The nth Power of a 2×2 Matrix

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Let α and β be the eigenvalues of the matrix

$$A=\begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

We present a very simple derivation of the formula for A^n (n is a positive integer) in terms of α and β. As A satisfies its characteristic polynomial, we have

$$A^2-(\alpha+\beta)A+\alpha\beta I=0, \quad \text{where } I=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{1}$$

The matrices X, Y, Z defined by

$$\begin{cases} X=\dfrac{A-\beta I}{\alpha-\beta},\ Y=\dfrac{A-\alpha I}{\beta-\alpha}, & \text{if } \alpha \neq \beta, \\ Z=A-\alpha I, & \text{if } \alpha=\beta, \end{cases}$$

satisfy

$$\begin{cases} X^2=X,\ XY=YX=0,\ Y^2=Y, & \text{if } \alpha \neq \beta, \\ Z^2=0, & \text{if } \alpha=\beta, \end{cases}$$

so that for $k \geqslant 2$

$$\begin{cases} X^k=X,\ Y^k=Y, & \text{if } \alpha \neq \beta, \\ Z^k=0, & \text{if } \alpha=\beta. \end{cases}$$

Hence we have

$$A^n=\begin{cases} (\alpha X+\beta Y)^n=\alpha^n X^n+\beta^n Y^n=\alpha^n X+\beta^n Y, & \text{if } \alpha \neq \beta, \\ (\alpha I+Z)^n=\alpha^n I+n\alpha^{n-1}Z, & \text{if } \alpha=\beta, \end{cases}$$

giving

$$A^n=\begin{cases} \alpha^n\left(\dfrac{A-\beta I}{\alpha-\beta}\right)+\beta^n\left(\dfrac{A-\alpha I}{\beta-\alpha}\right), & \text{if } \alpha \neq \beta, \\ \alpha^{n-1}(nA-(n-1)\alpha I), & \text{if } \alpha=\beta. \end{cases} \tag{2}$$

If the matrix A is invertible ($\alpha \neq 0, \beta \neq 0$), it is easy to see that (2) holds for all integral values of n.

If A is real but its eigenvalues $\alpha=p+iq$ and $\beta=p-iq$ are nonreal ($q \neq 0$) with some power of them real, say $(p+iq)^m=(p-iq)^m=r$, then, by (2), we have $A^m=rI$.

In the case of distinct eigenvalues, the reader will recognize the matrices X and Y as supplementary projections ($X+Y=I$) and

$$\text{eigenspace of } \alpha = \text{range of } X = \text{null space of } Y,$$
$$\text{eigenspace of } \beta = \text{null space of } X = \text{range of } Y.$$