A CLASS OF CHARACTER SUMS

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1. *Introduction*

Let p denote an odd prime, χ a multiplicative character modulo p,

$$e(t) = \exp\left(\frac{2\pi i t}{p}\right),$$

(t real) and $r_1(x)$, $r_2(x)$ rational functions of x with integral coefficients. The character sum $\sum_{x=0}^{p-1}{}^{*}\chi(r_1(x))\,e(r_2(x))$ has been estimated by Perel'muter [3]. (The asterisk (*) means that the singularities (mod p) of r_1 and r_2 are excluded and in the sum $1/v$ ($v \not\equiv 0 \pmod{p}$) is to be interpreted as the unique integer $w \pmod{p}$ such that $vw \equiv 1 \pmod{p}$). Perel'muter has given conditions under which this sum is $O(p^{\frac{1}{2}})$, thus generalizing the earlier deep work of Weil [4] and Carlitz and Uchiyama [2]. It is the purpose of this note to show that Perel'muter's work can be applied to estimate the character sum

$$S_p(k, r_1, r_2, \chi) = \sum_{x=0}^{p-1}{}^{*}\, x^k \chi(r_1(x))\, e(r_2(x)), \tag{1.1}$$

where $k = 1, 2, 3, \ldots$. To do this we introduce

$$S_p(r_1, r_2, \chi) = \sum_{x=1}^{p-1}{}^{*}\, \chi(r_1(x))\, e(r_2(x)), \tag{1.2}$$

and for any integer m we let $\bar{m}$ denote the function defined by $\bar{m}(x) = mx$. Then we set

$$\Phi_p = \Phi_p(r_1, r_2, \chi) = \max_m |S_p(r_1, r_2 + \bar{m}, \chi)| \tag{1.3}$$

and

$$L_p(k, r_1, r_2, \chi) = \sum_{m=-\infty}^{+\infty}{}' \frac{S_p(r_1, r_2 + \bar{m}, \chi)}{m^k}, \quad k = 1, 2, 3, \ldots, \tag{1.4}$$

where the maximum in (1.3) is taken over all integers m (or in view of the periodicity in m of $S_p(r_1, r_2 + \bar{m}, \chi)$ over those satisfying $0 \leqslant m \leqslant p-1$), the sum $\sum_{m=-\infty}^{+\infty}$ is taken in the narrow sense, that is as $\lim_{t \to +\infty} \sum_{m=-t}^{+t}$ and the dash ($'$) means that the term $m = 0$ is omitted. If r_1, $r_2 + \bar{m}$ ($m = 0, 1, \ldots, p-1$) satisfy the conditions given by Perel'muter [3] we know that $\Phi_p = O(p^{\frac{1}{2}})$. Trivially $\Phi_p < p$ so that $L_p(k, r_1, r_2, \chi)$

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is well-defined for $k \geqslant 2$ and in this case the sum can be taken in the wide sense, that is, as $\lim_{t_1, t_2 \to \infty} \sum_{m=-t_2}^{+t_1}$. However when $k = 1$ it is not obvious that the sum exists, that it does is shown in Lemma 1. Lemmas 2, 3, 4, 5 are devoted to estimating $L_p(k, r_1, r_2, \chi)$, $k \geqslant 1$. Lemma 6 gives the relationship between $S_p(k, r_1, r_2, \chi)$ and $S_p(r_1, r_2, \chi)$ and $L_p(n, r_1, r_2, \chi)$ $(n = 1, 2, \ldots, k)$ from which the final estimate $S_p(k, r_1, r_2, \chi) = O(\Phi_p p^k \log p)$ can be deduced.

2. *Existence of* $L_p(1, r_1, r_2, \chi)$

We prove

LEMMA 1. *$L_p(1, r_1, r_2, \chi)$ exists.*

Proof. For any positive integer t we have

$$\begin{aligned}\sum_{m=-t}^{+t}{}' \frac{S_p(r_1, r_2+\bar{m}, \chi)}{m} &= \sum_{m=1}^{t} \frac{S_p(r_1, r_2+\bar{m}, \chi)}{m} - \sum_{m=1}^{t} \frac{S_p(r_1, r_2-\bar{m}, \chi)}{m} \\ &= 2i \sum_{m=1}^{t} \frac{1}{m} \left\{ \sum_{x=1}^{p-1}{}^{*} \chi(r_1(x))\, e(r_2(x)) \sin \frac{2\pi m x}{p} \right\} \\ &= 2i \sum_{x=1}^{p-1}{}^{*} \chi(r_1(x))\, e(r_2(x)) \sum_{m=1}^{t} \frac{\sin(2\pi m x/p)}{m}.\end{aligned}$$

