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Amplification of completely bounded operators and Tomiyama's slice maps

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Abstract

Let $(\mathcal{M}, \mathcal{N})$ be a pair of von Neumann algebras, or of dual operator spaces with at least one of them having property S_σ, and let Φ be an arbitrary completely bounded mapping on $\mathcal{M}$. We present an explicit construction of an amplification of Φ to a completely bounded mapping on $\mathcal{M}\overline{\otimes}\mathcal{N}$. Our approach is based on the concept of slice maps as introduced by Tomiyama, and makes use of the description of the predual of $\mathcal{M}\bar{\otimes}\mathcal{N}$ given by Effros and Ruan in terms of the operator space projective tensor product (cf. Effros and Ruan (Internat. J. Math. 1(2) (1990) 163; J. Operator Theory 27 (1992) 179)).

We further discuss several properties of an amplification in connection with the investigations made in May et al. (Arch. Math. (Basel) 53(3) (1989) 283), where the special case $\mathcal{M} = \mathcal{B}(\mathcal{H})$ and $\mathcal{N} = \mathcal{B}(\mathcal{K})$ has been considered (for Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$). We will then mainly focus on various applications, such as a remarkable purely *algebraic* characterization of w^*-continuity using amplifications, as well as a generalization of the so-called Ge–Kadison Lemma (in connection with the uniqueness problem of amplifications). Finally, our study will enable us to show that the essential assertion of the main result in May et al. (1989) concerning completely bounded bimodule homomorphisms actually relies on a basic property of Tomiyama's slice maps.

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1. Introduction

The initial aim of this article is to present an *explicit* construction of the amplification of an arbitrary completely bounded mapping on a von Neumann algebra $\mathcal{M}$ to a completely bounded mapping on $\mathcal{M}\overline{\otimes}\mathcal{N}$, where $\mathcal{N}$ denotes another von Neumann algebra, and to give various applications.

Of course, an amplification of a completely bounded mapping on a von Neumann algebra is easily obtained if the latter is assumed to be *normal* (i.e., w^*-w^*-continuous); cf., e.g., [DEC–HAA85], Lemma 1.5(b). Our point is that we are dealing with not necessarily normal mappings and nevertheless even come up with an explicit formula for an amplification.

We further remark that, of course, as already established by Tomiyama, it is possible to amplify *completely positive* mappings on von Neumann algebras [STR81, Proposition 9.4]. One could then think of Wittstock's decomposition theorem to write an arbitrary completely bounded mapping on the von Neumann algebra $\mathcal{M}$ as a sum of four completely positive ones and apply Tomiyama's result—but the use of the decomposition theorem requires $\mathcal{M}$ to be injective. Moreover, we wish to stress that this procedure would be highly non-constructive: first, the proof of Tomiyama's result uses Banach limits in an essential way, and the decomposition theorem as well is an abstract existence result.

We even go on further to show that the same construction can actually be carried out in case $\mathcal{M}$ and $\mathcal{N}$ are only required to be dual operator spaces, where at least one of them shares property S_σ (the latter is satisfied, e.g., by all semidiscrete von Neumann algebras). In this abstract situation, we do not even have a version of the above-mentioned result of Tomiyama at hand.

Furthermore, our approach yields, in particular, a new and elegant construction of the usual amplification of an operator $\Phi\in\mathcal{CB}(\mathcal{B}(\mathcal{H}))$ to an operator in $\mathcal{CB}(\mathcal{B}(\mathcal{H}\otimes_2\mathcal{K}))=\mathcal{CB}(\mathcal{B}(\mathcal{H})\overline{\otimes}\mathcal{B}(\mathcal{K}))$, i.e., of the mapping

$$\Phi\mapsto\Phi^{(\infty)}$$

introduced in [EFF–KIS87, p. 265] and studied in detail in [MAY–NEU–WIT89] (cf. also [EFF–RUA88, p. 151, Theorem 4.2], where instead of $\mathcal{B}(\mathcal{H})$, an arbitrary dual operator space is considered as the first factor). But of course, the main interest of our approach lies in the fact that it yields an explicit and constructive description of the amplification in a much more general setting—where the second factor $\mathcal{B}(\mathcal{K})$ is replaced by an arbitrary von Neumann algebra or even a dual operator space. In this case, the very definition of the mapping $\Phi\mapsto\Phi^{(\infty)}$ does not make sense, since it is based on the representation of the elements in $\mathcal{B}(\mathcal{H})\overline{\otimes}\mathcal{B}(\mathcal{K})$ as infinite matrices with entries from $\mathcal{B}(\mathcal{H})$.

The crucial idea in our construction is to use the concept of slice maps as introduced by Tomiyama, in connection with the explicit description of the predual $(\mathcal{M}\overline{\otimes}\mathcal{N})_*$, given by Effros and Ruan—cf. [EFF–RUA90], for the case of von Neumann algebras, and [RUA92], for dual operator spaces with the first factor enjoying property S_σ. In these two cases, one has a canonical complete isometry

$$(\mathscr{M}\overline{\otimes}\mathscr{N})_* \overset{\text{cb}}{=} \mathscr{M}_*\hat{\otimes}\mathscr{N}_*, \tag{1}$$

where the latter denotes the projective operator space tensor product. The dichotomic nature of our statements precisely arises from this fact; at this point, we wish to emphasize that, as shown in [RUA92, Corollary 3.7], for a dual operator space $\mathscr{M}$, (1) holding for all dual operator spaces $\mathscr{N}$, is actually equivalent to $\mathscr{M}$ having property S_σ.

Hence, the combination of operator space theory and the classical operator algebraic tool provided by Tomiyama's slice maps enables us to show that the latter actually encodes all the essential properties of an amplification. Summarizing the major advantages of our approach to the amplification problem, we stress the following:

(a) The use of Tomiyama's slice maps gives rise to an explicit formula for the amplification of arbitrary completely bounded mappings with a simple structure.
(b) In particular, the approach is constructive.
(c) In various cases, it is easier to handle amplifications using our formula, since the construction does not involve any (w^*-)limits. As we shall see, the investigation of the amplification mapping is thus mainly reduced to purely algebraic considerations—in contrast to the somehow delicate analysis used in [EFF–RUA88] or [MAY–NEU–WIT89].
(d) Our framework—especially concerning the class of objects allowed as "second factors" of amplifications—is by far more general than what can be found in the literature.

We would finally like to point out that, in view of the above-mentioned characterization of dual operator spaces fulfilling Eq. (1), via property S_σ [RUA92, Corollary 3.7], our approach of the amplification problem seems "best possible" among the approaches satisfying (a)–(d).

The paper is organized as follows. First, we provide the necessary terminological background from the theory of operator algebras and operator spaces. We further give the exact definition of what we understand by an "amplification". The construction of the latter in our general situation is presented in Section 3.

Section 4 contains a discussion of this amplification mapping under various aspects, e.g., relating it to results from [DEC–HAA85,MAY–NEU–WIT89]. We derive a stability result concerning the passage to von Neumann subalgebras or dual operator subspaces, respectively. Furthermore, an alternative (non-constructive) description of our amplification mapping is given, which, in particular, immediately implies that the latter preserves complete positivity.

In Section 5, we introduce an analogous concept of "amplification in the first variable" which leads to an interesting comparison with the usual tensor product of operators on Hilbert spaces. The new phenomenon we encounter here is that the amplifications, one in the first, the other in the second variable, of two completely bounded mappings $\Phi \in \mathscr{CB}(\mathscr{M})$ and $\Psi \in \mathscr{CB}(\mathscr{N})$ do *not* commute in general. They do, though, if Φ or Ψ happens to be normal (cf. Theorem 5.1 below). But our main result of this section, which may even seem a little astonishing, states that for most von Neumann algebras $\mathscr{M}$ and $\mathscr{N}$, the converse holds true. This means that the apparently weak—and purely algebraic!—condition of commutation of the amplification of Φ with every amplified Ψ already suffices to ensure that Φ is normal (Theorem 5.4). The proof of the latter assertion is rather technical and involves the use of countable spiral nebulas, a concept first introduced by Hofmeier–Wittstock in [HOF–WIT97].

In the section thereafter, we turn to the uniqueness problem of amplifications. Of course, an amplification which only meets the obvious algebraic condition is far from being unique; but we shall show that with some (weak) natural assumptions, one can indeed establish uniqueness. One of our two positive results actually yields a considerable generalization (Proposition 6.2) of a theorem which has proved useful in the solution of the splitting problem for von Neumann algebras (cf. [GE–KAD96,STR–ZSI99]), and which is sometimes referred to as the Ge–Kadison Lemma.

Finally, Section 7 is devoted to a completely new approach to a theorem of May, Neuhardt and Wittstock regarding a module homomorphism property of the amplified mapping. We prove a version of their result in our context, which reveals that the essential assertion in fact relies on a fundamental and elementary module homomorphism property shared by Tomiyama's slice maps.

2. Preliminaries and basic definitions

Since a detailed account of the theory of operator spaces, as developed by Blecher–Paulsen, Effros–Ruan, Pisier et al., is by now available via different sources [EFF–RUA00,PIS00,WIT et al. 99], as for its basic facts, we shall restrict ourselves to fix the terminology and notation we use.

Let X and Y be operator spaces. We denote by $\mathscr{CB}(X, Y)$ the operator space of completely bounded maps from X to Y, endowed with the completely bounded norm $\|\cdot\|_{cb}$. We further write $\mathscr{CB}(X)$ for $\mathscr{CB}(X, X)$. If $\mathscr{M}$ and $\mathscr{R}$ are von Neumann algebras with $\mathscr{R} \subseteq \mathscr{M}$, then we denote by $\mathscr{CB}_{\mathscr{R}}(\mathscr{M})$ the space of all completely bounded $\mathscr{R}$-bimodule homomorphisms on $\mathscr{M}$. If two operator spaces X and Y are completely isometric, then we write $X \overset{cb}{=} Y$.

An operator space Y is called a *dual operator space* if there is an operator space X such that Y is completely isometric to X^*. In this case, X is called an *operator predual*, or *predual*, for short, of the operator space Y. In general, the predual of a dual operator space is not unique (up to complete isometry); see [RUA92, p. 180], for an easy example.

For a dual operator space Y with a given predual X, we denote by $\mathscr{CB}^{\sigma}(Y)$ the subspace of $\mathscr{CB}(Y)$ consisting of normal (i.e., $\sigma(Y, X)$-$\sigma(Y, X)$-continuous) mappings.

If $\mathscr{H}$ is a Hilbert space, then, of course, every w^*-closed subspace Y of $\mathscr{B}(\mathscr{H})$ is a dual operator space with a canonical predual $\mathscr{B}(\mathscr{H})_*/Y_\perp$, where $Y_\perp$ denotes the preannihilator of Y in $\mathscr{B}(\mathscr{H})_*$. We denote this canonical predual by Y_*. Conversely, if Y is a dual operator space with a given predual X, there is a w^*-homeomorphic complete isometry of Y onto a w^*-closed subspace $\tilde{Y}$ of $\mathscr{B}(\mathscr{H})$, for some suitable Hilbert space $\mathscr{H}$ (see [EFF–RUA90, Proposition 5.1]). Following the terminology of [EFF–KRA–RUA92, p. 3], we shall call such a map a *w^*-embedding*. In this case, X is completely isometric to the canonical predual $\tilde{Y}_*$, and we shall identify Y with $\tilde{Y}$ (cf. the discussion in [RUA92, p. 180]).

We denote by $\mathscr{H} \otimes_2 \mathscr{K}$ the Hilbert space tensor product of two Hilbert spaces $\mathscr{H}$ and $\mathscr{K}$. For operator spaces X and Y, we write $X \vee \otimes Y$ and $X \hat{\otimes} Y$ for the injective and projective operator space tensor product, respectively. We recall that, for operator spaces X and Y, we have a canonical complete isometry:

$$(X \hat{\otimes} Y)^* \overset{\mathrm{cb}}{=} \mathscr{CB}(X, Y^*).$$

If $\mathscr{M}$ and $\mathscr{N}$ are von Neumann algebras, we denote as usual by $\mathscr{M} \overline{\otimes} \mathscr{N}$ the von Neumann algebra tensor product. More generally, let V^* and W^* be dual operator spaces with given w^*-embeddings $V^* \subseteq \mathscr{B}(\mathscr{H})$ and $W^* \subseteq \mathscr{B}(\mathscr{K})$. Then the w^*-spatial tensor product of V^* and W^*, still denoted by $V^* \overline{\otimes} W^*$, is defined to be the w^*-closure of the algebraic tensor product $V^* \otimes W^*$ in $\mathscr{B}(\mathscr{H} \otimes_2 \mathscr{K})$. The corresponding w^*-embedding of $V^* \overline{\otimes} W^*$ determines a predual $(V^* \overline{\otimes} W^*)_*$. Since we have

$$(V^* \overline{\otimes} W^*)_* \overset{\mathrm{cb}}{=} V \otimes_{\mathrm{nuc}} W$$

completely isometrically (where $\otimes_{\mathrm{nuc}}$ denotes the nuclear operator space tensor product), the w^*-spatial tensor product of dual operator spaces does not depend on the given w^*-embedding (cf. [EFF–KRA–RUA93, p. 127]).

