EXERCISES 5, QUESTION 3

2. Prove that $x^2 - \sqrt{2}x + 1$ is irreducible in $\mathbb{Q}(\sqrt{2})[x]$, see Example 5.1.2.

Solution. Suppose that $x^2 - \sqrt{2}x + 1$ is reducible in $\mathbb{Q}(\sqrt{2})[x]$. Then $x^2 - \sqrt{2}x + 1$ has a root in $\mathbb{Q}(\sqrt{2})$, that is, there exists $a + b\sqrt{2} \in \mathbb{Q}(\sqrt{2})$ such that

$$(a + b\sqrt{2})^2 - \sqrt{2}(a + b\sqrt{2}) + 1 = 0.$$

Hence

$$(a^2 + 2b^2 - 2b + 1) + (2ab - a)\sqrt{2} = 0.$$

As $\sqrt{2}$ is irrational we have

$$a^2 + 2b^2 - 2b + 1 = 0, \tag{1}$$

$$2ab - a = 0. \tag{2}$$

From (2) we deduce that $a = 0$ or $b = 1/2$. If $a = 0$ then from (1) we obtain $2b^2 - 2b + 1 = 0$, which has no real roots, contradicting $b \in \mathbb{Q}$. If $b = 1/2$ Then (1) gives $a^2 + \frac{1}{2} = 0$, which has no real roots, contradicting $a \in \mathbb{Q}$. Hence $x^2 - \sqrt{2}x + 1$ is irreducible in $\mathbb{Q}(\sqrt{2})[x]$. ∎

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