New variational principles of symmetric boundary value problems*

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Abstract

The objective of this paper is to establish new variational principles for symmetric boundary value problems. Let V be a Banach space and V^* its topological dual. We shall consider problems of the type $\Lambda u = D\Phi(u)$ where $\Lambda : V \to V^*$ is a linear operator and $\Phi : V \to \mathbb{R}$ is a Gâteaux differentiable convex function whose derivative is denoted by $D\Phi$. It is established that solutions of the latter equation are associated with critical points of functions of the type

$$I_{\lambda,\mu}(u) := \mu\Phi^*(\Lambda u) - \lambda\Phi(u) - \frac{\mu-\lambda}{2}\langle \Lambda u, u\rangle,$$

where λ, μ are two real numbers, Φ^* is the Fenchel dual of the function Φ and $\langle .,.\rangle$ is the duality pairing between V and V^*. By assigning different values to λ and μ one obtains variety of new and classical variational principles associated to the equation $\Lambda u = D\Phi(u)$. Namely, Euler-Lagrange principle (for $\mu = 0, \lambda = 1$ and symmetric Λ), Clarke-Ekeland least action principle (for $\mu = 1, \lambda = 0$ and symmetric Λ), Brezis-Ekeland variational principle ($\mu = 1, \lambda = -1$) and of course many new variational principles such as

$$I_{1,1}(u) = \Phi^*(\Lambda u) - \Phi(u),$$

which corresponds to $\lambda = 1$ and $\mu = 1$. These new potential functions are quite flexible, and can be adapted to easily deal with both nonlinear and homogeneous boundary value problems.

1 Introduction

Let V be a reflexive Banach space, V^* its topological dual and $\langle .,.\rangle$ be the bi-linear duality pairing between V and V^*. Assume that $\Phi : V \to \mathbb{R}$ is convex, , Gâteaux differentiable and lower semi-continuous and that $\Lambda : Dom(\Lambda) \subset V \to V^*$ is a linear symmetric operator. Consider the equation

$$\Lambda u = D\Phi(u), \qquad u \in V \tag{1}$$

where $D\Phi(u)$ stands for the derivative of Φ at u. It follows that the Euler-Lagrange functional corresponding to equation (1) is of the form $F(u) = \frac{1}{2}\langle \Lambda u, u\rangle - \Phi(u)$ and critical points of F are weak solutions of these equation. F. Clarke and I. Ekeland established an interesting dual variational formulation for the Euler-Lagrange functional F where the operator Λ is not necessarily positive and may have an infinite sequence of

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eigenvalues ranging from $-\infty$ to ∞. In fact, they established a one-to-one correspondence between critical points of the functional F and the functional

$$F_{CE}(u) = \frac{1}{2}\langle \Lambda u, u\rangle - \Phi^*(\Lambda u),$$

where $\Phi^* : V^* \to (-\infty, \infty]$ is the Fenchel dual of Φ defined by

$$\Phi^*(p) = \sup_{u\in V}\{\langle u, p\rangle - \Phi(u)\}, \qquad \forall p \in V^*.$$

Recently, using the theory of non-convex self-duality [14, 15, 16, 17], the present author has established a new class of functionals for which their critical points are solutions of (1). Our aim in this work is to use a more direct approach towards establishing new variational principles. As a result, we shall show that one can indeed associate an infinity number of functionals to a given equation of the form (1) for which their critical points are solutions of this equation.
Recall that a linear operator $\Lambda : Dom(\Lambda) \subset V \to V^*$ is said to be positive (resp. negative) if $\langle \Lambda u, u\rangle \geq 0$ (resp. $\langle \Lambda u, u\rangle \leq 0$) for all $u \in Dom(\Lambda)$. Here we state our main theorem in this work.

Theorem 1.1 *Let V be a reflexive Banach space and V^* its topological dual. Let $\Phi : V \to \mathbb{R}$ be a Gâteaux differentiable convex and lower semi-continuous function, and let $\Lambda : Dom(\Lambda) \subset V \to V^*$ be a surjective linear operator such that its domain is dense in V. For each $\lambda, \mu \in \mathbb{R}$ and $u \in V$ define*

$$I_{\lambda,\mu}(u) := \mu\Phi^*(\Lambda u) - \lambda\Phi(u) - \frac{\mu - \lambda}{2}\langle \Lambda u, u\rangle.$$

Then, critical points of $I_{\lambda,\mu}$ are solutions of the equation

$$\Lambda u = D\Phi(u),$$

provided either of the following conditions hold,

1. *Λ is a symmetric operator, $\lambda \neq 0$ and $\mu = 0$;*
2. *Λ is a symmetric operator $\lambda = 0$ and $\mu \neq 0$;*
3. *Λ is a negative operator and $\mu = -\lambda \neq 0$;*
4. *Λ is a positive symmetric operator and $\lambda\mu > 0$;*
5. *Λ is a negative symmetric operator and $\lambda\mu < 0$.*

The surjectivity assumption on the linear operator $\Lambda : Dom(\Lambda) \subset V \to V^*$ in the theorem above can be weakened as explained in Remark 3.5.
We shall now discuss this result in more details by analyzing each item in Theorem 1.1.

Euler-Lagrange principle: In item 1) of Theorem 1.1, the operator Λ is symmetric, $\lambda \neq 0$ and $\mu = 0$. In this case the functional $I_{\lambda,\mu}$ reads as

$$I_{\lambda,0}(u) = -\lambda\Big(\Phi(u) - \frac{1}{2}\langle \Lambda u, u\rangle\Big),$$

and this is nothing but the Euler Lagrange functional corresponding to the equation $\Lambda u = D\Phi(u)$.

