

CHAPTER 2, QUESTION 10

10. Use Theorem 2.2.3 to prove that $\mathbb{Z} + \mathbb{Z}\left(\frac{1+\sqrt{77}}{2}\right)$ is not Euclidean.

Solution. We choose

$$m = 77, p = 5, q = 3, r = 49.$$

Then,

$$\begin{aligned} \left(\frac{m}{p}\right) &= \left(\frac{77}{5}\right) = \left(\frac{2}{5}\right) = -1, \\ \left(\frac{m}{q}\right) &= \left(\frac{77}{3}\right) = \left(\frac{2}{3}\right) = -1, \end{aligned}$$

$$\begin{aligned} (m-1)r^2 - 4m \left(\frac{(m-1)r^2}{4m}\right) &= 76 \cdot 49^2 - 4 \cdot 77 \left(\frac{76 \cdot 49^2}{4 \cdot 77}\right) \\ &= 182476 - 308 \cdot 592 \\ &= 140 \\ &= 5 \cdot 2^2 \cdot 7, \end{aligned}$$

$$(m-1)r^2 - 4m \left(\frac{(m-1)r^2}{4m}\right) - 4m = 140 - 308 = -168 = -3 \cdot 2^3 \cdot 7.$$

Hence, by Theorem 2.3.3, $\mathbb{Z} + \mathbb{Z}\left(\frac{1+\sqrt{77}}{2}\right)$ is not Euclidean with respect to ϕ_{77} . ■

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