Now for $x = 1, 2, \ldots, p-1$ we have

$$\lim_{t \to +\infty} \sum_{m=1}^{t} \frac{\sin(2\pi m x/p)}{m} = \sum_{m=1}^{\infty} \frac{\sin(2\pi m x/p)}{m} = \frac{\pi}{2} - \frac{\pi x}{p}, \tag{2.1}$$

so that

$$\sum_{m=-\infty}^{+\infty}{}' \frac{S_p(r_1, r_2+\bar{m}, \chi)}{m} = \frac{\pi i}{p} \sum_{x=1}^{p-1}{}^{*} (p-2x)\, \chi(r_1(x))\, e(r_2(x)), \tag{2.2}$$

which proves the result.

3. *Estimation of* $L_p(k, r_1, r_2, \chi)$, $k \geqslant 1$

We require a number of lemmas.

LEMMA 2. *For $x = 0, 1, 2, \ldots, p-1$ we have*

$$x = \frac{p-1}{2} + \sum_{m=1}^{p-1} \frac{e(mx)}{e(-m)-1}.$$

Proof. The polynomial $(z+i)^p-(z-i)^p$ is of degree $p-1$, its coefficient of z^{p-2} is 0 and it has the $p-1$ roots $\cot \frac{m\pi}{p}$ $(m = 1, 2, \ldots, p-1)$. Hence $\sum_{m=1}^{p-1} \cot \frac{m\pi}{p} = 0$, and as $\cot \dfrac{m\pi}{p} = -2i\left\{\tfrac{1}{2}+\dfrac{1}{e(-m)-1}\right\}$, we have

$$\frac{p-1}{2}+\sum_{m=1}^{p-1}\frac{1}{e(-m)-1}=0,$$

which proves the result for $x = 0$. For $x = 1, 2, \ldots, p-1$

$$\sum_{m=1}^{p-1}\frac{e(mx)-e(m(x-1))}{e(-m)-1}=-\sum_{m=1}^{p-1}e(mx)=1,$$

so that

$$\begin{aligned}\sum_{m=1}^{p-1}\frac{e(mx)}{e(-m)-1}&=1+\sum_{m=1}^{p-1}\frac{e(m(x-1))}{e(-m)-1}\\&=2+\sum_{m=1}^{p-1}\frac{e(m(x-2))}{e(-m)-1}\\&=\ldots\\&=x+\sum_{m=1}^{p-1}\frac{1}{e(-m)-1}\\&=x-\frac{p-1}{2},\end{aligned}$$

as required.

LEMMA 3. $|S_p(1, r_1, r_2, \chi)| \leqslant \Phi_p p \log p$.

Proof. We have by Lemma 2

$$\begin{aligned}S_p(1, r_1, r_2, \chi)&=\sum_{x=1}^{p-1}{}^{*}\, x\chi(r_1(x))\, e(r_2(x))\\&=\sum_{x=1}^{p-1}{}^{*}\left\{\frac{p-1}{2}+\sum_{m=1}^{p-1}\frac{e(mx)}{e(-m)-1}\right\}\chi(r_1(x))\, e(r_2(x))\\&=\frac{p-1}{2}S_p(r_1, r_2, \chi)+\sum_{m=1}^{p-1}\frac{1}{e(-m)-1}S_p(r_1, r_2+\overline{m}, \chi)\end{aligned}$$

so that

$$|S_p(1, r_1, r_2, \chi)| \leqslant \Phi_p\left\{\frac{p-1}{2}+\sum_{m=1}^{p-1}\frac{1}{|e(-m)-1|}\right\}.$$

Now

$$\sum_{m=1}^{p-1} \frac{1}{|e(-m)-1|} = \tfrac{1}{2}\sum_{m=1}^{p-1} \frac{1}{\sin(m\pi/p)}$$

$$= \sum_{m=1}^{\frac{1}{2}(p-1)} \frac{1}{\sin(m\pi/p)}$$

$$< \frac{p}{2} \sum_{m=1}^{\frac{1}{2}(p-1)} \frac{1}{m}$$

$$< \frac{p}{2} \log p,$$

giving

$$|S_p(1, r_1, r_2, \chi)| \leqslant \Phi_p \, p \log p.$$

LEMMA 4. *$L_p(1, r_1, r_2, \chi) = O(\Phi_p \log p)$, where the constant implied by the O-symbol is absolute.*