In the following, whenever we are speaking of "a dual operator space", we always mean "a dual operator space $\mathscr{M}$ with a given (operator) predual, denoted by $\mathscr{M}_*$".

We will now present, in the general framework of dual operator spaces, the construction of left and right slice maps, following the original concept first introduced by Tomiyama for von Neumann algebras ([TOM70, p. 4]; cf. also [KRA91, p. 119]). Let $\mathscr{M}$ and $\mathscr{N}$ be dual operator spaces. For any $\tau \in \mathscr{N}_*$ and any $u \in \mathscr{M} \overline{\otimes} \mathscr{N}$, the mapping

$$\mathscr{M}_* \ni \rho \mapsto \langle u, \rho \otimes \tau \rangle$$

is a continuous linear functional on $\mathscr{M}_*$, hence defines an element $L_\tau(u)$ of $\mathscr{M}$. As is easily verified, L_τ is the unique w^*-continuous linear map from $\mathscr{M} \overline{\otimes} \mathscr{N}$ to $\mathscr{M}$ such that

$$L_\tau(S \otimes T) = \langle \tau, T \rangle S \quad (S \in \mathscr{M},\ T \in \mathscr{N});$$

it will be called the left slice map associated with τ. In a completely analogous fashion, we see that for every $\rho \in \mathscr{M}_*$, there is a unique w^*-continuous linear map R_ρ

from $\mathcal{M}\overline{\otimes}\mathcal{N}$ to $\mathcal{N}$ such that

$$R_\rho(S\otimes T) = \langle \rho, S\rangle T \quad (S\in\mathcal{M}, T\in\mathcal{N}),$$

which we call the right slice map associated with ρ.

We note that, as an immediate consequence, we have for every $\rho\in\mathcal{M}_*$, $\tau\in\mathcal{N}_*$ and $u\in\mathcal{M}\overline{\otimes}\mathcal{N}$:

$$\langle L_\tau(u), \rho\rangle = \langle u, \rho\otimes\tau\rangle \quad \text{and} \quad \langle R_\rho(u), \tau\rangle = \langle u, \rho\otimes\tau\rangle.$$

Let us now briefly recall the definition of property S_σ for dual operator spaces, as introduced by Kraus [KRA83, Definition 1.4]. To this end, consider two dual operator spaces $\mathcal{M}$ and $\mathcal{N}$ with w^*-embeddings $\mathcal{M}\subseteq\mathcal{B}(\mathcal{H})$ and $\mathcal{N}\subseteq\mathcal{B}(\mathcal{K})$. Then the Fubini product of $\mathcal{M}$ and $\mathcal{N}$ with respect to $\mathcal{B}(\mathcal{H})$ and $\mathcal{B}(\mathcal{K})$ is defined through

$$\mathcal{F}(\mathcal{M},\mathcal{N},\mathcal{B}(\mathcal{H}),\mathcal{B}(\mathcal{K})) := \{u\in\mathcal{B}(\mathcal{H}\otimes_2\mathcal{K}) \mid L_\tau(u)\in\mathcal{M}, R_\rho(u)\in\mathcal{N} \text{ for all } \rho\in\mathcal{B}(\mathcal{H})_*, \tau\in\mathcal{B}(\mathcal{K})_*\}.$$

By the important result [RUA92, Proposition 3.3], we have

$$\mathcal{F}(\mathcal{M},\mathcal{N},\mathcal{B}(\mathcal{H}),\mathcal{B}(\mathcal{K})) \overset{\text{cb}}{=} (\mathcal{M}_*\hat{\otimes}\mathcal{N}_*)^*,$$

which in particular shows that the Fubini product actually does not depend on the particular choice of Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$ (cf. also [KRA83, Remark 1.2]). Hence, we may denote the Fubini product simply by $\mathcal{F}(\mathcal{M},\mathcal{N})$. Now, a dual operator space $\mathcal{M}$ is said to have property S_σ if

$$\mathcal{F}(\mathcal{M},\mathcal{N}) \overset{\text{cb}}{=} \mathcal{M}\overline{\otimes}\mathcal{N}$$

holds for every dual operator space $\mathcal{N}$.

The question of whether every von Neumann algebra has property S_σ has been answered in the negative by Kraus [KRA91, Theorem 3.3]; in fact, there exist even separably acting factors without property S_σ [KRA91, Theorem 3.12].

But every semidiscrete von Neumann algebra—and hence, every type I von Neumann algebra [EFF–LAN77, Proposition 3.5]—has property S_σ, as shown by Kraus in [KRA83, Theorem 1.9]. We finally refer to the very interesting results obtained by Effros et al. showing that property S_σ is actually equivalent to various operator space approximation properties (cf. [KRA91, Theorem 2.6; EFF–KRA–RUA93, Theorem 2.4]).

To avoid long paraphrasing, let us now introduce some terminology suitable for the statement of our results.

Definition 2.1. Let $\mathcal{M}$ and $\mathcal{N}$ be either

(i) arbitrary von Neumann algebras, or
(ii) dual operator spaces such that at least one of them has property S_σ.

Then, in both cases, we call ($\mathcal{M}$, $\mathcal{N}$) an *admissible pair*.

If $(\mathcal{M}, \mathcal{N})$ is an admissible pair, then we have a canonical complete isometry:

$$(\mathcal{M}\overline{\otimes}\mathcal{N})_* \overset{\text{cb}}{=} \mathcal{M}_*\hat{\otimes}\mathcal{N}_*.$$

In case $\mathcal{M}$ and $\mathcal{N}$ are both von Neumann algebras or dual operator spaces with $\mathcal{M}$ having property S_σ, this follows from [RUA92, Corollaries 3.6 and 3.7], respectively.

If $\mathcal{M}$ and $\mathcal{N}$ are dual operator spaces with $\mathcal{N}$ enjoying property S_σ, the above identification is seen as follows. (We remark that the argument below, together with [RUA92, Corollary 3.7], shows that property S_σ of a dual operator space $\mathcal{M}$—resp. $\mathcal{N}$—is actually characterized by the equality $(\mathcal{M}\overline{\otimes}\mathcal{N})_* \overset{\text{cb}}{=} \mathcal{M}_*\hat{\otimes}\mathcal{N}_*$ holding for all dual operator spaces $\mathcal{N}$—resp. $\mathcal{M}$.)

Let V be an arbitrary operator space. Then, by [KRA91, Theorem 2.6], V^* has property S_σ if and only if it has the weak* operator approximation property (in the terminology of [EFF–KRA–RUA93]). Owing to [EFF–KRA–RUA93, Theorem 2.4(6)], this is equivalent to

$$V\hat{\otimes}W \overset{\text{cb}}{=} V\otimes_{\text{nuc}}W$$

holding for all operator spaces W. But since both the projective and the injective operator space tensor product are symmetric, this in turn is equivalent to the canonical mapping

$$\Phi : W\hat{\otimes}V \to W \vee\otimes\ V$$

being one-to-one for all operator spaces W. But this is the case if and only if we have

$$W\otimes_{\text{nuc}}V \overset{\text{cb}}{=} W\hat{\otimes}V$$

for all operator spaces W, which finally is of course equivalent to

$$(W^*\overline{\otimes}V^*)_* \overset{\text{cb}}{=} W\hat{\otimes}V$$

holding for all operator spaces W.

We shall now make precise what we mean by an amplification, in particular, which topological requirements it should meet beyond the obvious algebraic ones.

Definition 2.2. Let $(\mathcal{M}, \mathcal{N})$ be an admissible pair. A completely bounded linear mapping

$$\chi : \mathcal{CB}(\mathcal{M}) \to \mathcal{CB}(\mathcal{M}\overline{\otimes}\mathcal{N}),$$

satisfying the algebraic amplification condition (AAC)

$$\chi(\Phi)(S\otimes T) = \Phi(S)\otimes T$$

for all $\Phi \in \mathcal{CB}(\mathcal{M})$, $S \in \mathcal{M}$, $T \in \mathcal{N}$, will be called an *amplification*, if in addition it enjoys the following properties:

(i) χ is a complete isometry,
(ii) χ is multiplicative (hence, by (i), an isometric algebra isomorphism onto the image),
(iii) χ is w^*-w^*-continuous,
(iv) $\chi(\mathcal{CB}^\sigma(\mathcal{M})) \subseteq \mathcal{CB}^\sigma(\mathcal{M} \overline{\otimes} \mathcal{N})$, i.e., normality of the mapping is preserved.

Remark 2.3. In particular, χ is unital. For $\chi(\mathrm{id}_{\mathcal{M}})$ coincides with $\mathrm{id}_{\mathcal{M} \overline{\otimes} \mathcal{N}}$ on all elementary tensors $S \otimes T \in \mathcal{M} \overline{\otimes} \mathcal{N}$ due to (AAC); furthermore, by (iv), it is normal, whence we obtain $\chi(\mathrm{id}_{\mathcal{M}}) = \mathrm{id}_{\mathcal{M} \overline{\otimes} \mathcal{N}}$ on the whole space $\mathcal{M} \overline{\otimes} \mathcal{N}$.

In [MAY–NEU–WIT89]—cf. also [EFF–KIS87, p. 265]—for every $\Phi \in \mathcal{CB}(\mathcal{B}(\mathcal{H}))$, the authors obtain a mapping $\Phi^{(\infty)} \in \mathcal{CB}(\mathcal{B}(\mathcal{H} \otimes_2 \mathcal{K})) = \mathcal{CB}(\mathcal{B}(\mathcal{H}) \overline{\otimes} \mathcal{B}(\mathcal{K}))$ which obviously fulfills the algebraic amplification condition (AAC). The definition of $\Phi^{(\infty)}$ is carried out in detail in [MAY–NEU–WIT89, p. 284, Section 2] (there, the Hilbert space $\mathcal{K}$ is written in the form $\ell_2(I)$ for some suitable set I). We restrict ourselves to recall that, given a representation of $u \in \mathcal{B}(\mathcal{H} \otimes_2 \mathcal{K})$ by an infinite matrix $[u_{i,j}]_{i,j}$, where $u_{i,j} \in \mathcal{B}(\mathcal{H})$, one has:

$$\Phi^{(\infty)}([u_{i,j}]_{i,j}) = [\Phi(u_{i,j})]_{i,j}.$$

In fact, as is well known, more than only (AAC) is satisfied. Indeed, we have

Theorem 2.4. *The mapping*

$$\mathcal{CB}(\mathcal{B}(\mathcal{H})) \to \mathcal{CB}(\mathcal{B}(\mathcal{H}) \overline{\otimes} \mathcal{B}(\mathcal{K}))$$

$$\Phi \mapsto \Phi^{(\infty)}$$

fulfills the above requirements of an amplification.

Proof. This follows from [MAY–NEU–WIT89, p. 284, Section 2, Proposition 2.2]; condition (iv) in Definition 2.2 is shown in [HOF95, Theorem 3.1]. $\square$

We close this section by remarking that several properties of the mapping $\Phi \mapsto \Phi^{(\infty)}$ are studied in a more general situation by Effros and Ruan in [EFF–RUA88], see p. 151 and Theorem 4.2.

3. The main construction

We now come to our central result concerning the existence of an amplification in the sense of Definition 2.2 and, which is most important, presenting its explicit form.

Theorem 3.1. *Let* $(\mathcal{M}, \mathcal{N})$ *be an admissible pair. Then there exists an amplification*

$$\chi : \mathcal{CB}(\mathcal{M}) \to \mathcal{CB}(\mathcal{M}\overline{\otimes}\mathcal{N})$$

in the sense of Definition 2.2. *The amplification is explicitly given by*

$$\langle \chi(\Phi)(u), \rho \otimes \tau \rangle = \langle \Phi(L_\tau(u)), \rho \rangle,$$

where $\Phi \in \mathcal{CB}(\mathcal{M})$, $u \in \mathcal{M}\overline{\otimes}\mathcal{N}$, $\rho \in \mathcal{M}_*$, $\tau \in \mathcal{N}_*$. *Here,* L_τ *denotes the left slice map associated with* τ.