Clarke-Ekeland least action principle: In the second item Λ is a symmetric operator, $\lambda = 0$ and $\mu \neq 0$. The functional $I_{\lambda,\mu}$ reads as

$$I_{0,\mu}(u) = \mu\Big(\Phi^*(\Lambda u) - \frac{1}{2}\langle \Lambda u, u\rangle\Big).$$

This is indeed the Clarke-Ekeland least action principle. In fact, Clarke and Ekeland introduced this interesting dual variational formulation for Hamiltonian systems associated with a convex Hamiltonian (see

[4, 5, 6, 7]). Such a duality principle has turned out to be extremely useful for various purposes such as existence of periodic solutions and solutions with minimum period.

Brezis-Ekeland variational principle: In item 3) the assumption is that Λ is a negative operator and $\mu = -\lambda \neq 0$. In this case the functional $I_{\lambda,\mu}$ has the following structure,

$$I_{-\mu,\mu}(u) = \mu\Big(\Phi^*(\Lambda u) + \Phi(u) - \langle \Lambda u, u\rangle\Big)$$

This principle was first proposed by H. Brezis and I. Ekeland for convex Gradient flows [2, 3]. Recently, there has been an extensive study to prove existence and uniqueness for certain partial differential equations using this principle. We refer the interested reader to [1, 11, 18, 19, 21] and references therein for more details on this principle (the bibliography is not exhaustive).

New variational principles: In items 4) and 5) of Theorem 1.1, Λ is assumed to be either a positive symmetric operator with $\lambda\mu > 0$, or a negative symmetric operator with $\lambda\mu < 0$. For example

$$I_{1,1}(u) = \Phi^*(\Lambda u) - \Phi(u) \quad \& \qquad I_{2,1}(u) = \Phi^*(\Lambda u) + \frac{1}{2}\langle \Lambda u, u\rangle - 2\Phi(u)$$

are just two new variational principles among many more for which their critical points are solutions of the equation $\Lambda u = D\Phi(u)$. As shown, these principles associate to an equation several potential functions which can often be used with relative ease compared to other methods such as the use of Euler-Lagrange functions. These potential functions are quite flexible, and can be adapted to easily deal with both nonlinear and homogeneous boundary value problems. Additionally, in most cases the solutions generated using this new method have greater regularity than the solutions obtained using the standard Euler-Lagrange function. In this work we mostly focus on Homogeneous boundary value problems and we address problems with non-linear boundary conditions in our forthcoming work. We refer the interested reader to [14, 15, 16, 17] where some particular cases of these principles were established. We also refer to [12, 13] for some applications in Partial differential equations and Dynamical systems.

Theorem 1.1 applies readily to many differential equations giving new formulations and resolutions. In the following two relatively simple examples we illustrate how the new functionals $I_{\lambda,\mu}$ given in Theorem 1.1 will be useful in the calculus of variations.

Example 1: A semi-linear Elliptic equation. Suppose Ω is a smooth bounded domain in $\mathbb{R}^N$ and consider the following problem,

$$\begin{cases} -\Delta u + u = |u|^{p-2}u + f(x), & x \in \Omega, \\ \frac{\partial u}{\partial n} = 0, & x \in \partial\Omega, \end{cases} \tag{2}$$

where $2 < p < \frac{2N}{N-2}$ and $f \in L^2(\Omega)$. We would like to apply Theorem 1.1 to this problem. Let $V = L^p(\Omega)$. It follows that $V^* = L^q(\Omega)$ where $\frac{1}{p} + \frac{1}{q} = 1$. Define $\Lambda : Dom(\Lambda) \subset V \to V^*$ by $\Lambda(u) = -\Delta u + u$ where

$$Dom(\Lambda) = \{u \in L^p(\Omega); -\Delta u + u \in L^q(\Omega) \text{ and } \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega\}.$$

Problem (2) can be rewritten as $\Lambda u = D\Phi(u)$ where the functional $\Phi : X \to \mathbb{R}$ is defined by

$$\Phi(u) = \frac{1}{p}\int_\Omega |u(x)|^p\, dx + \int_\Omega u(x)f(x)\, dx.$$

An easy calculation shows that Φ^* the Fenchel dual of Φ is of the form

$$\Phi^*(v) = \frac{1}{q}\int_\Omega |v(x) - f(x)|^q\, dx.$$

Also note that for each $u \in Dom(\Lambda)$ we have

$$\langle \Lambda u, u \rangle = \int_\Omega (-\Delta u + u) u \, dx = \int_\Omega \left(|\nabla u|^2 + u^2 \right) dx \geq 0,$$

which shows that Λ is a positive symmetric operator. Thus, Λ satisfies assumptions 1), 2) and 4) of Theorem 1.1. Therefore, critical points of either of the following functionals correspond to solutions of the problem (2):

I. Euler-Lagrange principle for $\lambda = 1, \mu = 0:$

$$I_{1,0} = \Phi(u) - \frac{1}{2}\langle \Lambda u, u \rangle = \frac{1}{p}\int_\Omega |u|^p \, dx + \int_\Omega uf \, dx - \frac{1}{2}\int_\Omega \left(|\nabla u|^2 + u^2 \right) dx;$$

II. Clarke-Ekeland principle for $\lambda = 0, \mu = 1:$

$$I_{0,1} = \Phi^*(\Lambda u) - \frac{1}{2}\langle \Lambda u, u \rangle = \frac{1}{q}\int_\Omega |-\Delta u + u - f|^q \, dx - \frac{1}{2}\int_\Omega \left(|\nabla u|^2 + u^2 \right) dx;$$

III. A new principle without the term $\langle \Lambda u, u \rangle$ for $\lambda = \mu = 1:$

$$I_{1,1} = \Phi^*(\Lambda u) - \Phi(u) = \frac{1}{q}\int_\Omega |-\Delta u + u - f|^q \, dx - \frac{1}{p}\int_\Omega |u|^p \, dx - \int_\Omega uf \, dx;$$

