On the Topological Centre Problem for Weighted Convolution Algebras

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Abstract

Let $G$ be a locally compact noncompact group. We show that under a very mild assumption on the weight function $w$, the weighted group algebra $L_1(G, w)$ is strongly Arens irregular in the sense of [Dal–Lam–Lau 01]. To this end, we first derive a general factorization theorem for bounded families in the $L_\infty (G, w^{-1})^*$-module $L_\infty (G, w^{-1})$.

1 Introduction

Let $G$ be a locally compact group, and let $w : G \rightarrow (0, \infty)$ be a weight function, i.e., a positive continuous function on $G$ such that $w(st) \leq w(s)w(t)$ for all $s, t \in G$; for convenience we shall assume that $w(e) = 1$, where $e$ is the neutral element of $G$. We will consider the following spaces, normed in such a way that multiplication resp. division by the weight becomes an isometry between the unweighted and the corresponding weighted space (whose norm we will denote by $\| \cdot \|_w$):

\begin{align*}
L_1(G, w) &= \{ f \mid wf \in L_1(G) \} \\
L_\infty (G, w^{-1}) &= \{ f \mid w^{-1}f \in L_\infty (G) \} \\
LUC (G, w^{-1}) &= \{ f \mid w^{-1}f \in LUC(G) \} \\
C_0 (G, w^{-1}) &= \{ f \mid w^{-1}f \in C_0(G) \} \\
M(G, w) &= \{ \mu \mid w\mu \in M(G) \}.
\end{align*}

Then we have $L_\infty (G, w^{-1}) = L_1(G, w)^*$ and $M(G, w) = C_0 (G, w^{-1})^*$. For every $y \in G$, we define $\tilde{\delta}_y := w(y)^{-1} \delta_y$, which is an element of norm one in $M(G, w)$.

Our aim is to show that for all locally compact noncompact groups, the weighted group algebra $L_1(G, w)$ is strongly Arens irregular in the sense of Dales–Lamb–Lau (see [DAL–LAM–LAU 01]), provided the weight satisfies some very mild boundedness condition. Here, strong Arens irregularity means that the topological centre of the bidual algebra $(L_1(G, w)^{**}, \circ)$, equipped with the first Arens product, precisely equals the algebra $L_1(G, w)$ itself, i.e., it is extremally small. This is a generalization of the main result, Thm. 1, of [LAU–LOS 88], where the corresponding assertion is proved for the (unweighted) group algebra $L_1(G)$, to the weighted situation. Although covering a

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by far more general case, our proof is not of higher complexity, if not even simpler, than the one
given in [Lau–Los 88].

In the following, we shall always regard \( L_1(\mathcal{G}, w)^{**} \) as endowed with the first Arens multiplication.
Let us briefly recall the three step construction of the latter, arising from the convolution product
(denoted by “\( \ast \)”) in \( L_1(\mathcal{G}, w) \) via various module actions. – For \( m, n \in L_1(\mathcal{G}, w)^{**}, h \in L_1(\mathcal{G}, w)^* \)
and \( f, g \in L_1(\mathcal{G}, w) \) one defines:
\[
\langle h \circ f, g \rangle := \langle h, f * g \rangle \\
\langle n \circ h, f \rangle := \langle n, h \circ f \rangle \\
\langle m \circ n, h \rangle := \langle m, n \circ h \rangle.
\]
A fairly comprehensive exposition of the basic theory of Arens products is given in [Pal 94], §1.4.
As for topological centres, an excellent source is [LAU–ÜLG 96]. We shall only need the definition
of the latter, which we briefly recall here:
\[
Z_t(L_1(\mathcal{G}, w)^{**}) := \{ m \in L_1(\mathcal{G}, w)^{**} \mid n \mapsto m \circ n \text{ is } w^* - w^* \text{ continuous on } L_1(\mathcal{G}, w)^{**} \}.
\]
We will use the fact (cf. [Grø 90], Prop. 1.3) that, with the natural module operation stemming from
the construction of the first Arens product on \( L_1(\mathcal{G}, w)^{**} \), the equality \( L_\infty(\mathcal{G}, w^{-1}) \odot L_1(\mathcal{G}, w) = \text{LUC (} \mathcal{G}, w^{-1}) \)
holds. Hence, a natural module operation of \( \text{LUC (} \mathcal{G}, w^{-1})^* \) on \( L_\infty(\mathcal{G}, w^{-1}) \) is given by
\[
\langle m \odot h, g \rangle = \langle m, h \odot g \rangle,
\]
where \( m \in \text{LUC (} \mathcal{G}, w^{-1})^* \), \( h \in L_\infty(\mathcal{G}, w^{-1}) \), \( g \in L_1(\mathcal{G}, w) \). It is readily verified that we have
\( m \odot h = \tilde{m} \circ h \), where \( \tilde{m} \) is an arbitrary Hahn-Banach extension of \( m \) to \( L_\infty(\mathcal{G}, w^{-1})^* \).

