Exact expressions for transient forced internal gravity waves and spatially-localized wave packets in a Boussinesq fluid

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Abstract

We re-examine a simple model describing the propagation of transient forced internal gravity waves in a Boussinesq fluid with constant horizontal mean velocity which was previously studied by Nadon and Campbell (Wave Motion, 2007). The waves are generated by a horizontally-periodic lower boundary condition and propagate upwards. We derive an alternative exact expression for the solution which more readily gives insight into the behaviour of the solution at high altitude. Some special cases of lower boundary conditions are considered to illustrate the features of the solution. This form of the solution allows us to use a Fourier transform to derive the solution for the more general situation where a wave packet is generated by a horizontally-localized lower boundary condition, comprising a continuous spectrum of horizontal wavenumbers or Fourier modes. This is a more realistic representation of internal gravity waves in the atmosphere and can be used as a starting point for investigating waves generated by an obstacle of finite horizontal extent such as an isolated mountain or a mountain range.

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1. Introduction

In Nadon and Campbell [7] an exact expression was derived to describe transient forced internal gravity waves propagating in a Boussinesq fluid under the assumptions of constant horizontal mean velocity and the long-wave limit of zero aspect ratio. The solution obtained comprises a part with steady amplitude and a transient part that goes to zero in the limit of infinite time.

In this paper, we use a different solution procedure and derive an alternative expression for the solution of the problem studied by [7]. From this alternative form, we are able to obtain further insight into the characteristics of the solution. In particular, we find that the form of the solution derived in the present paper is better suited for understanding the behaviour of the solution as the vertical coordinate goes to infinity, while that of [7] gives insight into the late-time behaviour of the solution. As an illustration, some special cases of lower boundary conditions are considered. We also note that in the case where the background fluid flow is zero, the solution reduces to a simple expression involving the Bessel function of order zero.

We extend this solution to the case of a horizontally-localized wave packet such as might be generated by an isolated mountain or a mountain range with multiple peaks. In that configuration, the solution is considered as a continuous spectrum of horizontal wavenumbers and an approximate asymptotic solution can be derived using a Fourier transform. We derive an exact solution for the special case of a bell-shaped forcing function where the horizontal
wavenumber spectrum is given by a function that decays exponentially from a single peak at the central wavenumber of zero. We then describe a more general procedure that can be used to obtain a solution in the form of an asymptotic series in powers of a small parameter and apply it to the configuration where the horizontal wavenumber spectrum takes the form of a Gaussian function centred at a non-zero wavenumber.

2. The time-dependent problem

2.1. Formulation of the problem

As in [7], the internal gravity waves are considered to be a small-amplitude perturbation to some basic state. The perturbation is given by

\[
\left( \frac{\partial}{\partial t} + \bar{u} \frac{\partial}{\partial x} \right)^2 \nabla^2 \psi + N^2 \psi_{xx} = 0, \tag{1}
\]

where \( \psi(x, z, t) \) is the streamfunction of the perturbation, \( x \) and \( z \) are horizontal and vertical Cartesian coordinates, \( t \) represents time and the subscripts denote partial differentiation. In addition, \( \bar{u} \) is the basic flow velocity and \( N \) is the Brunt-Väisälä frequency or buoyancy frequency, defined as

\[ N^2 = -g \left( \frac{d\bar{\rho}}{dz} \right) \bar{\rho}^{-1}, \]

with \( \bar{\rho}(z) \) being the basic flow density and \( g \) being the acceleration due to gravity. Both \( \bar{u} \) and \( N \) are taken to be constant.

We consider all the quantities appearing in (1) to have been made non-dimensional relative to some typical values of the corresponding quantities in the atmospheric context, i.e.,

\[
x = \frac{x^*}{L_x}, \quad z = \frac{z^*}{L_z}, \quad t = \frac{Ut^*}{L_x}, \quad \bar{u} = \frac{\bar{u}^*}{U}, \quad \psi = \frac{\psi^*}{\varphi}, \tag{2}
\]

where the asterisks denote the dimensional quantities and \( L_x, L_z, U \) and \( \varphi \) represent respectively typical values of the horizontal length scale, the
vertical length scale, the background horizontal wind speed, and the wave amplitude. For internal gravity waves in the lower and middle atmosphere, in a configuration where the Boussinesq approximation is applicable, $L_z$ is of the order of magnitude of the scale height, approximately 6-10 km, and is small relative to $L_x$, which could be as large as several hundred kilometres. The aspect ratio $L_z/L_x$ can thus be considered as a small parameter in the non-dimensional equation (1) and the non-dimensional Laplacian operator

$$\nabla^2 = \frac{L_z^2}{L_x^2} \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}$$

can be replaced by $\frac{\partial^2}{\partial z^2}$ to a first approximation. This is known as the long-wave limit and is the configuration that is considered here and in [7].