Proof. We construct a complete quotient mapping

$$\kappa : \mathcal{CB}(\mathcal{M}\overline{\otimes}\mathcal{N})_* \to \mathcal{CB}(\mathcal{M})_*$$

in such a way that $\chi := \kappa^*$ will enjoy the desired properties.

We first note that

$$\mathcal{CB}(\mathcal{M})_* \overset{\text{cb}}{=} \mathcal{M}_* \hat{\otimes} \mathcal{M} \tag{2}$$

and, analogously,

$$\mathcal{CB}(\mathcal{M}\overline{\otimes}\mathcal{N})_* \overset{\text{cb}}{=} (\mathcal{M}\overline{\otimes}\mathcal{N})_* \hat{\otimes} (\mathcal{M}\overline{\otimes}\mathcal{N}) \tag{3}$$

with completely isometric identifications.

We further write

$$W : \mathcal{CB}(\mathcal{M}\overline{\otimes}\mathcal{N}, \mathcal{CB}(\mathcal{N}_*, \mathcal{M})) \to \mathcal{CB}(\mathcal{N}_* \hat{\otimes} (\mathcal{M}\overline{\otimes}\mathcal{N}), \mathcal{M})$$

for the canonical complete isometry. For $\tau \in \mathcal{N}_*$, let us consider the left slice map

$$L_\tau : \mathcal{M}\overline{\otimes}\mathcal{N} \to \mathcal{M},$$

where

$$L_\tau(S \otimes T) = \langle \tau, T \rangle S$$

for $S \in \mathcal{M}$, $T \in \mathcal{N}$. In [RUA92, Theorem 3.4] (and Corollaries 3.6, 3.7), Ruan constructs a complete isometric isomorphism (cf. also [EFF-RUA90, Theorem 3.2])

$$\theta : \mathcal{M}_* \hat{\otimes} \mathcal{N}_* \to (\mathcal{M}\overline{\otimes}\mathcal{N})_*,$$

where for elementary tensors we have ($S \in \mathcal{M}$, $T \in \mathcal{N}$, $\rho \in \mathcal{M}_*$, $\tau \in \mathcal{N}_*$):

$$\langle \theta(\rho \otimes \tau), S \otimes T \rangle = \langle \rho, S \rangle \langle \tau, T \rangle. \tag{4}$$

Thus for all $\rho \in \mathcal{M}_*$ and $\tau \in \mathcal{N}_*$, we see that $\theta(\rho \otimes \tau) = \rho \otimes \tau$ on the algebraic tensor product $\mathcal{M} \otimes \mathcal{N}$; but since $\theta(\rho \otimes \tau), \rho \otimes \tau \in (\mathcal{M}\overline{\otimes}\mathcal{N})_*$ are normal functionals, on the whole space $\mathcal{M}\overline{\otimes}\mathcal{N}$ we obtain for all $\rho\in\mathcal{M}_*$, $\tau\in\mathcal{N}_*$:

$$\theta(\rho\otimes\tau) = \rho\otimes\tau. \tag{5}$$

Hence, in the sequel, we shall not explicitly note θ when applied on elementary tensors, except for special emphasis.

We now obtain a complete isometry:

$$\theta^* : \mathcal{M}\overline{\otimes}\mathcal{N} \to \mathcal{CB}(\mathcal{N}_*, \mathcal{M}).$$

For each $\tau\in\mathcal{N}_*$, we define a linear continuous mapping

$$\varphi_\tau : \mathcal{M}\overline{\otimes}\mathcal{N} \to \mathcal{M}$$

through

$$\varphi_\tau(u) := \theta^*(u)(\tau)$$

for $u\in\mathcal{M}\overline{\otimes}\mathcal{N}$. One immediately deduces from the definition that φ_τ is normal. Furthermore, φ_τ satisfies

$$\varphi_\tau(S\otimes T) = \langle \tau, T\rangle S$$

for all $S\in\mathcal{M}$, $T\in\mathcal{N}$. For if $\rho\in\mathcal{M}_*$, we have:

$$\begin{aligned}\langle \varphi_\tau(S\otimes T), \rho\rangle &= \langle \theta^*(S\otimes T)(\tau), \rho\rangle \\ &= \langle \theta^*(S\otimes T), \rho\otimes\tau\rangle \\ &= \langle \theta(\rho\otimes\tau), S\otimes T\rangle \\ &\overset{(4)}{=} \langle \rho, S\rangle\langle \tau, T\rangle.\end{aligned}$$

Hence, φ_τ coincides with the left slice map L_τ on all elementary tensors in $\mathcal{M}\overline{\otimes}\mathcal{N}$, and since both mappings are normal, we finally see that $\varphi_\tau = L_\tau$. Thus we obtain

$$\theta^*(u)(\tau) = L_\tau(u)$$

for all $u\in\mathcal{M}\overline{\otimes}\mathcal{N}$, $\tau\in\mathcal{N}_*$. We note that, of course,

$$\theta^*\in\mathcal{CB}(\mathcal{M}\overline{\otimes}\mathcal{N}, \mathcal{CB}(\mathcal{N}_*, \mathcal{M})).$$

Hence, by definition of W, we have:

$$W(\theta^*)\in\mathcal{CB}(\mathcal{N}_*\hat{\otimes}(\mathcal{M}\overline{\otimes}\mathcal{N}), \mathcal{M}).$$

Thanks to the functorial property of the projective operator space tensor product, we now get a completely bounded mapping

$$\mathrm{id}_{\mathcal{M}_*} \hat{\otimes} W(\theta^*) : \mathcal{M}_* \hat{\otimes} \mathcal{N}_* \hat{\otimes} (\mathcal{M} \overline{\otimes} \mathcal{N}) \rightarrow \mathcal{M}_* \hat{\otimes} \mathcal{M}.$$

What remains to be done, is the passage from $(\mathcal{M} \overline{\otimes} \mathcal{N})_* \hat{\otimes} (\mathcal{M} \overline{\otimes} \mathcal{N})$ to $\mathcal{M}_* \hat{\otimes} \mathcal{N}_* \hat{\otimes} (\mathcal{M} \overline{\otimes} \mathcal{N})$. This is accomplished by the complete quotient mapping

$$\theta^{-1} \hat{\otimes} \mathrm{id}_{\mathcal{M} \overline{\otimes} \mathcal{N}} : (\mathcal{M} \overline{\otimes} \mathcal{N})_* \hat{\otimes} (\mathcal{M} \overline{\otimes} \mathcal{N}) \rightarrow \mathcal{M}_* \hat{\otimes} \mathcal{N}_* \hat{\otimes} (\mathcal{M} \overline{\otimes} \mathcal{N}).$$

Now, defining

$$\kappa := (\mathrm{id}_{\mathcal{M}_*} \hat{\otimes} W(\theta^*)) \circ (\theta^{-1} \hat{\otimes} \mathrm{id}_{\mathcal{M} \overline{\otimes} \mathcal{N}}) : (\mathcal{M} \overline{\otimes} \mathcal{N})_* \hat{\otimes} (\mathcal{M} \overline{\otimes} \mathcal{N}) \rightarrow \mathcal{M}_* \hat{\otimes} \mathcal{M},$$

we obtain a completely bounded mapping which has the desired properties, as we shall now prove.

In view of (2) and (3), we see that we have indeed constructed a completely bounded mapping

$$\kappa : \mathcal{CB}(\mathcal{M} \overline{\otimes} \mathcal{N})_* \rightarrow \mathcal{CB}(\mathcal{M})_*,$$

and it remains to verify the properties of the mapping

$$\chi := \kappa^* : \mathcal{CB}(\mathcal{M}) \rightarrow \mathcal{CB}(\mathcal{M} \overline{\otimes} \mathcal{N}).$$

First, we remark the following:

$(*)$ Since the mapping $\theta : \mathcal{M}_* \hat{\otimes} \mathcal{N}_* \rightarrow (\mathcal{M} \overline{\otimes} \mathcal{N})_*$ is a complete isometry, it is readily seen that $\theta(\mathcal{M}_* \otimes \mathcal{N}_*) \otimes (\mathcal{M} \overline{\otimes} \mathcal{N})$ is norm dense in $(\mathcal{M} \overline{\otimes} \mathcal{N})_* \hat{\otimes} (\mathcal{M} \overline{\otimes} \mathcal{N})$. Every element of the space $\mathcal{CB}(\mathcal{M} \overline{\otimes} \mathcal{N}) \overset{\mathrm{cb}}{=} ((\mathcal{M} \overline{\otimes} \mathcal{N})_* \hat{\otimes} (\mathcal{M} \overline{\otimes} \mathcal{N}))^*$ is thus completely determined by its values on $\theta(\mathcal{M}_* \otimes \mathcal{N}_*) \otimes (\mathcal{M} \overline{\otimes} \mathcal{N})$.

By construction, we see that for all $\Phi \in \mathcal{CB}(\mathcal{M})$, $u \in \mathcal{M} \overline{\otimes} \mathcal{N}$, $\rho \in \mathcal{M}_*$ and $\tau \in \mathcal{N}_*$:

$$\begin{aligned} \langle \chi(\Phi), \theta(\rho \otimes \tau) \otimes u \rangle &= \langle \Phi, \kappa(\theta(\rho \otimes \tau) \otimes u) \rangle \\ &= \langle \Phi, \rho \otimes W(\theta^*)(\tau \otimes u) \rangle \\ &= \langle \Phi, \rho \otimes \theta^*(u)(\tau) \rangle \\ &= \langle \Phi, \rho \otimes L_\tau(u) \rangle. \end{aligned}$$

Hence we have:

$$\langle \chi(\Phi), \theta(\rho \otimes \tau) \otimes u \rangle = \langle \Phi, \rho \otimes L_\tau(u) \rangle,$$

and following $(*)$, by this $\chi(\Phi)$ is completely determined. Taking into account (5), we thus have that for all $\rho \in \mathcal{M}_*$, $\tau \in \mathcal{N}_*$, $u \in \mathcal{M} \overline{\otimes} \mathcal{N}$:

$$\langle \chi(\Phi), \rho \otimes \tau \otimes u \rangle = \langle \Phi, \rho \otimes L_\tau(u) \rangle, \tag{6}$$

which determines $\chi(\Phi)$ completely.

Now, let us verify the condition (AAC) and properties (i)–(iv).

(AAC) Fix $\Phi \in \mathcal{CB}(\mathcal{M})$, $S \in \mathcal{M}$, $T \in \mathcal{N}$, $\rho \in \mathcal{M}_*$ and $\tau \in \mathcal{N}_*$. We obtain:

$$\begin{aligned} \langle \rho \otimes \tau, \chi(\Phi)(S \otimes T) \rangle &= \langle \chi(\Phi), \rho \otimes \tau \otimes (S \otimes T) \rangle \\ &\overset{(6)}{=} \langle \Phi, \rho \otimes L_\tau(S \otimes T) \rangle \\ &= \langle \tau, T \rangle \langle \Phi, \rho \otimes S \rangle \\ &= \langle \rho, \Phi(S) \rangle \langle \tau, T \rangle \\ &= \langle \rho \otimes \tau, \Phi(S) \otimes T \rangle. \end{aligned}$$

Hence, for all $\Phi \in \mathcal{CB}(\mathcal{M})$, $S \in \mathcal{M}$ and $T \in \mathcal{N}$, we have:

$$\chi(\Phi)(S \otimes T) = \Phi(S) \otimes T,$$

as desired.

(i) To prove that χ is a complete isometry, we show that $\kappa = \chi_*$ is a complete quotient mapping. But this follows from the fact that $W(\theta^*)$ is a complete quotient mapping. (For if this is established, then the mapping $\mathrm{id}_{\mathcal{M}_*} \hat{\otimes} W(\theta^*) \in \mathcal{CB}(\mathcal{M}_* \hat{\otimes} \mathcal{N}_* \hat{\otimes} (\mathcal{M} \overline{\otimes} \mathcal{N}), \mathcal{M}_* \hat{\otimes} \mathcal{M})$ is a complete quotient map, and of course this is also the case for $\theta^{-1} \hat{\otimes} \mathrm{id}_{\mathcal{M} \overline{\otimes} \mathcal{N}} \in \mathcal{CB}((\mathcal{M} \overline{\otimes} \mathcal{N})_* \hat{\otimes} (\mathcal{M} \overline{\otimes} \mathcal{N}), \mathcal{M}_* \hat{\otimes} \mathcal{N}_* \hat{\otimes} (\mathcal{M} \overline{\otimes} \mathcal{N}))$.) This in turn is easily seen. Fix $n \in \mathbb{N}$. Let $S = [S_{i,j}]_{i,j} \in M_n(\mathcal{M})$, $||S|| < 1$. We are looking for an element $\varphi \in M_n(\mathcal{N}_* \hat{\otimes} (\mathcal{M} \overline{\otimes} \mathcal{N}))$, such that $||\varphi|| < 1$ and $W(\theta^*)^{(n)}(\varphi) = S$, where $W(\theta^*)^{(n)}$ denotes the nth amplification of the mapping $W(\theta^*)$.