VI. Another new principle for $\lambda = 2$ and $\mu = 1:$

$$\begin{aligned} I_{2,1} &= \Phi^*(\Lambda u) - 2\Phi(u) + \frac{1}{2}\langle \Lambda u, u \rangle \\ &= \frac{1}{q}\int_\Omega |-\Delta u + u - f|^q \, dx - \frac{2}{p}\int_\Omega |u|^p \, dx - 2\int_\Omega uf \, dx + \frac{1}{2}\int_\Omega \left(|\nabla u|^2 + u^2 \right) dx \end{aligned}$$

It is worth noting that in the above example the functionals $I_{1,1}$ and $I_{2,1}$ are continuously differentiable on the Sobolev space $W^{2,q}(\Omega)$ and therefore if $\widetilde{u}$ is a critical point of either $I_{1,1}$ or $I_{2,1}$ then $\widetilde{u} \in W^{2,q}(\Omega)$ is a solution of (2). Thus, any solution obtained by these functionals is slightly more regular than solutions obtained by the standard Euler-Lagrange functional which are at most $H^1(\Omega)$ regular.

Example 2: A semi-linear Parabolic equation. Let Ω be a smooth bounded domain in $\mathbb{R}^N$ and $T > 0$. Consider the following problem,

$$\begin{cases} u_t - \Delta u + |u|^{p-2}u = f(x,t), & (t,x) \in (0,T) \times \Omega, \\ u(x,t) = 0, & x \in \partial\Omega, t \in (0,T) \\ u(x,0) = u(x,T), & x \in \Omega, \end{cases} \tag{3}$$

where $1 < p < 2N/(N-2)$ and $f \in L^2\big(0,T; L^2(\Omega)\big)$. Let $V = L^2\big(0,T; H_0^1(\Omega)\big)$ and $V^* = L^2\big(0,T; H^{-1}(\Omega)\big)$. Define $\Phi : V \to \mathbb{R}$ by

$$\Phi(u) = \frac{1}{2}\int_\Omega \int_0^T |\nabla u|^2 \, dt \, dx + \frac{1}{p}\int_\Omega \int_0^T |u|^p \, dt \, dx - \int_\Omega \int_0^T fu \, dt \, dx,$$

and define $\Lambda : Dom(\Lambda) \subset V \to V^*$ by $\Lambda u = -u_t$ where

$$Dom(\Lambda) = \{u \in V; \, u_t \in V^* \text{ and } u(x,0) = u(x,T) \text{ for all } x \in \Omega\}.$$

Then problem (3) can be rewritten as $\Lambda u = D\Phi(u)$. For each $u \in Dom(\Lambda)$ we have

$$\langle \Lambda u, u \rangle_{V \times V^*} = -\int_\Omega \int_0^T u u_t \, dt \, dx = -\frac{1}{2}\int_\Omega \int_0^T \frac{d}{dt} u^2 \, dt \, dx = \frac{1}{2}\int_\Omega [u(x,0)^2 - u(x,T)^2] \, dx = 0.$$

Note that Λ is not symmetric so only item 3) of Theorem 1.1 applies to this problem. It then follows from item 3) in Theorem 1.1 for $\mu = 1$ and $\lambda = -1$ and Remark 3.5 that every critical point of the functional

$$I_{-1,1} = \Phi^*(\Lambda u) + \Phi(u) - \langle \Lambda u, u \rangle = \Phi^*(-u_t) + \Phi(u),$$

is a solution of the problem (3). On the other hand it is easily seen that the functional $I_{-1,1}$ admits a unique critical point (minimum) since the functional $I_{-1,1}$ is strictly convex, lower semi-continuous and coercive.

In the next section we gather some preliminary results required for the proofs. Section (3) is devoted to address some new variational principles and to prove Theorem 1.1.

2 Preliminaries

In this section we recall some important definitions and results from Convex Analysis and theory of linear operators used in this work. For more details we refer the interested reader to [9, 20].

Let V and W be two real Banach spaces and let $\langle ., . \rangle$ be a bi-linear form on the phase space $V \times W$.

Definition 2.1 *We say that a bi-linear form puts V and W in duality. This duality is said to be separating if,*
(1) for $0 \neq u \in V$, there exists an element $p \in W$ such that $\langle u, p \rangle \neq 0$,
(2) for $0 \neq p \in W$, there exists an element $u \in V$ such that $\langle u, p \rangle \neq 0$.

The weak topology on V induced by $\langle ., . \rangle$ is denoted by $\sigma(V, W)$ and analogously $\sigma(W, V)$ is the weak topology on W. It is known that $\sigma(V, W)$ and $\sigma(W, V)$ are Hausdorff topologies if and only if the duality between V and W is separating. A function $\Psi : V \to \mathbb{R}$ is said to be $\sigma(V, W)-$lower semi-continuous if

$$\Psi(u) \leq \liminf_{n \to \infty} \Psi(u_n),$$

for each $u \in V$ and any sequence u_n approaching u in the weak topology $\sigma(V, W)$. Let $\Psi : V \to \mathbb{R} \cup \{\infty\}$ be a proper convex function. The subdifferential $\partial \Psi$ of Ψ is defined to be the following set-valued operator: if $u \in Dom(\Psi)$, set

$$\partial \Psi(u) = \{p \in W; \langle p, v - u \rangle + \Psi(u) \leq \Psi(v) \text{ for all } v \in V\}$$

and if $u \notin Dom(\Psi)$, set $\partial \Psi(u) = \varnothing$. If Ψ is Gâteaux differentiable at u, denote by $D\Psi(u)$ the derivative of Ψ at u. In this case $\partial \Psi(u) = \{D\Psi(u)\}$.
The Fenchel dual of an arbitrary function Ψ is denoted by Ψ^* that is function on W and is defined by

$$\Psi^*(p) = \sup\{\langle p, u \rangle - \Psi(u); u \in V\}.$$

Clearly $\Psi^* : W \to \mathbb{R} \cup \{\infty\}$ is convex and $\sigma(W, V)-$lower semi-continuous. Consequently $\Psi^{**} : V \to \mathbb{R} \cup \{\infty\}$ is always convex and $\sigma(V, W)-$lower semi-continuous. The following standard result is crucial in the subsequent analysis.