In the sequel, we shall denote by \( \mathfrak{t}(\mathcal{G}) \) the compact covering number of the group \( \mathcal{G} \), i.e., the least
cardinality of a compact covering of \( \mathcal{G} \). For the sake of brevity, we further introduce the following
terminology (the first part of the definition also appears in [Dal–Lam–Lau 01]).

**Definition 1.1.** (i) A subset \( S \) of \( \mathcal{G} \) will be called dispersed if \( S \) is not contained in any union of a
family of compact subsets of \( \mathcal{G} \), the family having cardinality strictly less than \( \mathfrak{t}(\mathcal{G}) \).

(ii) For a subset \( S \subseteq \mathcal{G} \), we say that the weight \( w \) is diagonally bounded on \( S \) if we have:
\[
\sup_{s \in S} w(s)w\left( s^{-1} \right) < \infty.
\]

Now we can formulate the main result of the present note; we remark that it has very recently also
been obtained independently by Dales–Lamb–Lau in [Dal–Lam–Lau 01], though with a different
proof. In particular, our factorization result, Theorem 2.2, does not appear in [Dal–Lam–Lau 01].

**Theorem 1.2.** Let \( \mathcal{G} \) be a locally compact noncompact group with compact covering number \( \mathfrak{t}(\mathcal{G}) \).
Suppose that there is a dispersed set \( S \subseteq \mathcal{G} \) on which the weight function \( w \) is diagonally bounded.
Then \( L_1(\mathcal{G}, w) \) is strongly Arens irregular.

We wish to stress the following important points regarding our approach:

- We prove a (formal) sharpening of the interesting inclusion contained in Theorem 1.2. Namely,
we will show that for an element \( m \in L_\infty(\mathcal{G}, w^{-1})^* \) in order to belong to \( L_1(\mathcal{G}, w) \), it suffices
that left multiplication by \( m \) be \( w^*-w^* \)-continuous on the \( w^* \)-closure of the set of all Hahn-
Banach extensions of functionals in \( \overline{w} \{ w \} \subseteq \text{Ball (LUC (} \mathcal{G}, w^{-1})^* \} \) to \( L_\infty(\mathcal{G}, w^{-1})^* \). Instead,
the definition of the topological centre demands \( w^* \)-continuity on all of \( L_\infty(\mathcal{G}, w^{-1})^* \).
Then there exists a family \( S \subseteq \mathcal{G} \) on which the weight function \( w \) is diagonally bounded.

Next we present our crucial tool, which is a general factorization theorem for bounded families in \( L_\infty(\mathcal{G}, w^{-1}) \); it provides a generalization of Thm. 2.2 in [NEU 01a] to the weighted situation.

**Theorem 2.2.** Let \( \mathcal{G} \) be a locally compact noncompact group with compact covering number \( \mathfrak{t}(\mathcal{G}) \). Suppose that there exists a dispersed set \( S \subseteq \mathcal{G} \), and closed under finite unions; we denote the corresponding family of compacta by \( (K_\alpha)_{\alpha \in I} \). Set \( \tilde{I} := I \times I \). For \( \tilde{\alpha} = (\alpha, i) \in \tilde{I} \), put \( K_{\tilde{\alpha}} = K_{(\alpha, i)} := K_\alpha \). Then \( (K_{\tilde{\alpha}})_{\tilde{\alpha} \in \tilde{I}} \) is a covering of \( \mathcal{G} \) having the same properties than the original one. Since the set \( S \) is dispersed, by the same reasoning as in Lemma 3 of [Lau–Los 88], we see that there exists a family \( (y_{\tilde{\alpha}})_{\tilde{\alpha} \in \tilde{I}} \subseteq S \) such that

\[
K_{\tilde{\alpha}} y_{\tilde{\alpha}}^{-1} \cap K_{\tilde{\beta}} y_{\tilde{\beta}}^{-1} = \emptyset \quad \forall \tilde{\alpha}, \tilde{\beta} \in \tilde{I}, \tilde{\alpha} \neq \tilde{\beta}.
\]