Equation (1) is a simple model for the time evolution of small-amplitude internal gravity waves forced by flow over a wavy boundary in a density-stratified fluid. The domain is taken to be a semi-infinite region defined by $x_1 < x < x_2$, $z_1 < z < \infty$, with periodic boundary conditions at the left and right ends. At $z = z_1$, we apply a lower boundary condition of the form

$$\psi(x, z_1, t) = e^{ikx} + \text{c.c.},$$

where $k$ is the horizontal wavenumber and is taken to be positive and c.c. denotes the complex conjugate. Note that we could instead consider a forcing of the form $\psi(x, z_1, t) = e^{ik(x-ct)} + \text{c.c.}$, with $c$ being a non-zero (real) phase speed, as in [7]. In that case, the solutions are the same as those that we present here, but with $\bar{u}$ replaced by $\bar{u} - c$. The upper boundary condition is a radiation condition which specifies that only waves with upward group velocity are included in the solution and the initial condition is that the perturbation vorticity and its time derivative are zero at $t = 0$.  

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The form of the lower boundary condition tells us that we should seek a solution of the form

\[ \psi(x, z, t) = \phi(z, t)e^{ikx} + \text{c.c.} \quad (4) \]

When substituted into (1) and (3) with the long-wave limit, this gives

\[ \left( \frac{\partial}{\partial t} + ik\bar{u} \right)^2 \phi_{zz} - N^2k^2\phi = 0 \quad (5) \]

with the boundary condition

\[ \phi(z_1, t) = 1. \quad (6) \]

Defining \( \tilde{\phi}(z, s) \) to be the Laplace transform of \( \phi(z, t) \) we obtain

\[ \tilde{\phi}_{zz} - \frac{N^2k^2}{(s + ik\bar{u})^2} \tilde{\phi} = 0 \quad (7) \]

with the boundary condition

\[ \tilde{\phi}(z_1, s) = \frac{1}{s}. \quad (8) \]

The standard group velocity argument, first put forward by Booker and Bretherton [2], tells us that, with \( k > 0 \) and \( \bar{u} > 0 \), the solution that corresponds to a wave with upward group velocity is

\[ \tilde{\phi}(z, s) = \frac{1}{s}e^{-\frac{Nk(z-z_1)}{s+ik\bar{u}}} \quad (9) \]

and we then invert the Laplace transform to find \( \phi(z, t) \).

We first note that in the case where the mean flow velocity is zero,

\[ \tilde{\phi}(z, s) = \frac{1}{s}e^{-\frac{Nk(z-z_1)}{s}} \quad (10) \]
and so on inverting the transform, we find that

\[ \phi(z, t) = J_0(2\sqrt{Nk(z-z_1)t}), \]  

(11)

where \( J_0 \) is the Bessel function of order zero. The asymptotic behaviour of the Bessel function tells us that the wave amplitude oscillates in both \( t \) and \( z \) and goes to zero in the limit as \( t \to \infty \) and as \( z \to \infty \).

2.2. Time-dependent solution

The case where the mean flow velocity is nonzero was examined by Nadon and Campbell [7]. The solution is

\[ \phi(z, t) = e^{\frac{iN(z-z_1)}{u}} - e^{-ik\bar{u}t} \sum_{n=1}^{\infty} \left( \frac{i\sqrt{N(z-z_1)}}{\bar{u}\sqrt{kt}} \right)^n J_n \left( 2\sqrt{Nk(z-z_1)t} \right), \]  

(12)

where the functions \( J_n \) are Bessel functions of order \( n \).\(^1\) This series solution is an example of a general class of series called Neumann series whose elements are Bessel functions of increasing order. Neumann series and their relation to Laplace integrals are discussed in [10]. In [7] different procedures for inverting the Laplace transform are described, all of which lead to the expression (12).

