $\omega \equiv 2 \pmod{\rho}$, we have

$$ \left[ \frac{a}{\rho} \right]_3 = \begin{cases} 1, & \text{if } a \equiv \pm 1 \pmod{7}, \\ \omega, & \text{if } a \equiv \pm 3 \pmod{7}, \\ \omega^2, & \text{if } a \equiv \pm 2 \pmod{7}. \end{cases} $$
Thus the required $a$'s must satisfy
\[
\begin{cases}
a \equiv \pm 1 \pmod{9} \\
a \equiv \pm 1 \pmod{7}
\end{cases}
\lor
\begin{cases}
a \equiv \pm 2 \pmod{9} \\
a \equiv \pm 1 \pmod{7}
\end{cases}
\lor
\begin{cases}
a \equiv \pm 4 \pmod{9} \\
a \equiv \pm 2 \pmod{7}
\end{cases}.
\]

Hence $a_1 = 1$, $a_2 = 5$, $a_3 = 8$, $a_4 = 11$, $a_5 = 23$, $a_6 = 25$, $a_7 = 38$, $a_8 = 40$, $a_9 = 52$, $a_{10} = 55$, $a_{11} = 58$, $a_{12} = 62$. By Theorem (iii) there are no exceptional primes. Thus for all primes $p$ we have
\[
x^3 - 21x + 35 \equiv 0 \pmod{p}
\]
has 3 solutions $\iff p \equiv 1, 5, 8, 11, 23, 25, 38, 40, 52, 55, 58, 62 \pmod{63}$.

**Proof of Theorem.** We begin by noting the following easily proved consequences of (1), (2) and (6).

(13) If $p$ is a prime $\neq 3$ then $p^2 \nmid \lambda$.

(14) If $p$ is a prime such that $p \mid \lambda$ then $p \equiv 2 \pmod{3}$.

(15) If $p$ is a prime such that $p \mid A$ then $p \equiv 2 \pmod{3}$.

(16) If $p$ is a prime $\neq 3$ then
\[
p \mid A, \ p \mid B \iff p \mid \lambda.
\]

(17) If $p$ is a prime $\neq 3$ then
\[
p \mid A, \ p \nmid B \iff \text{there exists an Eisenstein prime } \pi \text{ dividing } p
\]
such that $\pi \mid \lambda, \overline{\pi} \mid \lambda$.

We also note that $\lambda$ is not the cube of an Eisenstein integer, otherwise,
\[
\frac{1}{2}(C + 3B\sqrt{-3}) = (\frac{1}{2}(g + h\sqrt{-3}))^3,
\]
for some integers $g$ and $h$, so that
\[
A = (-g^2 - 3h^2)/4, \quad B = (g^2h - h^3)/4, \quad C = (g^3 - 9gh^2)/4,
\]
and thus
\[
X^3 + AX + B = (X - h)(X^2 + hX + (h^2 - g^2)/4),
\]
contradicting that $X^3 + AX + B$ is irreducible in $\mathbb{Z}[X]$.

Proof of (i). Suppose $(\sqrt{-3})^e \mid \lambda$. Then $(\sqrt{-3})^e \mid \lambda$ and so $(\sqrt{-3})^{2e} \mid \lambda \overline{\lambda}$, that is $3^e \mid N(\lambda) = -A^3$, showing that $x = c$, as required.

We now prove (8). We let $\mu$ denote the product of primary Eisenstein primes such that $\lambda / (\sqrt{-3})^c \mu$ is a unit, say,
\[
\frac{\lambda / (\sqrt{-3})^c}{\mu} = (-1)^e \omega^e, \quad a = 0, 1, \quad e = 0, 1, 2.
\]
We first prove that $e=b$. We consider the Eisenstein integer $\lambda_1 = \frac{1}{2}(x+y\sqrt{-3})$ given by

$$
\lambda_1 = \lambda/((\sqrt{-3})^c = \begin{cases} 
\frac{1}{2}(C+3B\sqrt{-3}), & \text{if } 3 \mid A, \\
\frac{1}{2}\left(-B+\frac{C}{9}\sqrt{-3}\right), & \text{if } 3 \Vert A, \\
\frac{1}{2}\left(-\frac{C}{27}-\frac{B}{9}\sqrt{-3}\right), & \text{if } 3^2 \Vert A.
\end{cases}
$$

