Chapter 6, Question 5

5. Let m be a cubefree integer. Let $K = \mathbb{Q}(\sqrt[3]{m})$. Determine all the monomorphisms from K to $\mathbb{C}$.

Solution. As m is a cubefree integer, a simple argument shows that

$$\mathrm{irr}_{\mathbb{Q}}(\sqrt[3]{m}) = x^3 - m.$$

Hence

$$[K : \mathbb{Q}] = [\mathbb{Q}(\sqrt[3]{m}) : \mathbb{Q}]\deg(x^3 - m) = 3.$$

Thus there are exactly three monomorphisms : $K \to \mathbb{C}$. Now

$$\begin{aligned}
\sigma_1(a + b\sqrt[3]{m} + c(\sqrt[3]{m})^2) &= a + b\sqrt[3]{m} + c(\sqrt[3]{m})^2, \\
\sigma_2(a + b\sqrt[3]{m} + c(\sqrt[3]{m})^2) &= a + b\omega\sqrt[3]{m} + c\omega^2(\sqrt[3]{m})^2, \\
\sigma_3(a + b\sqrt[3]{m} + c(\sqrt[3]{m})^2) &= a + b\omega^2\sqrt[3]{m} + c\omega(\sqrt[3]{m})^2,
\end{aligned}$$

where $a, b, c \in \mathbb{Q}$ and ω is a complex cube root of unity, are easily checked to be distinct monomorphisms from K to $\mathbb{C}$. Thus $\sigma_1, \sigma_2, \sigma_3$ are all the monomorphisms from K to $\mathbb{C}$. ■

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