An alternative procedure is to write (9) as a Laurent series

\[ \tilde{\phi}(z, s) = \frac{1}{s} \sum_{j=0}^{\infty} \frac{(-1)^j}{j!} \left[ \frac{Nk(z-z_1)}{s + ik\bar{u}} \right]^j \]  

(13)

\(^1\)Note that there is a factor of \( i \) in the numerator of the fraction in the time-dependent series. In expressions (14) and (A.10) of [7] the factor of \( i \) was erroneously placed in the denominator of the fraction. The other equations in the derivation of the solution in [7], however, are correct.
and then invert it term by term to give

\[ \phi(z, t) = \sum_{j=0}^{\infty} \frac{(-1)^j [Nk(z - z_1)]^j}{j!} \mathcal{L}^{-1} \left\{ \frac{1}{s} \left( \frac{1}{s + ik\bar{u}} \right)^j \right\} \]

\[ = 1 + \sum_{j=1}^{\infty} \frac{(-1)^j [Nk(z - z_1)]^j}{j! \Gamma(j)} \int_0^t \tau^{j-1} e^{-ik\bar{u} \tau} d\tau. \quad (14) \]

The integral

\[ I_{j-1} = \int_0^t \tau^{j-1} e^{-ik\bar{u} \tau} d\tau, \quad j = 1, 2, \ldots \quad (15) \]

can be evaluated using the integration by parts formula for \( \int udv \) with either

(I) \( u = \tau^{n-1} \) and \( dv = e^{-ik\bar{u} \tau} d\tau \)

or

(II) \( u = e^{-ik\bar{u} \tau} \) and \( dv = \tau^{n-1} d\tau \).

Method I gives the expression (12) which was obtained by [7] using other methods.

Method II gives an alternative expression for the solution. With \( u = e^{-ik\bar{u} \tau} \) and \( dv = \tau^{n-1} d\tau \) we obtain

\[ I_{j-1} = \frac{t^j e^{-ik\bar{u} t}}{j} + \frac{ik\bar{u}}{j} \int_0^t \tau^j e^{-ik\bar{u} \tau} d\tau \]

\[ = \frac{t^j e^{-ik\bar{u} t}}{j} + \frac{(ik\bar{u}) t^{j+1} e^{-ik\bar{u} t}}{j(j+1)} + \frac{(ik\bar{u})^2}{j(j+1)} \int_0^t \tau^{j+1} e^{-ik\bar{u} \tau} d\tau \]

\[ = \ldots \]

\[ = \sum_{n=0}^{\infty} \frac{\Gamma(j)(ik\bar{u})^{n+j+n}}{\Gamma(n+j+1)}, \quad (16) \]
which gives

\[
\phi(z, t) = 1 + e^{-ik\bar{u}t} \sum_{j=1}^{\infty} \sum_{n=0}^{\infty} \frac{(-1)^j [Nk(z - z_1)t]^j \Gamma(j + n + 1)}{j! \Gamma(j + n + 1)}
\]

\[
= e^{-ik\bar{u}t} \sum_{j=0}^{\infty} \sum_{n=0}^{\infty} \frac{(-1)^j [Nk(z - z_1)t]^j \Gamma(j + n + 1)}{j! \Gamma(j + n + 1)}
\]

\[
= e^{-ik\bar{u}t} \sum_{n=0}^{\infty} \left( \frac{i\bar{u} \sqrt{kt}}{\sqrt{N(z - z_1)}} \right)^n \sum_{j=0}^{\infty} \frac{(-1)^j (\sqrt{Nk(z - z_1)t})^{2j+n}}{j! \Gamma(j + n + 1)}
\]

\[
= e^{-ik\bar{u}t} \sum_{n=0}^{\infty} \left( \frac{i\bar{u} \sqrt{kt}}{\sqrt{N(z - z_1)}} \right)^n J_n \left( \frac{2\sqrt{Nk(z - z_1)t}}{\sqrt{Nk(z - z_1)t}} \right). \tag{17}
\]

2.3. **Equivalence of the expressions for the solution**

We can relate the two expressions (12) and (17) using formula (9.1.41) of Abramowitz and Stegun [1],

\[
e^{\frac{1}{2}Z(T-1/T)} = \sum_{n=-\infty}^{\infty} T^n J_n(Z), \quad (T \neq 0), \tag{18}
\]

which can be written as

\[
e^{\frac{1}{2}Z(T-1/T)} - \sum_{n=1}^{\infty} T^n J_n(Z) = \sum_{n=-\infty}^{0} T^n J_n(Z)
\]

\[
= \sum_{n=0}^{\infty} T^{-n} J_{-n}(Z) = \sum_{n=0}^{\infty} (-T)^{-n} J_n(Z). \tag{19}
\]

Setting

\[
Z = 2\sqrt{Nk(z - z_1)t}, \quad T = \frac{i\sqrt{N(z - z_1)}}{\bar{u} \sqrt{kt}}
\]

and multiplying both sides by \(e^{-ik\bar{u}t}\) gives (12) on the left-hand side and (17) on the right-hand side of (19).
2.4. Asymptotic behaviour of the solution

For real $Z$, the Bessel functions $J_n$ behave like

$$J_n(Z) \sim Z^{-1/2}, \quad |Z| \gg 1.$$  

Thus, for each $n$, the respective term in the series in (12) is asymptotic to $t^{-n/2-1/4}$ for $t \gg 1$ and it is clear therefore that, as $t \to \infty$, the solution approaches the steady-state solution given by the first term in (12). However, the behaviour of the solution as $z \to \infty$ is not clearly evident from (12).