Let us choose $\tau \in \mathcal{N}_*$ with $||\tau|| = 1$. Then there exists $T \in \mathcal{N}$, $||T|| = 1$, such that $\langle \tau, T \rangle = 1$. Now $\varphi := [\tau \otimes (S_{i,j} \otimes T)]_{i,j} \in M_n(\mathcal{N}_* \hat{\otimes} (\mathcal{M} \overline{\otimes} \mathcal{N}))$ satisfies all our requirements. For we have:

$$\begin{aligned} ||\varphi||_{M_n(\mathcal{N}_* \hat{\otimes} (\mathcal{M} \overline{\otimes} \mathcal{N}))} &= ||\tau|| \, ||[S_{i,j} \otimes T]_{i,j}||_{M_n(\mathcal{M} \overline{\otimes} \mathcal{N})} \\ &= ||\tau|| \, ||[S_{i,j} \otimes T]_{i,j}||_{M_n(\mathcal{M} \vee \otimes \mathcal{N})} \\ &= ||\tau|| \, ||[S_{i,j}]_{i,j}||_{M_n(\mathcal{M})} ||T|| \\ &< 1. \end{aligned}$$

Furthermore, we see that:

$$\begin{aligned} W(\theta^*)^{(n)}(\varphi) &= W(\theta^*)^{(n)}([\tau \otimes (S_{i,j} \otimes T)]_{i,j}) \\ &= [W(\theta^*)(\tau \otimes (S_{i,j} \otimes T))]_{i,j} \\ &= [\theta^*(S_{i,j} \otimes T)(\tau)]_{i,j} \\ &= [L_\tau(S_{i,j} \otimes T)]_{i,j} \\ &= [\langle \tau, T \rangle S_{i,j}]_{i,j} \\ &= S, \end{aligned}$$

which establishes the claim.

(ii) We show that χ is an algebra homomorphism. To this end, let $\Phi, \Psi \in \mathcal{CB}(\mathcal{M})$. We have to show that

$$\chi(\Phi\Psi) = \chi(\Phi)\chi(\Psi),$$

as elements in

$$\mathcal{CB}(\mathcal{M} \overline{\otimes} \mathcal{N}) \overset{\text{cb}}{=} (\mathcal{M}_* \hat{\otimes} \mathcal{N}_* \hat{\otimes} (\mathcal{M} \overline{\otimes} \mathcal{N}))^*.$$

Fix $T \in \mathcal{M} \overline{\otimes} \mathcal{N}$, $\rho \in \mathcal{M}_*$ and $\tau \in \mathcal{N}_*$. It suffices to show that

$$\langle \chi(\Phi\Psi), \rho \otimes \tau \otimes T \rangle = \langle \chi(\Phi)\chi(\Psi), \rho \otimes \tau \otimes T \rangle.$$

On the left side, we obtain

$$\begin{aligned} \langle \chi(\Phi\Psi), \rho \otimes \tau \otimes T \rangle &\overset{(6)}{=} \langle \Phi\Psi, \rho \otimes L_\tau(T) \rangle \\ &= \langle \Phi(\Psi(L_\tau(T))), \rho \rangle \\ &= \langle \Phi, \rho \otimes \Psi(L_\tau(T)) \rangle. \end{aligned}$$

On the right, we have

$$\begin{aligned} \langle \chi(\Phi)\chi(\Psi), \rho \otimes \tau \otimes T \rangle &= \langle \chi(\Phi)[\chi(\Psi)(T)], \rho \otimes \tau \rangle \\ &= \langle \chi(\Phi), \rho \otimes \tau \otimes \chi(\Psi)(T) \rangle \\ &\overset{(6)}{=} \langle \Phi, \rho \otimes L_\tau(\chi(\Psi)(T)) \rangle. \end{aligned}$$

Thus we have to show that

$$L_\tau(\chi(\Psi)(T)) = \Psi(L_\tau(T)),$$

as elements in $\mathcal{M}$. To this end, let $\rho \in \mathcal{M}_*$. We then have

$$\begin{aligned}\langle L_\tau(\chi(\Psi)(T)), \rho \rangle &= \langle \chi(\Psi)(T), \rho \otimes \tau \rangle \\ &= \langle \chi(\Psi), \rho \otimes \tau \otimes T \rangle \\ &\overset{(6)}{=} \langle \Psi, \rho \otimes L_\tau(T) \rangle \\ &= \langle \Psi(L_\tau(T)), \rho \rangle,\end{aligned}$$

which yields the desired equality.

(iii) Being an adjoint mapping, χ is clearly w^*-w^*-continuous.

(iv) Fix $\Phi \in \mathcal{CB}^\sigma(\mathcal{M})$. We wish to prove that $\chi(\Phi) \in \mathcal{CB}^\sigma(\mathcal{M} \overline{\otimes} \mathcal{N})$. So let $(T_\alpha) \subseteq \mathrm{Ball}(\mathcal{M} \overline{\otimes} \mathcal{N})$ be a net such that $T_\alpha \overset{w^*}{\to} 0$. We claim that $\chi(\Phi)(T_\alpha) \to 0$ $(\sigma(\mathcal{M} \overline{\otimes} \mathcal{N}, \mathcal{M}_* \hat{\otimes} \mathcal{N}_*))$. It suffices to show that we have for arbitrary $\rho \in \mathcal{M}_*$ and $\tau \in \mathcal{N}_*$:

$$\langle \chi(\Phi)(T_\alpha), \rho \otimes \tau \rangle \to 0.$$

We obtain for all indices α:

$$\begin{aligned}\langle \chi(\Phi)(T_\alpha), \rho \otimes \tau \rangle &= \langle \chi(\Phi), \rho \otimes \tau \otimes T_\alpha \rangle \\ &\overset{(6)}{=} \langle \Phi, \rho \otimes L_\tau(T_\alpha) \rangle \\ &= \langle \Phi(L_\tau(T_\alpha)), \rho \rangle.\end{aligned}$$

Since Φ is normal, it thus remains to show that $L_\tau(T_\alpha) \overset{w^*}{\to} 0$; but this follows in turn from the normality of the slice map L_τ. $\square$

4. Discussion of the amplification mapping

In the sequel, for a mapping $\Phi \in \mathcal{CB}(\mathcal{M})$, we shall consider amplifications with respect to different von Neumann algebras or dual operator spaces $\mathcal{N}$. We will show that all of these amplifications are compatible with each other in a very natural way. For this purpose, we introduce the following

Notation 4.1. If $(\mathcal{M}, \mathcal{N})$ is an admissible pair, we will denote by $\chi_{\mathcal{N}}$ the amplification with respect to $\mathcal{N}$ as constructed in Theorem 3.1.

Taking the restriction of the mapping $\chi_{\mathcal{N}} : \mathcal{CB}(\mathcal{M}) \to \mathcal{CB}(\mathcal{M} \overline{\otimes} \mathcal{N})$ to the subalgebra $\mathcal{CB}^\sigma(\mathcal{M})$, by condition (iv) in Definition 2.2, we obtain a mapping $\chi_0 : \mathcal{CB}^\sigma(\mathcal{M}) \to \mathcal{CB}^\sigma(\mathcal{M} \overline{\otimes} \mathcal{N})$, which of course satisfies the algebraic condition

(AAC) on $\mathcal{CB}^\sigma(\mathcal{M})$. But this is the case as well for the amplification

$$H : \mathcal{CB}^\sigma(\mathcal{M}) \to \mathcal{CB}^\sigma(\mathcal{M}\overline{\otimes}\mathcal{N})$$

considered, for von Neumann algebras $\mathcal{M}$ and $\mathcal{N}$, by de Cannière and Haagerup in [DEC–HAA85, Lemma 1.5(b)]. Note that H is completely determined by (AAC), since the image of H consists of normal mappings. Hence we deduce that $\chi_0 = H$, thus showing that $\chi_{\mathcal{N}}$ is indeed a (w^*-w^*-continuous) extension of H to $\mathcal{CB}(\mathcal{M})$, where $\mathcal{M}$ and $\mathcal{N}$ are von Neumann algebras (cf. also [KRA91, p. 123], for a discussion of the mapping H in a more general setting).

Remark 4.2. Theorem 3.1 implies in particular (cf. (iv) in Definition 2.2) that whenever $(\mathcal{M}, \mathcal{N})$ is an admissible pair, then for every $\Phi \in \mathcal{CB}^\sigma(\mathcal{M})$, the mapping $\chi_{\mathcal{N}}(\Phi)$ is the unique normal map in $\mathcal{CB}^\sigma(\mathcal{M}\overline{\otimes}\mathcal{N})$ which satisfies (AAC). This entails that the last assumption made in [KRA83, Proposition 1.19] is always fulfilled (and even true in greater generality than actually needed there), which besides slightly simplifies the proof of [KRA83, Theorem 2.2].

Let us now briefly return to the special case where $\mathcal{M} = \mathcal{B}(\mathcal{H})$ and $\mathcal{N} = \mathcal{B}(\mathcal{K})$. We already noted in Theorem 2.4 that the mapping $\Phi \mapsto \Phi^{(\infty)}$, where $\Phi \in \mathcal{CB}(\mathcal{B}(\mathcal{H}))$, is an amplification in the sense of Definition 2.2. Since $\mathcal{B}(\mathcal{H})$ trivially is an injective factor, we conclude by Proposition 6.1 that

$$\chi_{\mathcal{B}(\mathcal{K})}(\Phi) = \Phi^{(\infty)} \tag{7}$$

for all $\Phi \in \mathcal{CB}(\mathcal{B}(\mathcal{H}))$. Thus, restricted to that special case, Theorem 3.1 provides a new construction of the mapping $\Phi \mapsto \Phi^{(\infty)}$ and for the first time shows the intimate relation to Tomiyama's slice maps.

We finally stress that our construction of $\chi_{\mathcal{B}(\mathcal{K})}(\Phi)$ is coordinate free—in contrast to the definition of $\Phi^{(\infty)}$ in [MAY–NEU–WIT89]: since our approach does not rely on an explicit representation of operators in $\mathcal{B}(\mathcal{H})\overline{\otimes}\mathcal{B}(\mathcal{K})$ as infinite matrices, we do not have to choose a basis in the Hilbert space $\mathcal{K}$, whereas in [MAY–NEU–WIT89], the latter is represented as $\ell_2(I)$ for some suitable I. The equality

$$\chi_{\mathcal{B}(\mathcal{K})}(\Phi) = \Phi^{(\infty)} \quad (\Phi \in \mathcal{CB}(\mathcal{B}(\mathcal{H}))),$$

obtained above, now immediately implies that the definition of the mapping $\Phi^{(\infty)}$ in [MAY–NEU–WIT89] does not depend on the particular choice of a basis. (This fact of course can also be obtained intrinsically from the construction of $\Phi^{(\infty)}$, but it comes for free in our approach.)

We now come to the result announced above which establishes the compatibility of our amplification with respect to different von Neumann algebras or dual operator spaces $\mathcal{N}$, respectively.

Theorem 4.3. *Let $(\mathcal{M}, \mathcal{N})$ be an admissible pair. Let further $\mathcal{N}_0 \subseteq \mathcal{N}$ be such that $(\mathcal{M}, \mathcal{N}_0)$ is admissible as well. Then, for all $\Phi \in \mathcal{CB}(\mathcal{M})$, we have*:

$$\chi_{\mathcal{N}}(\Phi)|_{\mathcal{M}\overline{\otimes}\mathcal{N}_0} = \chi_{\mathcal{N}_0}(\Phi).$$

Proof. Denote by $\iota : \mathcal{M}\overline{\otimes}\mathcal{N}_0 \hookrightarrow \mathcal{M}\overline{\otimes}\mathcal{N}$ the canonical embedding, and write $q : \mathcal{N}_* \twoheadrightarrow (\mathcal{N}_0)_*$ for the restriction map. Obviously, the pre-adjoint mapping $\iota_* : \mathcal{M}_* \hat{\otimes} \mathcal{N}_* \twoheadrightarrow \mathcal{M}_* \hat{\otimes} (\mathcal{N}_0)_*$ satisfies $\iota_* = \mathrm{id}_{\mathcal{M}_*} \hat{\otimes} q$.

