ROBUST MEAN FIELD LINEAR-QUADRATIC-GAUSSIAN GAMES WITH UNKNOWN L^2-DISTURBANCE*

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Abstract. This paper considers a class of mean field linear-quadratic-Gaussian (LQG) games with model uncertainty. The drift term in the dynamics of the agents contains a common unknown function. We take a robust optimization approach where a representative agent in the limiting model views the drift uncertainty as an adversarial player. By including the mean field dynamics in an augmented state space, we solve two optimal control problems sequentially, which combined with consistent mean field approximations provides a solution to the robust game. A set of decentralized control strategies is derived by use of forward-backward stochastic differential equations (FBSDE) and shown to be a robust ε-Nash equilibrium.

Key words. mean field game, model uncertainty, robust control, decentralized control, Nash equilibrium

AMS subject classifications. 91A10, 91A23, 91A25, 93E20

1. Introduction.

Mean field game theory provides an effective methodology for the analysis and strategy design in a large population of players which are individually insignificant but collectively have strong impact (see e.g. [29, 31, 32, 38]). A typical modeling analyzes a system of N players with mean field coupling in their dynamics or costs, or both. The linear-quadratic-Gaussian (LQG) framework is of particular interest since it allows an explicit solution procedure. Consider a large population of N agents. The dynamics of agent i are given by the stochastic differential equation (SDE)

$$dx_i(t) = (Ax_i(t) + Bu_i(t) + Gx^{(N)}(t))dt + DdW_i(t), \qquad t \geq 0, \tag{1}$$

where $x^{(N)} = (1/N)\sum_{i=1}^N x_i$ denotes the mean field coupling term. The cost of agent i is given by

$$\mathcal{J}_i(u_i, \ldots, u_N) = \mathbb{E}\Big[\int_0^T \Big(|x_i - \Gamma x^{(N)} - \eta|_Q^2 + u_i^T R u_i \Big)\, dt + x_i^T(T) H x_i(T) \Big], \tag{2}$$

where we denote $|z|_Q = (z^T Q z)^{\frac{1}{2}}$ and the symmetric matrices $Q \geq 0, H \geq 0$ and $R > 0$. The LQG modeling framework was first developed in [29, 31] to obtain a set of strategies $(\hat{u}_1, \ldots, \hat{u}_N)$ such that each $\hat{u}_i$ only uses the local sample path information of x_i and some deterministic functions reflecting the collective behavior of the agents and such that $(\hat{u}_1, \ldots, \hat{u}_N)$ is an ε-Nash equilibrium. There has existed a substantial body of literature adopting the LQG framework [4, 9, 35, 39, 47, 52].

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For further literature, the reader is referred to [13, 14, 15, 22, 32, 36] for nonlinear diffusion based games and the associated SDE analysis, [12, 38] for study of the coupled system of Hamilton-Jacobi-Bellman (HJB) and Fokker-Planck equations, [7, 10, 30, 45, 46] for models containing a major player, [8, 20] for time consistent strategies in mean field games, [57] for mean field oscillator games, [55] for Markovian switching mean field games, [16] for application to Bertrand and Cournot equilibrium models, and [1] for a related solution notion called stationary equilibrium where players optimize assuming a steady-state long-run average for the empirical distribution of others' states. For an overview on mean field game theory, see [7, 11, 23].

Within the traditional research on games, there has existed a fair amount of literature on model uncertainty. For an N player static game with finite action spaces and an uncertain payoff matrix, a robust-optimization equilibrium is introduced in [2] where each player optimizes its worst case payoff with respect to the uncertain set. A similar method is applied to hierarchical static games [25]. Robustness has been addressed in dynamic games as well. A linear-quadratic (LQ) game with system parameter uncertainties is presented in [33], and the deviation from the Nash equilibrium is estimated for a set of nominal strategies. Robust Nash equilibria are analyzed in [54] for an LQ game with an unknown time-varying disturbance signal as an adversarial player. In the first case, a soft-constrained game is solved where the cost includes a quadratic penalty term for the disturbance. The second case introduces a hard constraint by specifying an L^2 bound on the disturbance function. The work [34] deals with stochastic games where the payoff and state transition probabilities contain uncertainty. The solution is developed by letting each player solve a robust Markov decision problem to optimize its worst case cost while other players' strategies are fixed.

This paper aims to address model uncertainty in the mean field LQG game context. Specifically, we focus on drift uncertainty by adding to (1) a common unknown L^2-disturbance f. A practical motivation is that in many decision problems, a large number of agents can share a common uncertainty source fluctuating with time, and examples include taxation, subsidy, interest rates, and so on. A direct consequence of our modeling is that this disturbance has global influence on the population. To address robustness, each agent locally views the disturbance as an adversarial player, and for this purpose we incorporate into (2) an effort penalty term for the disturbance which in turn maximizes the resulting cost first. The agent minimizes subsequently. The framework of letting the disturbance maximize while its effort is penalized is called the soft-constraint approach [5, 21, 54]. It has the advantage of analytical tractability. When a hard constraint is considered, the robust mean field game is more difficult to tackle; see some preliminary analysis in [27]. Regarding robustness in mean field games, a related work is [51] where each agent is paired with its local disturbance as an adversarial player. The resulting solution is to replace the usual HJB equation by a Hamilton-Jacobi-Isaacs (HJI) equation in the solution.

To design the individual strategies it is necessary to build the dynamics of the mean field (i.e. state average of the agents) evolving under the disturbance. This technique shares its spirit with the state augmentation method in major player models [30, 45, 46]. The subsequent robust optimization problem, as a minimax control problem, leads to two optimal control problems with indefinite state weights [56]. They are different from the well known stochastic control problems with indefinite control weights [18, 41]. We will follow a convex optimization approach to solve the two control problems via variational analysis and forward-backward stochastic differential equations (FBSDE) [26, 44, 48]. Both the information structure and the

solution procedure for our model are different from [51] where each player and its local disturbance have access to its state and so dynamic programming is applicable. Our main contributions are summarized as follows:

- We formulate a class of mean field LQG games where the players face a common uncertainty source, and introduce the robust optimization approach to solve two convex optimal control problems.
- Decentralized strategies are obtained for the robust mean field game via a set of FBSDE.
- The performance of the decentralized strategies for the N players is characterized as a robust ε-Nash equilibrium.

The rest of this paper is organized as follows. Section 2 introduces the mean field LQG game with a common disturbance and defines the worst case cost for a player. Section 3 studies the limiting robust optimization problem which leads to two optimal control problems solved sequentially by the disturbance and the representative player. The solution equation system of the mean field game is obtained in Section 4 based on consistent mean field approximations. A key error estimate of the mean field approximation is developed in Section 5. Section 6 characterizes the set of decentralized strategies as a robust ε-Nash equilibrium, and Section 7 concludes the paper.

2. Mean Field LQG Games with Drift Uncertainty. Consider a finite time horizon $[0,T]$ for $T>0$. Suppose that $(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{0\le t\le T},\mathbb{P})$ is a complete filtered probability space. Throughout this paper, we denote by $\mathbb{R}^k$ the k-dimensional Euclidean space, $\mathbb{R}^{n\times k}$ the set of all $n\times k$ matrices. We use $|\cdot|$ to denote the norm of a Euclidean space, or the Frobenius norm for matrices. For a vector or matrix M, M^T denotes its transpose. Let $L^2_{\mathcal{F}}(0,T;\mathbb{R}^k)$ denote the space of all $\mathbb{R}^k$-valued $\mathcal{F}_t$-progressively measurable processes $x(\cdot)$ satisfying $\mathbb{E}\int_0^T|x(t)|^2dt<\infty$; $C([0,T];\mathbb{R}^k)$ (resp., $C^1([0,T];\mathbb{R}^k)$) is the space of all $\mathbb{R}^k$-valued functions $h(\cdot)$ defined on $[0,T]$ which are continuous (resp., continuously differentiable); $L^2(0,T;\mathbb{R}^k)$ is the space of all $\mathbb{R}^k$-valued measurable functions $h(\cdot)$ on $[0,T]$ satisfying $\int_0^T|h(t)|^2dt<\infty$, and we denote the norm $\|h\|_{L^2}=(\int_0^T|h(t)|^2dt)^{1/2}$. Throughout the paper, we use C (or $C_1,C_2,\ldots$) to denote a generic constant which does not depend on the population size N and may vary from place to place.