Proof. From the proof of Lemma 1 (2.2)

$$L_p(1, r_1, r_2, \chi) = \pi i S_p(r_1, r_2, \chi) - \frac{2\pi i}{p} S_p(1, r_1, r_2, \chi),$$

so that by Lemma 3

$$|L_p(1, r_1, r_2, \chi)| \leqslant \pi\Phi_p + 2\pi\Phi_p \log p$$

$$= O(\Phi_p \log p).$$

LEMMA 5. *For $n \geqslant 2$, $L_p(n, r_1, r_2, \chi) = O(\Phi_p)$, where the constant implied by the O-symbol is absolute.*

Proof. This is clear as

$$|L_p(n, r_1, r_2, \chi)| \leqslant \Phi_p \sum_{m=-\infty}^{+\infty}{}' \frac{1}{m^n} = O(\Phi_p).$$

4. *Estimation of $S_p(k, r_1, r_2, \chi)$.*

We begin by relating $S_p(k, r_1, r_2, \chi)$ to $S_p(r_1, r_2, \chi)$ and $L_p(n, r_1, r_2, \chi)$ $(n = 1, 2, \ldots, k)$,

(Lemma 6). This lemma was suggested by [1]. (Note that the 2 appearing in the expression $\frac{p^3\sqrt{p}}{2\pi}$ (last line, p. 153) should be omitted).

LEMMA 6. *For* $k \geqslant 1$

$$S_p(k, r_1, r_2, \chi) = \frac{p^k}{k+1} S_p(r_1, r_2, \chi) - p^k \sum_{n=1}^{k} \frac{k(k-1)\dots(k-(n-2))}{(2\pi i)^n} L_p(n, r_1, r_2, \chi).$$

Proof. By Jordan's test x^k has a convergent Fourier series for $0 < x < p$, say

$$x^k = \sum_{m=-\infty}^{+\infty} w_m e(mx),$$

where the sum is taken in the narrow sense and the coefficients are given by

$$w_m = \frac{1}{p} \int_0^p x^k e(-mx)\, dx \quad (m = 0, \pm 1, \pm 2, \dots).$$

Taking $m = 0$ we have

$$w_0 = \frac{p^k}{k+1}$$

and for $m \neq 0$ we have

$$w_m = -p^k \sum_{r=1}^{k} \frac{k(k-1)\dots(k-(r-2))}{(2\pi i m)^r}.$$

Thus

$$-p^k \sum_{r=1}^{k} \frac{k(k-1)\dots(k-(r-2))}{(2\pi i)^r} L_p(r, r_1, r_2, \chi)$$

$$= -p^k \sum_{r=1}^{k} \frac{k(k-1)\dots(k-(r-2))}{(2\pi i)^r} \sum_{m=-\infty}^{+\infty}{}' \frac{S_p(r_1, r_2+\overline{m}, \chi)}{m^r}$$

$$= \sum_{m=-\infty}^{+\infty}{}' w_m S_p(r_1, r_2+\overline{m}, \chi)$$

$$= \sum_{m=-\infty}^{+\infty}{}' w_m \sum_{x=1}^{p-1}{}^{*} \chi(r_1(x))\, e(r_2(x)+mx)$$

$$= \sum_{x=1}^{p-1}{}^{*} \chi(r_1(x))\, e(r_2(x)) \sum_{m=-\infty}^{+\infty}{}' w_m e(mx)$$

$$= \sum_{x=1}^{p-1}{}^{*} \chi(r_1(x))\, e(r_2(x)) \{x^k - w_0\}$$

$$= S_p(k, r_1, r_2, \chi) - \frac{p^k}{k+1} S_p(r_1, r_2, \chi),$$

as required.

THEOREM 1. *$S_p(k, r_1, r_2, \chi) = O(\Phi_p p^k \log p)$, where the implied constant depends only on k.*

Proof. This is immediate from Lemmas 4, 5, 6.

5. *An example*

We finish by giving one simple application of Theorem 1.

THEOREM 2.

$$\sum_{x=1}^{p-1} x^k \left(\frac{x}{p}\right) = O(p^{k+\frac{1}{2}} \log p),$$

$k = 1, 2, \ldots$.

Proof. Here $r_1(x) = x, r_2(x) = 0, \chi(x) = \left(\frac{x}{p}\right)$ so that

$$S_p(r_1, r_2+\bar{m}, \chi) = \sum_{x=1}^{p-1} \left(\frac{x}{p}\right) e(mx) = \left(\frac{m}{p}\right) i^{\frac{1}{4}(p-1)^2} p^{\frac{1}{2}},$$

giving

$$\Phi_p = \max_m \left| \left(\frac{m}{p}\right) i^{\frac{1}{4}(p-1)^2} p^{\frac{1}{2}} \right| = p^{\frac{1}{2}},$$

as required.

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