Proposition 2.1 *Let $\Psi : V \to \mathbb{R} \cup \{\infty\}$ be an arbitrary function. The following statements hold:*
(1) $\Psi^{**}(u) \leq \Psi(u)$ for all $u \in V$.
(2) $\Psi(u) + \Psi^*(p) \geq \langle p, u \rangle$ for all $u \in V$ and $p \in W$.
(3) If Ψ is convex and lower-semi continuous then $\Psi^{**} = \Psi$ and the following assertions are equivalent:

- $\Psi(u) + \Psi^*(p) = \langle u, p \rangle$.
- $p \in \partial \Psi(u)$.
- $u \in \partial \Psi^*(p)$.

The following is a crucial property of convex functions. A proof is given in [15] and for the convenience of the reader we shall also sketch the proof here.

Proposition 2.2 *Let V and W be in separating duality and $\Psi : V \to \mathbb{R} \cup \{\infty\}$ be a proper convex function. Suppose Ψ is sub-differentiable at $u, v \in V$. If there exist $p \in \partial\Psi(u)$ and $q \in \partial\Psi(v)$ with*

$$\langle p - q, u - v\rangle = 0 \tag{4}$$

then $p, q \in \partial\Psi(u) \cap \partial\Psi(v)$.

Proof. It follows from $p \in \partial\Psi(u)$ and $q \in \partial\Psi(v)$ that

$$\begin{aligned} \Psi(u) + \Psi^*(p) &= \langle p, u\rangle \\ \Psi(v) + \Psi^*(q) &= \langle q, v\rangle. \end{aligned}$$

Adding up this equalities, we obtain

$$\langle p, u\rangle + \langle q, v\rangle = \Psi(u) + \Psi^*(p) + \Psi(v) + \Psi^*(q)$$

It also follows from (4) that $\langle p, u\rangle + \langle q, v\rangle = \langle p, v\rangle + \langle q, u\rangle$, which together with the above equation imply that

$$\begin{aligned} \langle p, v\rangle + \langle q, u\rangle &= \Psi(u) + \Psi^*(p) + \Psi(v) + \Psi^*(q) \\ &= \Psi(v) + \Psi^*(p) + \Psi(u) + \Psi^*(q) \end{aligned}$$

and therefore

$$\Psi(v) + \Psi^*(p) - \langle p, v\rangle + \Psi(u) + \Psi^*(q) - \langle q, u\rangle = 0.$$

This together with the fact that

$$\Psi(v) + \Psi^*(p) - \langle p, v\rangle \geq 0, \quad \Psi(u) + \Psi^*(q) - \langle q, u\rangle \geq 0,$$

imply that both terms are indeed zero,

$$\begin{aligned} \Psi(v) + \Psi^*(p) - \langle p, v\rangle &= 0, \\ \Psi(u) + \Psi^*(q) - \langle q, u\rangle &= 0. \end{aligned}$$

Thus, it follows from Proposition 2.1 that $p \in \partial\Psi(v)$ and $q \in \partial\Psi(u)$. □

As an important and straightforward consequence of the above Proposition we have the following.

Theorem 2.2 *Let V and W be in separating duality and $\Psi : V \to \mathbb{R} \cup \{\infty\}$ be a proper convex function. Suppose Ψ is Gâteaux differentiable at $u, v \in X$. Then*

$$\langle D\Psi(u) - D\Psi(v), u - v\rangle = 0$$

if and only if $D\Psi(u) = D\Psi(v)$.

Proof. Since Ψ is Gâteaux differentiable at $u, v \in X$, we have $\partial\Psi(u) = \{D\Psi(u)\}$ and $\partial\Psi(v) = \{D\Psi(v)\}$. Set $p = D\Psi(u)$ and $q = D\Psi(v)$. If $\langle D\Psi(u) - D\Psi(v), u - v\rangle = 0$, it follows from the above Proposition that $p, q \in \partial\Psi(u) \cap \partial\Psi(v)$. This implies $D\Psi(u) = D\Psi(v)$. □

For the reader's convenience, we also recall some standard notions about linear operators.

Definition 2.3 *Let V be a reflexive Banach space and V^* its topological dual. A linear operator $\Lambda : Dom(\Lambda) \subset V \to V^*$ is called symmetric if $Dom(\Lambda)$ is dense in V and $\langle \Lambda u, v\rangle = \langle u, \Lambda v\rangle$ for all elements u and v in the domain of Λ. The operator Λ is said to be positive (resp. negative) if $\langle \Lambda u, u\rangle \geq 0$ (resp. $\langle \Lambda u, u\rangle \leq 0$) for all $u \in Dom(\Lambda)$.*

Definition 2.4 *Let V be a reflexive Banach space and V^* its topological dual. Let $\Lambda : Dom(\Lambda) \subset V \to V^*$ be a linear operator. The adjoint Λ^* of Λ is a linear operator from V to V^* defined by*

$$\langle \Lambda^* u, v\rangle = \langle u, \Lambda v\rangle, \qquad \forall v \in Dom(\Lambda),$$

with

$$Dom(\Lambda^*) = \Big\{u \in V;\ sup\{\langle u, \Lambda v\rangle;\ v \in Dom(\Lambda), \|v\|_V \leq 1\} < \infty\Big\}.$$

Note that for a symmetric operator Λ, in general $\Lambda \neq \Lambda^*$ unless Λ is self-adjoint. Throughout this paper we always assume that every linear operator has a dense domain.