Set \( S' := \{ y_{\tilde{\alpha}} \mid \tilde{\alpha} \in \tilde{I} \} \). We define, for \( (\alpha, i), (\beta, j) \in \tilde{I} \):

\[
(\alpha, i) \preceq (\beta, j) \iff K_{(\alpha, i)} \subseteq K_{(\beta, j)} \iff K_\alpha \subseteq K_\beta \iff : \alpha \preceq' \beta.
\]

Let \( \mathcal{F} \) be an ultrafilter on \( I \) which dominates the order filter. Define, for \( j \in I \),

\[
\psi_j' := w^*(-\lim_{\beta \to \infty} y_{(\beta,j)}^{-1} y_{\tilde{\beta}}^\sim) \in \overline{w^*}(\mathcal{G}) \subseteq \text{Ball}(LUC(\mathcal{G}, w^{-1})),
\]
and let \( \psi_j \) be arbitrary Hahn-Banach extensions of \( \psi_j' \) to \( L_\infty (\mathcal{G}, w^{-1})^* \).

Since \( w \) is diagonally bounded on \( S' \), we have:

\[
\sup_{s \in S'} \| w \left( s^{-1} \right) \delta_s \|_w = \sup_{s \in S'} w(s)w \left( s^{-1} \right) < \infty.
\]

Thus, the family of functions

\[
H_{(\alpha, i)} := \left( w \left( y_{(\alpha, i)}^{-1} \right) \delta_{y_{(\alpha, i)}} \right) \circ \left( \chi_{K_{(\alpha, i)}} \right) h_i = w \left( y_{(\alpha, i)}^{-1} \right) r_{y_{(\alpha, i)}} \left( \chi_{K_{(\alpha, i)}} \right) h_i
\]

is bounded in \( L_\infty (\mathcal{G}, w^{-1}) \), whence \( (w^{-1}H_{(\alpha, i)}) \) is a bounded family in \( L_\infty (\mathcal{G}) \). By (1), the projections \( r_{y_{(\alpha, i)}} \chi_{K_{(\alpha, i)}} = \chi_{K_{(\alpha, i)}y_{(\alpha, i)}^{-1}} \) are pairwise orthogonal, so that

\[
H := \sum_{\alpha \in I} \sum_{i \in I} w^{-1}H_{(\alpha, i)} \quad (w^* \text{- limits})
\]

defines a function in \( L_\infty (\mathcal{G}) \). Hence, we have

\[
h := \sum_{\alpha \in I} \sum_{i \in I} H_{(\alpha, i)} \in L_\infty (\mathcal{G}, w^{-1}).
\]

Using (1), we obtain for all \((\alpha, i), (\beta, j), (\gamma, k) \in \widetilde{I}\), where \((\gamma, k) \preceq (\beta, j)\):

\[
\chi_{K_{(\gamma, k)}} r_{y_{(\beta, j)}}^{-1} r_{y_{(\alpha, i)}} \left( \chi_{K_{(\alpha, i)}} \right) h_i = \chi_{K_{(\gamma, k)}} \chi_{K_{(\beta, j)}} r_{y_{(\beta, j)}}^{-1} r_{y_{(\alpha, i)}} \left( \chi_{K_{(\alpha, i)}} \right) h_i
\]

\[
= \chi_{K_{(\gamma, k)}} \left[ r_{y_{(\beta, j)}}^{-1} \left( r_{y_{(\beta, j)}} \chi_{K_{(\beta, j)}} \right) r_{y_{(\alpha, i)}} \left( \chi_{K_{(\alpha, i)}} \right) h_i \right]
\]

\[
= \delta_{(\alpha, i), (\beta, j)} \chi_{K_{(\gamma, k)}} h_j.
\]

