Chapter 7, Question 12

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12. Prove from first principles that $K = \mathbb{Q}(\theta)$, $\theta^3 + 30\theta + 90 = 0$, is a pure cubic field, and express K in the form $K = \mathbb{Q}(m^{1/3})$ for some cubefree integer m.

Solution. We use Cardan's method of solving a cubic equation. If $a, b, c \in \mathbb{C}$ are such that

$$a^3 + b^3 = -90, \ ab = -10, \tag{1}$$

then

$$(a+b)^3 + 30(a+b) + 90 = (a^3 + b^3 + 90) + 3(ab + 10)(a+b) = 0,$$

showing that $x = a + b$ is a root of $x^3 + 30x + 90 = 0$. We now find a and b satisfying (1). From (1) we see that

$$a^3 + b^3 = -90, \ a^3 b^3 = -100,$$

so that a^3, b^3 are the roots of the quadratic equation

$$t^2 + 90t - 1000 = 0.$$

Solving the quadratic equation, we obtain without loss of generality

$$a^3 = \frac{-90 + \sqrt{12100}}{2} = 10$$

and

$$b^3 = \frac{-90 - \sqrt{12100}}{2} = -100,$$

so that

$$a = \sqrt[3]{10}, \ b = -(\sqrt[3]{10})^2.$$

Thus

$$\theta = \sqrt[3]{10} - (\sqrt[3]{10})^2$$

is a root of $\theta^3 + 30\theta + 90 = 0$. Hence

$$K = \mathbb{Q}(\theta) = \mathbb{Q}(\sqrt[3]{10} - (\sqrt[3]{10})^2) \subseteq \mathbb{Q}(\sqrt[3]{10}).$$

Now

$$\theta^2 = \left(\sqrt[3]{10} - (\sqrt[3]{10})^2\right)^2 = -20 + 10\sqrt[3]{10} + \left(\sqrt[3]{10}\right)^2$$

so that

$$20 + \theta + \theta^2 = 20 + (\sqrt[3]{10} - (\sqrt[3]{10})^2) + (-20 + 10\sqrt[3]{10} + (\sqrt[3]{10})^2) = 11\sqrt[3]{10}$$

and thus

$$\sqrt[3]{10} = \frac{20}{11} + \frac{1}{11}\theta + \frac{1}{11}\theta^2 \in K$$

and

$$\mathbb{Q}(\sqrt[3]{10}) \subseteq K.$$

This completes the proof that $K = \mathbb{Q}(\sqrt[3]{10})$, so K is a pure cubic field. ∎

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