From (18) we have

$$(20) \quad \lambda = (-1)^a \omega^e (\sqrt{-3})^c \mu,$$

and as $\mu$ is a product of primary Eisenstein integers we have

$$(21) \quad \mu \equiv \pm 1 \pmod{3},$$

and

$$(22) \quad \lambda_1 = (-1)^a \omega^e \mu \equiv \pm \omega^e \pmod{3}.$$  

Then, as $3 \nmid x$, we have

$$
\begin{align*}
& \begin{cases} 
\quad e=0 \iff 3 \mid y, \\
\quad e=1 \iff 3 \mid x+y, \ 3 \nmid y, \\
\quad e=2 \iff 3 \mid x-y, \ 3 \nmid y,
\end{cases} \\
\end{align*}
$$

and appealing to (4) and (19) we obtain $e=b$ as asserted. By the definition of $\tau$ we have $\tau^3 \mid \mu$ and $\mu/\tau^3$ is cubefree. We let $F_1$ denote the largest positive integer dividing $\mu/\tau^3$, and set

$$(24) \quad \rho = \mu/(\tau^3 F_1).$$

Clearly $\rho$ is a product of primary Eisenstein primes, and

$$(25) \quad \lambda = (-1)^a \omega^b (\sqrt{-3})^c \mu, \quad \mu = F_1 \rho \tau^3.$$  

We show that $\rho$ is the unique Eisenstein integer satisfying (8). This will be done in four steps:

(a) $N(\rho) = F_1,$

(b) $F_1 = \prod_{q(\text{prime}) \equiv 1 \pmod{3}} q,$

(c) $\rho N(\rho) \mid \lambda/((\sqrt{-3})^c \tau^3),$  

(d) $\rho$ is the unique product of primary Eisenstein primes having property (8).
Proof of (a). From (25) we have \( N(\mu) = F_1^2 N(\rho) N(\tau)^3 \). As \( N(\mu) \) is a cube, \( F_1^2 N(\rho) \) is also a cube. Clearly \( F_1 \) is cubefree, so that to prove \( N(\rho) = F_1 \) it suffices to prove that \( N(\rho) \) is cubefree. Suppose not. Then there exists a prime \( p \) such that

\[
p^3 | N(\rho) \mid N(\mu) = -A^3/3^c ,
\]

so that \( p \mid A \) and \( p \neq 3 \). Hence, by (15), we have \( p \equiv 1 \pmod{3} \), say \( p = \pi \overline{\pi} \), where \( \pi \) and \( \overline{\pi} \) are conjugate Eisenstein primes. Then \( \pi^3 \overline{\pi}^3 \mid p \rho \), and as \( \rho \) is not divisible by a rational integer, we have \( \pi^3 \mid \rho \) or \( \overline{\pi}^3 \mid \rho \), contradicting that \( \rho \) is cubefree. Thus we have \( F_1 = N(\rho) \), which is (a), and by (25)

(26) \[ \mu = \rho N(\rho) \tau^3 . \]

Proof of (b). We begin by showing that \( F_1 = N(\rho) \) is squarefree. Suppose not. Then, by an argument similar to that in the proof of (a), there is an Eisenstein prime \( \pi \) such that \( \pi^2 \mid \rho \). Hence \( \pi^4 \mid \rho N(\rho) \), contradicting that \( F_1 \rho \) is cubefree. Next we show that for any prime \( p \), we have

\[
p \mid A, \ p \mid B, \ p \equiv 1 \pmod{3} \iff p \mid N(\rho) ,
\]

completing the proof of (b) as \( F_1 \) is squarefree.