We shall prove that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{M}\overline{\otimes}\mathcal{N} & \xrightarrow[\chi_{\mathcal{N}}(\Phi)]{} & \mathcal{M}\overline{\otimes}\mathcal{N} \\ \uparrow \iota & & \iota \uparrow \\ \mathcal{M}\overline{\otimes}\mathcal{N}_0 & \xrightarrow{\chi_{\mathcal{N}_0}(\Phi)} & \mathcal{M}\overline{\otimes}\mathcal{N}_0 \end{array}$$

This yields the desired equality (and shows at the same time that $\chi_{\mathcal{N}}(\Phi)|_{\mathcal{M}\overline{\otimes}\mathcal{N}_0}$ indeed leaves the space $\mathcal{M}\overline{\otimes}\mathcal{N}_0$ invariant).

To this end, fix $u \in \mathcal{M}\overline{\otimes}\mathcal{N}_0$. We have to show that

$$\chi_{\mathcal{N}}(\Phi)(\iota(u)) = \iota(\chi_{\mathcal{N}_0}(\Phi)(u)).$$

Let $\rho \in \mathcal{M}_*$ and $\tau \in \mathcal{N}_*$. It is sufficient to verify that

$$\langle \chi_{\mathcal{N}}(\Phi)(\iota(u)), \rho \otimes \tau \rangle = \langle \iota(\chi_{\mathcal{N}_0}(\Phi)(u)), \rho \otimes \tau \rangle.$$

For the left-hand expression, we get:

$$\langle \chi_{\mathcal{N}}(\Phi)(\iota(u)), \rho \otimes \tau \rangle = \langle \chi_{\mathcal{N}}(\Phi), \rho \otimes \tau \otimes \iota(u) \rangle = \langle \Phi, \rho \otimes L_\tau(\iota(u)) \rangle.$$

On the right-hand side, we find:

$$\begin{aligned} \langle \iota(\chi_{\mathcal{N}_0}(\Phi)(u)), \rho \otimes \tau \rangle &= \langle \chi_{\mathcal{N}_0}(\Phi)(u), \rho \otimes q(\tau) \rangle \\ &= \langle \chi_{\mathcal{N}_0}(\Phi), \rho \otimes q(\tau) \otimes u \rangle \\ &= \langle \Phi, \rho \otimes L_{q(\tau)}(u) \rangle. \end{aligned}$$

It thus remains to be shown that

$$L_\tau(\iota(u)) = L_{q(\tau)}(u).$$

But for $\varphi \in \mathcal{M}_*$, we obtain:

$$\begin{aligned}\langle L_\tau(\imath(u)), \varphi \rangle &= \langle \imath(u), \varphi \otimes \tau \rangle \\ &= \langle u, \varphi \otimes q(\tau) \rangle \\ &= \langle L_{q(\tau)}(u), \varphi \rangle,\end{aligned}$$

which yields the claim. □

We shall now present another very natural, though highly non-constructive, approach to the amplification problem, and discuss its relation to our concept. Let $(\mathcal{M}, \mathcal{N})$ be an admissible pair, with $\mathcal{M} \subseteq \mathcal{B}(\mathcal{H})$ and $\mathcal{N} \subseteq \mathcal{B}(\mathcal{K})$. An element $\Phi \in \mathcal{CB}(\mathcal{M})$ may be considered as a completely bounded mapping from $\mathcal{M}$ with values in $\mathcal{B}(\mathcal{H})$. Hence we can assign to Φ a Wittstock–Hahn–Banach extension $\tilde{\Phi} \in \mathcal{CB}(\mathcal{B}(\mathcal{H}))$. Then the mapping

$$\tilde{\Phi}^{(\infty)}|_{\mathcal{M} \overline{\otimes} \mathcal{N}} : \mathcal{M} \overline{\otimes} \mathcal{N} \to \mathcal{B}(\mathcal{H}) \overline{\otimes} \mathcal{B}(\mathcal{K})$$

is completely bounded and of course satisfies the algebraic amplification condition (AAC). As we shall now see, this (non-constructive) abstract procedure yields indeed a mapping which takes values in $\mathcal{M} \overline{\otimes} \mathcal{N}$ and does not depend on the choice of the Wittstock–Hahn–Banach extension—namely, $\tilde{\Phi}^{(\infty)}|_{\mathcal{M} \overline{\otimes} \mathcal{N}}$ is nothing but $\chi_{\mathcal{N}}(\Phi)$. This will be proved in a similar fashion as Theorem 4.3.

Theorem 4.4. *Let $(\mathcal{M}, \mathcal{N})$ be an admissible pair, where $\mathcal{M} \subseteq \mathcal{B}(\mathcal{H})$ and $\mathcal{N} \subseteq \mathcal{B}(\mathcal{K})$. Let further $\Phi \in \mathcal{CB}(\mathcal{M})$. Then for an arbitrary Wittstock–Hahn–Banach extension $\tilde{\Phi} \in \mathcal{CB}(\mathcal{B}(\mathcal{H}))$ obtained as above, we have*:

$$\tilde{\Phi}^{(\infty)}|_{\mathcal{M} \overline{\otimes} \mathcal{N}} = \chi_{\mathcal{N}}(\Phi).$$

Proof. We write $\imath : \mathcal{M} \overline{\otimes} \mathcal{N} \hookrightarrow \mathcal{B}(\mathcal{H}) \overline{\otimes} \mathcal{B}(\mathcal{K})$ for the canonical embedding. It is sufficient to show that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{B}(\mathcal{H}) \overline{\otimes} \mathcal{B}(\mathcal{K}) & \xrightarrow[\tilde{\Phi}^{(\infty)}]{} & \mathcal{B}(\mathcal{H}) \overline{\otimes} \mathcal{B}(\mathcal{K}) \\ \uparrow \imath & & \uparrow \imath \\ \mathcal{M} \overline{\otimes} \mathcal{N} & \xrightarrow{\chi_{\mathcal{N}}(\Phi)} & \mathcal{M} \overline{\otimes} \mathcal{N} \end{array}$$

Fix $u \in \mathcal{M} \overline{\otimes} \mathcal{N}$, $\rho \in \mathcal{B}(\mathcal{H})_*$, $\tau \in \mathcal{B}(\mathcal{K})_*$. We have only to show that

$$\langle \imath[\chi_{\mathcal{N}}(\Phi)(u)], \rho \otimes \tau \rangle = \langle \tilde{\Phi}^{(\infty)}(\imath(u)), \rho \otimes \tau \rangle,$$

which by Eq. (7) is equivalent to

$$\langle \imath[\chi_{\mathcal{N}}(\Phi)(u)], \rho \otimes \tau \rangle = \langle \chi_{\mathcal{B}(\mathcal{K})}(\tilde{\Phi})(\imath(u)), \rho \otimes \tau \rangle. \tag{8}$$

On the left-hand side, we obtain

$$\begin{aligned} \langle \imath[\chi_{\mathcal{N}}(\Phi)(u)], \rho \otimes \tau \rangle &= \langle \chi_{\mathcal{N}}(\Phi)(u), \rho|_{\mathcal{M}} \otimes \tau|_{\mathcal{N}} \rangle \\ &= \langle \Phi, \rho|_{\mathcal{M}} \otimes L_{\tau|_{\mathcal{N}}}(u) \rangle. \end{aligned}$$

Now, writing $\imath' : \mathcal{M} \hookrightarrow \mathcal{B}(\mathcal{H})$ for the canonical embedding, we note that

$$\begin{aligned} \langle L_\tau(\imath(u)), \rho \rangle &= \langle \imath(u), \rho \otimes \tau \rangle \\ &= \langle u, \rho|_{\mathcal{M}} \otimes \tau|_{\mathcal{N}} \rangle \\ &= \langle L_{\tau|_{\mathcal{N}}}(u), \rho|_{\mathcal{M}} \rangle \\ &= \langle \imath'(L_{\tau|_{\mathcal{N}}}(u)), \rho \rangle, \end{aligned}$$

whence we have

$$L_\tau(\imath(u)) = \imath'(L_{\tau|_{\mathcal{N}}}(u)).$$

Thus, we see that

$$\tilde{\Phi}[L_\tau(\imath(u))] = \tilde{\Phi}[\imath'(L_{\tau|_{\mathcal{N}}}(u))] = \imath'[\Phi(L_{\tau|_{\mathcal{N}}}(u))]. \tag{9}$$

Now we find for the right-hand side term of Eq. (8):

$$\begin{aligned} \langle \chi_{\mathcal{B}(\mathcal{K})}(\tilde{\Phi})(\imath(u)), \rho \otimes \tau \rangle &= \langle \tilde{\Phi}, \rho \otimes L_\tau(\imath(u)) \rangle \\ &= \langle \tilde{\Phi}[L_\tau(\imath(u))], \rho \rangle \\ &\overset{(9)}{=} \langle \imath'[\Phi(L_{\tau|_{\mathcal{N}}}(u))], \rho \rangle \\ &= \langle \Phi(L_{\tau|_{\mathcal{N}}}(u)), \rho|_{\mathcal{M}} \rangle \\ &= \langle \Phi, \rho|_{\mathcal{M}} \otimes L_{\tau|_{\mathcal{N}}}(u) \rangle, \end{aligned}$$

whence we deduce the desired equality. □

Remark 4.5. If $\mathcal{M} \subseteq \mathcal{B}(\mathcal{H})$, $\mathcal{N} \subseteq \mathcal{B}(\mathcal{K})$ are von Neumann algebras and $\Phi \in \mathcal{CP}(\mathcal{M})$ is completely positive, then, using an arbitrary Arveson extension of Φ to a completely positive map $\tilde{\Phi} \in \mathcal{CP}(\mathcal{B}(\mathcal{H}))$, we obtain by the same reasoning as above that $\tilde{\Phi}^{(\infty)}|_{\mathcal{M}\overline{\otimes}\mathcal{N}} = \chi_{\mathcal{N}}(\Phi)$. But it is easy to see from the definition of the mapping $\tilde{\Phi} \mapsto \tilde{\Phi}^{(\infty)}$ (cf. [MAY–NEU–WIT89, p. 284, Section 2]) that $\tilde{\Phi}^{(\infty)}$ still is completely positive, whence $\chi_{\mathcal{N}}(\Phi)$ also is. This shows that our amplification $\chi_{\mathcal{N}}$ preserves complete positivity.

5. An algebraic characterization of normality

In the following, let still $(\mathcal{M}, \mathcal{N})$ be an admissible pair. Using Tomiyama's left slice map, for every completely bounded mapping $\Phi \in \mathcal{CB}(\mathcal{M})$, we have constructed an amplification $\chi_{\mathcal{N}}(\Phi) =: \Phi \otimes \mathrm{id}_{\mathcal{N}} \in \mathcal{CB}(\mathcal{M} \overline{\otimes} \mathcal{N})$. In an analogous fashion, replacing in the above construction Tomiyama's left by the appropriate right slice maps, we obtain—*mutatis mutandis*—an amplification of completely bounded mappings $\Psi \in \mathcal{CB}(\mathcal{N})$ "in the left variable". This yields an amplification $\mathrm{id}_{\mathcal{M}} \otimes \Psi \in \mathcal{CB}(\mathcal{M} \overline{\otimes} \mathcal{N})$, which is completely determined by the equation:

$$\langle \mathrm{id}_{\mathcal{M}} \otimes \Psi, \rho \otimes \tau \otimes u \rangle = \langle \Psi, \tau \otimes R_\rho(u) \rangle, \tag{10}$$

where $u \in \mathcal{M} \overline{\otimes} \mathcal{N}$, $\rho \in \mathcal{M}_*$ and $\tau \in \mathcal{N}_*$ are arbitrary. Of course, $\mathrm{id}_{\mathcal{M}} \otimes \Psi$ shares analogous properties to those of $\Phi \otimes \mathrm{id}_{\mathcal{N}}$.

