Congruences modulo 16 for the Class Numbers of Complex Quadratic Fields

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Communicated by H. Zassenhaus

Received April 28, 1986

Let \( h(d) \) denote the class number of the quadratic field \( \mathbb{Q}(\sqrt{d}) \) of discriminant \( d \). A number of new determinations of \( h(d) \) modulo 16 are proved. For example, it is shown that if \( p \) and \( q \) are primes satisfying \( p \equiv q \equiv 5 \pmod{8} \), \( (p/9) = 1 \), then

\[
\begin{align*}
4 \pmod{16} & \quad \text{if} \quad \left( \frac{aA+bB}{p} \right) = (-1)^{b+B+4k}, \\
12 \pmod{16} & \quad \text{if} \quad \left( \frac{aA+bB}{p} \right) = (-1)^{b+B+4k},
\end{align*}
\]

where \( a \) and \( b \) are unique integers such that \( p = a^2 + b^2 \), \( a \equiv 1 \pmod{4} \), \( b \equiv ((p-1)/2)! \pmod{a} \), and \( A \) and \( B \) are the unique integers such that \( q = A^2 + B^2 \), \( A \equiv 1 \pmod{4} \), \( B \equiv ((q-1)/2)! \pmod{A} \).

1. Introduction

As usual we denote the class number of the quadratic field \( \mathbb{Q}(\sqrt{d}) \) of discriminant \( d \) by \( h(d) \). When \( d = (-1)^n p_1 \cdots p_s q_{s+1} \cdots q_n \), where \( n \) is a positive integer, \( p_1, \ldots, p_s \) are \( s \) (\( \geq 0 \)) distinct primes \( \equiv 1 \pmod{4} \), and \( q_{s+1}, \ldots, q_n \) are \( n-s \) (\( \geq 0 \)) distinct primes \( \equiv 3 \pmod{4} \), the authors [8] have proved a congruence of the form

* Research supported by Natural Sciences and Engineering Research Council of Canada, Grant A-8049.
† Research supported by Natural Sciences and Engineering Research Council of Canada, Grant A-7233.

0022-314X/87 $3.00
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\[
\sum_{e \mid d, e > 0, e \equiv 1 \pmod{4}} (c_1(d, e) h(-4e) + c_2(d, e) h(-8e)) \\
+ \sum_{e \mid d, e < 0, e \equiv 1 \pmod{4}} (c_3(d, e) h(e) + c_4(d, e) h(8e)) \\
+ \frac{(-1)^n}{2} \prod_{i=1}^{n} (|p_i| - 1) \equiv c_5(d) + c_6(d) \pmod{2^{n+2}}, \quad (1.1)
\]

where

\[
c_1(d, e) = (\frac{e}{2}) \prod_{p \mid d/e} \left( \left( \frac{e}{p} \right) - \left( \frac{-1}{p} \right) \right),
\]

\[
c_2(d, e) = \prod_{p \mid d/e} \left( \left( \frac{e}{p} \right) - \left( \frac{-2}{p} \right) \right),
\]

\[
c_3(d, e) = \left( 5 - (\frac{e}{2}) \right) \prod_{p \mid d/e} \left( \left( \frac{e}{p} \right) - 1 \right),
\]

\[
c_4(d, e) = - \prod_{p \mid d/e} \left( \left( \frac{e}{p} \right) - \left( \frac{2}{p} \right) \right),
\]

\[
c_5(d) = \begin{cases} 
2^{n-1} & \text{if } d \text{ is divisible only by primes } \equiv 3 \pmod{4}, \\
0 & \text{otherwise},
\end{cases}
\]

\[
c_6(d) = \begin{cases} 
0 & \text{if } 3 \nmid d, \\
4 & \text{if } d = -3, \\
0 & \text{if } d \neq -3, 3 \mid d, \text{ and } p \mid d/3 \\
& \text{for some prime } p \equiv 1 \pmod{3}, \\
2^{n+1} & \text{if } d \neq -3, 3 \mid d, \text{ and all primes } \\
& p \mid d/3 \text{ satisfy } p \equiv 2 \pmod{3}.
\end{cases}
\]

This congruence contains as special cases many known congruences, for example, those of Pizer [13] and those of Kenku [11].