Finally, since the functional proposed in Theorems 1.1 may not be Gâteaux differentiable, we are required to give a meaning to a critical point of such a functional.

Definition 2.5 *Let $\Psi : V^* \to \mathbb{R} \cup \{+\infty\}$ be a convex and lower semi-continuous function and let $F : V \to \mathbb{R}$ be Gâteaux differentiable. Assume that $\Lambda : Dom(\Lambda) \subset V \to V^*$ is a linear operator and $\lambda \in \mathbb{R}$ is a scalar. Say that $u \in V$ is a critical point of*

$$I(w) := \Psi(\Lambda w) + F(w) + \lambda\langle \Lambda w, w\rangle,$$

if $I(u)$ is finite and there exists $v \in \partial\Psi(\Lambda u)$ such that

$$\langle v, \Lambda\eta\rangle + \langle DF(u), \eta\rangle + \lambda\langle \Lambda\eta, u\rangle + \lambda\langle \Lambda u, \eta\rangle = 0, \qquad \text{for all } \eta \in Dom(\Lambda).$$

3 Variational principles

In this section we first establish a variational principle for symmetric boundary value problems similar to the one in item 4) of Theorem 1.1 where we replace the sign condition on the operator with a new condition so called the symmetry-condition. We shall also prove Theorem 1.1 at the end of this section. Throughout this section we shall always assume that V is a reflexive Banach space and V^* is its topological dual.

Definition 3.1 *Let $\Phi : V \to \mathbb{R}$ be a Gâteaux differentiable convex function and $\Lambda : Dom(\Lambda) \subset V \to V^*$ be a symmetric operator. Say that the pair (Λ, Φ) satisfies the symmetry-condition at $\tilde{u} \in V$ if for every $\tilde{v} \in \partial\Phi^*(\Lambda\tilde{u}) \cap Dom(\Lambda)$ with $\Lambda\tilde{v} = D\Phi(\tilde{u})$ one can conclude that $\Lambda\tilde{u} = \Lambda\tilde{v}$.*
We say that the pair (Λ, Φ) satisfies the symmetry-condition globally if for each $\tilde{u}, \tilde{v} \in V$ satisfying

$$\begin{cases} \Lambda\tilde{u} = D\Phi(\tilde{v}) \\ \Lambda\tilde{v} = D\Phi(\tilde{u}), \end{cases}$$

one can conclude that $\Lambda\tilde{u} = \Lambda\tilde{v}$.

It is easily seen that if the pair (Λ, Φ) satisfies the symmetry-condition globally then it satisfies the symmetry condition at each $u \in V$. Here is our result for situations in which the symmetry-condition is fulfilled.

Theorem 3.2 *Let $\Phi : V \to \mathbb{R}$ be a Gâteaux differentiable convex and lower semi-continuous function, and let the linear operator $\Lambda : Dom(\Lambda) \subset V \to V^*$ be symmetric and onto. Assume that the following properties hold,*

1. *u is a critical point of*

$$I(w) = \Phi^*(\Lambda w) - \Phi(w).$$

2. *The pair (Λ, Φ) satisfies the symmetry-condition at u.*

Then u is a solution of the equation

$$\Lambda w = D\Phi(w). \tag{5}$$

Conversely, solutions of the equation (5) are always critical points of the functional $I(w)$.

Proof. Suppose u is a critical point of I. Thus, there exists $v \in \partial\Phi^*(\Lambda u)$ such that

$$\langle v, \Lambda\eta\rangle - \langle D\Phi(u), \eta\rangle = 0 \qquad \text{for all } \eta \in Dom(\Lambda). \tag{6}$$

Since Λ is onto, there exists $w \in Dom(\Lambda)$ such that $\Lambda w = D\Phi(u)$. This together with (6) imply that

$$\langle v, \Lambda\eta\rangle = \langle \Lambda w, \eta\rangle = \langle w, \Lambda\eta\rangle \qquad \text{for all } \eta \in Dom(\Lambda), \tag{7}$$

from which we obtain

$$\langle v - w, \Lambda\eta\rangle = 0 \qquad \text{for all } \eta \in Dom(\Lambda).$$

Since Λ is onto the latter expression implies that $w = v$ and therefore $\Lambda v = \Lambda w = D\Phi(u)$. It also follows from $v \in \partial\Phi^*(\Lambda u)$ that $\Lambda u = D\Phi(v)$. Thus,

$$\Lambda v = D\Phi(u) \quad \& \quad \Lambda u = D\Phi(v). \tag{8}$$

Considering (8) and the fact that (Λ, Φ) satisfies the symmetry-condition at u we obtain that $\Lambda u = \Lambda v$. This yields that $\Lambda u = D\Phi(u)$ as desired.

Conversely, suppose u is a solution of problem (5). It follows from $\Lambda u = D\Phi(u)$ that $u \in \partial\Phi^*(\Lambda u)$ and therefore for $\eta \in Dom(\Lambda)$ we obtain

$$\begin{aligned} \langle u, \Lambda\eta\rangle - \langle D\Phi(u), \eta\rangle &= \langle \Lambda u - D\Phi(u), \eta\rangle \\ &= 0, \end{aligned}$$

thereby giving that u is a critical point of I. □

We now discuss some cases where the symmetry-condition holds.

