A CLASS OF CHARACTER SUMS

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1. Introduction

Let $p$ denote an odd prime, $\chi$ a multiplicative character modulo $p$,

$$e(t) = \exp\left(\frac{2\pi it}{p}\right),$$

($t$ real) and $r_1(x), r_2(x)$ rational functions of $x$ with integral coefficients. The character sum $\sum_{x=0}^{p-1} \chi(r_1(x)) e(r_2(x))$ has been estimated by Perel'muter [3]. (The asterisk (*) means that the singularities (mod $p$) of $r_1$ and $r_2$ are excluded and in the sum $1/v(v \neq 0 \mod p)$ is to be interpreted as the unique integer $w \mod p$ such that $vw \equiv 1 \mod p$). Perel'muter has given conditions under which this sum is $O(p^\lambda)$, thus generalizing the earlier deep work of Weil [4] and Carlitz and Uchiyama [2]. It is the purpose of this note to show that Perel'muter's work can be applied to estimate the character sum

$$S_p(k, r_1, r_2, \chi) = \sum_{x=0}^{p-1} x^k \chi(r_1(x)) e(r_2(x)), \quad (1.1)$$

where $k = 1, 2, 3, \ldots$. To do this we introduce

$$S_p(r_1, r_2, \chi) = \sum_{x=1}^{p-1} \chi(r_1(x)) e(r_2(x)), \quad (1.2)$$

and for any integer $m$ we let $\overline{m}$ denote the function defined by $\overline{m}(x) = mx$. Then we set

$$\Phi_p = \Phi_p(r_1, r_2, \chi) = \max_m |S_p(r_1, r_2 + \overline{m}, \chi)| \quad (1.3)$$

and

$$L_p(k, r_1, r_2, \chi) = \sum_{m=-\infty}^{+\infty} \frac{S_p(r_1, r_2 + \overline{m}, \chi)}{m^k}, \quad k = 1, 2, 3, \ldots, \quad (1.4)$$

where the maximum in (1.3) is taken over all integers $m$ (or in view of the periodicity in $m$ of $S_p(r_1, r_2 + \overline{m}, \chi)$ over those satisfying $0 \leq m \leq p-1$), the sum is taken in the narrow sense, that is as $\lim_{t \to +\infty} \sum_{m=-t}^{+t}$ and the dash (') means that the term $m = 0$ is omitted. If $r_1, r_2 + \overline{m}$ ($m = 0, 1, \ldots, p-1$) satisfy the conditions given by Perel'muter [3] we know that $\Phi_p = O(p^\lambda)$. Trivially $\Phi_p < p$ so that $L_p(k, r_1, r_2, \chi)$

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is well-defined for \( k \geq 2 \) and in this case the sum can be taken in the wide sense, that is, as \( \lim_{t_1, t_2 \to \infty} \sum_{m = -t}^{+t} \). However when \( k = 1 \) it is not obvious that the sum exists, that it does is shown in Lemma 1. Lemmas 2, 3, 4, 5 are devoted to estimating \( L_p(k, r_1, r_2, \chi) \), \( k \geq 1 \). Lemma 6 gives the relationship between \( S_p(k, r_1, r_2, \chi) \) and \( L_p(n, r_1, r_2, \chi) \) \( (n = 1, 2, \ldots, k) \) from which the final estimate \( S_p(k, r_1, r_2, \chi) = O(\Phi_p p^k \log p) \) can be deduced.

2. Existence of \( L_p(1, r_1, r_2, \chi) \)

We prove

**Lemma 1.** \( L_p(1, r_1, r_2, \chi) \) exists.

**Proof.** For any positive integer \( t \) we have

\[
\sum_{m = -t}^{+t} \frac{S_p(r_1, r_2 + \bar{m}, \chi)}{m} = \sum_{m = 1}^{t} \frac{S_p(r_1, r_2 + \bar{m}, \chi)}{m} - \sum_{m = 1}^{t} \frac{S_p(r_1, r_2 - \bar{m}, \chi)}{m} = 2i \sum_{m = 1}^{t} \frac{1}{m} \left( \sum_{x = 1}^{p-1} \chi(r_1(x)) e(r_2(x)) \sin \frac{2\pi mx}{p} \right)
\]

\[
= 2i \sum_{x = 1}^{p-1} \chi(r_1(x)) e(r_2(x)) \sum_{m = 1}^{t} \sin \frac{2\pi mx}{p}.
\]

