Chapter 9, Question 8

8. Let K be an algebraic number field and O_K its ring of integers. Let I be an integral ideal of O_K such that $N(I) = |N(a)|$ for some $a \in I$. Prove that $I =< a >$.

Solution. If $a = 0$, then $N(I) = 0$ so $I =< 0 >=< a >$. Thus we may suppose that $a \neq 0$. Hence $N(I) \neq 0$. As $a \in I$, we have $< a >\subseteq I$ so that $I \,|< a >$. Thus $< a >= IJ$ for some integral ideal J of O_K. Hence

$$N(I) = |N(a)| = N(< a >) = N(IJ) = N(I)N(J)$$

so that as $N(I) \neq 0$ we have

$$N(J) = 1$$

and thus

$$J =< 1 > .$$

Hence

$$I = I < 1 >= IJ =< a > . \qquad \blacksquare$$

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