Lemma 3.3 *The pair (Λ, Φ) satisfy the symmetry-condition globally provided either of the following conditions hold.*

1. *The operator Λ is positive.*
2. *The map $w \to \Lambda w + D\Phi(w)$ is injective.*
3. *There exists a constant $M > 0$ such that $\|\Lambda w\|_{V^*} \geq M\|w\|_V$ for all $w \in V$ and $D\Phi$ is Lipschitz with a Lipschitz constant K such that $K < M$.*

Proof. Part 1): it follows from $\Lambda u = D\Phi(v)$ and $\Lambda v = D\Phi(u)$ that $\Lambda u - \Lambda v = D\Phi(v) - D\Phi(u)$. Thus,

$$\langle \Lambda u - \Lambda v, u - v\rangle = \langle D\Phi(v) - D\Phi(u), u - v\rangle = -\langle D\Phi(v) - D\Phi(u), v - u\rangle.$$

Since Λ is positive we have $\langle \Lambda u - \Lambda v, u - v\rangle \geq 0$ and therefore

$$\langle D\Phi(v) - D\Phi(u), v - u\rangle \leq 0. \tag{9}$$

On the other hand Φ is convex and therefore $D\Phi$ is monotone, i.e.

$$\langle D\Phi(v) - D\Phi(u), v - u\rangle \geq 0.$$

The latter inequality with (9) imply that

$$\langle D\Phi(v) - D\Phi(u), v - u\rangle = 0.$$

It now follows from Theorem 2.2 that $D\Phi(v) = D\Phi(u)$ and therefore $\Lambda u = \Lambda v$.
Part 2): it follows from $\Lambda u = D\Phi(v)$ and $\Lambda v = D\Phi(u)$ that $\Lambda u + D\Phi(u) = \Lambda v + D\Phi(v)$ and by the injectivity assumption we obtain $u = v$.
Part 3): it follows that

$$\|\Lambda u - \Lambda v\|_{V^*} \geq M\|v - u\|_V \geq K\|v - u\|_V \geq \|D\Phi(u) - D\Phi(v)\|_{V^*}. \tag{10}$$

Since $\Lambda u = D\Phi(v)$ and $\Lambda v = D\Phi(u)$ we have $\Lambda u - \Lambda v = D\Phi(v) - D\Phi(v)$. Therefore, substituting $\Lambda u - \Lambda v = D\Phi(v) - D\Phi(v)$ in (10) yields that

$$\|\Lambda u - \Lambda v\|_{V^*} \geq M\|v-u\|_V \geq K\|v-u\|_V \geq \|\Lambda u - \Lambda v\|_{V^*},$$

from which we obtain

$$M\|v-u\|_V = K\|v-u\|_V.$$

Since, $M > K$ we must have $u = v$. □

Example: System of Transport equations. Let $a : \Omega \to \mathbb{R}^N$ be a smooth function on a bounded domain Ω of $\mathbb{R}^N$. Consider the first order operator $Aw = a.\nabla w = \Sigma_{i=1}^N a_i \frac{\partial w_i}{\partial x_i}$. Assume that the vector field $\Sigma_{i=1}^N a_i \frac{\partial w_i}{\partial x_i}$ is actually the restriction of a smooth vector field $\Sigma_{i=1}^N \bar{a}_i \frac{\partial w_i}{\partial x_i}$ defined on an open neighborhood of $\bar{\Omega}$ and each $\bar{a}_i$ is a $C^{1,1}$ function on that neighborhood. Consider the system

$$\begin{cases} \epsilon a.\nabla u = \Delta v + |v|^{p-2}v, & x \in \Omega \\ -\epsilon a.\nabla v = \Delta u + |u|^{q-2}, & x \in \Omega \\ u = v = 0, & x \in \partial\Omega \end{cases} \tag{11}$$

We can apply Theorem 3.2 to establish the following result.

Theorem 3.4 *Assume that $div(a) = 0$ on Ω, $1 < p, q < \frac{2N}{N-2}$, $p' = p/(p-1)$ and $q' = q/(q-1)$. Then there exists $\epsilon_0 > 0$ such that for $0 < \epsilon < \epsilon_0$ every critical point of the functional*

$$I(u,v) = \frac{1}{p'}\int_\Omega |\epsilon a.\nabla u - \Delta v|^{p'}\,dx + \frac{1}{q'}\int_\Omega |\epsilon a.\nabla v + \Delta u|^{q'}\,dx - \frac{1}{p}\int_\Omega |v|^p\,dx - \frac{1}{q}\int_\Omega |u|^q\,dx$$

is a solution of the system (11).

Proof. Set $V = L^q(\Omega) \times L^p(\Omega)$ and $V^* = L^{q'}(\Omega) \times L^{p'}(\Omega)$. Define $\Phi : V \to \mathbb{R}$ by

$$\Phi(u,v) = \frac{1}{q}\int_\Omega |u|^q\,dx + \frac{1}{p}\int_\Omega |v|^p\,dx.$$

Let $\Lambda : Dom(\Lambda) \subset V \to V^*$ be the operator $\Lambda(u,v) = (-\epsilon a.\nabla v - \Delta u, \epsilon a.\nabla u - \Delta v)$ with

$$Dom(\Lambda) = \{(u,v) \in V; \Lambda(u,v) \in V^* \quad \& \quad u = v = 0, \quad x \in \partial\Omega\}.$$

Note that Λ is a symmetric operator and for each $(u,v) \in Dom(\Lambda)$ we have

$$\begin{aligned} \langle \Lambda(u,v), (u,v)\rangle &= \int_\Omega |\nabla v|^2\,dx + \int_\Omega |\nabla u|^2\,dx + 2\epsilon\int_\Omega (a.\nabla u)v\,dx \\ &\geq \int_\Omega |\nabla v|^2\,dx + \int_\Omega |\nabla u|^2\,dx - \frac{2\epsilon\|a\|_\infty}{\lambda 1}\|\nabla u\|_{L^2(\Omega;\mathbb{R}^N)}\|\nabla v\|_{L^2(\Omega;\mathbb{R}^N)}, \end{aligned}$$

where λ_1 is the first eigenvalue of $-\Delta$ with Dirichlet boundary condition. The above estimate indeed shows that Λ is positive provided $\epsilon \leq \epsilon_0 := \lambda_1/\|a\|_\infty$. The result now follows from Theorem 3.2 and Lemma 3.3. □

We remark that by standard methods in the calculus of variations like the mountain pass lemma one can prove that the functional I in Theorem 3.4 has at least one critical point when $|1/p - 1/q| < 1/N$. This critical point then will be a solution of the system (11) due to Theorem 3.4. Since in this work we are concerned with new variational principles we do not elaborate about existence results. We conclude this section by proving the main theorem stated in the introduction.