It is natural to ask for some relation between the operators in $\mathcal{CB}(\mathcal{M} \overline{\otimes} \mathcal{N})$ which arise by the two different kinds of amplification. Let us briefly compare our situation with the classical setting of amplification of operators on Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$. For operators $S \in \mathcal{B}(\mathcal{H})$ and $T \in \mathcal{B}(\mathcal{K})$, we consider the amplifications $S \otimes \mathrm{id}_{\mathcal{K}}$ and $\mathrm{id}_{\mathcal{H}} \otimes T$ in $\mathcal{B}(\mathcal{H}) \overline{\otimes} \mathcal{B}(\mathcal{K}) = \mathcal{B}(\mathcal{H} \otimes_2 \mathcal{K})$. Then it is evident that we have

$$(S \otimes \mathrm{id}_{\mathcal{K}})(\mathrm{id}_{\mathcal{H}} \otimes T) = (\mathrm{id}_{\mathcal{H}} \otimes T)(S \otimes \mathrm{id}_{\mathcal{K}}), \tag{11}$$

and the tensor product of S and T is defined precisely to be this operator.

Back in our situation, for $\Phi \in \mathcal{CB}(\mathcal{M})$ and $\Psi \in \mathcal{CB}(\mathcal{N})$, the corresponding amplifications $\Phi \otimes \mathrm{id}_{\mathcal{N}}$ and $\mathrm{id}_{\mathcal{M}} \otimes \Psi$ will commute provided at least one of the involved mappings Φ or Ψ is normal—but not in general. In fact, we will even show that for most von Neumann algebras $\mathcal{M}$ and $\mathcal{N}$, $\Phi \in \mathcal{CB}(\mathcal{M})$ satisfying this innocent looking commutation relation for all $\Psi \in \mathcal{CB}(\mathcal{N})$ forces Φ to be normal! This phenomenon, which may be surprising at first glance, is due to the fact that, in contrast to the multiplication *in* a von Neumann algebra $\mathcal{R}$ (in our case $\mathcal{R} = \mathcal{M} \overline{\otimes} \mathcal{N}$), the multiplication in $\mathcal{CB}(\mathcal{R})$ is not w^*-continuous in both variables, but only in the left one. In fact, the subset of $\mathcal{CB}(\mathcal{R}) \times \mathcal{CB}(\mathcal{R})$ on which the product is w^*-continuous is precisely $\mathcal{CB}(\mathcal{R}) \times \mathcal{CB}^\sigma(\mathcal{R})$. This is reflected in the following result which parallels relation (11) in our context.

Theorem 5.1. *Let $(\mathcal{M}, \mathcal{N})$ be an admissible pair. If $\Psi \in \mathcal{CB}^\sigma(\mathcal{N})$ is a normal mapping, then we have for every $\Phi \in \mathcal{CB}(\mathcal{M})$:*

$$(\Phi \otimes \mathrm{id}_{\mathcal{N}})(\mathrm{id}_{\mathcal{M}} \otimes \Psi) = (\mathrm{id}_{\mathcal{M}} \otimes \Psi)(\Phi \otimes \mathrm{id}_{\mathcal{N}}).$$

Proof. Fix $u \in \mathcal{M} \overline{\otimes} \mathcal{N}$, $\rho \in \mathcal{M}_*$ and $\tau \in \mathcal{N}_*$. We first remark that

$$\begin{aligned} \langle \mathrm{id}_{\mathcal{M}} \otimes \Psi, \rho \otimes \tau \otimes u \rangle &\overset{(10)}{=} \langle \Psi, \tau \otimes R_\rho(u) \rangle \\ &= \langle \Psi(R_\rho(u)), \tau \rangle \\ &= \langle R_\rho(u), \Psi_*(\tau) \rangle \\ &= \langle u, \rho \otimes \Psi_*(\tau) \rangle, \end{aligned} \tag{12}$$

where Ψ_* denotes the pre-adjoint mapping of Ψ.

To establish the claim, it suffices to prove that

$$\langle (\Phi \otimes \mathrm{id}_{\mathcal{N}})(\mathrm{id}_{\mathcal{M}} \otimes \Psi), \rho \otimes \tau \otimes u \rangle = \langle (\mathrm{id}_{\mathcal{M}} \otimes \Psi)(\Phi \otimes \mathrm{id}_{\mathcal{N}}), \rho \otimes \tau \otimes u \rangle.$$

The left side takes on the following form:

$$\begin{aligned} \langle (\Phi \otimes \mathrm{id}_{\mathcal{N}})(\mathrm{id}_{\mathcal{M}} \otimes \Psi), \rho \otimes \tau \otimes u \rangle &= \langle \chi_{\mathcal{N}}(\Phi), \rho \otimes \tau \otimes (\mathrm{id}_{\mathcal{M}} \otimes \Psi)(u) \rangle \\ &= \langle \Phi, \rho \otimes L_\tau[(\mathrm{id}_{\mathcal{M}} \otimes \Psi)(u)] \rangle, \end{aligned}$$

whereas on the right side, we obtain

$$\begin{aligned} \langle (\mathrm{id}_{\mathcal{M}} \otimes \Psi)(\Phi \otimes \mathrm{id}_{\mathcal{N}}), \rho \otimes \tau \otimes u \rangle &= \langle (\mathrm{id}_{\mathcal{M}} \otimes \Psi)(\chi_{\mathcal{N}}(\Phi)(u)), \rho \otimes \tau \rangle \\ &\overset{(12)}{=} \langle \chi_{\mathcal{N}}(\Phi)(u), \rho \otimes \Psi_*(\tau) \rangle \\ &= \langle \chi_{\mathcal{N}}(\Phi), \rho \otimes \Psi_*(\tau) \otimes u \rangle \\ &= \langle \Phi, \rho \otimes L_{\Psi_*(\tau)}(u) \rangle. \end{aligned}$$

Hence, it remains to be shown that

$$L_\tau[(\mathrm{id}_{\mathcal{M}} \otimes \Psi)(u)] = L_{\Psi_*(\tau)}(u).$$

But for arbitrary $\rho' \in \mathcal{M}_*$, we get:

$$\begin{aligned} \langle L_\tau[(\mathrm{id}_{\mathcal{M}} \otimes \Psi)(u)], \rho' \rangle &= \langle (\mathrm{id}_{\mathcal{M}} \otimes \Psi)(u), \rho' \otimes \tau \rangle \\ &\overset{(12)}{=} \langle u, \rho' \otimes \Psi_*(\tau) \rangle \\ &= \langle L_{\Psi_*(\tau)}(u), \rho' \rangle, \end{aligned}$$

which yields the claim. □

We now come to the main result of this section, which establishes a converse of the above theorem for a very wide class of von Neumann algebras. It may be interpreted as a result on "automatic normality", i.e., some simple, purely algebraic condition—namely commuting of the associated amplifications—automatically implies the strong topological property of normality. Besides one very mild technical condition, the first von Neumann algebra, $\mathscr{M}$, can be completely arbitrary, the second one, $\mathscr{N}$, is only assumed to be properly infinite; we will show by an example that without the latter condition, the conclusion does not hold in general. The weak technical assumption mentioned says, roughly speaking, that the von Neumann algebras involved should not be acting on Hilbert spaces of pathologically large dimension. We remark that for separably acting von Neumann algebras $\mathscr{M}$ and $\mathscr{N}$, or, more generally, in case $\mathscr{M}\overline{\otimes}\mathscr{N}$ is countably decomposable, our assumption is trivially satisfied. In order to formulate the condition precisely, we use the following natural terminology which has been introduced in [NEU02].

Definition 5.2. Let $\mathscr{R}\subseteq\mathscr{B}(\mathscr{H})$ be a von Neumann algebra. Then we define the *decomposability number* of $\mathscr{R}$, denoted by $\mathrm{dec}(\mathscr{R})$, to be the smallest cardinal number κ such that every family of non-zero pairwise orthogonal projections in $\mathscr{R}$ has at most cardinality κ.

Remark 5.3. Of course, a von Neumann algebra $\mathscr{R}$ is countably decomposable if and only if $\mathrm{dec}(\mathscr{R})\leqslant\aleph_0$.

We are now ready to state the main theorem of this section.

Theorem 5.4. *Let $\mathscr{M}$ and $\mathscr{N}$ be von Neumann algebras such that $\mathrm{dec}(\mathscr{M}\overline{\otimes}\mathscr{N})$ is a non-measurable cardinal, and suppose $\mathscr{N}$ is properly infinite. If $\Phi\in\mathscr{CB}(\mathscr{M})$ is such that*

$$(\Phi\otimes\mathrm{id}_{\mathscr{N}})(\mathrm{id}_{\mathscr{M}}\otimes\Psi)=(\mathrm{id}_{\mathscr{M}}\otimes\Psi)(\Phi\otimes\mathrm{id}_{\mathscr{N}})$$

for all $\Psi\in\mathscr{CB}(\mathscr{N})$, then Φ must be normal.

Remark 5.5. The assumption that the decomposability number of $\mathscr{M}\overline{\otimes}\mathscr{N}$ be non-measurable, is very weak and in fact just excludes a set-theoretic pathology. We briefly recall that a cardinal κ is said to be (real-valued) measurable if for every set Γ of cardinality κ, there exists a diffused probability measure on the power set $\mathfrak{P}(\Gamma)$. Measurability is a property of "large" cardinals ($\aleph_0$ is of course not measurable). In order to demonstrate how weak the restriction to non-measurable cardinals is, let us just mention the fact (cf. [KAN–MAG78, Section 1, p. 106, 108]) that the existence of measurable cardinals cannot be proved in ZFC (= the axioms of Zermelo–Fraenkel + the axiom of choice), and that it is consistent with ZFC to assume the non-existence of measurable cardinals [GAR–PFE84, Section 4, Theorem 4.14, p. 972]. For a further discussion, we refer to [NEU02], and the references therein.

We begin now with the preparations needed for the proof of Theorem 5.4. We first note that, even though requiring a non-measurable decomposability number means only excluding exotic von Neumann algebras, it is already enough to ensure a very pleasant property of the predual. The latter is actually of great (technical) importance in the proof of Theorem 5.4 and is therefore stated explicitly in the following.

Proposition 5.6. *Let $\mathscr{R}$ be a von Neumann algebra. Then* $\mathrm{dec}(\mathscr{R})$ *is non-measurable if and only if the predual $\mathscr{R}_*$ has Mazur's property (i.e., w^*-sequentially continuous functionals on $\mathscr{R}$ are w^*-continuous—and hence belong to $\mathscr{R}_*$).*

Proof. This is Theorem 3.11 in [NEU02]. □

The crucial idea in the proof of Theorem 5.4 is to use a variant of the concept of a *countable spiral nebula*, which has been introduced in [HOF–WIT97, Section 1.1] (cf. ibid., Lemma 1.5).

Definition 5.7. Let $\mathscr{R}$ be a von Neumann algebra. A sequence $(\alpha_n, e_n)_{n\in\mathbb{N}}$ of *-automorphisms α_n on $\mathscr{R}$ and projections e_n in $\mathscr{R}$ will be called a *countable reduced spiral nebula* on $\mathscr{R}$ if the following holds true:

$$e_n \leqslant e_{n+1}, \quad \mathrm{WOT} - \lim_n e_n = 1, \ \alpha_m(e_m) \perp \alpha_n(e_n) \quad \text{for all } m, n \in \mathbb{N}_0, \ m \neq n.$$

The following proposition is the crucial step in order to establish Theorem 5.4. It shows that a countable reduced spiral nebula may be used to derive a result on automatic normality—and not only on automatic boundedness, as is done in [HOF–WIT97, Lemma 1.5].

Proposition 5.8. *Let $\mathscr{R}$ be a von Neumann algebra such that* $\mathrm{dec}(\mathscr{R})$ *is non-measurable, and suppose that there is a countable reduced spiral nebula (α_n, e_n) on $\mathscr{R}$.*

(i) *Put* $l_k := \frac{k(k+1)}{2}$, and let $\mathfrak{F}$ *be a free ultrafilter on $\mathbb{N}$. Then the operators*

$$\Psi_n := w^* - \lim_{k\to\mathfrak{F}} \alpha^{-1}_{l_k+n}$$

are completely positive and unital.

(ii) *If $\Phi \in \mathscr{B}(\mathscr{R})$ commutes with all elements of the set* $S := \overline{\{\alpha_n^{-1} \mid n \in \mathbb{N}_0\}}^{w^*} \subseteq \mathscr{CP}(\mathscr{R})$, *then Φ is normal.*

Proof. Statement (i) is clear. In order to prove (ii), in view of Proposition 5.6, we only have to show that if $(x_n)_{n\in\mathbb{N}_0} \subseteq \mathrm{Ball}(\mathscr{R})$ converges w^* to 0, then the same is true for the sequence $(\Phi(x_n))_n$. It suffices to prove that if $(\Phi(x_{n_i}))_i$ is a w^*-convergent subnet, then its limit must be 0.