Examining (17), on the other hand, we see that for each $n$, the respective term in the series is asymptotic to $|z - z_1|^{-n/2-1/4}$ for $|z - z_1| \gg 1$. Thus, (17) shows that the solution approaches zero as $z \to \infty$ even if $t$ is finite, which is consistent with the finite propagation speed of the waves.

A graphical representation of the solution and its temporal evolution towards the steady-state is given in Figure 1. The expression (12) was computed with a sum of 50 terms using the approximations for the Bessel functions in MATLAB. The values of the (non-dimensional) input parameters are $\bar{u} = 1$, $k = 2$ and $N = \sqrt{2}$. As $t$ increases, the transients die out and the solution tends to a steady state given by $\exp(i(kx + mz)) + \exp(-i(kx + mz))$. Since $\bar{u}$ is positive, the vertical wavenumber $m = N/\bar{u}$ of each of these upward-propagating modes is of the same sign as the horizontal wavenumber, according to Booker and Bretherton’s group velocity argument [2]. This gives phase lines with a negative slope given by $-k\bar{u}/N$, as seen in Figure 1(d).

3. Time-dependent lower boundary conditions

By following either of the two solution procedures outlined here, we can derive exact solutions for (5) under other more general conditions, e.g. for
a configuration where the lower boundary condition depends on $t$, provided that the boundary function is one whose the Laplace transform is known. In this section, we consider, as examples, two different lower boundary conditions which depend on time.

### 3.1. Lower boundary condition given by a step function in time

We consider the case where the lower boundary condition for (1) is

$$\psi(x, z_1, t) = [H(t - t_1) - H(t - t_2)]e^{ikx} + \text{c.c.}, \quad \text{(20)}$$

where $t_1 < t_2 > t_1 \geq 0$ and $u$ is the Heaviside step function defined by

$$H(t - t_1) = \begin{cases} 
0, & t < t_1, \\
1, & t > t_1.
\end{cases} \quad \text{(21)}$$

In that case, the lower boundary condition for (7) is

$$\tilde{\phi}(z_1, s) = \frac{e^{-t_1 s} - e^{-t_2 s}}{s}. \quad \text{(22)}$$

For $k > 0$, the solution of (7) corresponding to an upward-propagating wave is then

$$\tilde{\phi}(z, s) = e^{-t_1 s}e^{-Nk(z-z_1)}_{s+i\bar{u}} + e^{-t_2 s}e^{-Nk(z-z_1)}_{s+i\bar{u}}. \quad \text{(23)}$$

Inverting the Laplace transform using method I gives

$$\phi(z, t) = H(t - t_1)e^{iN(z-z_1)}_{u} - H(t - t_1)e^{-ik\bar{u}(t-t_1)} \sum_{n=0}^{\infty} \left( \frac{\sqrt{N(z-z_1)}}{i\bar{u}\sqrt{k(t-t_1)}} \right)^n J_n \left( 2\sqrt{Nk(z-z_1)(t-t_1)} \right) - H(t - t_2)e^{iN(z-z_1)}_{u}.$$
\[- H(t - t_2)e^{-ik\bar{u}(t-t_2)}\sum_{n=0}^{\infty} \left(\sqrt{N(z-z_1)}\right)^n \mathcal{J}_n \left(2\sqrt{Nk(z-z_1)(t-t_2)}\right) \]

and using method II gives

\[
\phi(z, t) = H(t-t_1)e^{-ik\bar{u}(t-t_1)}\sum_{n=0}^{\infty} \left(\frac{i\bar{u}\sqrt{k(t-t_1)}}{\sqrt{N(z-z_1)}}\right)^n \mathcal{J}_n \left(2\sqrt{Nk(z-z_1)(t-t_1)}\right) \\
- H(t - t_2)e^{-ik\bar{u}(t-t_2)}\sum_{n=0}^{\infty} \left(\frac{i\bar{u}\sqrt{k(t-t_2)}}{\sqrt{N(z-z_1)}}\right)^n \mathcal{J}_n \left(2\sqrt{Nk(z-z_1)(t-t_2)}\right).
\]