Now for \( x = 1, 2, \ldots, p-1 \) we have

\[
\lim_{t \to \infty} \sum_{m = 1}^{t} \frac{\sin (2\pi mx/p)}{m} = \sum_{m = 1}^{\infty} \frac{\sin (2\pi mx/p)}{m} = \frac{\pi}{2} - \frac{\pi x}{p}, \tag{2.1}
\]

so that

\[
\sum_{m = -\infty}^{+\infty} \frac{S_p(r_1, r_2 + \bar{m}, \chi)}{m} = \frac{\pi i}{p} \sum_{x = 1}^{p-1} (p - 2x) \chi(r_1(x)) e(r_2(x)), \tag{2.2}
\]

which proves the result.

3. Estimation of \( L_p(k, r_1, r_2, \chi) \), \( k \geq 1 \)

We require a number of lemmas.

**Lemma 2.** For \( x = 0, 1, 2, \ldots, p-1 \) we have

\[
x = \frac{p-1}{2} + \sum_{m = 1}^{p-1} \frac{e(mx)}{e(-m) - 1}.
\]

**Proof.** The polynomial \((z+i)^p - (z-i)^p\) is of degree \( p-1 \), its coefficient of \( z^{p-2} \) is 0 and it has the \( p-1 \) roots \( \cot \frac{m\pi}{p} \) \( (m = 1, 2, \ldots, p-1) \). Hence \( \sum_{m = 1}^{p-1} \cot \frac{m\pi}{p} = 0 \),
and as \( \cot \frac{m \pi}{p} = -2i \left( \frac{1}{2} + \frac{1}{e(-m)-1} \right) \), we have

\[
\frac{p-1}{2} + \sum_{m=1}^{p-1} \frac{1}{e(-m)-1} = 0,
\]

which proves the result for \( x = 0 \). For \( x = 1, 2, \ldots, p-1 \)

\[
\sum_{m=1}^{p-1} \frac{e(mx) - e(m(x-1))}{e(-m)-1} = -\sum_{m=1}^{p-1} e(mx) = 1,
\]

so that

\[
\sum_{m=1}^{p-1} \frac{e(mx)}{e(-m)-1} = 1 + \sum_{m=1}^{p-1} \frac{e(m(x-1))}{e(-m)-1}
\]

\[
= 2 + \sum_{m=1}^{p-1} \frac{e(m(x-2))}{e(-m)-1}
\]

\[
= \ldots
\]

\[
= x + \sum_{m=1}^{p-1} \frac{1}{e(-m)-1}
\]

\[
= x - \frac{p-1}{2},
\]

as required.

**Lemma 3.** \( |S_p(1, r_1, r_2, \chi)| \leq \Phi_p \, p \log p \).

**Proof.** We have by Lemma 2

\[
S_p(1, r_1, r_2, \chi) = \sum_{x=1}^{p-1} x\chi(r_1(x))e(r_2(x))
\]

\[
= \sum_{x=1}^{p-1} \left( \frac{p-1}{2} + \sum_{m=1}^{p-1} \frac{e(mx)}{e(-m)-1} \right) \chi(r_1(x))e(r_2(x))
\]

\[
= \frac{p-1}{2} S_p(r_1, r_2, \chi) + \sum_{m=1}^{p-1} \frac{1}{e(-m)-1} S_p(r_1, r_2 + \overline{m}, \chi)
\]

so that

\[
|S_p(1, r_1, r_2, \chi)| \leq \Phi_p \left( \frac{p-1}{2} + \sum_{m=1}^{p-1} \frac{1}{|e(-m)-1|} \right).
\]
Now
\[\sum_{m=1}^{p-1} \frac{1}{|e(-m)-1|} = \frac{1}{2} \sum_{m=1}^{p-1} \frac{1}{\sin (m\pi/p)} = \sum_{m=1}^{\frac{p(p-1)}{2}} \frac{1}{\sin (m\pi/p)} < \frac{p}{2} \sum_{m=1}^{\frac{p(p-1)}{2}} \frac{1}{m} < \frac{p}{2} \log p,\]
giving
\[|S_p(1, r_1, r_2, \chi)| \leq \Phi_p p \log p.\]

**Lemma 4.** $L_p(1, r_1, r_2, \chi) = O(\Phi_p \log p)$, where the constant implied by the $O$-symbol is absolute.

**Proof.** From the proof of Lemma 1 (2.2)
\[L_p(1, r_1, r_2, \chi) = \pi i S_p(r_1, r_2, \chi) - \frac{2\pi i}{p} S_p(1, r_1, r_2, \chi),\]
so that by Lemma 3
\[|L_p(1, r_1, r_2, \chi)| \leq \pi \Phi_p + 2\pi \Phi_p \log p = O(\Phi_p \log p).\]

**Lemma 5.** For $n \geq 2$, $L_p(n, r_1, r_2, \chi) = O(\Phi_p)$, where the constant implied by the $O$-symbol is absolute.