The projections $\alpha_n(e_n)$ being pairwise orthogonal, noting that $l_{\kappa+1} = l_\kappa + \kappa + 1$, we see that

$$\tilde{x} := \sum_{\kappa=0}^{\infty} \sum_{\nu=0}^{\kappa} \alpha_{l_\kappa+\nu}(e_{l_\kappa+\nu} x_\nu e_{l_\kappa+\nu}) \in \mathscr{R}.$$

Now we deduce (cf. the proof of [HOF–WIT97, Lemma 1.5]) that

$$\Psi_n(\tilde{x}) = x_n \quad \text{for all } n \in \mathbb{N}_0. \tag{13}$$

Let $\Psi := w^* - \lim_j \Psi_{n_{i_j}} \in \mathrm{Ball}(\mathscr{CB}(\mathscr{R}))$ be a w^*-cluster point of the net $(\Psi_{n_i})_i$. Then $\Psi \in S$, and we obtain, using (13):

$$\begin{aligned}
\lim_i \Phi(x_{n_i}) &= \lim_i \Phi(\Psi_{n_i}(\tilde{x})) = \lim_i \Psi_{n_i}(\Phi(\tilde{x})) \\
&= \lim_j \Psi_{n_{i_j}}(\Phi(\tilde{x})) = \Psi(\Phi(\tilde{x})) \\
&= \Phi(\Psi(\tilde{x})) = \Phi\left(\lim_j \Psi_{n_{i_j}}(\tilde{x})\right) \\
&= \Phi\left(\lim_j x_{n_{i_j}}\right) = \Phi(0) = 0,
\end{aligned}$$

which finishes the proof. $\square$

We are now sufficiently prepared for Theorem 5.4.

Proof. Since $\mathscr{N}$ is properly infinite, it is isomorphic to $\mathscr{N} \overline{\otimes} \mathscr{B}(\ell_2(\mathbb{Z}))$; cf., e.g., Appendix C, "Theorem", in [VAN78]. By applying the argument in the first part of the proof of Proposition 2.2 in [HOF–WIT97], we can construct a countable reduced spiral nebula (α_n, e_n) on $\mathscr{B}(\ell_2(\mathbb{Z}))$. Hence, $(\tilde{\alpha}_n, \tilde{e}_n) := (\mathrm{id}_{\mathscr{M}} \otimes \mathrm{id}_{\mathscr{N}} \otimes \alpha_n, 1_{\mathscr{M}} \otimes 1_{\mathscr{N}} \otimes e_n)$ is a countable reduced spiral nebula on $\mathscr{M} \overline{\otimes} \mathscr{N} \overline{\otimes} \mathscr{B}(\ell_2(\mathbb{Z})) = \mathscr{M} \overline{\otimes} \mathscr{N}$. We will now apply Proposition 5.8 to the von Neumann algebra $\mathscr{M} \overline{\otimes} \mathscr{N}$ and the mapping $\Phi \otimes \mathrm{id}_{\mathscr{N}} \in \mathscr{CB}(\mathscr{M} \overline{\otimes} \mathscr{N})$ to show that the latter, and hence Φ itself, is normal. To this end, we only have to prove that $\Phi \otimes \mathrm{id}_{\mathscr{N}}$ commutes with all elements of the set $S := \overline{\{\tilde{\alpha}_n^{-1} \mid n \in \mathbb{N}_0\}}^{w^*}$. But the amplification in the first variable, $\chi_{\mathscr{M}}$, is w^*-w^*-continuous (this is established in the same fashion as we did for $\chi_{\mathscr{M}}$ in Theorem 3.1), and this entails that every element of S is of the form $\mathrm{id}_{\mathscr{M}} \otimes \Psi$ for some $\Psi \in \mathscr{CB}(\mathscr{N})$. $\square$

Remark 5.9. The assumption of $\mathscr{N}$ being properly infinite may be interpreted as a certain richness condition which is necessary to be imposed on the second von Neumann algebra. Indeed, if in contrast $\mathscr{N}$ is, e.g., a finite type I factor, the statement is wrong in general. This is easily seen as follows: In this case, $\mathscr{CB}(\mathscr{N}) = \mathscr{CB}^\sigma(\mathscr{N})$. Now let $\mathscr{M}$ be an arbitrary von Neumann algebra such that the singular part of the Tomiyama–Takesaki decomposition of $\mathscr{CB}(\mathscr{M})$ is non-trivial, i.e.,

$\mathcal{CB}^s(\mathcal{M}) \neq (0)$. Then by Theorem 5.1, every $\Phi \in \mathcal{CB}^s(\mathcal{M})$ will satisfy the commutation relation in Theorem 5.4 for all $\Psi \in \mathcal{CB}(\mathcal{N}) = \mathcal{CB}^\sigma(\mathcal{N})$, even though it is singular.

We finish our discussion by considering again an arbitrary admissible pair $(\mathcal{M}, \mathcal{N})$. In view of Theorem 5.1, it would be natural, following the model of the tensor product of bounded operators on Hilbert spaces, to define a tensor product of $\Phi \in \mathcal{CB}(\mathcal{M})$ and $\Psi \in \mathcal{CB}^\sigma(\mathcal{N})$ by setting:

$$\Phi \overline{\otimes} \Psi := (\Phi \otimes \mathrm{id}_{\mathcal{N}})(\mathrm{id}_{\mathcal{M}} \otimes \Psi) = (\mathrm{id}_{\mathcal{M}} \otimes \Psi)(\Phi \otimes \mathrm{id}_{\mathcal{N}}),$$

which defines an operator in $\mathcal{CB}(\mathcal{M} \overline{\otimes} \mathcal{N})$. It would be interesting to go even further and consider, for arbitrary $\Phi \in \mathcal{CB}(\mathcal{M})$ and $\Psi \in \mathcal{CB}(\mathcal{N})$, the two "tensor products" $\Phi \tilde{\otimes}_1 \Psi$ and $\Phi \tilde{\otimes}_2 \Psi$, defined respectively through

$$\Phi \tilde{\otimes}_1 \Psi := (\Phi \otimes \mathrm{id}_{\mathcal{N}})(\mathrm{id}_{\mathcal{M}} \otimes \Psi) \in \mathcal{CB}(\mathcal{M} \overline{\otimes} \mathcal{N})$$

and

$$\Phi \tilde{\otimes}_2 \Psi := (\mathrm{id}_{\mathcal{M}} \otimes \Psi)(\Phi \otimes \mathrm{id}_{\mathcal{N}}) \in \mathcal{CB}(\mathcal{M} \overline{\otimes} \mathcal{N}).$$

By Theorem 5.4, we know that these are different in general, even in the case of von Neumann algebras $\mathcal{M}$ and $\mathcal{N}$. We wish to close this section by stressing the resemblance of these considerations to the well-known construction of the two Arens products on the bidual of a Banach algebra, which in general are different. From this point of view, the subalgebra $\mathcal{CB}^\sigma(\mathcal{N})$ could play the role of what in the context of Banach algebras is known as the topological centre (for the latter term, see, e.g., [DAL00, Definition 2.6.19]).

6. Uniqueness of the amplification and a generalization of the Ge–Kadison Lemma

In the following, we will show that under different natural conditions, an algebraic amplification is uniquely determined. This leads in particular to a remarkable generalization of the so-called Ge–Kadison Lemma (see Proposition 6.2), which is obtained as an application of our Theorem 5.1.

But before establishing positive results, we briefly point out why in general, an algebraic amplification is highly non-unique. In [TOM70, p. 28], Tomiyama states, in the context of von Neumann algebras, that the "product projection" of two projections of norm one "might not be unique" (of course, in our case, the second projection of norm one would be the identity mapping). The following simple argument shows that whenever $\mathcal{M}$ and $\mathcal{N}$ are infinite-dimensional dual operator spaces, there are uncountably many different algebraic amplifications of a map $\Phi \in \mathcal{CB}(\mathcal{M})$ to a completely bounded map on $\mathcal{M} \overline{\otimes} \mathcal{N}$, regardless of Φ being normal or not. Namely, for any non-zero functional $\varphi \in (\mathcal{M} \overline{\otimes} \mathcal{N})^*$ which vanishes on

$\mathcal{M} \vee \otimes \mathcal{N}$, and any non-zero vector $v \in \mathcal{M}\overline{\otimes}\mathcal{N}$,

$$\chi_{\mathcal{N}}^{\varphi,v}(\Phi) := \chi_{\mathcal{N}}(\Phi) - \langle \varphi, \chi_{\mathcal{N}}(\Phi)(\cdot) \rangle v$$

defines such an algebraic amplification. Furthermore, the amplification $\chi_{\mathcal{N}}^{\varphi,v}$ can of course be taken arbitrarily close to an isometry: for every $\varepsilon > 0$, an obvious choice of φ and v yields an algebraic amplification $\chi_{\mathcal{N}}^{\varphi,v} : \mathcal{CB}(\mathcal{M}) \to \mathcal{CB}(\mathcal{M}\overline{\otimes}\mathcal{N})$ such that

$$(1-\varepsilon)\|\Phi\|_{\mathrm{cb}} \leqslant \|\chi_{\mathcal{N}}^{\varphi,v}(\Phi)\|_{\mathrm{cb}} \leqslant (1+\varepsilon)\|\Phi\|_{\mathrm{cb}}$$

for all $\Phi \in \mathcal{CB}(\mathcal{M})$.

We begin our investigation by noting a criterion which establishes the uniqueness of amplification for a large class of von Neumann algebras, though only assuming part of the requirements listed in Definition 2.2.

Proposition 6.1. *Let $\mathcal{M}$ and $\mathcal{N}$ be von Neumann algebras, where $\mathcal{M}$ is an injective factor. If $\chi' : \mathcal{CB}(\mathcal{M}) \to \mathcal{CB}(\mathcal{M}\overline{\otimes}\mathcal{N})$ is a bounded linear mapping which satisfies the algebraic amplification condition* (AAC) *and properties* (iii) *and* (iv) *in Definition* 2.2, *then $\chi' = \chi_{\mathcal{N}}$.*

Proof. Owing to [CHA–SMI93, Theorem 4.2], we have the density relation:

$$\overline{\mathcal{CB}^{\sigma}(\mathcal{M})}^{w^*} = \mathcal{CB}(\mathcal{M}).$$

Hence, due to their w^*-w^*-continuity (condition (iii)), $\chi_{\mathcal{N}}$ and χ' will be identical if they only coincide on $\mathcal{CB}^{\sigma}(\mathcal{M})$. But if $\Phi \in \mathcal{CB}^{\sigma}(\mathcal{M})$, by condition (iv), the mappings $\chi_{\mathcal{N}}(\Phi)$ and $\chi'(\Phi)$ are normal, so that to establish their equality, it suffices to show that they coincide on all elementary tensors $S \otimes T \in \mathcal{M}\overline{\otimes}\mathcal{N}$. But this is true since they both fulfill condition (AAC). $\square$

We will now present our generalization of the Ge–Kadison Lemma which was first obtained in [GE–KAD96, Lemma F] and used in the solution of the splitting problem for tensor products of von Neumann algebras (in the factor case). Furtheron, a version of the lemma was again used by Strătilă and Zsidó in [STR–ZSI99] in order to derive an extremely general commutation theorem which extends both Tomita's classical commutant theorem and the above mentioned splitting result [STR–ZSI99, Theorem 4.7]. The result we give below generalizes the lemma in the version as stated (and proved) in [STR–ZSI99, Section 3.4], which the authors refer to as a "smart technical device". In [STR–ZSI99], it is applied to *normal* conditional expectations of von Neumann algebras [STR–ZSI99, Section 3.5]; of course, these are in particular *completely positive*. Our generalization yields an analogous result which in turn can be applied to *arbitrary* completely bounded mappings. This fact may be of interest in the further development of the subject treated in [GE–KAD96,STR–ZSI99]. Furthermore, we prove that the lemma not only holds for von Neumann algebras but for arbitrary admissible pairs, a situation which is not considered in the latter articles. Hence, our version is likely to prove useful in the interesting problem of obtaining a splitting theorem in the more general context of w^*-spatial tensor products of ultraweakly closed subspaces—a question which is actually hinted at in [GE–KAD96, p. 457].

Proposition 6.2. *Let* $(\mathscr{M}, \mathscr{N})$ *be an admissible pair, and* $\Phi \in \mathscr{CB}(\mathscr{M})$ *an arbitrary completely bounded operator. Suppose* $\Theta : \mathscr{M}\overline{\otimes}\mathscr{N} \to \mathscr{M}\overline{\otimes}\mathscr{N}$ *is* any *map which satisfies, for some non-zero element* $n \in \mathscr{N}$:

(i) Θ *commutes with the slice maps* $\mathrm{id}_{\mathscr{M}} \otimes \tau n$ $(\tau \in \mathscr{N}_*)$
(ii) Θ *coincides with* $\Phi \otimes \mathrm{id}_{\mathscr{N}}$ *on* $\mathscr{M} \otimes n$.