Proof of Theorem 1.1. Let u be a critical point of $I_{\lambda,\mu}$. Then there exists $v \in \partial\Phi^*(\Lambda u)$ such that

$$\mu\langle v, \Lambda\eta\rangle - \lambda\langle D\Phi(u), \eta\rangle - \frac{\mu-\lambda}{2}\langle u, \Lambda\eta\rangle - \frac{\mu-\lambda}{2}\langle \Lambda u, \eta\rangle = 0, \qquad \forall \eta \in Dom(\Lambda). \tag{12}$$

Set $w := 2v - u$ and therefore $v = (u+w)/2$. This together with $v \in \partial\Phi^*(\Lambda u)$ give that

$$\Lambda u = D\Phi(v) = D\Phi(\frac{u+w}{2}). \tag{13}$$

Substituting $v = (u+w)/2$ in (12) implies that

$$\frac{\mu}{2}\langle w, \Lambda\eta\rangle + \frac{\lambda}{2}\langle u, \Lambda\eta\rangle - \lambda\langle D\Phi(u), \eta\rangle - \frac{\mu-\lambda}{2}\langle \Lambda u, \eta\rangle = 0, \qquad \forall \eta \in Dom(\Lambda). \tag{14}$$

If Λ is symmetric we then have

$$\frac{\mu}{2}\langle w, \Lambda\eta\rangle - \frac{\mu}{2}\langle u, \Lambda\eta\rangle + \lambda\langle u, \Lambda\eta\rangle - \lambda\langle D\Phi(u), \eta\rangle = 0, \qquad \forall \eta \in Dom(\Lambda). \tag{15}$$

On the other hand, $v \in \partial\Phi^*(\Lambda u)$ and therefore $\Lambda u = D\Phi(v)$. Plugging this into (15) implies that

$$\frac{\mu}{2}\langle w, \Lambda\eta\rangle - \frac{\mu}{2}\langle u, \Lambda\eta\rangle + \lambda\langle D\Phi(v), \eta\rangle - \lambda\langle D\Phi(u), \eta\rangle = 0, \qquad \forall \eta \in Dom(\Lambda). \tag{16}$$

Proof under assumption 1): by assuming $\mu = 0$ and $\lambda \neq 0$ in (16) we have

$$\langle D\Phi(v), \eta\rangle - \langle D\Phi(u), \eta\rangle = 0, \qquad \forall \eta \in Dom(\Lambda).$$

Since, $Dom(\Lambda)$ is dense, this implies that $D\Phi(u) = D\Phi(v)$ for which together with (13) we obtain $\Lambda u = D\Phi(u)$ and we are done.

Proof under assumption 2): by assuming $\mu \neq 0$ and $\lambda = 0$ in (16) we have

$$\langle w, \Lambda\eta\rangle - \langle u, \Lambda\eta\rangle = 0, \qquad \forall \eta \in Dom(\Lambda).$$

This shows that $w = u$ as Λ is surjective. The result now follows from (13).

Proof under assumption 3): by assuming $\mu = -\lambda \neq 0$ in (14) we obtain

$$\frac{1}{2}\langle w, \Lambda\eta\rangle - \frac{1}{2}\langle u, \Lambda\eta\rangle + \langle D\Phi(u), \eta\rangle - \langle \Lambda u, \eta\rangle = 0, \qquad \forall \eta \in Dom(\Lambda). \tag{17}$$

By substituting $\Lambda u = D\Phi(v)$ and $w = 2v - u$ in (17) we have

$$\langle v - u, \Lambda\eta\rangle = \langle D\Phi(v) - D\Phi(u), \eta\rangle, \qquad \forall \eta \in Dom(\Lambda). \tag{18}$$

Since Λ is surjective, there exists $\eta_0 \in Dom(\Lambda)$ such that $\Lambda\eta_0 = D\Phi(v) - D\Phi(u)$. Plugging η_0 into (18) implies that

$$\langle v - u, \Lambda\eta_0\rangle = \langle D\Phi(v) - D\Phi(u), \eta_0\rangle,$$

from which we have

$$\langle v - u, D\Phi(v) - D\Phi(u)\rangle = \langle \Lambda\eta_0, \eta_0\rangle. \tag{19}$$

Since Λ is a negative operator the right hand side of (19) is non-positive and since $D\Phi$ is monotone the left hand side of (19) is non-negative from which we obtain

$$\langle D\Phi(v) - D\Phi(u), v - u\rangle = 0.$$

It now follows from Theorem 2.2 that $D\Phi(u) = D\Phi(v)$ from which together with (13) we obtain $\Lambda u = D\Phi(u)$.