Taking into account (2), we deduce that for all \( j \in I \) and \((\gamma, k) \in \widetilde{I}\):

\[
\chi_{K_{(\gamma, k)}} (\psi_j \circ h) = w^* - \lim_{\beta \to \infty} \sum_{\alpha \in I} \sum_{i \in I} w \left( y_{(\alpha, i)}^{-1} \right) w \left( y_{(\beta, j)}^{-1} \right)^{-1} \chi_{K_{(\gamma, k)}} r_{y_{(\beta, j)}}^{-1} r_{y_{(\alpha, i)}} \left( \chi_{K_{(\alpha, i)}} \right) h_i
\]

\[
= \chi_{K_{(\gamma, k)}} h_j,
\]

whence the desired factorization formula follows.

\[\square\]

### 3 Strong Arens irregularity of \( L_1 (\mathcal{G}, w) \)

We now come to the proof of Theorem 1.2. – To establish the nontrivial inclusion, let \( m \in Z_t (L_1 (\mathcal{G}, w)^{**}) \). The group \( \mathcal{G} \) being noncompact, we infer from Proposition 2.1 that \( L_1 (\mathcal{G}, w) \) has Mazur’s property of level \( \mathfrak{p} (\mathcal{G}) \). So in order to prove that \( m \in L_1 (\mathcal{G}, w) \), let \( (h_\alpha)_{\alpha \in I} \subseteq L_\infty (\mathcal{G}, w^{-1}) \) be a bounded net converging \( w^* \) to 0, where \(|I| = \mathfrak{p} (\mathcal{G}) \). Thanks to Theorem 2.2, we have the factorization

\[
h_\alpha = \psi_\alpha \circ h = \tilde{\psi}_\alpha \circ h \quad (\alpha \in I)
\]
with \( \psi_\alpha \in \delta_\mathcal{G} \subseteq \text{Ball}(\mathcal{LUC}(\mathcal{G}, w^{-1})^*) \) and \( h \in L_\infty(\mathcal{G}, w^{-1}) \). Here, \( \tilde{\psi}_\alpha \) denotes some arbitrarily chosen Hahn-Banach extension of \( \psi_\alpha \) to \( L_\infty(\mathcal{G}, w^{-1})^* \). We have to show that \( a_\alpha := \langle m, h_\alpha \rangle \xrightarrow{\alpha} 0 \).

Due to the boundedness of \( (h_\alpha)_\alpha \), it suffices to prove that every convergent subnet of \( (a_\alpha)_\alpha \) tends to 0. Let \( (\langle m, h_{\alpha, \beta} \rangle)_\beta \) be such a convergent subnet. Furthermore, let

\[
E := w^* - \lim_{\gamma} \tilde{\psi}_{\alpha, \beta, \gamma} \in \text{Ball} \left( L_\infty(\mathcal{G}, w^{-1})^* \right)
\]

be a \( w^* \)-cluster point of the net \( \left( \tilde{\psi}_{\alpha, \beta} \right)_\beta \subseteq \text{Ball} \left( L_\infty(\mathcal{G}, w^{-1})^* \right) \).

We first note that \( E \odot h = 0 \), since for arbitrary \( g \in L_1(\mathcal{G}, w) \) we obtain:

\[
\langle E \odot h, g \rangle = \langle E, h \odot g \rangle = \lim_{\gamma} \langle \psi_{\alpha, \beta, \gamma}, h \odot g \rangle = \lim_{\gamma} \langle h_{\alpha, \beta, \gamma}, g \rangle = 0.
\]

Now we conclude, using the fact that \( m \in Z_t(L_1(\mathcal{G}, w)^{**}) \):

\[
\lim_{\beta} \langle m, h_{\alpha, \beta} \rangle = \lim_{\gamma} \langle m, h_{\alpha, \beta, \gamma} \rangle = \lim_{\gamma} \langle m, \tilde{\psi}_{\alpha, \beta, \gamma} \odot h \rangle = \lim_{\gamma} \langle m \circ \tilde{\psi}_{\alpha, \beta, \gamma}, h \rangle = \langle m \circ E, h \rangle = \langle m, E \circ h \rangle = 0,
\]

which yields the desired convergence.

References


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