We have appealing to (13), (16), (20) and (26)

\[
p \mid A, \ p \mid B, \ p \equiv 1 \pmod{3}
\Rightarrow p \mid \lambda , \ p^3 \mid \lambda
\Rightarrow \exists \ some \ Eisenstein \ prime \ \pi \ dividing \ p \ with \ \pi \mid \lambda , \ \pi^3 \mid \lambda
\Rightarrow \pi \mid \rho N(\rho)
\Rightarrow p \mid N(\rho)^3
\Rightarrow p \mid N(\rho) ,
\]

and appealing to (14), (16), (20), (26)

\[
p \mid N(\rho) \Rightarrow p \mid \mu , \ p \neq 3 \Rightarrow p \mid \lambda \Rightarrow p \equiv 1 \pmod{3} , p \mid A, p \mid B .
\]

This completes the proof of (b). From (9) and (b) we see that

(27) \[ F = 3^c F_1 . \]

Proof of (c). From (18) we have \( \mu \mid \lambda / (\sqrt{-3})^c \). But by (26) \( \mu = \rho N(\rho) \tau^3 \) so that \( \rho N(\rho) \mid \lambda / ((\sqrt{-3})^c \tau^3) \), which is (c).

Proof of (d). Suppose that \( \rho_1 \) is a product of primary Eisenstein primes such that

\[
\rho_1 N(\rho_1) \mid \lambda / ((\sqrt{-3})^c \tau^3), \quad N(\rho_1) = F_1 .
\]

As
\begin{align*}
\lambda &= (-1)^a \omega^b (\sqrt{-3})^c \rho N(\rho) \tau^3,
\end{align*}

we have

\begin{align*}

\rho_1 N(\rho_1) | \rho N(\rho), & \quad N(\rho_1) = N(\rho),
\end{align*}

so that \( \rho_1 | \rho \), say, \( \rho = \kappa \rho_1 \). As \( N(\rho) = N(\rho_1) \), \( \kappa \) is a unit, and so as both \( \rho \) and \( \rho_1 \) are products of primary Eisenstein primes we have \( \rho = \rho_1 \). This completes the proof of (d).

Proof of (ii). We first prove that \( F \neq 1 \). Suppose on the contrary that \( F = 1 \). Then, by (9), we see that \( \alpha = 0 \) and \( N(\rho) = 1 \). As \( \alpha = 0 \), by (10), we have \( b = 0 \) and so by (4)

either

(I) \( 3 \nmid A \),

or

(II) \( 3 \parallel A, 3 \nmid B, 3^3 | C \).

As \( N(\rho) = 1 \), by (8), we see that

either

(III) there are no primes \( q \equiv 1 \pmod{3} \) dividing \( A \),

or

(IV) there are primes \( q \equiv 1 \pmod{3} \) dividing \( A \) none of which divide \( B \).

Recall that \( A < 0 \) and that by (15) \( A \) has no prime divisors \( \equiv 2 \pmod{3} \). Also recall that \( C > 0 \).

If (I) and (III) hold then \( A = -1 \). By (2) we see that \( B = 0, C = 2 \), which contradicts \( B \neq 0 \).

If (II) and (III) hold then \( A = -3 \). By (2) we see that \( B = \pm 1, C = 9 \), which contradicts \( 3^3 | C \).

If (I) and (IV) hold then \( A = -q_1 \cdots q_s \), where the \( q_i \) are \( s (\geq 1) \) primes \( \equiv 1 \pmod{3} \) which do not divide \( B \). We have \( q_i = \pi_i \overline{\pi}_i \), where \( \pi_i \) and \( \overline{\pi}_i \) are distinct conjugate primary Eisenstein primes. Now

\begin{align*}
\pi_i^3 \overline{\pi}_i^3 | q_i^3 | A^3 | \frac{1}{2} (C + 3B \sqrt{-3}) \times \frac{1}{2} (C - 3B \sqrt{-3})
\end{align*}

