Chapter 6, Question 2

2. Prove that the cubic equation $x^3 + ax + b = 0$, where $a, b \in \mathbb{R}$, has three distinct real roots if $-4a^3 - 27b^2 > 0$, one real and two nonreal complex conjugate roots if $-4a^3 - 27b^2 < 0$, and at least two equal real roots if $-4a^3 - 27b^2 = 0$.

Solution. Let $x_1, x_2, x_3 \in \mathbb{C}$ be the three roots of $x^3 + ax + b = 0$. The discriminant of $x^3 + ax + b$ is

$$-4a^3 - 27b^2 = (x_1 - x_2)^2 (x_1 - x_3)^2 (x_2 - x_3)^2.$$

If x_1, x_2, x_3 are all real and distinct then $(x_1 - x_2)^2 (x_1 - x_3)^2 (x_2 - x_3)^2 > 0$ so that $-4a^3 - 27b^2 > 0$. If x_1, x_2, x_3 are all real and not all distinct then $(x_1 - x_2)^2 (x_1 - x_3)^2 (x_2 - x_3)^2 = 0$ so that $-4a^3 - 27b^2 = 0$. If x_1, x_2, x_3 are not all real then two of them,say x_1 and x_2, are nonreal and complex conjugates of one another (and thus unequal) and the third of them, namely x_3, is real, so all three are distinct. Hence $x_1 = r + is,\ x_2 = r - is,\ x_3 = t$, where $r, s, t \in \mathbb{R}$ and $s \neq 0$. Thus

$$x_1 - x_2 = 2is,\ x_1 - x_3 = r - t + is,\ x_2 - x_3 = r - t - is,$$

so that

$$\begin{aligned}(x_1 - x_2)^2 (x_1 - x_3)^2 (x_2 - x_3)^2 &= (2is)^2 (r - t + is)^2 (r - t - is)^2 \\ &= -4s^2 \left((r - t)^2 + s^2\right)^2 < 0,\end{aligned}$$

and thus $-4a^3 - 27b^2 < 0$. ∎

February 24, 2004