A graphical representation of the solution (24) is given in Figure 2. As before, \(\bar{u} = 1\), \(k = 2\) and \(N = \sqrt{2}\). The wave forcing is switched on at \(t = 0\) at the level \(z_1 = 0\), kept fixed at an amplitude of 1 and then switched off at \(t = 2.5\). This gives a wave packet which is localized in the vertical direction. By \(t = 20\) the centre of the wave packet has propagated out of the vertical domain shown and only small amplitude oscillations remain within this range of altitudes. As \(t\) increases the solution continues to approach zero in the domain shown.

### 3.2. Lower boundary condition given by an exponential function of time

Another example of a time-dependent lower boundary condition is

\[
\psi(x, z_1, t) = e^{-\gamma t}e^{ikx} + \text{c.c.}, \text{ where } \gamma \text{ is a positive constant.} \quad (26)
\]

In that case, the lower boundary condition for (7) is

\[
\tilde{\phi}(z_1, s) = \frac{1}{s + \gamma}. \quad (27)
\]
For \( k > 0 \), the solution corresponding to an upward-propagating wave is

\[
\tilde{\phi}(z, s) = e^{-\frac{Nk(z-z_1)}{s+\gamma}}.
\]  

(28)

Inverting the Laplace transform using method I gives

\[
\phi(z, t) = e^{-\gamma t} \mathcal{L}^{-1}\left\{ \frac{e^{-\frac{Nk(z-z_1)}{s+\gamma+ik\tilde{u}}}}{s} \right\}
\]

\[
= e^{-\gamma t} e^{\frac{Nk(z-z_1)}{\gamma + ik\tilde{u}}} - e^{-ik\tilde{u}t} \sum_{n=0}^{\infty} \left( \frac{\sqrt{Nk(z-z_1)}}{(\gamma - ik\tilde{u})\sqrt{t}} \right)^n J_n\left( 2\sqrt{Nk(z-z_1)t} \right)
\]

\[
= e^{-\gamma t} e^{\frac{aNk(z-z_1)}{\gamma^2 + k^2\tilde{u}^2}} e^{\frac{(Nk^2\tilde{u}(z-z_1))}{\gamma^2 + k^2\tilde{u}^2}}
\]

\[- e^{-ik\tilde{u}t} \sum_{n=0}^{\infty} \left( \frac{\sqrt{Nk(z-z_1)}}{(\gamma - ik\tilde{u})\sqrt{t}} \right)^n J_n\left( 2\sqrt{Nk(z-z_1)t} \right) \]  

(29)

and using method II gives

\[
\phi(z, t) = e^{-ik\tilde{u}t} \sum_{n=0}^{\infty} \left( \frac{(ik\tilde{u} - \gamma)\sqrt{t}}{Nk(z-z_1)} \right)^n J_n\left( 2\sqrt{Nk(z-z_1)t} \right).
\]  

(30)

A graphical representation of the solution (29) is given in Figure 3. As before, \( \tilde{u} = 1 \), \( k = 2 \) and \( N = \sqrt{2} \) and we also set the exponential decay rate \( \gamma = 0.005 \). In this case, the wave forcing has an amplitude of 1 at \( t = 0 \) and decays to zero as \( t \) increases. The wave packet is localized in the vertical direction but its vertical extent is not as well-defined as in the case of the step function. Again, by \( t = 20 \) the centre of the wave packet has propagated out of the vertical domain shown and as \( t \) increases the solution continues to approach zero in the domain shown.

### 4. Horizontally-localized wave packet solutions

The solutions obtained in sections 2 and 3 satisfy a horizontally-periodic monochromatic lower boundary condition. A wave forcing of this form is
justified if the wavelength of the disturbance is assumed to be of the order of magnitude of the circumference of the earth. However, for a disturbance that is not on this scale, there is no reason to assume periodicity. This is especially true when considering internal gravity waves forced by topography; in reality, a mountain range would be of finite length only and would not be monochromatic. A more realistic representation can be obtained by imposing a lower boundary condition in the form of a horizontally-localized wave packet with an amplitude that varies slowly in space [5, 6]. This type of configuration was considered by [5] and [6] for some cases where there is vertical shear and a critical level in the background flow and numerical solutions and approximate asymptotic solutions were obtained.