In this paper we analyze (1.1) in the case \( n = 2 \) in order to obtain new congruences modulo 16 involving \( h(-8pq) \) and \( h(-4pq) \), when \( pq \equiv 1 \pmod{4} \), and \( h(-pq) \), when \( pq \equiv 3 \pmod{4} \), where \( p \) and \( q \) are distinct odd primes. We begin by giving a summary of references to known results (see Table I). In cases 1–3, 7, 9, 10, 13, and 18, (1.1) does not give new information. In case 4, (1.1) is used, together with the conjecture given in [16].
to conjecture the value of $h(-4pq)$ (mod 16) (see Section 2, Conjecture). In cases 5, 6, 8, 11, 12, and 14, (1.1) is used in conjunction with results specified in the table to obtain the value of $h(-8pq)$ (mod 16) (see Section 3, Theorem 1; Section 4, Theorem 2; Section 5, Theorem 3; Section 6, Theorem 4; Section 7, Theorem 5; Section 8, Theorem 6). In cases 15 and 16, as $h(-8pq)$ is known modulo 16, (1.1) gives $h(-4pq)$ (mod 16) (see Section 9, Theorem 7; Section 10, Theorem 8). In case 17, since neither $h(-4pq)$ nor $h(-8pq)$ is known individually (mod 16), (1.1) just gives $h(-4pq) + h(-8pq)$ (mod 16) (see Section 11, Theorem 9).

In proving our results we shall need the classical congruences (see, e.g., [13, Propositions 1 and 2]).
CONGRUENCES MODULO 16

\[ h(-4p) \equiv \frac{1}{2}(p-1) \pmod{4} \quad \text{if } p \equiv 1 \pmod{4}, \quad (1.2) \]

\[ h(-p) \equiv 1 \pmod{2} \quad \text{if } p \equiv 3 \pmod{4}, \quad (1.3) \]

\[ h(-4p) + h(-8p) \equiv \frac{1}{2}(p-1) \pmod{8} \quad \text{if } p \equiv 1 \pmod{8}, \quad (1.4) \]

\[ 2h(-p) + h(-8p) \equiv \begin{cases} \frac{1}{2}(p-3) & \text{if } p \equiv 3 \pmod{8}, \ p > 3, \\ 4 & \text{if } p = 3, \end{cases} \]

\[ h(-4p) + h(-8p) \equiv \frac{1}{2}(p+3) \pmod{8} \quad \text{if } p \equiv 5 \pmod{8}, \quad (1.6) \]

\[ h(-8p) \equiv \frac{1}{2}(p+1) \pmod{8} \quad \text{if } p \equiv 7 \pmod{8}. \quad (1.7) \]

We will also use the following notation. If \( p \) is a prime \( \equiv 1 \pmod{4} \) we let \( a \) and \( b \) be the unique integers such that

\[ p = a^2 + b^2, \quad a \equiv 1 \pmod{4}, \quad b \equiv ((p-1)/2)! \pmod{p}. \quad (1.8) \]

Similarly if \( q \) is a prime \( \equiv 1 \pmod{4} \), we define integers \( A \) and \( B \) uniquely by replacing \( p \) by \( q \), \( a \) by \( A \), \( b \) by \( B \) in (1.8). Frequent use will be made of the congruence (and the similar one involving \( q \) and \( A \))

\[ p \equiv \begin{cases} 2a - 1 & \text{if } p \equiv 1 \pmod{16}, \\ 2a + 3 & \text{if } p \equiv 5 \pmod{16}. \end{cases} \quad (1.9) \]

This is given in [16, p. 972] and is a straightforward deduction from (1.8). We will also use the congruence

\[ h(-4p) \equiv -a + b + 1 \pmod{8}. \quad (1.10) \]

This congruence was given by Gauss in a letter to Dirichlet dated 30 May 1828 [5], [6, p.287]. A proof by Dedekind is given in [6, pp. 299–301; 7, pp. 692–693] (see also [1, 16, 18]).

From (1.4), (1.6), (1.8), (1.9), and (1.10) (see also [1]), we obtain

\[ h(-8p) \equiv b \pmod{8}. \quad (1.11) \]

In addition, if \( (p/q) = +1 \), Burde's rational biquadratic reciprocity law [4] asserts that

\[ \left( \frac{p}{q} \right)_4 \left( \frac{q}{p} \right)_4 = \left( \frac{aA + bB}{p} \right) = \left( \frac{aA + bB}{q} \right) \quad (1.12) \]