2.1. The game with a finite population. Consider N agents (or players) denoted by $\mathcal{A}_i$, $1\le i\le N$, respectively. The state x_i of $\mathcal{A}_i$ is $\mathbb{R}^n$-valued and satisfies the linear SDE

$$dx_i(t)=(Ax_i(t)+Bu_i(t)+Gx^{(N)}(t)+f(t))dt+DdW_i(t),\quad 1\le i\le N, \tag{3}$$

where $x^{(N)}=(1/N)\sum_{j=1}^N x_j$. The control u_i takes its value in $\mathbb{R}^{n_1}$. The $\mathbb{R}^{n_2}$-valued standard Brownian motions $\{W_i(t),1\le i\le N\}$ are independent. The initial states $\{x_i(0),1\le i\le N\}$ are deterministic and their empirical mean has the limit $\lim_{N\to\infty}(1/N)\sum_{i=1}^N x_i(0)=m_0$. We take $\{\mathcal{F}_t\}_{0\le t\le T}$ as the natural filtration generated by the Nn_2-dimensional Brownian motion $(W_1(t),\ldots,W_N(t))$, and $\mathcal{F}=\mathcal{F}_T$. The admissible control set $\mathcal{U}$ of $\mathcal{A}_i$ is

$$\mathcal{U}:=\left\{u_i(\cdot):u_i\in L^2_{\mathcal{F}}(0,T;\mathbb{R}^{n_1})\right\}.$$

Denote $u=(u_1,\ldots,u_N)$ and $u_{-i}=(u_1,\ldots,u_{i-1},u_{i+1},\ldots,u_N)$.

The function $f\in L^2(0,T;\mathbb{R}^n)$ is an unknown disturbance to characterize the model uncertainty, and represents an influence from the common environment for

decision-making. A natural motivation for considering deterministic disturbance is the following. Although each player $\mathcal{A}_i$ regards the disturbance as adversarial, it *should not* be excessively pessimistic by assuming that the latter will use the sample path information of W_i to play against it, and instead only considers a deterministic f. We note that dynamical systems with both deterministic disturbance and stochastic noise have been considered in the control literature to account for different uncertainty causes (see e.g. [6, 24, 49]), often leading to robust controller design problems.

The cost functional of $\mathcal{A}_i$ is

$$J_i(u_i,u_{-i},f)=\mathbb{E}\bigg[\int_0^T\Big(|x_i-(\Gamma x^{(N)}+\eta)|_Q^2+u_i^TRu_i-\frac{1}{\gamma}|f(t)|^2\Big)\,dt + x_i^T(T)Hx_i(T)\bigg], \tag{4}$$

where the symmetric matrices $Q\geq 0$, $R>0$, $H\geq 0$ and the constant $\gamma>0$. We assume uniform agents in the sense that they share the same parameter datum $(A,B,G,D;\Gamma,\eta,Q,R,\gamma,H)$. Also, to simplify the analysis, we consider constant parameters.

Due to the unknown function f, $\mathcal{A}_i$ cannot evaluate its cost even if all control policies $(u_1,\dots,u_N)$ are known. To address this indeterminacy, we approach the game from a robust optimization point of view where each agent takes f as an adversarial player. Here a soft-constraint [5, 21, 54] for the disturbance is adopted in that the term $-\frac{1}{\gamma}|f(t)|^2$ is included in (4) while f attempts to maximize. For given (u_i,u_{-i}), define the worst case cost of $\mathcal{A}_i$ as

$$J_i^{\rm wo}(u_i,u_{-i})=\sup_{f\in L^2(0,T;\mathbb{R}^n)}J_i(u_i,u_{-i},f).$$

A set of strategies $(\hat u_1,\dots,\hat u_N)$ is a robust ε-Nash equilibrium for the N players if for $\varepsilon\geq 0$,

$$J_i^{\rm wo}(\hat u_i,\hat u_{-i})-\varepsilon\leq\inf_{u_i\in\mathcal{U}}J_i^{\rm wo}(u_i,\hat u_{-i})\leq J_i^{\rm wo}(\hat u_i,\hat u_{-i}). \tag{5}$$

Our central objective is to design decentralized strategies based on the above solution notion.

3. The Limiting Robust Optimization Problem. We start by making an appropriate approximation of the coupling term $x^{(N)}$. Adding up the N equations in (3) and normalizing by $1/N$, we obtain

$$dx^{(N)}=[(A+G)x^{(N)}+Bu^{(N)}+f]dt+D(1/N)\sum_{j=1}^N dW_j,$$

where $u^{(N)}=(1/N)\sum_{j=1}^N u_j$. Intuitively, from the point of view of $\mathcal{A}_i$, $u^{(N)}$ may be approximated by a deterministic function $\bar u$. Moreover, $(1/N)\sum_{j=1}^N dW_j$ vanishes as $N\to\infty$ due to the law of large numbers. In turn, a deterministic function m can be used to approximate $x^{(N)}$. The above reasoning suggests to introduce the limiting ordinary differential equation (ODE)

$$\dot m=(A+G)m+B\bar u+f,\quad m(0)=m_0. \tag{6}$$

3.1. The limiting model of the mean field game. Consider the optimization problem of a representative agent $\mathcal{A}_i$:

$$\begin{cases} dx_i = (Ax_i + Bu_i + Gm_i + f)dt + DdW_i, \\ \dot{m}_i = (A+G)m_i + B\bar{u} + f, \end{cases} \tag{7}$$

where the second equation is motivated from (6) and $m_i(0) = m_0$. For the limiting model (7), $(W_i, x_i(0))$ is the same as in (3). We reuse $(x_i, \mathcal{A}_i)$ to denote the state and the corresponding agent. This shall cause no risk of confusion. Since f will be determined as its worst case form depending on $x_i(0)$, m_i is associated with the agent index i so that it is ready as an appropriate notation for the subsequent closed-loop dynamics. The cost functional is given by

$$\bar{J}_i(u_i, f) = \mathbb{E}\int_0^T \left\{ |x_i - (\Gamma m_i + \eta)|_Q^2 + u_i^T R u_i - \frac{1}{\gamma}|f(t)|^2 \right\} dt + \mathbb{E}x_i^T(T) H x_i(T).$$

We aim to find a solution pair $(\hat{f}, \hat{u}_i)$ such that

$$\bar{J}_i(\hat{u}_i, \hat{f}) = \min_{u_i \in \mathcal{U}} \max_{f \in L^2(0,T;\mathbb{R}^n)} \bar{J}_i(u_i, f). \tag{8}$$

Finally, we need a consistency condition, i.e., $\frac{1}{N}\sum_{i=1}^N \hat{u}_i$ converges to $\bar{u}$ in some sense (this will be made precise in Section 4) and we look for $\bar{u} \in C([0,T];\mathbb{R}^{n_1})$; the feasibility of doing so will be clear from our solution procedure. The next part of our plan is to show that such strategies have the property in (5) when applied in the game of N agents. In the following, we solve the optimization problem (8) in two steps.