and

\begin{align*}
\pi_i, \overline{\pi}_i | \text{GCD} \left( \frac{1}{2} (C + 3B \sqrt{-3}), \frac{1}{2} (C - 3B \sqrt{-3}) \right),
\end{align*}

so we can choose \( \pi_i \) without loss of generality such that \( \pi_i^3 | \frac{1}{2} (C + 3B \sqrt{-3}) \). Hence

\begin{align*}
\frac{1}{2} (C + 3B \sqrt{-3}) = \epsilon \pi_1^3 \cdots \pi_s^3, \quad \frac{1}{2} (C - 3B \sqrt{-3}) = \overline{\epsilon} \pi_1^3 \cdots \pi_s^3,
\end{align*}

where \( \epsilon \) is a unit. As the \( \pi_i \) are primary and \( \frac{1}{2} (C + 3B \sqrt{-3}) \equiv \pm 1 \pmod{3} \) we have \( \epsilon \equiv \pm 1 \pmod{3} \) so that \( \epsilon = \pm 1 \). Set \( \Omega = \pi_1 \cdots \pi_s \). Then

\begin{align*}
A = -\Omega \overline{\Omega}, \quad B = \epsilon (\Omega^3 - \overline{\Omega}^3) / 3 \sqrt{-3},
\end{align*}

and thus

\begin{align*}
X^3 + AX + B = X^3 - \Omega \overline{\Omega} X + \frac{\epsilon}{3 \sqrt{-3}} (\Omega^3 - \overline{\Omega}^3)
\end{align*}
which contradicts that $X^3 + AX + B$ is irreducible.

If (II) and (IV) hold then $A = -3q_1 \cdots q_s$, where the $q_i$ are $s \geq 1$ primes $\equiv 1 \pmod{3}$ which do not divide $B$. Arguing as in the previous case, we see that

$$A = -3\Omega\overline{\Omega}, \quad \frac{1}{2}(C + 3B\sqrt{-3}) = \varepsilon(\sqrt{-3})^3\Omega^3, \quad \frac{1}{2}(C - 3B\sqrt{-3}) = -\varepsilon(\sqrt{-3})^3\overline{\Omega}^3,$$

where $\varepsilon = \pm 1$ and $\Omega = \pi_1 \cdots \pi_s$. Hence $B = -\varepsilon(\Omega^3 + \overline{\Omega}^3)$ and so

$$X^3 + AX + B = X^3 - 3\Omega\overline{\Omega}X - \varepsilon(\Omega^3 + \overline{\Omega}^3) = (X - \varepsilon(\Omega + \overline{\Omega}))(X^2 + \varepsilon(\Omega + \overline{\Omega})X + (\Omega^2 - \Omega\overline{\Omega} + \overline{\Omega}^2)),$$

which contradicts that $X^3 + AX + B$ is irreducible.

This completes the proof that $F \neq 1$. Then, from (8), (9) and (10), we see that $\phi(F) \equiv 0 \pmod{3}$.

Next we suppose that there are $t$ integers satisfying (11), say $a_1, \cdots, a_t$, and show that $t = \phi(F)/3$. Let $G$ denote the multiplicative group of reduced residue classes modulo $F$ and $H$ the multiplicative group of cube roots of unity. We consider the homomorphism $\theta : G \to H$ given by

$$\theta([k]_3) = \left[\frac{k}{\rho}\right]_3 \omega^{-bk'k''},$$

where $[k]_3$ denotes the residue class modulo $F$ of the integer $k$ coprime with $F$. If $b = 0$, $\theta$ is onto since $\rho \neq 1$ is cubefree. If $b \neq 0$, $\theta$ is onto since for $v = 3F_1 \pm 1$, $\theta([v]_3) = \omega^{-bF_1} \neq 1$. Hence $t = \text{card} \{[a_1], \cdots, [a_t]\} = |\ker \theta| = |G|/|H| = \phi(F)/3$ as asserted.

This completes the proof of (ii).

