Chapter 6, Question 22

22. Prove that the discriminant of the trinomial polynomial $x^n+ax+b \in \mathbb{Z}[x]$, where n is an integer ≥ 2, is

$$(-1)^{(n-1)(n-2)/2}(n-1)^{n-1}a^n + (-1)^{n(n-1)/2}n^n b^{n-1}.$$

Solution. Let $f(x) \in \mathbb{Z}[x]$. If $f(x)$ is monic and $x_0 \in \mathbb{C}$ we show that

$$\operatorname{disc}((x-x_0)f(x)) = f(x_0)^2 \operatorname{disc}(f(x)). \tag{1}$$

Let $x_1, \ldots, x_n \in \mathbb{C}$ be the roots of $f(x)$. Then the roots of $(x-x_0)f(x)$ are $x_0, x_1, \ldots, x_n$ and

$$\begin{aligned}\operatorname{disc}((x-x_0)f(x)) &= \prod_{0\leq i<j\leq n}(x_i-x_j)^2 \\ &= \prod_{1\leq j\leq n}(x_0-x_j)^2 \prod_{1\leq i<j\leq n}(x_i-x_j)^2 \\ &= f(x_0)^2\operatorname{disc}(f(x)),\end{aligned}$$

which is (1). Next we show that

$$\operatorname{disc}(x^m+c) = (-1)^{\frac{m(m-1)}{2}} m^m c^{m-1}, \ m \in \mathbb{N}, \ c \in \mathbb{Z}. \tag{2}$$

Let $g(x) = x^m + c$. Let $\theta_1, \ldots, \theta_m \in \mathbb{C}$ be the roots of $g(x)$ so that

$$\theta_1 \cdots \theta_m = (-1)^m c.$$

Then

$$\begin{aligned}(-1)^{\frac{m(m-1)}{2}}\operatorname{disc}(x^m+c) &= \prod_{i=1}^m g'(\theta_i) = \prod_{i=1}^m m\theta_i^{m-1} \\ &= m^m(\theta_1\cdots\theta_m)^{m-1} = m^m(-1)^{m(m-1)}c^{m-1} \\ &= m^m c^{m-1},\end{aligned}$$

which is (2).

We now determine $\operatorname{disc}(x^n+ax+b)$, $n \in \mathbb{N}$, $n \geq 2$, $a \in \mathbb{Z}$, $b \in \mathbb{Z}$. First we treat the case $b = 0$. In this case we have

$$\begin{aligned}
\operatorname{disc}(x^n+ax+b) &= \operatorname{disc}(x^n+ax) = \operatorname{disc}(x(x^{n-1}+a)) \\
&= a^2\operatorname{disc}(x^{n-1}+a) \qquad \text{by (1)} \\
&= a^2(-1)^{\frac{(n-1)(n-2)}{2}}(n-1)^{n-1}a^{n-2} \text{ by (2)} \\
&= (-1)^{\frac{(n-1)(n-2)}{2}}(n-1)^{n-1}a^n \\
&= (-1)^{\frac{(n-1)(n-2)}{2}}(n-1)^{n-1}a^n + (-1)^{\frac{(n-1)(n-2)}{2}}n^n b^{n-1}.
\end{aligned}$$

Now we turn to the case $b \neq 0$. Let $\theta_1, \ldots, \theta_n \in \mathbb{C}$ be the roots of $f(x) = x^n + ax + b \in \mathbb{Z}[x]$ so that

$$\theta_i^n + a\theta_i + b = 0, \ i = 1, 2, \ldots, n$$

and

$$\theta_1 \cdots \theta_n = (-1)^n b.$$

As $b \neq 0$ we have $\theta_i \neq 0, \ i = 1, 2, \ldots, n$. Now

$$\begin{aligned}
&(-1)^{\frac{(n-1)(n-2)}{2}}\operatorname{disc}(x^n+ax+b) \\
&\qquad = \prod_{i=1}^n f'(\theta_i) \\
&\qquad = \prod_{i=1}^n (n\theta_i^{n-1}+a) \\
&\qquad = \prod_{i=1}^n \frac{(n\theta_i^n + a\theta_i)}{\theta_i} \\
&\qquad = \frac{(-1)^n}{b}\prod_{i=1}^n (n\theta_i^n + a\theta_i) \\
&\qquad = \frac{(-1)^n}{b}\prod_{i=1}^n (n(-a\theta_i - b) + a\theta_i) \\
&\qquad = \frac{(-1)^n}{b}\prod_{i=1}^n (-bn - a(n-1)\theta_i) \\
&\qquad = \frac{(-1)^n}{b}a^n(n-1)^n\prod_{i=1}^n \left(\frac{-n}{a(n-1)} - \theta_i\right)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{(-1)^n}{b} a^n (n-1)^n f\left(\frac{-bn}{a(n-1)}\right) \\
&= \frac{(-1)^n}{b} a^n (n-1)^n \left(\left(\frac{-bn}{a(n-1)}\right)^n + a\left(\frac{-bn}{a(n-1)}\right) + b\right) \\
&= \frac{(-1)^n}{b} \left((-1)^n b^n n^n - a^n bn(n-1)^{n-1} + a^n b(n-1)^n\right) \\
&= (-1)^n \left((-1)^n b^{n-1} n^n - a^n n(n-1)^{n-1} + a^n (n-1)^n\right) \\
&= (-1)^n ((-1)^n b^{n-1} n^n - a^n (n-1)^{n-1}) \\
&= n^n b^{n-1} - (-1)^n (n-1)^{n-1} a^n \\
&= (-1)^{n-1} (n-1)^{n-1} a^n + n^n b^{n-1}
\end{aligned}
$$

so that

$$
\operatorname{disc}(x^n + ax + b) = (-1)^{(n-1)(n-2)/2} (n-1)^{n-1} a^n + (-1)^{n(n-1)/2} n^n b^{n-1}. \qquad \blacksquare
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