Chapter 9, Question 7

7. Let $K = \mathbb{Q}(\sqrt{-23})$. Let $I =< 2, \frac{1}{2}(1+\sqrt{-23}) >$.

(a) Prove that $N(I) = 2$.

(b) Prove that $I^3 =< \frac{-3+\sqrt{-23}}{2} >$.

(c) Use (a) and (b) to prove that I is not a principal ideal.

Solution. (a) Let

$$I =< 2, \frac{1}{2}(1+\sqrt{-23}) >, \ J =< 2, \frac{1}{2}(1-\sqrt{-23}) > .$$

Then

$$\begin{aligned} IJ &=< 2, \frac{1}{2}(1+\sqrt{-23}) >< 2, \frac{1}{2}(1-\sqrt{-23}) > \\ &=< 4, 1+\sqrt{-23}, 1-\sqrt{-23}, 6 > \\ &=< 2 >< 2, \frac{1+\sqrt{-23}}{2}, \frac{1-\sqrt{-23}}{2}, 3 > . \end{aligned}$$

Now

$$1 = 3 - 2 \in< 2, \frac{1+\sqrt{-23}}{2}, \frac{1-\sqrt{-23}}{2}, 3 >$$

so that

$$< 2, \frac{1+\sqrt{-23}}{2}, \frac{1-\sqrt{-23}}{2}, 3 >=< 1 > .$$

Thus

$$< 2 >= IJ.$$

Hence, by Theorems 9.3.2 and 9.2.5, we have

$$N(I)N(J) = N(IJ) = N(< 2 >) = |N(2)| = 2^2$$

so that

$$N(I) = 1, , 2 \text{ or } 2^2.$$

If $N(I) = 1$ then $I =< 1 >$. Hence there exist $\alpha, \beta \in O_K$ such that

$$2\alpha + \frac{1}{2}(1+\sqrt{-23})\beta = 1.$$

As $O_K = \mathbb{Z} + \mathbb{Z}(\frac{1+\sqrt{-23}}{2})$ (Theorem 5.4.2), there exist $a, b, c, d \in \mathbb{Z}$ such that

$$\alpha = a + b\left(\frac{1+\sqrt{-23}}{2}\right), \ \beta = c + d\left(\frac{1+\sqrt{-23}}{2}\right).$$

Hence

$$2\left(a + b\left(\frac{1+\sqrt{-23}}{2}\right)\right) + \frac{1}{2}\left(1+\sqrt{-23}\right)\left(c + d\left(\frac{1+\sqrt{-23}}{2}\right)\right) = 1.$$

Equating real and imaginary parts, we obtain

$$8a + 4b + 2c - 22d = 4$$

and

$$2b + c + d = 0.$$

Eliminating d, we obtain

$$8a + 48b + 24c = 4,$$

which is clearly impossible as the left hand side is divisible by 8 and the right hand side is not. Hence $N(I) \neq 1$.

If $N(I) = 2^2$ then $N(J) = 1$ and by exactly the same kind of argument, we get a contradiction. Hence $N(I) \neq 2^2$.

This proves that $N(I) = 2$.

(b) We have

$$\begin{aligned} I^2 &=< 2, \frac{1}{2}(1+\sqrt{-23}) >^2 =< 4, 1+\sqrt{-23}, \left(\frac{1+\sqrt{-23}}{2}\right)^2 > \\ &=< 4, 1+\sqrt{-23}, \frac{-11+\sqrt{-23}}{2} > \\ &=< 4, \frac{-11+\sqrt{-23}}{2} >, \end{aligned}$$

as

$$1+\sqrt{-23} = 3\cdot 4 + 2\left(\frac{-11+\sqrt{-23}}{2}\right).$$

Then

$$\begin{aligned}
I^3 = I^2 I &=< 4, \frac{-11+\sqrt{-23}}{2} >< 2, \frac{1+\sqrt{-23}}{2} > \\
&=< 8, -11+\sqrt{-23}, 2+2\sqrt{-23}, \frac{-17-5\sqrt{-23}}{2} > \\
&=< 8, -11+\sqrt{-23}, \frac{-17-5\sqrt{-23}}{2} > \\
&\quad (\text{as } 2+2\sqrt{-23} = 3\cdot 8 + 2(-11+\sqrt{-23})) \\
&=< 8, -11+\sqrt{-23}, \frac{-17-5\sqrt{-23}}{2} + 3(-11+\sqrt{-23}) > \\
&=< 8, -11+\sqrt{-23}, \frac{-83+\sqrt{-23}}{2} > \\
&=< 8, 8+(-11+\sqrt{-23}), 5\cdot 8 + \left(\frac{-83+\sqrt{-23}}{2}\right) > \\
&=< 8, -3+\sqrt{-23}, \frac{-3+\sqrt{-23}}{2} > \\
&=< 8, \frac{-3+\sqrt{-23}}{2} > \\
&=< \frac{-3+\sqrt{-23}}{2} >
\end{aligned}$$

as

$$8 = \left(\frac{-3-\sqrt{-23}}{2}\right)\left(\frac{-3+\sqrt{-23}}{2}\right).$$

(c) Suppose I is a principal ideal, say

$$I =< \frac{a+b\sqrt{-23}}{2} >,$$

where $a, b \in \mathbb{Z}$ are such that $a \equiv b \pmod 2$.

From (a) we have

$$2 = N(I) = N\left(< \frac{a+b\sqrt{-23}}{2} >\right) = \left|N\left(< \frac{a+b\sqrt{-23}}{2} >\right)\right| = \frac{a^2+23b^2}{4},$$

so that

$$a^2 + 23b^2 = 8.$$

This equation has no solutions in integers a and b, a contradiction. Thus I is not principal.

From (b) we have

$$< \frac{-3+\sqrt{-23}}{2} >= I^3 =< \frac{a+b\sqrt{-23}}{2} >^3=< \left(\frac{a+b\sqrt{-23}}{2}\right)^3 >$$

so that

$$\frac{-3+\sqrt{-23}}{2} = \pm\left(\frac{a+b\sqrt{-23}}{2}\right)^3$$

as the only units in $\mathbb{Z} + \mathbb{Z}(\frac{1+\sqrt{-23}}{2})$ are ± 1. Replacing (a, b) by $(-a, -b)$, if necessary, we may suppose that the $+$ sign holds. Hence

$$-12 + 4\sqrt{-23} = (a + b\sqrt{-23})^3 = (a^3 - 69ab^2) + (3a^2b - 23b^3)\sqrt{-23}.$$

Thus

$$a^3 - 69ab^2 = -12, \qquad (1)$$

$$3a^2b - 23b^3 = 4. \qquad (2)$$

From (2) we deduce that $b \mid 4$ so that $b = \pm 1, \pm 2, \pm 4$. If $b = \pm 1$ then (2) gives

$$3a^2 - 23 = \pm 4,$$

so that

$$a = \pm 3, \; b = \pm 1,$$

but these values do not satisfy (1). If $b = \pm 2$ then (2) gives

$$3a^2 - 92 = \pm 2,$$

which has no integral solutions. If $b = \pm 4$ then (2) gives

$$3a^2 - 368 = \pm 1,$$

which has no integral solutions. Hence I is not principal. ∎