Proof of (iii). Let $p$ denote a prime such that $p \mid 3C$, and let $\pi$ be an Eisenstein prime such that $p \mid \pi, \pi \not\mid \lambda$. By class field theory, or appealing to [2], we know that $N_p(A, B) = 3 \Leftrightarrow \left[\frac{\lambda}{\pi}\right]_3 = 1$. From (25) we see that

$$\left[\frac{\lambda}{\pi}\right]_3 = \left[\frac{\omega}{\pi}\right]_3 \left[\frac{\mu}{\pi}\right]_3 = \omega^{b(N(\pi) - 1)/3} \left[\frac{\mu}{\pi}\right]_3 = \omega^{bp} \left[\frac{\mu}{\pi}\right]_3.$$

As $\mu = \rho N(\rho) \pi^3$ we have (appealing to the law of cubic reciprocity)

$$\left[\frac{\mu}{\pi}\right]_3 = \left[\frac{\rho^2 \overline{\rho} \pi^3}{\pi}\right]_3 = \left[\frac{\rho^2 \overline{\rho}}{\pi}\right]_3 = \left[\frac{\rho}{\pi}\right]_3 \left[\frac{\overline{\rho}}{\pi}\right]_3 = \left[\frac{\rho}{\pi}\right]_3 \left[\frac{\overline{\rho}}{\pi}\right]_3 = \left[\frac{N(\pi)}{\overline{\rho}}\right]_3 \left[\frac{p}{\rho}\right]_3.$$
\[
\left[ \frac{\lambda}{\pi} \right]_3 = 1 \iff \omega^{bp} \left[ \frac{p}{\rho} \right]_3^{-h} = 1 \iff \left[ \frac{p}{\rho} \right]_3^h = \omega^{bp'}
\]
\[
\iff \left[ \frac{p}{\rho} \right]_3^{p''} = \omega^{bp'} \iff \left[ \frac{p}{\rho} \right]_3 = \omega^{bp' p''}.
\]
Since \( \rho \) is not divisible by a rational prime, \( N(\rho) \) is squarefree, and \( 3 \nmid N(\rho) \), an easy calculation shows that the value of the quantity \( \left[ k/\rho \right]_3 \omega^{-bk'k}' \), where \( k \) is a fixed integer coprime with \( 9N(\rho)N(\rho) \)

\[
is determined by the residue class of \( k \) modulo \( F \). Hence \( \left[ \lambda/\pi \right]_3 = 1 \) if and only if \( p \equiv a_i \pmod{F} \) for some \( i, 1 \leq i \leq \phi(F)/3 \). Thus for a prime \( p \) not dividing \( 3C \), we have \( N_p(A, B) = 3 \) if and only if \( p \equiv a_i \pmod{F} \) for some \( i, 1 \leq i \leq \phi(F)/3 \).

It remains to determine the set of exceptional primes, that is the set \( E(A, B) \) given by

\[
E(A, B) = \{ p \text{ (prime) } \mid N_p(A, B) \neq 3, p \equiv a_i \pmod{F} \text{ for some } i, \text{ or } N_p(A, B) = 3, p \not\equiv a_i \pmod{F} \text{ for any } i \}.
\]

It suffices to consider the primes \( p \) dividing \( 3C \). First we consider the prime \( 3 \). We observe that \( X^3 + AX + B \) splits modulo 3 if and only if \( A \equiv -1 \pmod{3} \), \( B \equiv 0 \pmod{3} \), that is, if and only if \( 3 \nmid A, 3 \mid B \).

If \( b = 0 \) we see from (4) that \( N_3(A, B) = 3 \) if and only if \( 3 \mid B \). Next, by (11), (25), (19), and the result

\[
\left[ \frac{3}{\beta} \right]_3 = \omega^{\mp 2y/3}, \quad \text{if } \beta = \frac{1}{2}(x+y\sqrt{-3}) \equiv \pm 1 \pmod{3},
\]

we have

\[
3 \equiv a_i \pmod{F} \text{ for some } i \iff \left[ \frac{3}{\rho} \right]_3 = 1 \iff \left[ \frac{3}{\mu} \right]_3 = 1 \iff \left[ \frac{3}{\lambda_1} \right]_3 = 1
\]
\[
\iff \begin{cases} 3 \mid B, & \text{if } 3 \nmid A, \\ 81 \mid C, & \text{if } 3 \mid A. \end{cases}
\]

If \( 3 \nmid A \) we have \( 3 \notin E(A, B) \). From (4) we see that if \( 3 \mid A \), then \( 3 \nmid B \), so \( 3 \in E(A, B) \iff 81 \mid C \).