(see also [14]).
On the other hand if \( p \equiv 3 \pmod{4} \) then by a result of Mordell [12] we have for \( p > 3 \),

\[
h(-p) \equiv \begin{cases} 
1 \pmod{4} & \text{if } \left( \frac{p-1}{2} \right) ! \equiv -1 \pmod{p}, \\
3 \pmod{4} & \text{if } \left( \frac{p-1}{2} \right) ! \equiv 1 \pmod{p}.
\end{cases}
\] (1.13)

Then, from (1.5) and (1.13), we obtain for \( p \equiv 3 \pmod{8} \) and \( p > 3 \),

\[
h(-8p) \equiv \begin{cases} 
2 \pmod{8} & \text{if } \left( \frac{p-1}{2} \right) ! \equiv (-1)^{\frac{p-3}{8}} \pmod{p}.
\end{cases}
\] (1.14)

2. \( p \equiv 1 \pmod{8} \), \( q \equiv 5 \pmod{8} \), \( (p/q) = -1 \)

In this case (1.1) gives

\[ h(-8pq) + h(-4pq) + 2h(-4p) \equiv q + 3 \pmod{16} \] (2.1)

(cf. [13, Proposition 5, Eq. (21)].

Kaplan [10, Cas 1a, p. 347; Cas 2a, p. 350] gives

\[ h(-4pq) \equiv h(-8pq) \equiv 4 \pmod{8}. \] (2.2)

Combining (2.1) with a conjecture of Williams and Currie [16] for the value of \( h(-8pq) \) modulo 16, and using (1.10), we obtain a conjecture for \( h(-4pq) \) modulo 16.

**Conjecture.** Let \( p \) and \( q \) be primes such that

\[ p \equiv 1 \pmod{8}, \quad q \equiv 5 \pmod{8}, \quad \left( \frac{p}{q} \right) = -1. \]

and define integers \( a, b, A, B \) uniquely using (1.8). Then we have

\[
h(-4pq) \equiv \begin{cases} 
4 \pmod{16} & \text{if } \left( \frac{a+bi}{a-bi} \right)^{q} \equiv (-1)^{a+A+b+B/4} i \pmod{q}, \\
12 \pmod{16} & \text{if } \left( \frac{a+bi}{a-bi} \right)^{q} \equiv (-1)^{a+A+b-B/4} i \pmod{q}.
\end{cases}
\]
The following table illustrates the conjecture. We write

\[ X \equiv \left( \frac{a + bi}{a - bi} \right)^{(q - 1)/4} \pmod{q}, \quad Y = (-1)^{(a + A + b + B)/4} \cdot i. \]

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3. \( p = q = 5 \pmod{8}, (p/q) = 1 \)

In this case (1.1) gives

\[ h(-8pq) + h(-4pq) + 2h(-8p) + 2h(-8q) \equiv 4 \pmod{16} \quad (3.1) \]

(cf. [13, Proposition 5, Eq. (29)])

We have

**Theorem 1.** Let \( p \) and \( q \) be primes such that

\[ p \equiv q \equiv 5 \pmod{8}, \quad \left( \frac{p}{q} \right) = 1, \]

and define integers \( a, b, A, B \) uniquely using (1.8). Then we have

\[ h(-8pq) \equiv \begin{cases} 4 \pmod{16} & \text{if } \left( \frac{aA + bB}{p} \right) = (-1)^{(h + B + A)/4}, \\
12 \pmod{16} & \text{if } \left( \frac{aA + bB}{p} \right) = (-1)^{(h + B)/4}. \end{cases} \]

**Proof.** From Kaplan [10, Proposition \( B'_4 \)] we have

\[ \left( \frac{p}{q} \right)_4 \left( \frac{q}{p} \right)_4 = (-1)^{h(-4pq) + 8}/8. \]  

(3.2)
Hence, from (1.12) and (3.2), we obtain
\[
\left( \frac{aA + bB}{p} \right) = (-1)^{(h_1 - 4pq) + 8\frac{1}{8}}. \tag{3.3}
\]

Next, by (1.11), we have
\[
h(-8p) \equiv b \pmod{8}, \quad h(-8q) \equiv B \pmod{8},
\]
so that
\[
(-1)^{b + B/4} = (-1)^{(h(-8p) + h(-8q))/4}. \tag{3.4}
\]

Multiplying (3.3) and (3.4) together, we obtain
\[
\left( \frac{aA + bB}{p} \right) (-1)^{(b + B/4)} = (-1)^{(h(-8p) + h(-8q))/4}. \tag{3.5}
\]

Using \( h(-8pq) \equiv 4 \pmod{8} \) [10, Cas 2a, p. 350], in (3.1), we obtain
\[
h(-8pq) \equiv h(-4pq) + 2h(-8p) + 2h(-8q) + 4 \pmod{16}. \tag{3.6}
\]

Putting (3.6) into (3.5), we get
\[
(-1)^{(h(-8pq) + 4)/8} = \left( \frac{aA + bB}{p} \right) (-1)^{(b + B)/4},
\]
which is the required result.