In the simpler configuration examined in the present study, the background flow is constant, there is no critical level and an exact solution has been obtained for the case of a horizontally-periodic monochromatic wave forcing. We can use the exact solution written in the form 17 to derive solutions in certain configurations where a wave packet is generated by a horizontally-localized lower boundary condition comprising a continuous spectrum of horizontal wavenumbers or Fourier modes.

We consider the linear equation (1) in the domain given by $-\infty < x < \infty$, $z_1 < z < \infty$, with the requirement that $|\psi| \rightarrow 0$ as $x \rightarrow \pm \infty$. As before, we also impose zero initial conditions and the condition that only waves with upward group velocity are present. To represent a horizontally-localized wave packet, we consider the lower boundary condition

$$\psi(x, z_1, t) = A(x)e^{i k_0 x} + \text{c.c.},$$

where $k_0$ is an $O(1)$ constant, $A$ is a slowly-varying function of $x$ and $A \rightarrow 0$
as $x \to \pm \infty$. Subject to these conditions, equation (1) can be solved using a Fourier transform, defined by

$$\hat{\psi}(k, z, t) = \int_{-\infty}^{\infty} \psi(x, z, t) e^{-ikx} dx, \quad \hat{A}(k) = \int_{-\infty}^{\infty} A(x) e^{-ikx} dx.$$  \(32\)

The solution of (1) is then the inverse transform of

$$\hat{\psi}(k, z, t) = e^{-ik\bar{u}t} \hat{A}(k) \sum_{n=0}^{\infty} G_n(k, z, t),$$  \(33\)

where

$$G_n(k, z, t) = \left( \frac{\{\text{sgn}(k)\}i\bar{u}\sqrt{|k|t}}{\sqrt{N(z - z_1)}} \right)^n J_n \left( 2\sqrt{Nk(z - z_1)t} \right).$$  \(34\)

For positive $k$ this is the expression in (17); for negative $k$ it is obtained by taking the positive sign in the exponent in (9) in order to have upward group velocity.

We first examine the special case where $k_0 = 0$ and

$$A(x) = \frac{a}{a^2 + x^2},$$  \(35\)

which can be considered to represent a bell-shaped isolated mountain. This gives

$$\psi(x, z, t) = \frac{1}{2} \sum_{n=0}^{\infty} \left\{ \int_0^{\infty} G_n(k, z, t) e^{-ik(x - \bar{u}t) - ak} dk + \int_{-\infty}^{0} G_n(k, z, t) e^{-ik(x - \bar{u}t) + ak} dk \right\} + \text{c.c.}$$  \(36\)

For $z \neq z_1$, we define $Z = 2\sqrt{N|k|(z - z_1)t}$ and write the integrals in terms of $Z$ to give

$$\psi(x, z, t) = \frac{1}{4N(z - z_1)t} \sum_{n=0}^{\infty} \left( \frac{i\bar{u}}{N(z - z_1)} \right)^n.$$
\[ \left\{ \int_{0}^{\infty} e^{-\alpha Z^2} Z^{n+1} J_n(Z) dZ + (-1)^n \int_{0}^{\infty} e^{-\alpha^* Z^2} Z^{n+1} J_n(Z) dZ \right\} + \text{c.c.,} \]  

(37)

where

\[ \alpha = \frac{a - i(x - \bar{u} t)}{4N(z - z_1)} \]

and \( \alpha^* \) is the complex conjugate of \( \alpha \). According to formula (11.4.29) in [1], this gives

\[ \psi(x, z, t) = \frac{1}{2} e^{-\frac{N(z-z_1)t}{2N(x-z_1)}} \sum_{n=0}^{\infty} \frac{(2i\bar{u} t)^n}{(a - i(x - \bar{u} t))^{n+1}} + \text{c.c.} \]  

(38)

When \( z \neq z_1 \) and \( a \) is sufficiently large and \( \bar{u} \) sufficiently small that the series converges, this gives an exact solution for a wave packet generated by the forcing function (35). The solution is shown in Figure 4. The lower boundary is at \( z_1 = 0 \), but the plot is shown for \( z > 0.5 \) since the expression (38) is not valid at \( z = 0 \). At early time, the solution comprises modes with phase lines with both positive and negative slope emanating from the source centred at \( x = 0 \), but as \( t \) increases it tends to a state where only the phase lines with negative slope are present. A wave packet with oscillations in the vertical direction develops over the source region with a similar form to that seen in other investigations with this type of horizontally-localized forcing, e.g. in the numerical simulations of [6]).