Then we must have $\Theta = \Phi \otimes \mathrm{id}_{\mathscr{N}}$.

In particular, if $\Phi \in \mathscr{CB}(\mathscr{M})$ *and* $(\Phi_\alpha)_\alpha \subseteq \mathscr{CB}(\mathscr{M})$ *is a bounded net converging* w^* *to* Φ, *then* $\Phi_\alpha \otimes \mathrm{id}_{\mathscr{N}} \xrightarrow{w^*} \Phi \otimes \mathrm{id}_{\mathscr{N}}$.

Remark 6.3. (i) The known version of the lemma supposes that $\mathscr{M}$ and $\mathscr{N}$ are von Neumann algebras, and that the mapping Φ is normal and completely positive. As shown by the above, all these assumptions can be either weakened or even just dropped.

(ii) The last statement in the proposition is even true *without* assuming the net $(\Phi_\alpha)_\alpha$ to be bounded, and in this generality follows directly from our Theorem 3.1 (property (iii) in Definition 2.2). Nevertheless, this "unbounded" version does not appear in the literature. (We remark that the assumption of boundedness is missing in the statement of Lemma 3.4 in [STR–ZSI99], though needed in the proof presented there.) We shall point out below how the "bounded" version can be easily obtained by using the first part of the Proposition (cf. [STR–ZSI99, Section 3.4]).

Proof. Using the first result of the last section, our argument follows the same lines as the one given in [STR–ZSI99]; we present the proof because of its brevity. Fix a non-zero element $m \in \mathscr{M}$. We obtain for every $u \in \mathscr{M}\overline{\otimes}\mathscr{N}$, $\rho \in \mathscr{M}_*$ and $\tau \in \mathscr{N}_*$:

$$\begin{aligned}\langle \Theta(u), \rho \otimes \tau \rangle m \otimes n &= [(\rho m \otimes \mathrm{id}_{\mathscr{N}})(\mathrm{id}_{\mathscr{M}} \otimes \tau n)\Theta](u) \\ &= [(\rho m \otimes \mathrm{id}_{\mathscr{N}})\Theta(\mathrm{id}_{\mathscr{M}} \otimes \tau n)](u) \quad \text{by (i)} \\ &= [(\rho m \otimes \mathrm{id}_{\mathscr{N}})(\Phi \otimes \mathrm{id}_{\mathscr{N}})(\mathrm{id}_{\mathscr{M}} \otimes \tau n)](u) \quad \text{by (ii)} \\ &= [(\rho m \otimes \mathrm{id}_{\mathscr{N}})(\mathrm{id}_{\mathscr{M}} \otimes \tau n)(\Phi \otimes \mathrm{id}_{\mathscr{N}})](u) \quad \text{by Theorem 5.1} \\ &= \langle (\Phi \otimes \mathrm{id}_{\mathscr{N}})(u), \rho \otimes \tau \rangle m \otimes n,\end{aligned}$$

which proves the first statement in the proposition. In order to prove the second assertion, we only have to show that if Θ is any w^*-cluster point of the net $(\Phi_\alpha \otimes \mathrm{id}_{\mathscr{N}})_\alpha$, then $\Theta = \Phi \otimes \mathrm{id}_{\mathscr{N}}$. But Θ trivially satisfies condition (ii) in our proposition, and using the normality of the mappings $\mathrm{id}_{\mathcal{M}} \otimes \tau n$, another application of Theorem 5.1 shows that Θ also meets condition (i), whence the first part of the proposition finishes the proof. $\square$

7. Completely bounded module homomorphisms and Tomiyama's slice maps

The main objective of [MAY–NEU–WIT89] was to establish the following remarkable property of the amplification mapping

$$\mathcal{CB}(\mathcal{B}(\mathcal{H})) \ni \Phi \mapsto \Phi^{(\infty)} \in \mathcal{CB}(\mathcal{B}(\mathcal{H}) \overline{\otimes} \mathcal{B}(\mathcal{K})).$$

If $\mathcal{R} \subseteq \mathcal{B}(\mathcal{H})$ is a von Neumann subalgebra, and if $\Phi \in \mathcal{CB}_{\mathcal{R}}(\mathcal{B}(\mathcal{H}))$ is an $\mathcal{R}$-bimodule homomorphism, then $\Phi^{(\infty)}$ is an $\mathcal{R} \overline{\otimes} \mathcal{B}(\mathcal{K})$-bimodule homomorphism, i.e., $\Phi^{(\infty)} \in \mathcal{CB}_{\mathcal{R} \overline{\otimes} \mathcal{B}(\mathcal{K})}(\mathcal{B}(\mathcal{H}) \overline{\otimes} \mathcal{B}(\mathcal{K}))$; see [MAY–NEU–WIT89, Proposition 2.2] (and [EFF–RUA88, Theorem 4.2], for a generalization).

An important special situation, which nevertheless shows the strength and encodes the essential statement of the result, is the case where $\mathcal{R} = \mathbb{C}\, \mathbb{1}_{\mathcal{B}(\mathcal{H})}$. Then the above theorem reads as follows:

(∗∗) Every $\Phi \in \mathcal{CB}(\mathcal{B}(\mathcal{H}))$ can be amplified to an $\mathbb{1}_{\mathcal{B}(\mathcal{H})} \otimes \mathcal{B}(\mathcal{K})$-bimodule homomorphism $\Phi^{(\infty)} \in \mathcal{CB}(\mathcal{B}(\mathcal{H}) \overline{\otimes} \mathcal{B}(\mathcal{K}))$.

The aim of this section is to prove an analogous result where now $\mathcal{B}(\mathcal{H})$ and $\mathcal{B}(\mathcal{K})$ are replaced by arbitrary von Neumann algebras $\mathcal{M}$ and $\mathcal{N}$, using our amplification $\chi_{\mathcal{N}}$. In this section, $\mathcal{M}$ and $\mathcal{N}$ will always denote von Neumann algebras.

The proofs of [MAY–NEU–WIT89, Proposition 2.2] and [EFF–RUA88, Theorem 4.2] use either the Wittstock decomposition theorem or a corresponding result by Paulsen [PAU86, Lemma 7.1] to reduce to the case of a completely positive mapping, where the assertion is then obtained by a fairly subtle analysis involving the Schwarz inequality. In contrast to this, the explicit form of $\chi_{\mathcal{N}}$ provides us with a direct route to the above announced goal; indeed, as we shall see, the statement asserting the module homomorphism property of the amplified mapping reduces to a corresponding statement about Tomiyama's slice maps which in turn can be verified in an elementary fashion (see Lemma 7.1 below).

In order to prove our result, a little preparation is needed. In the sequel, we restrict ourselves to left slice maps; analogous statements hold of course in the case of right slice maps. As is well-known (and easy to see), Tomiyama's left slice maps satisfy the following module homomorphism property:

$$L_\tau((a \otimes \mathbb{1}_{\mathcal{N}})u(b \otimes \mathbb{1}_{\mathcal{N}})) = aL_\tau(u)b,$$

where $u \in \mathcal{M} \overline{\otimes} \mathcal{N}$, $a, b \in \mathcal{M}$, $\tau \in \mathcal{N}_*$. In the following (elementary) lemma, we shall establish an analogous property involving elementary tensors of the form $\mathbb{1}_{\mathcal{M}} \otimes a$ and $\mathbb{1}_{\mathcal{M}} \otimes b$ for $a, b \in \mathcal{N}$. To this end, we briefly recall that for every von Neumann algebra $\mathcal{N}$, the predual $\mathcal{N}_*$ is an $\mathcal{N}$-bimodule in a very natural manner, the actions being given by

$$\langle \tau \cdot a, b \rangle = \langle \tau, ab \rangle \quad \text{and} \quad \langle a \cdot \tau, b \rangle = \langle \tau, ba \rangle,$$

for $a, b \in \mathcal{N}$, $\tau \in \mathcal{N}_*$. For later purposes, we note the (easily verified) equation:

$$(\mathbb{1}_{\mathcal{M}} \otimes a) \cdot (\rho \otimes \tau) \cdot (\mathbb{1}_{\mathcal{M}} \otimes b) = \rho \otimes (a \cdot \tau \cdot b), \tag{14}$$

where $a, b \in \mathcal{N}$, $\rho \in \mathcal{M}_*$, $\tau \in \mathcal{N}_*$. We now come to

Lemma 7.1. *Let $\mathcal{M}$ and $\mathcal{N}$ be von Neumann algebras. Then Tomiyama's left slice map satisfies*:

$$L_\tau((\mathbb{1}_{\mathcal{M}} \otimes a)u(\mathbb{1}_{\mathcal{M}} \otimes b)) = L_{b \cdot \tau \cdot a}(u),$$

where $u \in \mathcal{M} \overline{\otimes} \mathcal{N}$, $a, b \in \mathcal{N}$, $\tau \in \mathcal{N}_*$.

Proof. Due to the normality of left slice maps, it suffices to prove the statement for elementary tensors $u = S \otimes T \in \mathcal{M} \overline{\otimes} \mathcal{N}$. But in this case, we obtain:

$$\begin{aligned} L_\tau((\mathbb{1}_{\mathcal{M}} \otimes a)u(\mathbb{1}_{\mathcal{M}} \otimes b)) &= L_\tau(S \otimes aTb) \\ &= \langle \tau, aTb \rangle S \\ &= \langle b \cdot \tau \cdot a, T \rangle S \\ &= L_{b \cdot \tau \cdot a}(u), \end{aligned}$$

whence the desired equality follows. □

The above statement will now be used to derive a corresponding module homomorphism property shared by the amplification of a completely bounded mapping, as announced at the beginning of this section.

Theorem 7.2. *Let $\mathcal{M}$* and *$\mathcal{N}$ be von Neumann algebras. Then for every $\Phi \in \mathcal{CB}(\mathcal{M})$, the amplified mapping $\chi_{\mathcal{N}}(\Phi) \in \mathcal{CB}(\mathcal{M} \overline{\otimes} \mathcal{N})$ is an $\mathbb{1}_{\mathcal{M}} \otimes \mathcal{N}$-bimodule homomorphism.*

Proof. As in the proof of Theorem 3.1, we denote by κ the pre-adjoint mapping of $\chi_{\mathcal{N}}$. Fix $u \in \mathcal{M} \overline{\otimes} \mathcal{N}$, $a, b \in \mathcal{N}$, $\rho \in \mathcal{M}_*$ and $\tau \in \mathcal{N}_*$. We write $a' := \mathbb{1}_{\mathcal{M}} \otimes a$, $b' := \mathbb{1}_{\mathcal{M}} \otimes b$.

We first note that

$$\kappa(\rho \otimes \tau \otimes (a'ub')) = \kappa(\rho \otimes (b \cdot \tau \cdot a) \otimes u). \tag{15}$$

For by Eq. (6), in order to establish (15), we only have to show that

$$L_\tau(a'ub') = L_{b\cdot\tau\cdot a}(u),$$

which in turn is precisely the statement of Lemma 7.1.

Now we obtain for $\Phi \in \mathcal{CB}(\mathcal{M})$:

$$\begin{aligned}\langle \chi_{\mathcal{N}}(\Phi)(a'ub'), \rho\otimes\tau\rangle &= \langle \Phi, \kappa(\rho\otimes\tau\otimes(a'ub'))\rangle \\ &\overset{(15)}{=} \langle \Phi, \kappa(\rho\otimes(b\cdot\tau\cdot a)\otimes u)\rangle \\ &\overset{(14)}{=} \langle \Phi, \kappa[(b'\cdot(\rho\otimes\tau)\cdot a')\otimes u]\rangle \\ &= \langle \chi_{\mathcal{N}}(\Phi)(u), b'\cdot(\rho\otimes\tau)\cdot a'\rangle \\ &= \langle a'\chi_{\mathcal{N}}(\Phi)(u)b', \rho\otimes\tau\rangle,\end{aligned}$$

which finishes the proof. □

We finish by remarking that Theorem 7.2 yields in particular the statement $(**)$ given above; for if $\Phi \in \mathcal{CB}(\mathcal{B}(\mathcal{H}))$, we have $\chi_{\mathcal{B}(\mathcal{H})}(\Phi) = \Phi^{(\infty)}$, as noted in Section 4, Eq. (7).

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