The following table illustrates Theorem 1. We write \( X = (-1)^{(b + B)/4}, \)
\( Y = ((aA + bB)/p). \)

<table>
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<th>( q )</th>
<th>( a )</th>
<th>( b )</th>
<th>( A )</th>
<th>( B )</th>
<th>( X )</th>
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<td>1</td>
<td>1</td>
<td>12</td>
</tr>
</tbody>
</table>

4. \( p = q = 5 \pmod{8}, \ (p/q) = -1 \)

In this case (1.1) gives
\[
h(-8pq) + h(-4pq) + 2h(-8p) + 2h(-4q) \equiv 4 \pmod{16} \tag{4.1}
\]
(cf. [13, Proposition 5, Eq. (30)]). This congruence can be used to determine \( h(-8pq) \pmod{16} \). We prove
THEOREM 2. Let $p$ and $q$ be primes such that

$$p \equiv q \equiv 5 \pmod{8}, \quad \left( \frac{p}{q} \right) = -1,$$

and define integers $a, b, A, B$ uniquely using (1.8). Then we have

$$h(-8pq) \equiv \begin{cases} 
4 \pmod{16} & \text{if } \left( \frac{aA + (-1)^{(b+B)/4} bB}{p} \right) = (-1)^{(b+B+4)/4}, \\
12 \pmod{16} & \text{if } \left( \frac{aA + (-1)^{(b+B)/4} bB}{p} \right) = (-1)^{(b+B)/4}.
\end{cases}$$

Proof. From Kaplan \[lo, Proposition B',\] we have

$$\left( \frac{pq}{2} \right)_4 \left( \frac{2p}{q} \right)_4 \left( \frac{2q}{p} \right)_4 = (-1)^{h(-4pq)/8}. \quad (4.2)$$

Appealing to \[15, Problem 323,\] we have

$$\left( \frac{2p}{q} \right)_4 \left( \frac{2q}{p} \right)_4 = \left( \frac{aA + (-1)^{(b+B)/4} bB}{p} \right). \quad (4.3)$$

Using (4.3) in (4.2), we obtain

$$(-1)^{(pq+7)/8} \left( \frac{aA + (-1)^{(b+B)/4} bB}{p} \right) = (-1)^{h(-4pq)/8}. \quad (4.4)$$

Next from (1.9) and (1.10) we get

$$2h(-4p) + 2h(-4q) \equiv p + q + 6 + 2h + 2B \pmod{16},$$

that is,

$$2h(-4p) + 2h(-4q) \equiv pq + 7 + 2b + 2B \pmod{16},$$

and so

$$(-1)^{(pq+7)/8 + (b+B)/4} = (-1)^{(h(-4p) + h(-4q))/4}. \quad (4.5)$$

Multiplying (4.4) and (4.5) together, we get

$$(-1)^{(h(-4pq) + 2h(-4p) + 2h(-4q))/8} = \left( - \left( \frac{aA + (-1)^{(b+B)/4} bB}{p} \right) \right) (-1)^{(b+B)/4}. \quad (4.6)$$

Now from (1.1) we have, as $h(-8pq) \equiv 4 \pmod{8}$ \[10, Case 2a, p. 350,\]

$$h(-8pq) \equiv h(-4pq) + 2h(-4p) + 2h(-4q) + 4 \pmod{16}. \quad (4.7)$$
Using (4.7) in (4.6), we obtain
\[
(-1)^{(h(-8pq)+4)/8} = -\left(\frac{aA + (-1)^{(h+B)/4}bB}{p}\right)(-1)^{(h+B)/4},
\]
which gives the required result.

The following table illustrates Theorem 2. We write
\[
X = (-1)^{(h+B)/4}, \quad Y = ((aA+XhB)/p).
\]

<table>
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<th>h</th>
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<td>44</td>
</tr>
</tbody>
</table>

5. \(p \equiv 1 \pmod{8}, q \equiv 3 \pmod{8}, (p/q) = -1\)

In this case (1.1) gives
\[
h(-8pq) + 2h(-pq) + 2h(-4p) \equiv q - 3 \pmod{16} \quad (5.1)
\]
(cf. [13, Proposition 5, Eq. (19)]). This congruence can be used to determine \(h(-8pq)\) modulo 16.