If \( b \neq 0 \), then, by (4), we see that \( N_3(A, B) \neq 3 \). Moreover, by (4), (10) and (9), we have \( 9 \mid F \), so that \( 3 \nmid a_i \pmod{F} \) for any \( i \). Hence, in this case, we have \( 81 \nmid C \) and \( 3 \notin E(A, B) \).

Combining cases we see that
Next we consider primes $p \neq 3$ dividing $C$. If $p \mid A$ (so that $p \equiv 1 \pmod{3}$) then $p \mid F$ showing that $p \not\equiv a_i \pmod{F}$ for any $i$. Clearly $N_p(A, B) \neq 3$ in this case, so that $p \notin E(A, B)$.

If $p \nmid A$ then $p \nmid F$. As $p \mid C$ we have $p \mid \text{disc}(X^3 + AX + B)$ and so $N_p(A, B) \neq 3$. However, we show that $[p/\rho]_3 = \omega^{b \rho^p \rho'}$ so that $p \equiv a_i \pmod{F}$ for some $i$ and thus $p \in E(A, B)$. Since $p \nmid A$, we have $\gcd(p, \lambda) = 1$ as $N(\lambda) = -A^3$, and

$$\left[ \frac{p}{\rho} \right]_3 = \left[ \frac{\rho}{p} \right]_3 = \left[ \frac{F_1 \rho \tau^3}{p} \right]_3 = \left[ \frac{\mu}{p} \right]_3 = \left[ \frac{\omega^{-b} \lambda}{p} \right]_3 = \omega^{bp^p} \left[ \frac{1}{p} \right]_3 \omega^{b \frac{3B + C}{p}} = \omega^{bp^p \rho'}$$

as asserted. This completes the proof of the theorem. \[ \square \]

Let $L$ denote the cubic field $Q(\theta)$, where $\theta$ is any root of the cubic equation $x^3 + Ax + B = 0$. By a result of Llorente and Nart [5: Theorem 2] the discriminant $d(L)$ of $L$ is given by

$$d(L) = 3^{2a} \prod_{q \mid (\text{primc}) \equiv 1 \pmod{3}} q^2.$$

Further, by the conductor-discriminant formula for a cyclic cubic field [1: Corollary 17.29], we have $d(L) = f(L)^2$, where $f(L)$ is the conductor of $L$, that is, the conductor of $X^3 + AX + B$. This shows that $F$ (as in (9)) is the conductor of $X^3 + AX + B$.

We conclude by remarking that if $F$ is a prime, the set $\{a_1, \ldots, a_{\phi(F)/3}\}$ consists precisely of the nonzero cubes modulo $F$. This is clear, for if $F$ is a prime, we have $\alpha = b = 0$ and as $\rho$ is an Eisenstein prime of norm $F_1$, $[a_i/\rho]_3 = 1$ if and only if $a_i$ is a nonzero cube modulo $F$.

For example consider the irreducible abelian cubic $X^3 - 31X + 62$. We have $A = -31, B = 62, C = 124, b = 0, \alpha = 0, F = 31, E(A, B) = \{2\}$, so that for $p \neq 2$

$$x^3 - 31x + 62 \equiv 0 \pmod{p}$$

has 3 solutions

$$\iff p \equiv \text{nonzero cube} \pmod{31}\,$$

$$\iff p \equiv 1, 2, 4, 8, 15, 16, 23, 27, 29, 30 \pmod{31}.$$


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