3.2. The control problem with respect to the disturbance. Let $u_i \in \mathcal{U}$ and $\bar{u} \in C([0,T];\mathbb{R}^{n_1})$ be fixed. The optimal control problem is

$$(\mathbf{P1}) \quad \text{maximize}_{f \in L^2(0,T;\mathbb{R}^n)} \bar{J}_i(u_i, f). \tag{9}$$

Clearly (P1) is equivalent to the following problem

$$\begin{aligned} (\mathbf{P1a}) \quad \text{minimize}_{f \in L^2(0,T;\mathbb{R}^n)} \bar{J}_i'(u_i, f) = \mathbb{E}\int_0^T \left\{ -|x_i - (\Gamma m_i + \eta)|_Q^2 + \frac{1}{\gamma}|f(t)|^2 \right\} dt \\ - \mathbb{E}x_i^T(T) H x_i(T). \end{aligned}$$

(P1a) is an optimal control problem with negative semi-definite state weights. We are interested in the situation where (P1a) is a strictly convex problem with a coercivity property. This ensures that the worse case disturbance is uniquely determined by $\mathcal{A}_i$. The procedure below to identify conditions for ensuring convexity is similar to [41].

To study the convexity of $\bar{J}_i'$ in f, we construct a simpler auxiliary optimal control problem. Denote

$$\widehat{Q} = (I - \Gamma)^T Q (I - \Gamma).$$

Consider the dynamics

$$\dot{z} = (A+G)z + g, \quad z(0) = 0, \tag{10}$$

where $g \in L^2(0,T;\mathbb{R}^n)$. The optimal control problem is

$$(\mathbf{P1b}) \quad \text{minimize} \quad \bar{J}_i''(g) = \int_0^T \left\{ -z^T \widehat{Q} z + \frac{1}{\gamma}|g(t)|^2 \right\} dt - z^T(T) H z(T).$$

For any $s \in \mathbb{R}$, we have $\bar{J}_i''(sg) = s^2 \bar{J}_i''(g)$, and so view $\bar{J}_i''$ as a quadratic functional of g.

DEFINITION 3.1. *Let $F(g)$ be a real-valued functional of $g \in L^2(0,T;\mathbb{R}^n)$. If $F(g) \geq 0$ for all g, F is said to be positive semi-definite. If furthermore, $F(g) > 0$ for all $g \neq 0$, F is said to be positive definite.*

LEMMA 3.2. *$\bar{J}_i'(u_i, f)$ is convex (resp., strictly convex) in f if and only if $\bar{J}_i''(g)$ is positive semi-definite (resp., positive definite).*

Proof. Let (x_i, m_i) and (x_i', m_i') be the state processes of (7) corresponding to (u_i, f) and (u_i, f'), respectively. Take any $\lambda_1 \in [0,1]$ and denote $\lambda_2 = 1 - \lambda_1$. Then

$$\begin{aligned} &\lambda_1 \bar{J}_i'(u_i, f) + \lambda_2 \bar{J}_i'(u_i, f') - \bar{J}_i'(u_i, \lambda_1 f + \lambda_2 f') \\ &= \lambda_1 \lambda_2 \mathbb{E} \int_0^T \left\{ |x_i - x_i' - \Gamma(m_i - m_i')|_Q^2 + \frac{1}{\gamma}|f(t) - f'(t)|^2 \right\} dt \\ &\quad - \lambda_1 \lambda_2 \mathbb{E}|x_i(T) - x_i'(T)|_H^2. \end{aligned}$$

Denote $g = f - f'$, and $z = x_i - x_i'$. Therefore, z is deterministic and satisfies (10). In addition, $m_i - m_i' = z$ for $t \in [0,T]$. Hence

$$\lambda_1 \bar{J}_i'(u_i, f) + \lambda_2 \bar{J}_i'(u_i, f') - \bar{J}_i'(u_i, \lambda_1 f + \lambda_2 f') = \lambda_1 \lambda_2 \bar{J}_i''(g)$$

and the lemma follows. □

For our further existence analysis, we need to ensure $\bar{J}_i'(u_i, f)$ to be both strictly convex and coercive in f. For this purpose, we introduce the following assumption.

(**H1**) There exists a small $\epsilon_0 > 0$ such that $\bar{J}_i''(g) - \epsilon_0 \|g\|_{L^2}^2$ is positive semi-definite.

Note that (H1) is completely determined by the parameters $(\widehat{Q}, \gamma, \epsilon_0, H, T)$, and does not depend on u_i. Concerning (H1), we have the following result.

PROPOSITION 3.3. *The following statements are equivalent:*
(i) (H1) *holds true on* $[0,T]$.
(ii) *The Riccati equation*

$$\dot{P} + (A+G)^T P + P(A+G) - \gamma P^2 - \widehat{Q} = 0, \quad P(T) = -H \tag{11}$$

has a unique solution on $[0,T]$.
(iii) *For any* $t \in [0,T]$,

$$\det\{[(0,I)e^{\mathcal{A}t}(0,I)^T]\} > 0,$$

where

$$\mathcal{A} = \begin{pmatrix} A+G+\gamma H & -\gamma I \\ \breve{Q} & -(A+G+\gamma H)^T \end{pmatrix}$$

and $\breve{Q} = \gamma H^2 + \widehat{Q} + (A+G)^T H + H(A+G)$.

Proof. In fact, (H1) is the uniform convexity condition proposed in [50], and the equivalence between (i) and (ii) is a corollary of Theorem 4.5 of [50]. Moreover, (iii) $\Longrightarrow$ (ii) is given in [44, Theorem 4.3, p. 48]. On the other hand, (ii) $\Longrightarrow$ (iii) is implied by Theorems 2.7 and 2.9 of [59, Ch. 6]. $\square$

For illustration of condition (ii), we give the following example.

EXAMPLE 3.4. *Consider system (3)-(4) with parameters $A = 0.5$, $B = 1$, $G = 0.25$, $Q = 1$, $\Gamma = 0.8$, $R = 1.5$, $H = 0$, $\gamma = 1$. Denote $\widehat{A} = A + G$. We solve (11) to obtain*

$$P(t) = \frac{-\widehat{Q}(e^{\alpha(t-T)} - e^{-\alpha(t-T)})}{\lambda_2 e^{\alpha(t-T)} - \lambda_1 e^{-\alpha(t-T)}}, \tag{12}$$

where

$$\lambda_1 = -\widehat{A} + \sqrt{\widehat{A}^2 - \gamma\widehat{Q}} = -0.027158, \quad \lambda_2 = -\widehat{A} - \sqrt{\widehat{A}^2 - \gamma\widehat{Q}} = -1.472842,$$
$$\alpha = \sqrt{\widehat{A}^2 - \gamma\widehat{Q}} = 0.722842.$$

If $0 < T < T_{\max} = \frac{1}{2\alpha}\ln(\lambda_2/\lambda_1) = 2.752198$, $P(t)$ given by (12) is well defined on $[0, T]$. By the local Lipschitz continuity property of the vector field in (11), $P(t)$ is the unique solution.

Note that (11) is not a standard Riccati equation since the state weight matrix $-\widehat{Q}$ is not positive semi-definite. In general, the solvability of (11) cannot be ensured on an arbitrary time horizon. Condition (iii) enables us to determine the solvability of (11) on a given time horizon. Note that condition (iii) is equivalent to $\det\{[(0, I)e^{\mathcal{A}t}(0, I)^T]\} \neq 0, \forall t \in [0, T]$ by noting $\det\{[(0, I)e^{\mathcal{A}t}(0, I)^T]\}_{t=0} = 1$. Condition (iii) is more checkable as illustrated by the following example.