We prove

**Theorem 3.** Let \(p\) and \(q\) be primes such that
\[
p \equiv 1 \pmod{8}, \quad q \equiv 3 \pmod{8}, \quad \left(\frac{p}{q}\right) = -1,
\]
and define \(a, b\) by (1.8). Then, for \(q > 3\), we have

\[
h(-8pq) \equiv \begin{cases} 
4 \pmod{16} & \text{if } \left(\frac{a-bi}{a+bi}\right)^{(q+1)/4} \\
\equiv (-1)^{(2a-2b+q+3)/8} \left(\frac{q-1}{2}\right)!i \pmod{q} \\
12 \pmod{16} & \text{if } \left(\frac{a-bi}{a+bi}\right)^{(q+1)/4} \\
\equiv (-1)^{(2a-2b+q+11)/8} \left(\frac{q-1}{2}\right)!i \pmod{q},
\end{cases}
\]
and, for \( q = 3 \), we have

\[
h(-24p) \equiv \begin{cases} 2a - 2b + 2 \pmod{16} & \text{if } a = b \pmod{3}, \\ 2a - 2b + 10 \pmod{16} & \text{if } a \equiv -b \pmod{3}.
\end{cases}
\]

**Proof.** From (1.13) and [16, Theorem (b)(i), (ii)] we have for \( q > 3 \)

\[
h(-pq) \equiv \begin{cases} 2 \pmod{8} & \text{if } \left(\frac{a - bi}{a + bi}\right)^{(q+1)/4} \equiv \left(\frac{q-1}{2}\right)^i \pmod{q}, \\ 6 \pmod{8} & \text{if } \left(\frac{a - bi}{a + bi}\right)^{(q+1)/4} \equiv -\left(\frac{q-1}{2}\right)^i \pmod{q}.
\end{cases}
\]

(5.2)

The required result now follows from (1.10), (5.1), and (5.2).

For \( q = 3 \) the result follows from (1.10), (5.1), and [16, Theorem (b)(iii)]. We remark that in this case (5.1) is equivalent to (11.9) and (11.10) of Corollary 11.4 of [2]. The following short tables illustrate Theorem 3. We write

\[
X \equiv \left(\frac{a - bi}{a + bi}\right)^{(q+1)/4} \pmod{q},
\]

\[
Y \equiv (-1)^{(2a - 2b + q + 3)/8} \left(\frac{q-1}{2}\right)^i \pmod{q}.
\]

### \( q > 3 \):

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<th>( p )</th>
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</table>

### \( q = 3 \):

<table>
<thead>
<tr>
<th>( p )</th>
<th>( a )</th>
<th>( b )</th>
<th>( a \equiv b \pmod{3} )</th>
<th>( 2a - 2b + 2 \pmod{16} )</th>
<th>( h(-24p) )</th>
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</thead>
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<td>4</td>
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</tr>
</tbody>
</table>
6. \( p \equiv 5 \pmod{8}, \ q \equiv 3 \pmod{8}, \ (p/q) = 1. \)

In this case (1.1) gives

\[
h(-8pq) + 2h(-4p) + 2h(-8q) \equiv p - 1 \pmod{16}\]  

(6.1) (cf. [13, Proposition 5, Eq. (25)]). This congruence enables us to determine \( h(-8pq) \) modulo 16.

**Theorem 4.** Let \( p \) and \( q \) be primes such that

\[
p \equiv 5 \pmod{8}, \ q \equiv 3 \pmod{8}, \ \left(\frac{p}{q}\right) = 1,
\]

and define \( a \) and \( b \) by (1.8). Then, for \( q > 3 \), we have

\[
h(-8pq) \equiv \begin{cases} 4 \pmod{16} & \text{if } \left(\frac{q-1}{2}\right)! \equiv (-1)^{(q+2b+1)/8} \pmod{q}, \\ 12 \pmod{16} & \text{if } \left(\frac{q-1}{2}\right)! \equiv (-1)^{(q+2b+9)/8} \pmod{q}; \end{cases}
\]

and, for \( q = 3 \), we have

\[
h(-24p) = -2b \pmod{16}.
\]

**Proof.** For \( q > 3 \) the result follows from (1.9), (1.10), (1.14), and (6.1). For \( q = 3 \) the result follows from (1.9), (1.10), (6.1) and the fact that \( h(-24) = 2 \). We remark that in this case (6.1) is equivalent to (11.7) and (11.8) of Corollary 11.4 of [2].