Other horizontally-localized lower boundary conditions may be considered. While it is not in general possible to derive an exact solution in closed form, we can in some cases obtain a solution in the form of an asymptotic series by following a procedure that was applied by [4] to examine the behaviour of Rossby wave packets. Consider, for example, the case where the
boundary condition is (31) with an amplitude of Gaussian form

\[ A(x) = e^{-\mu^2 x^2}. \]  

(39)

With \( k_0 > 0 \), the wave packet then comprises a spectrum of horizontal wavenumbers with peaks at \( k = \pm k_0 \). We can then evaluate the inverse Fourier transform of (32) by expanding \( G_n(k, z, t) \), for each \( n \), in powers of \( (k - k_0) \) to give

\[
\psi(x, z, t) = \frac{1}{2} \sqrt{\frac{\pi}{\mu}} e^{-ik_0(x - \bar{u}t)} e^{-\mu^2 x^2} \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} \frac{1}{p!} \frac{\partial^p G_n(k_0, z, t)}{\partial k^p} I_p + \text{c.c.} \]  

(40)

where

\[
I_p = \int_{-\infty}^{\infty} (k - k_0)^p e^{-\frac{1}{4\mu^2} [k - k_0 - 2i\mu^2(x - \bar{u}t)]^2} dk, \quad p = 0, 1, 2, \ldots
\]

Introducing the new variable of integration \( \lambda = (2\mu)^{-1} [k - k_0 - 2i\mu^2(x - \bar{u}t)] \) and then deforming the contour of integration back onto the real axis in the complex \( \lambda \) plane gives

\[
I_p = (2\mu)^{p+1} \int_{-\infty}^{\infty} [\lambda + i\mu(x - \bar{u}t)]^p e^{-\lambda^2} d\lambda \]  

(41)

\[
= 2^{p+1} \mu^{p+2} [i(x - \bar{u}t)]^p \sum_{l=0}^{p} \frac{p!}{2^l (p-l)! (l/2)!} \frac{\sqrt{\pi}}{[i(x - \bar{u}t)]^l} \]  

(42)

This gives an asymptotic series in even powers of \( \mu \),

\[
\psi(x, z, t) = e^{-ik_0(x - \bar{u}t)} e^{-\mu^2(x - \bar{u}t)^2} \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} \frac{\partial^p G_n(k_0, z, t)}{\partial k^p} \sum_{l=0}^{p} \frac{\mu^{2p-l} [2i(x - \bar{u}t)]^{p-l}}{(p-l)! (l/2)!} + \text{c.c.} \]  

(43)
Expanding the $t$-dependent part of the exponential function $\exp \{-\mu^2(x - \bar{u}t)^2\}$ in powers of $\mu$ gives the leading behaviour of this expansion,

$$\psi(x, z, t) \sim e^{-ik_0(x-\bar{u}t)} e^{-\mu^2x^2} [G_n(k_0, z, t) + O(\mu^2)] + \text{c.c.},$$

which satisfies the specified lower boundary condition. This is a valid asymptotic approximation for $x \sim O(\mu^{-1})$ and $t \sim O(\mu^{-1})$. Higher-order corrections to this approximation are readily obtained by adding more terms from the full expansion. Further investigations could be carried out using multiple-scale analysis in both space and time \cite{4, 3, 6}. A contour plot of this leading term is shown in Figure 5. This procedure can be applied to find an asymptotic approximation for a wave packet solution to any desired order of accuracy for any lower boundary condition of the form \eqref{31}, provided the Fourier transform of the function $A(x)$ is known in a relatively simple closed form.

5. Discussion

We examined a simple model for the temporal evolution of long internal gravity waves forced at low altitude by a horizontally-periodic lower boundary condition and propagating upwards in a Boussinesq fluid flow with constant background velocity. We first considered the case where the amplitude of the waves at the lower boundary was kept constant. Two different procedures were used to solve the problem resulting in two different expressions, the first of which is the solution derived in \cite{7}. Both expressions take the form of Neumann series, i.e., series of Bessel functions of increasing order. As noted in \cite{7}, these exact solutions can be used as leading-order solutions in weakly-nonlinear analyses and other investigations using more sophisticated models.
They are also useful for testing and validating the results of time-dependent numerical simulations [8, 11].

Depending on what space and time regime is of interest, one expression may be more suitable than the other. If the late-time evolution of the solution is of interest, then the form (12) is more useful, since it clearly indicates the asymptotic behaviour of the solution for $t \gg 1$ and its evolution to the steady state. If the behaviour of the solution at high altitude and finite time is of interest, on the other hand, the form (17) may be more useful. We also considered two cases where the amplitude of the waves at the lower boundary is a function of time, a step function or an exponentially-decaying function.