EXAMPLE 3.5. *Consider system (3)-(4) with parameters $A = -0.5$, $G = 0.25$, $Q = 1$, $\Gamma = 0.8$, $H = 0$, $\gamma = 1$. We obtain*

$$\mathcal{A} = \begin{pmatrix} -0.25 & -1 \\ 0.04 & 0.25 \end{pmatrix},$$
$$e^{\mathcal{A}t} = \begin{pmatrix} -\frac{1}{3}e^{\frac{3}{20}t} + \frac{4}{3}e^{-\frac{3}{20}t} & -\frac{10}{3}e^{\frac{3}{20}t} + \frac{10}{3}e^{-\frac{3}{20}t} \\ \frac{2}{15}e^{\frac{3}{20}t} - \frac{2}{15}e^{-\frac{3}{20}t} & \frac{4}{3}e^{\frac{3}{20}t} - \frac{1}{3}e^{-\frac{3}{20}t} \end{pmatrix},$$
$$\det\{[(0, 1)e^{\mathcal{A}t}(0, 1)^T]\} = \frac{4}{3}e^{\frac{3}{20}t} - \frac{1}{3}e^{-\frac{3}{20}t} > 0, \qquad \forall t \geq 0.$$

Thus for any $T > 0$, (11) admits a unique solution on $[0, T]$. Therefore, (H1) *holds true on $[0, T]$.*

LEMMA 3.6. *Assume* (H1). *Then $\bar{J}_i'(u_i, f)$ is strictly convex in f. Moreover, $\bar{J}_i'(u_i, f)$ is coercive in f and, in particular, there exists a constant $C_{u_i, x_i(0)}$ depending on $(u_i, x_i(0))$ such that*

$$\bar{J}_i'(u_i, f) \geq \frac{\epsilon_0}{2}\|f\|_{L^2}^2 - C_{u_i, x_i(0)}.$$

Proof. Since $\bar{J}_i''(g) - \epsilon_0\|g\|_{L^2}^2$ is positive semi-definite by (H1), $\bar{J}_i''(g)$ is positive definite. By Lemma 3.2, $\bar{J}_i'(u_i, f)$ is strictly convex in f. Following the method in proving Lemma 3.2, we can further show that $\chi(f) := \bar{J}_i'(u_i, f) - \epsilon_0\|f\|_{L^2}^2$ is convex in f. By (7) and direct estimates, we can show

$$\sup_{\|f\|_{L^2} \leq 1} |\chi(f)| \leq C_{0, u_i, x_i(0)},$$

where the constant $C_{0,u_i,x_i(0)}$ depends on $(u_i, x_i(0))$. Now consider f with $\|f\|_{L^2} \geq 1$. Define $f_1 = \frac{f}{\|f\|_{L^2}}$. The convexity of $\chi(f)$ implies

$$\chi(f_1) \leq \frac{1}{\|f\|_{L^2}}\chi(f) + \frac{\|f\|_{L^2}-1}{\|f\|_{L^2}}\chi(0) \leq \frac{1}{\|f\|_{L^2}}\chi(f) + C_{0,u_i,x_i(0)}. \tag{13}$$

Consequently, for $\|f\|_{L^2} \geq 1$, (13) gives

$$\chi(f) \geq -2C_{0,u_i,x_i(0)}\|f\|_{L^2}.$$

Hence for any f, $\chi(f) \geq -C_{0,u_i,x_i(0)}(2\|f\|_{L^2}+1)$. It follows that

$$\begin{aligned}\bar{J}_i'(u_i,f) &= \chi(f) + \epsilon_0\|f\|_{L^2}^2 \\ &\geq \epsilon_0\|f\|_{L^2}^2 - C_{0,u_i,x_i(0)}(2\|f\|_{L^2}+1) \\ &\geq \frac{\epsilon_0}{2}\|f\|_{L^2}^2 - C_{u_i,x_i(0)}\end{aligned}$$

for some constant $C_{u_i,x_i(0)}$. □

THEOREM 3.7. *Suppose that* (H1) *holds and let $u_i \in \mathcal{U}$ and $\bar{u}$ be fixed. Then*

(i) *$\bar{J}_i'(u_i,f)$ has a unique minimizer $\hat{f}$, or equivalently, $\bar{J}_i(u_i,f)$ has a unique maximizer $\hat{f}$;*

(ii) *there exists a unique solution $(x_i, m_i, p_i) \in L^2_{\mathcal{F}}(0,T;\mathbb{R}^n) \times L^2(0,T;\mathbb{R}^{2n})$ to the equation system*

$$\begin{cases} dx_i = (Ax_i + Bu_i + Gm_i + \gamma p_i)dt + DdW_i, \\ \dot{m}_i = (A+G)m_i + B\bar{u} + \gamma p_i, \\ \dot{p}_i = -(A+G)^T p_i - (I-\Gamma)^T Q[\mathbb{E}x_i - (\Gamma m_i + \eta)], \end{cases} \tag{14}$$

where $m_i(0) = m_0$ and $p_i(T) = H\mathbb{E}x_i(T)$, and furthermore $\hat{f} = \gamma p_i$.

Proof. (i) By Lemma 3.6, $\bar{J}_i'$ is strictly convex and coercive. In addition, $\bar{J}_i'$ is continuous in f. Hence there exists a unique $\hat{f}$ such that $\bar{J}_i'(u_i,\hat{f}) = \inf_f \bar{J}_i'(u_i,f)$ [37, Chap. 7], [43].

(ii) We start by establishing existence. Let the optimal state-control pair be denoted by $(x_i, m_i, \hat{f})$, which is uniquely determined. We have the relation

$$dx_i = (Ax_i + Bu_i + Gm_i + \gamma\hat{f})dt + DdW_i, \tag{15}$$

$$\dot{m}_i = (A+G)m_i + B\bar{u} + \gamma\hat{f}, \tag{16}$$

where $m_i(0) = m_0$. By using (x_i, m_i), we obtain a unique solution p_i from

$$\dot{p}_i = -(A+G)^T p_i - (I-\Gamma)^T Q[\mathbb{E}x_i - (\Gamma m_i + \eta)], \tag{17}$$

where $p_i(T) = H\mathbb{E}x_i(T)$.

Now we consider another control $f = \hat{f} + \tilde{f} \in L^2(0,T;\mathbb{R}^n)$ in place of $\hat{f}$. Let $\tilde{x}_i$ and $\tilde{m}_i$ be the first variations of x_i and m_i, respectively, which result from the variation $\tilde{f}$ for $\hat{f}$. Then we have $\tilde{x}_i = \tilde{m}_i$ for all $t \in [0,T]$ and

$$\frac{d\tilde{x}_i}{dt} = (A+G)\tilde{x}_i + \tilde{f}, \quad \tilde{x}_i(0) = 0.$$

Since $\bar{J}_i'$ has a minimum at $(x_i, m_i, \hat{f})$, the first variation of the cost satisfies

$$0 = \frac{\delta \bar{J}_i'}{2} = \mathbb{E}\int_0^T \left\{ -[x_i - (\Gamma m_i + \eta)]^T Q(I-\Gamma)\tilde{x}_i + \frac{1}{\gamma}\hat{f}^T\tilde{f} \right\} dt \\ - \mathbb{E}x_i^T(T)H\tilde{x}_i(T). \tag{18}$$

On the other hand,

$$\begin{aligned} \frac{d}{dt}(p_i^T\tilde{x}_i) &= \tilde{x}_i^T\dot{p}_i + p_i^T\frac{d\tilde{x}_i}{dt} \\ &= -[\mathbb{E}x_i - (\Gamma m_i + \eta)]^T Q(I-\Gamma)\tilde{x}_i + p_i^T\tilde{f}. \end{aligned} \tag{19}$$

Integrating both sides of (19) and invoking (18), we obtain

$$p_i^T(T)\tilde{x}_i(T) = \int_0^T \left(p_i^T\tilde{f} - \frac{1}{\gamma}\hat{f}^T\tilde{f} \right) dt + \mathbb{E}x_i^T(T)H\tilde{x}_i(T). \tag{20}$$

Recalling $p_i(T) = H\mathbb{E}x_i(T)$, since $\tilde{f}$ is arbitrary, it follows from (20) that

$$\hat{f} = \gamma p_i$$

for a.e. $t \in [0,T]$. Therefore, (x_i, m_i, p_i) determined by (15)-(17) is a solution to (14).