The following tables illustrate Theorem 4. We write

\[
X \equiv ((q - 1)/2)! \pmod{q}, \quad Y = (-1)^{(q + 2b + 1)/8}.
\]

\[
\begin{array}{cccccc}
q > 3: \\
p & q & b & X & Y & h(-8pq) \\
5 & 11 & 2 & -1 & 1 & 12 \\
5 & 59 & 2 & 1 & 1 & 20 \\
53 & 11 & -2 & -1 & -1 & 36 \\
53 & 59 & -2 & 1 & -1 & 140 \\
\end{array}
\]
7. $p \equiv 5 \pmod{8}, \ q \equiv 3 \pmod{8}, \ (p/q) = -1$

In this case (1.1) gives

$$h(-8pq) + 2h(-8p) + 4h(-q) \equiv \begin{cases} p-1 \pmod{16} & \text{if } q > 3, \\ p + 7 \pmod{16} & \text{if } q = 3 \end{cases}$$

(7.1)

(cf. [13, Proposition 5, Eq. (26)]). This congruence enables us to determine $h(-8pq)$ modulo 16.

**Theorem 5.** Let $p$ and $q$ be primes such that

$$p \equiv 5 \pmod{8}, \ q \equiv 3 \pmod{8}, \ \left(\frac{p}{q}\right) = -1,$$

and define $a$ and $b$ by (1.8). Then, for $q > 3$, we have

$$h(-8pq) \equiv \begin{cases} 4 \pmod{16} & \text{if } \left(\frac{q-1}{2}\right)! \equiv (-1)^{(a-b+1)/4} \pmod{q}, \\ 12 \pmod{16} & \text{if } \left(\frac{q-1}{2}\right)! \equiv (-1)^{(a-b-3)/4} \pmod{q} \end{cases}$$

and, for $q = 3$, we have

$$h(-24p) \equiv 2a - 2b + 6 \pmod{16}.$$
The following tables illustrate Theorem 5. We write

$$X \equiv \left( \frac{q-1}{2} \right)! \pmod{q}, \quad Y = (-1)^{(a-b+1)/4}.$$ 

$q > 3$:

<table>
<thead>
<tr>
<th>$p$</th>
<th>$q$</th>
<th>$a$</th>
<th>$b$</th>
<th>$X$</th>
<th>$Y$</th>
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<tr>
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<td>59</td>
<td>5</td>
<td>-6</td>
<td>1</td>
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</tr>
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</table>

$q = 3$:

<table>
<thead>
<tr>
<th>$p$</th>
<th>$a$</th>
<th>$b$</th>
<th>$2a-2b+6$</th>
<th>$h(-24p)$</th>
</tr>
</thead>
<tbody>
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<td>5</td>
<td>1</td>
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<td>4</td>
<td>4</td>
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<tr>
<td>29</td>
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<td>197</td>
<td>1</td>
<td>-14</td>
<td>36</td>
<td>20</td>
</tr>
</tbody>
</table>

8. $p \equiv 5 \pmod{8}$, $q \equiv 7 \pmod{8}$, $(p/q) = -1$

In this case (1.1) gives

$$h(-8pq) + 2h(-pq) \equiv p + 3 \pmod{16} \quad (8.1)$$

(cf. [13, Proposition 5, Eq. (32)]). This congruence enables us to determine $h(-8pq)$ modulo 16. We have

**Theorem 6.** Let $p$ and $q$ be primes such that

$$p \equiv 5 \pmod{8}, \quad q \equiv 7 \pmod{8}, \quad \left( \frac{p}{q} \right) = -1,$$

and define $a$ and $b$ as in (1.8). Then we have

$$h(-8pq) = \begin{cases} 4 \pmod{16} & \text{if } \left( \frac{a-bi}{a+bi} \right)^{(q+1)/4} \equiv (-1)^{(q-1)/4} \left( \frac{q-1}{2} \right)! \pmod{q}, \\ 12 \pmod{16} & \text{if } \left( \frac{a-bi}{a+bi} \right)^{(q+1)/4} \equiv (-1)^{(q+3)/4} \left( \frac{q-1}{2} \right)! \pmod{q}. \end{cases}$$
Proof. The result follows from (1.9), (1.13), (8.1), and [16, Theorem (b) (i), (ii)].