**Proof.** This is clear as
\[|L_p(n, r_1, r_2, \chi)| \leq \Phi_p \sum_{m=-\infty}^{+\infty} \frac{1}{m^n} = O(\Phi_p).\]

4. Estimation of $S_p(k, r_1, r_2, \chi)$.

We begin by relating $S_p(k, r_1, r_2, \chi)$ to $S_p(r_1, r_2, \chi)$ and $L_p(n, r_1, r_2, \chi)$ ($n = 1, 2, ..., k$),

(Lemma 6). This lemma was suggested by [1]. (Note that the 2 appearing in the expression $\frac{p^3}{2\pi}$ (last line, p. 153) should be omitted).
Lemma 6. For $k \geq 1$

$$S_p(k, r_1, r_2, \chi) = \frac{p^k}{k+1} S_p(r_1, r_2, \chi) - p^k \sum_{n=1}^{k} \frac{k(k-1) \ldots (k-(n-2))}{(2\pi i)^n} L_p(n, r_1, r_2, \chi).$$

Proof. By Jordan's test $x^k$ has a convergent Fourier series for $0 < x < p$, say

$$x^k = \sum_{m=-\infty}^{+\infty} w_m e(mx),$$

where the sum is taken in the narrow sense and the coefficients are given by

$$w_m = \frac{1}{p} \int_0^p x^k e(-mx) \, dx \quad (m = 0, \pm 1, \pm 2, \ldots).$$

Taking $m = 0$ we have

$$w_0 = \frac{p^k}{k+1}$$

and for $m \neq 0$ we have

$$w_m = -p^k \sum_{r=1}^{k} \frac{k(k-1) \ldots (k-(r-2))}{(2\pi i m)^r}.$$

Thus

$$-p^k \sum_{r=1}^{k} \frac{k(k-1) \ldots (k-(r-2))}{(2\pi i)^r} L_p(r, r_1, r_2, \chi) = -p^k \sum_{r=1}^{k} \frac{k(k-1) \ldots (k-(r-2))}{(2\pi i)^r} \sum_{m=-\infty}^{+\infty} \frac{S_p(r_1, r_2 + \overline{m}, \chi)}{m^r}.$$

$$= \sum_{m=-\infty}^{+\infty} w_m S_p(r_1, r_2 + \overline{m}, \chi)$$

$$= \sum_{m=-\infty}^{+\infty} w_m \sum_{x=1}^{p-1} \chi(r_1(x)) e(r_2(x) + mx)$$

$$= \sum_{x=1}^{p-1} \chi(r_1(x)) e(r_2(x)) \sum_{m=-\infty}^{+\infty} w_m e(mx)$$

$$= \sum_{x=1}^{p-1} \chi(r_1(x)) e(r_2(x))(x^k - w_0)$$

$$= S_p(k, r_1, r_2, \chi) - \frac{p^k}{k+1} S_p(r_1, r_2, \chi),$$

as required.
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THEOREM 1. $S_p(k, r_1, r_2, \chi) = O(\Phi_p p^k \log p)$, where the implied constant depends only on $k$.

Proof. This is immediate from Lemmas 4, 5, 6.

5. An example

We finish by giving one simple application of Theorem 1.

THEOREM 2.

$$\sum_{x=1}^{p-1} x^k \left( \frac{x}{p} \right) = O(p^{k+\frac{1}{2}} \log p),$$

$k = 1, 2, \ldots$.

Proof. Here $r_1(x) = x$, $r_2(x) = 0$, $\chi(x) = \left( \frac{x}{p} \right)$ so that

$$S_p(r_1, r_2 + \bar{m}, \chi) = \sum_{x=1}^{p-1} \left( \frac{x}{p} \right) e(mx) = \left( \frac{m}{p} \right) i^{k(p-1)2} p^k,$$

giving

$$\Phi_p = \max_m \left| \left( \frac{m}{p} \right) i^{k(p-1)2} p^k \right| = p^k,$$

as required.

References


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