Chapter 6, Question 7

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7. Let θ be a root of the equation $x^6+2x^2+2=0$. Let $K=\mathbb{Q}(\theta)$. How many distinct elements are there in the complete set of conjugates of $\alpha=\theta^2+\theta^4$ relative to K?

Solution. As θ is a root of x^6+2x^2+2 we have $\theta^6+2\theta^2+2=0$ so that

$$\theta^6=-2-2\theta^2,\ \theta^8=-2\theta^2-2\theta^4.$$

Let $\alpha=\theta^2+\theta^4$. Then

$$\begin{aligned}\alpha^2=\left(\theta^2+\theta^4\right)^2&=\theta^4+2\theta^6+\theta^8\\&=\theta^4+2(-2-2\theta^2)+(-2\theta^2-2\theta^4)\\&=\theta^4-4-4\theta^2-2\theta^2-2\theta^4\\&=-4-6\theta^2-\theta^4.\end{aligned}$$

Further

$$\begin{aligned}\alpha^3=\alpha\cdot\alpha^2&=(\theta^2+\theta^4)(-4-6\theta^2-\theta^4)\\&=-4\theta^2-4\theta^4-6\theta^4-6\theta^6-\theta^6-\theta^8\\&=-4\theta^2-10\theta^4-7\theta^6-\theta^8\\&=-4\theta^2-10\theta^4-7(-2-2\theta^2)-(-2\theta^2-2\theta^4)\\&=-4\theta^2-10\theta^4+14+14\theta^2+2\theta^2+2\theta^4\\&=14+12\theta^2-8\theta^4.\end{aligned}$$

We now seek $A,B,C\in\mathbb{Q}$ such that

$$\alpha^3+A\alpha^2+B\alpha+C=0.$$

We want

$$(14+12\theta^2-8\theta^4)+A(-4-6\theta^2-\theta^4)+B(\theta^2+\theta^4)+C=0,$$

that is

$$(14-4A+C)+(12-6A+B)\theta^2+(-8-A+B)\theta^4=0.$$

Since the polynomial x^6+2x^2+2 is 2-Eisenstein, it is irreducible and therefore it is the minimal polynomial of θ. Hence θ cannot satisfy a nontrivial quartic polynomial over $\mathbb{Q}$ and so

$$\begin{aligned} 14 - 4A + C &= 0, \\ 12 - 6A + B &= 0, \\ -8 - A + B &= 0. \end{aligned}$$

Solving these equations for A, B, C we obtain

$$A = 4,\ \ B = 12,\ \ C = 2.$$

Thus $\theta^2 + \theta^4$ is a root of the cubic polynomial $x^3 + 4x^2 + 12x + 2$. This polynomial is 2-Eisenstein so it is irreducible. Hence

$$[\mathbb{Q}(\theta^2 + \theta^4) : \mathbb{Q}] = \deg(x^3 + 4x^2 + 12x + 2) = 3.$$

Thus there are three distinct conjugates in the complete set of six conjugates of α relative to K, each repeated twice. ■

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