The solution expressed in the form (17) allows us to obtain asymptotic solutions in the more general situation where a wave packet is generated by a horizontally-localized lower boundary condition comprising a continuous spectrum of horizontal wavenumbers or Fourier modes. In that case, we can evaluate the inverse Fourier transform of (17) to obtain an asymptotic solution. This gives a more realistic representation of internal gravity waves in the atmosphere and can be used as a starting point for investigating waves generated by an obstacle of finite horizontal extent such as an isolated mountain or a mountain range.

Another problem involving wave propagation in geophysical fluid dynamics that gives two equivalent solutions by using two different integration procedures for evaluating an inverse Laplace transform is described by [9]. That problem involves Rossby waves which arise in a geophysical fluid flow from the effects of the Coriolis force. In the configuration studied by [9], the Rossby waves are generated at the northern boundary of a rectangular do-
main on a horizontal plane by a boundary condition analogous to (3) and
propagate southwards and the solution is expressed in two alternative forms
in terms of generalized hypergeometric functions.

We note however that the solutions obtained here are based on a number
of assumptions, namely that there is no shear in the background flow, that
the Boussinesq approximation can be made and that the long-wave limit
can be taken. In the case where there is vertical shear, i.e. where \( \bar{u} \) is a
function of \( z \), there is the possibility of a critical level, an altitude at which
\( \bar{u}(z) \) is equal to \( c \) the phase speed of the waves. Critical level phenomena
have been studied extensively in both linear and nonlinear configurations
and using both analytical and numerical methods. Approximate analytical
solutions valid near the critical level are readily obtained using the method
of Frobenius in the steady linear case and approximate asymptotic solutions
can be obtained in the time-dependent linear case [2], but in general there
are no closed form exact solutions analogous to those obtained here for the
case with constant mean flow speed.

Without the long-wave limit, we would not have been able to obtain
exact time-dependent solutions for the configurations studied here; only ap-
proximate asymptotic expressions would have been possible. However, the
qualitative behaviour of the solutions would be the same and, in particular,
the time-dependent terms would still decay to zero and the solution would
reach a steady state corresponding to the residue at the pole \( s = 0 \) of \( \tilde{\phi}(z, s) \).
The steady-state solution would be proportional to \( \exp(ikx + imz) \), with
\( m = \pm(\frac{N^2}{\bar{u}^2} - \delta k^2)^{1/2} \), where \( \delta = \frac{L_z^2}{L_x^2} \) is the square of the aspect ratio
and \( \text{sgn}(m) = \text{sgn}(k\bar{u}) \). Campbell and Nikitina [6] showed that in a steady
problem with constant mean flow speed and in a time-dependent problem with shear, with a Gaussian forcing function $\exp(-\mu^2 x^2)$, taking into account a non-zero aspect ratio adds terms of $O(\delta \mu^2)$. It is quite straightforward to show that this order of correction would be obtained if a non-zero aspect ratio were to be added to the time-dependent problem with constant mean flow speed in section 4.

References


Figure 1: The solution $\psi(x, z, t) = \phi(z, t) e^{ikx} + \text{c.c.}$ with $\phi(z, t)$ given by (12) at (a) $t = 1$, (b) $t = 5$, (c) $t = 10$, (d) $t = 100$. 
Figure 2: Solution $\psi(x, z, t) = \phi(z, t)e^{ikx} + \text{c.c.}$ obtained with lower boundary condition (20) with $\phi(z, t)$ given by (24) at (a) $t = 5$, (b) $t = 10$, (c) $t = 15$, (d) $t = 20$. 
Figure 3: Solution $\psi(x,z,t) = \phi(z,t)e^{ikx} + \text{c.c.}$ obtained with lower boundary condition $(26)$ ($\gamma = 0.005$) with $\phi(z,t)$ given by $(29)$ at (a) $t=5$, (b) $t=10$, (c) $t=15$, (d) $t=20$. 
Figure 4: Exact solution \( \psi(x, z, t) \) given by (38) obtained with the spatially-localized lower boundary condition (35) with \( a = 2 \) and \( \bar{u} = 0.01 \) at (a) \( t = 10 \), (b) \( t = 20 \), (c) \( t = 90 \).
Figure 5: Leading-order term in the asymptotic series solution for $\psi(x, z, t)$ obtained with the spatially-localized lower boundary condition (31) with amplitude (39) at (a) $t = 1$, (b) $t = 5$, (c) $t = 10$. 

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