We proceed to show uniqueness. Suppose that (x_i', m_i', p_i') is another solution of (14). Set the control $f' = \gamma p_i'$. It is straightforward to show that the first variation of $\bar{J}_i'$ at the state control pair (x_i', m_i', f') is zero. Since $\bar{J}_i'$ is strictly convex, this implies that (x_i', m_i', f') is the unique optimal state-control pair and so coincides with $(x_i, m_i, \hat{f})$ where (x_i, m_i) is the optimal state process determined from (15)-(17). This further implies $p_i' = p_i$. So uniqueness follows. The last part of (ii) is now obvious. □

3.3. The control problem of player $\mathcal{A}_i$. Assume that (H1) holds. This will ensure that all the equation systems in this section have a well defined solution. The dynamics are given by

$$\begin{cases} dx_i = (Ax_i + Bu_i + Gm_i + \gamma p_i)dt + DdW_i, \\ \dot{m}_i = (A+G)m_i + B\bar{u} + \gamma p_i, \\ \dot{p}_i = -(A+G)^T p_i - (I-\Gamma)^T Q[\mathbb{E}x_i - (\Gamma m_i + \eta)], \end{cases} \tag{21}$$

where $m_i(0) = m_0$ and $p_i(T) = H\mathbb{E}x_i(T)$. The optimal control problem is

$$\begin{aligned} (\mathbf{P2}) \quad \text{minimize}_{u_i \in L^2_{\mathcal{F}}(0,T;\mathbb{R}^{n_1})} \bar{J}_i(u_i, \hat{f}_{u_i}) &= \mathbb{E}\int_0^T \big\{ |x_i - (\Gamma m_i + \eta)|_Q^2 \\ &\quad + u_i^T R u_i - \gamma |p_i(t)|^2 \big\} dt + \mathbb{E}x_i^T(T)Hx_i(T). \end{aligned}$$

Here we have taken $\hat{f}_{u_i} = \gamma p_i$ which depends on u_i. We may simply write $\bar{J}_i(u_i)$. This is again a linear quadratic optimal control problem with indefinite weight for the state vector (x_i, m_i, p_i). Note that a perturbation in u_i will cause a change of the mean term $\mathbb{E}x_i$. So this is essentially a mean field type optimal control problem; see related work [3, 58].

We continue to identify conditions under which (P2) is strictly convex and coercive. These conditions will be characterized by using an auxiliary control problem with dynamics

$$\begin{cases} \dot{z}_i = Az_i + B\nu_i + Gz + \gamma q, \\ \dot{z} = (A+G)z + \gamma q, \\ \dot{q} = -(A+G)^T q - (I-\Gamma)^T Q(z_i - \Gamma z), \end{cases} \tag{22}$$

where $z_i(0) = z(0) = 0$ and $q(T) = Hz_i(T)$. The control $\nu_i \in L^2(0,T;\mathbb{R}^{n_1})$. The optimal control problem is

$$\textbf{(P2a)} \quad \text{minimize} \quad \bar{J}_i^a(\nu_i) = \int_0^T \left\{ |z_i - \Gamma z|_Q^2 + \nu_i^T R \nu_i - \gamma |q(t)|^2 \right\} dt + |z_i(T)|_H^2.$$

We may view this as a deterministic optimal control problem with two point boundary value conditions for the state trajectory. We say $\bar{J}_i^a$ is positive semi-definite if $\bar{J}_i^a(\nu_i) \geq 0$ for all ν_i; if furthermore, $\bar{J}_i^a(\nu_i) > 0$ whenever $\nu_i \neq 0$, we say $\bar{J}_i^a$ is positive definite. In order to have a well defined optimal control problem, we need to show that (22) has a unique solution.

LEMMA 3.8. *Assume* (H1). *For each ν_i, there exists a unique solution $(z_i, z, q) \in C^1([0,T];\mathbb{R}^{3n})$ to* (22).

Proof. Indeed, by taking $u_i = 0$ and $u_i = \nu_i \in L^2(0,T;\mathbb{R}^{n_1})$ in (21), we obtain two solutions (x_i^0, m_i^0, p_i^0) and $(x_i^{\nu_i}, m_i^{\nu_i}, p_i^{\nu_i})$, respectively. It is easy to show that $(z_i, z, q) := (x_i^{\nu_i} - x_i^0, m_i^{\nu_i} - m_i^0, p_i^{\nu_i} - p_i^0)$ is a solution of (22) by observing that $x_i^{\nu_i} - x_i^0$ is deterministic.

If there exist two different solutions to (22) for some ν_i, then we can construct two different solutions to (21) for a given u_i, which is a contradiction to Theorem 3.7. □

LEMMA 3.9. *$\bar{J}_i(u_i)$ is convex (resp., strictly convex) in $u_i \in \mathcal{U}$ if and only if $\bar{J}_i^a(\nu_i)$ is positive semi-definite (resp., positive definite).*

Proof. See appendix A. □

We introduce the following assumption.

(H2) There exists a small constant $\delta_0 > 0$ such that $\bar{J}_i^a(\nu_i) - \delta_0 \|\nu_i\|^2 \geq 0$ for all $\nu_i \in L^2(0,T;\mathbb{R}^{n_1})$.

3.4. Representation of the quadratic functional. We intend to find an expression of $\bar{J}_i^a(\nu_i)$ so that (H2) can be characterized in a more explicit form. A change of coordinates will make the computation more convenient. Define $\check{z} = z_i - z$. Then (22) becomes

$$\begin{cases} \dot{\check{z}} = A\check{z} + B\nu_i, \\ \dot{z} = (A+G)z + \gamma q, \\ \dot{q} = -\widehat{Q}z - (A+G)^T q - (I-\Gamma)^T Q\check{z}, \end{cases} \tag{23}$$

where $\check{z}(0) = z(0) = 0$ and $q(T) = H(\check{z}(T) + z(T))$.

Define the Hamiltonian matrix

$$\mathcal{H} = \begin{bmatrix} A+G & \gamma I \\ -\widehat{Q} & -(A+G)^T \end{bmatrix}$$

and the matrix ODE $\dot{\Phi}(t) = \mathcal{H}\Phi(t)$ where $\Phi(0) = I$. Denote the partition

$$\Phi(t) = \begin{bmatrix} \Phi_{11}(t) & \Phi_{12}(t) \\ \Phi_{21}(t) & \Phi_{22}(t) \end{bmatrix},$$

where each submatrix Φ_{ij} is an $n \times n$ matrix function.

