Uncoverings-by-bases for graphic matroids

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Matroids

- A matroid is a pair $\mathcal{M} = (E, I)$, where $E$ is a finite set and $I$ is a family of subsets of $E$ (called independent sets) satisfying the following axioms:

  (i) $\emptyset \in I$;

  (ii) if $A \in I$ and $B \subseteq A$, then $B \in I$;

  (iii) if $A, B \in I$ and $|A| < |B|$, then $\exists x \in B$ such that $A \cup \{x\} \in I$.

- The maximal independent sets are called bases.

- By axiom (iii), bases must all have the same size.
Uncoverings-by-bases, graphic matroids

• Let $\mathcal{M} = (E, I)$ be a matroid.

• An uncovering-by-bases for $\mathcal{M}$ (or a $t$-UBB for $\mathcal{M}$) is a set $\mathcal{U}$ of bases for $\mathcal{M}$ such that any $t$-subset of $E$ is disjoint from at least one base in $\mathcal{U}$.

We’ll be considering the following class of matroids:

• Let $G = (V, E)$ be a graph. The graphic matroid $M(G)$ has ground set $E$, and a subset of $E$ is independent iff it contains no cycle.

• If $G$ is connected, then the bases for $M(G)$ are precisely the spanning trees for $G$. 
UBBs for graphic matroids

• The *edge connectivity* \( \kappa(G) \) of a connected graph \( G \) is the size of the smallest set of edges whose removal disconnects \( G \).

• Thus is fewer that \( \kappa(G) \) edges are removed from \( G \), the resulting graph contains a spanning tree.

• So, for \( t \leq \kappa(G) - 1 \), there exists a \( t \)-UBB for \( M(G) \).

• From now on, assume \( t = \kappa(G) - 1 \).
Complete bipartite graphs

- Consider the complete bipartite graph $K_{m,n}$, where $n \geq m \geq 2$, with vertex set $X \cup Y$ where $|X| = m$ and $|Y| = n$.

- $\kappa(K_{m,n}) = \min\{m, n\} = m$, so let $t = m - 1$.

- Let $A$ be an arbitrary $t$-set of edges. Since $|A| < m$, there exists $u \in X$ incident with no edge of $A$. Similarly, since $|A| < n$, there exists $v \in Y$ incident with no edge of $A$.

- Construct a spanning tree using $uv$ and all other edges incident with each of $u$ and $v$.

- The set of all such spanning trees forms a $t$-UBB for $M(K_{m,n})$. 
Complete graphs (odd order)

- Consider the complete graph $K_n$, where $n$ is odd (set $n = 2k + 1$).

- $\kappa(K_n) = n - 1 = 2k$, so let $t = n - 2 = 2k - 1$.

- Again, let $A$ denote an arbitrary $t$-set of edges.

- Idea: build a $t$-UBB where the spanning trees are paths.

- Why? They are contained inside Hamilton cycles, and it follows from Ore’s Theorem that $K_n \setminus A$ is Hamiltonian.
Complete graphs (odd order)

- Lucas (1891) showed that $K_n$ ($n = 2k + 1$) has a decomposition into $k$ disjoint Hamilton cycles.

- Let $D = \{C_1, \ldots, C_k\}$ be such a decomposition, and for each $C_i \in D$, form $n = 2k + 1$ paths $C \setminus e$ (for each edge $e \in C$), giving $k(2k + 1)$ paths altogether.

- We claim that this is a $t$-UBB for $M(K_n)$:

  - If $\exists C_i \in D$ with $A \cap C_i = \emptyset$, take any path in $C_i$.

  - If not, then $A$ meets every cycle, so the $2k - 1$ edges are spread across all $k$ cycles.
    \[ \Rightarrow \exists \text{ cycle } C_j \text{ containing just one edge } e \in A. \]
    \[ \Rightarrow \text{ Use the path } C_j \setminus e. \]
Graphs with Hamiltonian decompositions

• In fact, this same construction works for any graph $G$ with a Hamiltonian decomposition.

• Suppose the decomposition has $k$ Hamilton cycles.

• $G$ must be $2k$-regular, so $\kappa(G) \leq 2k$.

• Also, for any edge-cut of $G$, each of the $k$ Hamilton cycles must cross it at least twice, so $\kappa(G) \geq 2k$.

• Hence $\kappa(G) = 2k$, so take $t = 2k - 1$ and the same construction can be used.
Complete graphs (even order)

- Now consider $K_n$ where $n = 2k$ is even.

- $\kappa(G) = n = 2k - 1$, so let $t = n - 2 = 2k - 2$.

- In this case, there is no Hamiltonian decomposition. However, there is a decomposition into $k - 1$ Hamilton cycles and a 1-factor.

- Why don’t we try using the same construction as before, using this “near-decomposition”? 
Complete graphs (even order)

- Because it doesn’t quite work!

- For each of the Hamilton cycles $C_1, \ldots, C_{k-1}$, we can construct $n = 2k$ paths. This would be a $t$-UBB except for one problem case: where the set $A$ of $2k - 2$ “bad” edges consists of two edges from each of our $k - 1$ cycles.

- Fortunately, this can be fixed, by taking some extra cycles which use the edges in the 1-factor.