CHAPTER 2, QUESTION 17

17. Prove that if p is a prime with $p \equiv 3,\ 5,\ 6 \pmod 7$ then there do not exist integers x and y such that $p = x^2 + xy + 2y^2$.

Solution. Let p be a prime with $p \equiv 3,\ 5$ or $6 \pmod 7$. Suppose that there exist integers x and y such that $p = x^2 + xy + 2y^2$. From this equation it is clear that $p \nmid x$ and $p \nmid y$. Then $4p = (2x + y)^2 + 7y^2$, so that $p \nmid 2x + y$. Hence,

$$\left(\frac{-7}{p}\right) = \left(\frac{-7y^2}{p}\right) = \left(\frac{(2x+y)^2 - 4p}{p}\right) = \left(\frac{(2x+y)^2}{p}\right) = 1.$$

But $\left(\frac{-7}{p}\right) = -1$ as $p \equiv 3,\ 5,\ 6 \pmod 7$, a contradiction. Hence no such integers x and y exist. ∎

June 20, 2004