We have

$$\check{z}(t) = \int_0^t e^{A(t-\tau)} B\nu_i(\tau) d\tau. \tag{24}$$

By solving (z, q) in (23), we obtain

$$\begin{aligned} z(t) &= \Phi_{12}(t)q(0) - \int_0^t \Phi_{12}(t-s)(I-\Gamma)^T Q\check{z}(s)ds, \\ q(t) &= \Phi_{22}(t)q(0) - \int_0^t \Phi_{22}(t-s)(I-\Gamma)^T Q\check{z}(s)ds, \end{aligned}$$

where $q(0)$ is to be determined. At the terminal time,

$$z(T) = \Phi_{12}(T)q(0) - \int_0^T \Phi_{12}(T-s)(I-\Gamma)^T Q\check{z}(s)ds$$

and

$$\begin{aligned} q(T) &= \Phi_{22}(T)q(0) - \int_0^T \Phi_{22}(T-s)(I-\Gamma)^T Q\check{z}(s)ds \\ &= H\check{z}(T) + H\Phi_{12}(T)q(0) - H\int_0^T \Phi_{12}(T-s)(I-\Gamma)^T Q\check{z}(s)ds, \end{aligned}$$

where the second equality is due to the terminal condition of q. It follows that

$$\begin{aligned} &[\Phi_{22}(T) - H\Phi_{12}(T)]q(0) \\ &= H\check{z}(T) + \int_0^T [\Phi_{22}(T-s) - H\Phi_{12}(T-s)](I-\Gamma)^T Q\check{z}(s)ds. \end{aligned} \tag{25}$$

PROPOSITION 3.10. *If* (H1) *holds,* $\Phi_{22}(T) - H\Phi_{12}(T)$ *is nonsingular.*

Proof. Under (H1), (23) has a unique solution by Lemma 3.8, and accordingly, $q(0)$ is uniquely determined. If $\Phi_{22}(T) - H\Phi_{12}(T)$ is singular, we may find two different solutions of $q(0)$ from (25) which further give two different solutions to (23), leading to a contradiction. Hence, $\Phi_{22} - H\Phi_{12}(T)$ is nonsingular. □

By solving $q(0)$ in (25) and further eliminating $\check{z}$, we write z and q as integrals depending on ν_i. Define the linear operator

$$[\mathcal{L}(\nu_i)](t) = \begin{bmatrix} \check{z}(t) \\ z(t) \\ q(t) \end{bmatrix}.$$

By standard estimates we can show that $\mathcal{L}$ is a linear and bounded operator from $L^2(0, T; \mathbb{R}^{n_1})$ to $L^2(0, T; \mathbb{R}^{3n})$. Let $\mathcal{L}^*$ be its adjoint operator from $L^2(0, T; \mathbb{R}^{3n})$ to $L^2(0, T; \mathbb{R}^{n_1})$. Define the operator

$$\mathcal{L}_T\nu_i = \check{z}(T) + z(T).$$

It can be shown that $\mathcal{L}_T$ is a linear and bounded operator from $L^2(0,T;\mathbb{R}^{n_1})$ to $\mathbb{R}^n$. Let $\mathcal{L}_T^*$ be its adjoint operator. Now $\bar{J}_i^a$ may be represented in terms of the inner product on $L^2(0,T;\mathbb{R}^{n_1})$:

$$\bar{J}_i^a(\nu_i) = \langle \Theta\nu_i, \nu_i\rangle + \langle R\nu_i, \nu_i\rangle + \langle \Theta_T\nu_i, \nu_i\rangle, \tag{26}$$

where

$$\Theta\nu_i = \mathcal{L}^* \begin{bmatrix} Q & Q(I-\Gamma) & 0 \\ (I-\Gamma)^T Q & \widehat{Q} & 0 \\ 0 & 0 & -\gamma I \end{bmatrix} \mathcal{L}\nu_i, \quad \Theta_T\nu_i = \mathcal{L}_T^* H \mathcal{L}_T \nu_i.$$

Proposition 3.11.

(i) *$\bar{J}_i(u_i)$ is convex in $u_i \in \mathcal{U}$ if and only if $\langle(\Theta+\Theta_T+R)\nu_i,\nu_i\rangle \geq 0$ for all $\nu_i \in L^2(0,T;\mathbb{R}^{n_1})$.*

(ii) (H2) *holds if and only if there exists $\delta_0 > 0$ such that $\langle(\Theta+\Theta_T+R)\nu_i,\nu_i\rangle \geq \delta_0\|\nu_i\|_{L^2}^2$ for all $\nu_i \in L^2(0,T;\mathbb{R}^{n_1})$.*

Proof. (i) follows from Lemma 3.9 and the representation (26). (ii) follows from (26). □

The criterion in part (ii) of Proposition 3.11 still involves the operators Θ and Θ_T on an infinite dimensional space. Here we give a sufficient condition to endure (H2) based on some more computable parameters. It is clear that $\langle(\Theta+\Theta_T+R)\nu_i,\nu_i\rangle \geq \int_0^T(|\nu_i(t)|_R^2 - \gamma|q(t)|^2)dt$. For simplicity, we only consider the case $H=0$, and simple computations lead to

$$\begin{aligned} q(t) &= \Phi_{22}(t)\Phi_{22}^{-1}(T)\int_0^T \Phi_{22}(T-s)(I-\Gamma)^T Q \int_0^s e^{A(s-\tau)}B\nu_i(\tau)d\tau ds \\ &\quad - \int_0^t \Phi_{22}(t-s)(I-\Gamma)^T Q \int_0^s e^{A(s-\tau)}B\nu_i(\tau)d\tau ds =: q_1(t) - q_2(t). \end{aligned}$$

Denote $b_1 = \sup_{0\leq t\leq T}|\Phi_{22}(t)|$, $b_2 = \sup_{0\leq t\leq T}|\Phi_{22}(t)\Phi_{22}^{-1}(T)|$, $b_3 = |Q(I-\Gamma)|$, $b_4 = \int_0^T |e^{As}B|ds$ and $b_5 = \sup_{0\leq t\leq T}|e^{At}B|$. By exchanging the order of integration in q_1 and q_2, it is easy to show

$$|q_1(t)|^2 \leq (b_1b_2b_3b_4)^2 T\int_0^T \nu_i^2(s)ds, \quad |q_2(t)| \leq b_1b_3b_5\int_0^t (t-\tau)|\nu_i(\tau)|d\tau,$$

which further gives

$$\int_0^T |q(t)|^2 dt \leq C_q \int_0^T |\nu_i(t)|^2 dt, \tag{27}$$

where $C_q = 2(b_1b_2b_3b_4)^2T^2 + \frac{1}{6}(b_1b_3b_5)^2T^4$. For the case $H=0$, (H2) holds whenever $R > \gamma C_q I$.

3.5. The solution of (P2). Let $\bar{u} \in C([0,T];\mathbb{R}^{n_1})$ be fixed.

Lemma 3.12. *Assume* (H1)-(H2). *Then* (P2) *has a unique optimal state-control pair of the form $(x_i, m_i, p_i, \hat{u}_i)$ satisfying*

$$\begin{cases} dx_i = (Ax_i + B\hat{u}_i + Gm_i + \gamma p_i)dt + DdW_i, \\ \dot{m}_i = (A+G)m_i + B\bar{u} + \gamma p_i, \\ \dot{p}_i = -(A+G)^T p_i - (I-\Gamma)^T Q[\mathbb{E}x_i - (\Gamma m_i + \eta)], \end{cases} \tag{28}$$

where $p_i(T) = H\mathbb{E}x_i(T)$. *Furthermore, the backward stochastic differential equation* (BSDE)

$$\begin{cases} dy_i = \left\{-A^T y_i + Q[x_i - (\Gamma m_i + \eta)]\right\} dt + \zeta_i dW_i, \\ y_i(T) = -Hx_i(T) \end{cases} \tag{29}$$

has a unique solution $(y_i, \zeta_i) \in L^2_{\mathcal{F}}(0, T; \mathbb{R}^{2n})$ *and*

$$\hat{u}_i = R^{-1}B^T y_i. \tag{30}$$

Proof. Under (H2), by adapting Lemma 3.9 to the auxiliary control problem with cost functional $\bar{J}_i(u_i) - \delta_0\mathbb{E}\int_0^T |u_i|^2 dt$, we can show that $\bar{J}_i(u_i) - \delta_0\mathbb{E}\int_0^T |u_i|^2 dt$ is convex in u_i. By the method in proving Lemma 3.6, we can further show that $\bar{J}_i$ is strictly convex and coercive in u_i. Hence (P2) has a unique optimal state-control pair $(x_i, m_i, p_i, \hat{u}_i)$ which minimizes $\bar{J}_i(u_i)$.