The following table illustrates Theorem 6. We write

\[ X \equiv \left( \frac{a - bi}{a + bi} \right)^{(q + 1)/4} \quad (\text{mod } q), \quad Y \equiv (-1)^{(a - 1)/4}((q - 1)/2)! \quad (\text{mod } q). \]

<table>
<thead>
<tr>
<th>p</th>
<th>q</th>
<th>a</th>
<th>b</th>
<th>X</th>
<th>Y</th>
<th>h(−8pq)</th>
</tr>
</thead>
<tbody>
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<td>−i</td>
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<tr>
<td>13</td>
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<td>−2</td>
<td>−i</td>
<td>i</td>
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</tr>
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</tr>
<tr>
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<td>i</td>
<td>i</td>
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<td>157</td>
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<td>−11</td>
<td>−6</td>
<td>i</td>
<td>−i</td>
<td>44</td>
</tr>
</tbody>
</table>

9. \( p \equiv q \equiv 3 \pmod{8}, \ (p/q) = 1 \)

In this case (1.1) gives

\[
h(-8pq) + h(-4pq) + 4h(-p) + 2h(-8q) \\
\equiv \begin{cases} 
  p + q - 2 \pmod{16} & \text{if } p > 3, \\
  q + 9 \pmod{16} & \text{if } p = 3 
\end{cases} \quad (9.1)
\]

(cf. [13, Proposition 5, Eq. (24)]). As \( h(-8pq) \) is known modulo 16 [10, Proposition B'15], (9.1) allows us to determine \( h(-4pq) \) modulo 16. We have

**Theorem 7.** Let \( p \) and \( q \) be distinct primes satisfying

\[ p \equiv q \equiv 3 \pmod{8}, \quad \left( \frac{p}{q} \right) = 1. \]

There exist integers \( x, y, k, l \) and \( m \) such that

\[ p = l^2 - 2k^2m, \quad 2q = k^2x^2 + 2lxy + 2my^2. \]
(see [10, p. 356]). Define $\varepsilon_p = \pm 1$ and $\varepsilon_q = \pm 1$ by

$$\left(\frac{p-1}{2}\right)! \equiv \varepsilon_p \pmod{p}, \quad \left(\frac{q-1}{2}\right)! \equiv \varepsilon_q \pmod{q}.$$ 

Then, for $p > 3$ and $q > 3$, we have

$$h(-4pq) \equiv 4 \pmod{16} \quad \text{if } \varepsilon_p \varepsilon_q = (-1)^{\frac{p}{4}} \frac{-2}{|k^2x + ly|},$$

$$h(-4pq) \equiv 12 \pmod{16} \quad \text{if } \varepsilon_p \varepsilon_q = (-1)^{\frac{p}{4} + \frac{q}{4}} \frac{-2}{|k^2x + ly|}.$$ 

If $p = 3$ and $q > 3$ we have

$$h(-12q) \equiv 4 \pmod{16} \quad \text{if } \varepsilon_q = \left(\frac{-2}{|k^2x + ly|}\right),$$

$$h(-12q) \equiv 12 \pmod{16} \quad \text{if } \varepsilon_q = -\left(\frac{-2}{|k^2x + ly|}\right).$$ 

If $p > 3$ and $q = 3$ we have

$$h(-12p) \equiv p - 7 \pmod{16} \quad \text{if } \varepsilon_p = -\left(\frac{-2}{|k^2x + ly|}\right),$$

$$h(-12p) \equiv p + 1 \pmod{16} \quad \text{if } \varepsilon_p = \left(\frac{-2}{|k^2x + ly|}\right).$$ 

Proof. From [10, Proposition B15] we have

$$h(-8pq) \equiv 0 \pmod{16} \quad \text{if } \left(\frac{-2}{|k^2x + ly|}\right) = 1,$$

$$h(-8pq) \equiv 8 \pmod{16} \quad \text{if } \left(\frac{-2}{|k^2x + ly|}\right) = -1. \quad (9.2)$$ 

For $p > 3$ and $q > 3$ the result follows from (1.13), (1.14), (9.1), and (9.2). For $p = 3$ and $q > 3$ the result follows from (1.14), (9.1), and (9.2). For $p > 3$ and $q = 3$ the result follows from (1.13), (9.1), and (9.2).

