

EXERCISES 4, QUESTION 5

5. Express the algebraic number

$$\sqrt[3]{\frac{1 + \sqrt{17}}{5}} + \sqrt[3]{\frac{1 - \sqrt{17}}{5}}$$

as the quotient of an algebraic integer and an ordinary integer.

Solution. Let

$$\alpha = \sqrt[3]{\frac{1 + \sqrt{17}}{5}} + \sqrt[3]{\frac{1 - \sqrt{17}}{5}}.$$

Cubing α , we obtain

$$\begin{aligned} \alpha^3 &= \frac{1 + \sqrt{17}}{5} + 3 \left(\frac{1 + \sqrt{17}}{5} \right)^{2/3} \left(\frac{1 - \sqrt{17}}{5} \right)^{1/3} \\ &\quad + 3 \left(\frac{1 + \sqrt{17}}{5} \right)^{1/3} \left(\frac{1 - \sqrt{17}}{5} \right)^{2/3} + \frac{1 - \sqrt{17}}{5} \\ &= \frac{2}{5} + 3 \left(\frac{1 + \sqrt{17}}{5} \right)^{1/3} \left(\frac{1 - \sqrt{17}}{5} \right)^{1/3} \left(\left(\frac{1 + \sqrt{17}}{5} \right)^{1/3} \left(\frac{1 - \sqrt{17}}{5} \right)^{1/3} \right) \\ &= \frac{2}{5} + 3 \left(\left(\frac{1 + \sqrt{17}}{5} \right) \left(\frac{1 - \sqrt{17}}{5} \right) \right)^{1/3} \alpha \\ &= \frac{2}{5} + 3 \left(\frac{-16}{25} \right)^{1/3} \alpha \\ &= \frac{2}{5} - \frac{6}{5} 10^{1/3} \alpha \end{aligned}$$

so that

$$6 \cdot 10^{1/3} \alpha = 2 - 5\alpha^3.$$

Cubing again we obtain

$$2160\alpha^3 = (2 - 5\alpha^3)^3 = 8 - 60\alpha^3 + 150\alpha^6 - 125\alpha^9$$

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so that

$$125\alpha^9 - 150\alpha^6 + 2220\alpha^3 - 8 = 0.$$

Thus

$$5^3\alpha^9 - 2 \cdot 3 \cdot 5^2\alpha^6 + 2^2 \cdot 3 \cdot 5 \cdot 37\alpha^3 - 2^3 = 0.$$

Let $\beta = 5\alpha$. Then

$$\begin{aligned}\beta^9 - 150\beta^6 + 7500\beta^3 - 15625 &= 5^9\alpha^9 - 2 \cdot 3 \cdot 5^8\alpha^6 + 2^2 \cdot 3 \cdot 5^7 \cdot 37\alpha^3 - 2^3 \cdot 5^6 \\ &= 5^6 (5^3\alpha^9 - 2 \cdot 3 \cdot 5^2\alpha^6 + 2^2 \cdot 3 \cdot 5 \cdot 37\alpha^3 - 2^3) \\ &= 0.\end{aligned}$$

Hence

$$\begin{aligned}\beta &= 5 \left(\left(\frac{1 + \sqrt{17}}{5} \right)^{1/3} + \left(\frac{1 - \sqrt{17}}{5} \right)^{1/3} \right) \\ &= (25 + 25\sqrt{17})^{1/3} + (25 - 25\sqrt{17})^{1/3}\end{aligned}$$

is an algebraic integer. Finally

$$\alpha = \frac{(25 + 25\sqrt{17})^{1/3} + (25 - 25\sqrt{17})^{1/3}}{5},$$

where the numerator is an algebraic integer and the denominator an ordinary integer. ■

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