Given $(x_i, m_i, p_i, \hat{u}_i)$, (29) is a standard linear BSDE and so has a unique solution (y_i, ζ_i). Further define the BSDE

$$dy = \left\{-G^T y_i - (A+G)^T y - \Gamma^T Q[x_i - (\Gamma m_i + \eta)]\right\} dt + \zeta dW_i,$$

where $y(T) = 0$. It also has a unique solution $(y, \zeta) \in L^2_{\mathcal{F}}(0, T; \mathbb{R}^{2n})$. It can be checked that

$$\frac{d}{dt}[\mathbb{E}(y + y_i) + p_i] = -(A+G)^T[\mathbb{E}(y + y_i) + p_i]$$

and $\mathbb{E}(y(T) + y_i(T)) + p_i(T) = 0$. So

$$\mathbb{E}(y_i + y) + p_i = 0 \tag{31}$$

for all $t \in [0, T]$.

Let $\hat{u}_i$ be replaced by $\hat{u}_i + \tilde{u}_i \in L^2_{\mathcal{F}}(0, T; \mathbb{R}^{n_1})$ in (28), and the resulting solution be denoted by $(x_i + \tilde{x}_i, m_i + \tilde{m}_i, p_i + \tilde{p}_i)$, which exists and is unique by Theorem 3.7. It follows that

$$\begin{cases} \dot{\tilde{x}}_i = A\tilde{x}_i + B\tilde{u}_i + G\tilde{m}_i + \gamma\tilde{p}_i, \\ \dot{\tilde{m}}_i = (A+G)\tilde{m}_i + \gamma\tilde{p}_i, \\ \dot{\tilde{p}}_i = -(A+G)^T\tilde{p}_i - (I-\Gamma)^T Q(\mathbb{E}\tilde{x}_i - \Gamma\tilde{m}_i), \end{cases}$$

where $\tilde{x}_i(0) = \tilde{m}_i(0) = 0$ and $\tilde{p}_i(T) = H\mathbb{E}\tilde{x}_i(T)$. The first variation of $\bar{J}_i$ about $\hat{u}_i$ satisfies

$$\begin{aligned} 0 = \frac{\delta \bar{J}_i}{2} = \mathbb{E}\int_0^T \left\{(\tilde{x}_i - \Gamma\tilde{m}_i)^T Q[x_i - (\Gamma m_i + \eta)] + \tilde{u}_i^T R\hat{u}_i - \gamma\tilde{p}_i^T p_i\right\} dt \\ + \mathbb{E}\tilde{x}_i^T(T)Hx_i(T). \end{aligned} \tag{32}$$

By applying Ito's formula to $\tilde{x}_i^T y_i$, we obtain

$$\begin{aligned} &\mathbb{E}\tilde{x}_i^T(T)y_i(T) - \mathbb{E}\tilde{x}_i^T(0)y_i(0) \\ &= \mathbb{E}\int_0^T \left\{\tilde{x}_i^T Q[x_i - (\Gamma m_i + \eta)] + y_i^T(B\tilde{u}_i + G\tilde{m}_i + \gamma\tilde{p}_i)\right\} dt. \end{aligned}$$

Similarly,

$$\begin{aligned}&\mathbb{E}\tilde{m}_i^T(T)y(T)-\mathbb{E}\tilde{m}_i^T(0)y(0)\\&=\mathbb{E}\int_0^T\left\{\gamma y^T\tilde{p}_i-\tilde{m}_i^T(G^Ty_i+\Gamma^TQ[x_i-(\Gamma m_i+\eta)])\right\}dt.\end{aligned}$$

Therefore, adding up the two equations yields

$$\begin{aligned}&-\mathbb{E}\tilde{x}_i^T(T)Hx_i(T)\\&=\mathbb{E}\int_0^T\left\{(\tilde{x}_i-\Gamma\tilde{m}_i)^TQ[x_i-(\Gamma m_i+\eta)]+y_i^TB\tilde{u}_i+\gamma(y+y_i)^T\tilde{p}_i\right\}dt.\end{aligned}\tag{33}$$

By (32) and (33),

$$\mathbb{E}\int_0^T[\tilde{u}_i^TR\hat{u}_i-\gamma\tilde{p}_i^Tp_i-\tilde{u}_i^TB^Ty_i-\gamma\tilde{p}_i^T(y+y_i)]dt=0.$$

Note that by (31),

$$\mathbb{E}\int_0^T\tilde{p}_i^T(p_i+y+y_i)dt=\int_0^T\tilde{p}_i^T[p_i+\mathbb{E}(y+y_i)]dt=0.$$

Hence,

$$\mathbb{E}\int_0^T\tilde{u}_i^T(R\hat{u}_i-B^Ty_i)dt=0.$$

Since $\tilde{u}_i\in L^2_{\mathcal{F}}(0,T;\mathbb{R}^{n_1})$ is arbitrary, (30) follows. □

After substituting $\hat{u}_i=R^{-1}B^Ty_i$ into (28), we form the equation system

$$\begin{cases}dx_i=(Ax_i+BR^{-1}B^Ty_i+Gm_i+\gamma p_i)dt+DdW_i,\\\dot{m}_i=(A+G)m_i+B\bar{u}+\gamma p_i,\\\dot{p}_i=-(A+G)^Tp_i-(I-\Gamma)^TQ[\mathbb{E}x_i-(\Gamma m_i+\eta)],\\dy_i=\left\{-A^Ty_i+Q[x_i-(\Gamma m_i+\eta)]\right\}dt+\zeta_idW_i,\end{cases}\tag{34}$$

where $x_i(0)$ is given, $m_i(0)=m_0$, $p_i(T)=H\mathbb{E}x_i(T)$, and $y_i(T)=-Hx_i(T)$. This equation system consists of 2 forward equations and 2 backward equations. It is clear that the solution of the optimal control problem (P2) satisfies the above FBSDE. A natural question is whether this FBSDE's solution completely determines the optimal control. This is answered by the next theorem. Denote

$$S[0,T]=L^2_{\mathcal{F}}(0,T;\mathbb{R}^n)\times C^1([0,T];\mathbb{R}^{2n})\times L^2_{\mathcal{F}}(0,T;\mathbb{R}^{2n}).$$

THEOREM 3.13. *Assume* (H1)-(H2). *Then the* FBSDE (34) *has a unique solution* $(x_i,m_i,p_i,y_i,\zeta_i)\in S[0,T]$ *and the optimal control for* (P2) *is given by* $\hat{u}_i=R^{-1}B^Ty_i$.

Proof. We solve (P1) and next (P2) to determine $\hat{u}_i$. By Lemma 3.12, we obtain $(x_i,m_i,p_i,y_i,\zeta_i)$ to satisfy (28)-(29) and $\hat{u}_i=R^{-1}B^Ty_i$. Then $(x_i,m_i,p_i,y_i,\zeta_i)$ satisfies (34).