We remark that (9.1) is equivalent to the appropriate congruences of Corollary 11.6 of [2] when $p$ or $q = 3$. 
The following tables illustrate Theorem 7. We write

\[ X = \left( \frac{-2}{k^2 + ly} \right), \quad Y = (-1)^{(p - 3)/8}. \]

### $p > 3, q > 3$:

<table>
<thead>
<tr>
<th>$p$</th>
<th>$q$</th>
<th>$k$</th>
<th>$l$</th>
<th>$m$</th>
<th>$x$</th>
<th>$y$</th>
<th>$X$</th>
<th>$Y$</th>
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### $p = 3, q > 3$:

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<th>$k$</th>
<th>$l$</th>
<th>$m$</th>
<th>$x$</th>
<th>$y$</th>
<th>$X$</th>
<th>$h(-12q)$</th>
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<td>11</td>
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<td>3</td>
<td>1</td>
<td>12</td>
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</table>

### $p > 3, q = 3$:

<table>
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<th>$p(\text{mod } 16)$</th>
<th>$e_p$</th>
<th>$k$</th>
<th>$l$</th>
<th>$m$</th>
<th>$x$</th>
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<th>$X$</th>
<th>$h(-12p)$</th>
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<td>36</td>
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</tbody>
</table>

10. $p \equiv 3 \pmod 8, \; q \equiv 7 \pmod 8, \; (p/q) = 1$

In this case (1.1) gives

\[ h(-8pq) + h(-4pq) \equiv 0 \pmod{16} \] (10.1)
(cf. [13, Proposition 5, Eq. (27)]). We remark that, when \( p = 3 \), (10.1) is equivalent to the last two congruences of Corollary 11.6 of [2].

As \( p \equiv 3 \pmod{8} \) and \( (q/p) = -1 \) there exist integers \( x, y, k, l, \) and \( m \) such that

\[ 2q = k^2x^2 + 2lx + 2my^2, \quad p = l^2 - 2k^2m, \tag{10.2} \]

[10, p. 356]. Moreover, as \( q \equiv 7 \pmod{8} \), by [10, Proposition B''], we have

\[ h(-8pq) \equiv 0 \pmod{16} \quad \Leftrightarrow \quad \left( \frac{|k^2x + ly^2|}{q} \right) = 1. \tag{10.3} \]

From (10.1) and (10.3) we obtain

**THEOREM 8.** Let \( p \) and \( q \) be primes satisfying

\[ p \equiv 3 \pmod{8}, \quad q \equiv 7 \pmod{8}, \quad \left( \frac{p}{q} \right) = 1. \]

Then, with \( x, y, l, m \) as defined in (10.2), we have

\[ h(-4pq) \equiv \begin{cases} 0 \pmod{16} & \text{if} \quad \left( \frac{|k^2x + ly^2|}{q} \right) = 1, \\ 8 \pmod{16} & \text{if} \quad \left( \frac{|k^2x + ly^2|}{q} \right) = -1. \end{cases} \]

The following table illustrates Theorem 8. We write \( Z = (|k^2x + ly^2|/q) \).

<table>
<thead>
<tr>
<th>( p )</th>
<th>( q )</th>
<th>( k )</th>
<th>( l )</th>
<th>( m )</th>
<th>( x )</th>
<th>( y )</th>
<th>( h(-4pq) )</th>
<th>( Z )</th>
</tr>
</thead>
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</tbody>
</table>

11. \( p \equiv 3, q \equiv 7 \pmod{8}, \left( \frac{p}{q} \right) = -1 \)

In this case, from [10, Cas 5a, p. 354; Cas 7a, p. 356] (see also [3]), we have \( h(-4pq) \equiv h(-8pq) \equiv 4 \pmod{8} \), so that \( h(-4pq) + h(-8pq) \equiv 0 \pmod{8} \). However, \( h(-4pq) \) and \( h(-8pq) \) are not known individually modulo 16, so that (1.1) in this case just gives
Theorem 9. [13, Proposition 5, Eq. (28)]. Let $p$ and $q$ be primes satisfying

$$p \equiv 3 \pmod{8}, \quad q \equiv 7 \pmod{8}, \quad \left(\frac{p}{q}\right) = -1.$$

Then

$$h(-8pq) + h(-4pq) = p + q - 2 \pmod{16}.$$

References

12. L. J. Mordell, The congruence $(p - 1/2)! \equiv \pm 1 \pmod{p}$, Amer. Math. Monthly 68 (1961), 145–146.