Proof under assumption 4) and 5): It follows from (16) that

$$\frac{\mu}{2}\langle w - u, \Lambda\eta\rangle = \lambda\langle D\Phi(u) - D\Phi(v), \eta\rangle, \qquad \forall \eta \in Dom(\Lambda). \tag{20}$$

Substituting $w = 2v - u$ in (20) we obtain

$$\frac{\mu}{\lambda}\langle v - u, \Lambda\eta\rangle = \langle D\Phi(u) - D\Phi(v), \eta\rangle, \qquad \forall \eta \in Dom(\Lambda). \tag{21}$$

Since Λ is surjective, there exists $\eta_0 \in Dom(\Lambda)$ such that $\Lambda\eta_0 = D\Phi(v) - D\Phi(u)$. Therefore, it follows from (21) for η_0 that

$$\langle v-u, D\Phi(v) - D\Phi(u)\rangle = -\frac{\lambda}{\mu}\langle \Lambda\eta_0, \eta_0\rangle. \tag{22}$$

Under either assumptions 4) or 5) in the theorem the right hand side of (22) is non-positive and since Φ is convex the left hand side of (22) is non-negative. Therefore,

$$\langle D\Phi(u) - D\Phi(v), v-u\rangle = 0.$$

Again using Theorem 2.2 we have $D\Phi(u) = D\Phi(v)$ and the result follows from (13). $\square$

Remark 3.5 *One can drop the surjectivity assumption on Λ in items 3), 4) and 5) and replace it by a condition on the sign of the operator Λ^*, the adjoint of Λ, as follows:*

- *In item 3), the surjectivity of Λ can be replaced by the condition that Λ^* is a negative operator. Indeed, it follows from (18) in the proof and Definition 2.4 that $v-u \in Dom(\Lambda^*)$. Thus, form (18) we have that $\Lambda^*(v-u) = D\Phi(v) - D\Phi(u)$ and therefore*

$$\langle v-u, \Lambda^*(v-u)\rangle = \langle D\Phi(v) - D\Phi(u), v-u\rangle,$$

from which we obtain $\langle D\Phi(v) - D\Phi(u), v-u\rangle = 0$ and consequently $D\Phi(v) = D\Phi(u)$.

- *In item 4) (resp. item 5)), the surjectivity of Λ can be replaced by the condition that Λ^* is a positive (resp. negative) operator. Indeed, it follows from (21) in the proof that $v-u \in Dom(\Lambda^*)$ and therefore,*

$$-\frac{\mu}{\lambda}\langle v-u, \Lambda^*(v-u)\rangle = \langle D\Phi(v) - D\Phi(u), v-u\rangle$$

This together with the sign condition on Λ^* imply that $\langle v-u, D\Phi(v) - D\Phi(u)\rangle = 0$ and, consequently $D\Phi(v) = D\Phi(u)$.

It is also evident that the surjectivity assumption is not needed in item 1). In item 2) it can be replaced by the condition that Λ has a dense range in V^*.

4 A variational principle on convex sets

Let V be a reflexive Banach space, V^* its topological dual and K be a closed convex subset of V. Assume that $\Phi : V \to \mathbb{R}$ is convex, Gâteaux differentiable and lower semi-continuous and that $\Lambda : Dom(\Lambda) \subset V \to V^*$ is a linear symmetric operator. Our objective is to provide a variational principle for the following equation.

$$\Lambda u = D\Phi(u), \qquad u \in K. \tag{23}$$

Let Φ^* be the Fenchel dual of Φ. Define the function $\Psi_K : V \to (-\infty, +\infty]$ by

$$\Psi_K(u) = \begin{cases} \Phi^*(\Lambda u), & u \in K, \\ +\infty, & u \notin K. \end{cases} \tag{24}$$

Consider the functional $I : V \to (-\infty, +\infty]$ defined by

$$I(w) := \Psi_K(w) - \Phi(w).$$

A point $u \in Dom(\Psi_K)$ is said to be a critical point of I if $D\Phi(u) \in \partial\Psi_K(u)$ or equivalently,

$$\Psi_K(v) - \Psi_K(u) \geq \langle D\Phi(u), v-u\rangle, \qquad \forall v \in V.$$

The above definition for the critical points of I is equivalent to the one given in Definition 2.5. We have the following result.

Theorem 4.1 *Let V be a reflexive Banach space and K be a closed convex subset of V. Let $\Phi : V \to \mathbb{R}$ be a Gâteaux differentiable convex and lower semi-continuous function, and let the linear operator $\Lambda : Dom(\Lambda) \subset V \to V^*$ be symmetric and positive. Assume that u is a critical point of $I(w) = \Psi_K(w) - \Phi(w)$, and that there exists $v \in K$ satisfying the linear equation,*

$$\Lambda v = D\Phi(u).$$

Then $u \in K$ is a solution of the equation

$$\Lambda u = D\Phi(u).$$

Proof. Since u is a critical point of I, it follows from the definition that

$$\Psi_K(w) - \Psi_K(u) \geq \langle D\Phi(u), w - u\rangle, \qquad \forall w \in V. \tag{25}$$

Since $I(u)$ is a finite number we have that $u \in K$ and $\Psi_K(u) = \Phi^*(\Lambda u)$. By assumption, there exists $v \in K$ satisfying $\Lambda v = D\Phi(u)$. Substituting $w = v$ in (25) yields that

$$\Phi^*(\Lambda v) - \Phi^*(\Lambda u) = \Psi_K(v) - \Psi_K(u) \geq \langle D\Phi(u), v - u\rangle = \langle \Lambda v, v - u\rangle. \tag{26}$$

On the other hand it follows from $\Lambda v = D\Phi(u)$ that $u \in \partial\Phi^*(\Lambda v)$ and therefore

$$\Phi^*(\Lambda u) - \Phi^*(\Lambda v) \geq \langle u, \Lambda u - \Lambda v\rangle. \tag{27}$$

Adding up (26) with (27) we obtain

$$0 \geq \langle u, \Lambda u - \Lambda v\rangle + \langle \Lambda v, v - u\rangle.$$

Since Λ is symmetric we obtain that $\langle u - v, \Lambda u - \Lambda v\rangle \leq 0$ from which together with the fact that Λ is non-negative we obtain

$$\langle u - v, \Lambda u - \Lambda v\rangle = 0. \tag{28}$$

By applying Theorem 2.2 to the convex function $w \to \langle \Lambda w, w\rangle$ and taking into account (28), we obtain that $\Lambda u = \Lambda v$. Therefore,

$$\Lambda u = \Lambda v = D\Phi(u),$$

as desired. □

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