We continue to show uniqueness. Let $(x_i,m_i,p_i,y_i,\zeta_i)$ and $(x_i',m_i',p_i',y_i',\zeta_i')$ be two solutions of (34). Define $\check{u}_i=R^{-1}B^Ty_i$ and $u_i'=R^{-1}B^Ty_i'$ which are both

well-determined elements in $L^2_{\mathcal{F}}(0,T;\mathbb{R}^{n_1})$. In particular, we have

$$\begin{cases} dx_i = (Ax_i + B\check{u}_i + Gm_i + \gamma p_i)dt + DdW_i, \\ \dot{m}_i = (A+G)m_i + B\bar{u} + \gamma p_i, \\ \dot{p}_i = -(A+G)^T p_i - (I-\Gamma)^T Q[\mathbb{E}x_i - (\Gamma m_i + \eta)], \\ dy_i = \left\{-A^T y_i + Q[x_i - (\Gamma m_i + \eta)]\right\} dt + \zeta_i dW_i, \end{cases} \tag{35}$$

where $x_i(0)$ is given, $m_i(0) = m_0$, $p_i(T) = H\mathbb{E}x_i(T)$, and $y_i(T) = -Hx_i(T)$.

As in the proof of Lemma 3.12, we evaluate the first variation of $\bar{J}_i(u_i)$ at $(x_i, m_i, p_i, \check{u}_i)$ and can show $\delta\bar{J}_i = 0$. Since $\bar{J}_i$ is convex, this zero first variation condition implies that $\check{u}_i$ is an optimal control of (P2). By the same reasoning, u_i' is also an optimal control. By strict convexity, we have $\check{u}_i = u_i'$. Subsequently, we have $(x_i, m_i, p_i) = (x_i', m_i', p_i')$ by Theorem 3.7. This further implies $(y_i, \zeta_i) = (y_i', \zeta_i')$. □

4. The Solution of the Robust Game. Note that Theorem 3.13 determines the strategy of a representative agent when $\bar{u}$ is fixed. Denote

$$x^{(N)} = \frac{1}{N}\sum_{i=1}^N x_i, \quad y^{(N)} = \frac{1}{N}\sum_{i=1}^N y_i, \quad m^{(N)} = \frac{1}{N}\sum_{i=1}^N m_i, \quad p^{(N)} = \frac{1}{N}\sum_{i=1}^N p_i. \tag{36}$$

By (34), we obtain

$$\begin{cases} dx^{(N)} = \left(Ax^{(N)} + BR^{-1}B^T y^{(N)} + Gm^{(N)} + \gamma p^{(N)}\right) dt + \frac{D}{N}\sum_{i=1}^N dW_i, \\ \frac{dm^{(N)}}{dt} = (A+G)m^{(N)} + B\bar{u} + \gamma p^{(N)}, \\ \frac{dp^{(N)}}{dt} = -(A+G)^T p^{(N)} - (I-\Gamma)^T Q\left[\mathbb{E}x^{(N)} - (\Gamma m^{(N)} + \eta)\right], \\ dy^{(N)} = \left\{-A^T y^{(N)} + Q[x^{(N)} - (\Gamma m^{(N)} + \eta)]\right\} dt + \frac{1}{N}\sum_{i=1}^N \zeta_i dW_i, \end{cases} \tag{37}$$

where $x^{(N)}(0) = (1/N)\sum_{i=1}^N x_i(0)$, $m^{(N)}(0) = m_0$, $p^{(N)}(T) = H\mathbb{E}x^{(N)}(T)$, and $y^{(N)}(T) = -Hx^{(N)}(T)$.

As an approximation to (37), we construct the following limiting system

$$\begin{cases} \dot{\mathbf{x}} = A\mathbf{x} + BR^{-1}B^T\mathbf{y} + G\mathbf{m} + \gamma\mathbf{p}, \\ \dot{\mathbf{m}} = (A+G)\mathbf{m} + B\bar{u} + \gamma\mathbf{p}, \\ \dot{\mathbf{p}} = -(A+G)^T\mathbf{p} - (I-\Gamma)^T Q[\mathbf{x} - (\Gamma\mathbf{m} + \eta)], \\ \dot{\mathbf{y}} = -A^T\mathbf{y} + Q[\mathbf{x} - (\Gamma\mathbf{m} + \eta)], \end{cases} \tag{38}$$

where $\mathbf{x}(0) = \mathbf{m}(0) = m_0$, $\mathbf{p}(T) = H\mathbf{x}(T)$, and $\mathbf{y}(T) = -H\mathbf{x}(T)$. This is a two point boundary value problem.

Note that $\mathbf{y}$ is intended as an approximation of $y^{(N)}$ when $N \to \infty$. The consistency requirement imposes

$$\bar{u} = R^{-1}B^T\mathbf{y}. \tag{39}$$

Under the condition (39), the first two equations in (38) coincide to give $\mathbf{x} = \mathbf{m}$ for all $t \in [0,T]$. Consequently, we eliminate the equation of $\mathbf{x}$ and introduce the new system

$$\begin{cases} \dot{\mathbf{m}} = (A+G)\mathbf{m} + BR^{-1}B^T\mathbf{y} + \gamma\mathbf{p}, \\ \dot{\mathbf{p}} = -(A+G)^T\mathbf{p} - (I-\Gamma)^T Q[\mathbf{m} - (\Gamma\mathbf{m} + \eta)], \\ \dot{\mathbf{y}} = -A^T\mathbf{y} + Q[\mathbf{m} - (\Gamma\mathbf{m} + \eta)], \end{cases} \tag{40}$$

where $\mathbf{m}(0) = m_0$, $\mathbf{p}(T) = H\mathbf{m}(T)$, and $\mathbf{y}(T) = -H\mathbf{m}(T)$. This is still a two point boundary value problem. The next corollary follows from Theorem 3.13.

COROLLARY 4.1. *Assume* (H1)-(H2). *Suppose that* (40) *has a unique solution* $(\mathbf{m}, \mathbf{p}, \mathbf{y}) \in C^1([0,T]; \mathbb{R}^{3n})$ *and take* $\bar{u} = R^{-1}B^T\mathbf{y}$ *in* (34). *Then* (34) *has a unique solution* $(x_i, m_i, p_i, y_i, \zeta_i) \in S[0,T]$.

4.1. The special case of same initial conditions. Consider the special case where all agents have the same initial condition $x_i(0) = m_0$ for all $i \geq 1$. The FBSDE (34) defines a mapping

$$\Lambda(\bar{u}) = R^{-1}B^T\mathbb{E}y_i,$$

where $\bar{u} \in C([0,T]; \mathbb{R}^{n_1})$. Clearly $R^{-1}B^T\mathbb{E}y_i$ is an $\mathbb{R}^{n_1}$-valued continuous function of $t \in [0,T]$.

By the consistency requirement $\bar{u} = \Lambda(\bar{u})$, we set $\bar{u} = R^{-1}B^T\mathbb{E}y_i$ in the second equation of (34) to obtain the equation system of the mean field game:

$$\begin{cases} dx_i = (Ax_i + BR^{-1}B^Ty_i + Gm_i + \gamma p_i)dt + DdW_i, \\ \dot{m}_i = (A+G)m_i + BR^{-1}B^T\mathbb{E}y_i + \gamma p_i, \\ \dot{p}_i = -(A+G)^Tp_i - (I-\Gamma)^TQ[\mathbb{E}x_i - (\Gamma m_i + \eta)], \\ dy_i = \left\{-A^Ty_i + Q[x_i - (\Gamma m_i + \eta)]\right\}dt + \zeta_i dW_i, \end{cases} \tag{41}$$

where $x_i(0) = m_i(0) = m_0$, $p_i(T) = H\mathbb{E}x_i(T)$, and $y_i(T) = -Hx_i(T)$.

REMARK 1. For this special case, it can be shown that (41) has a unique solution $(x_i, m_i, p_i, y_i, \zeta_i) \in S[0,T]$ if and only if (40) has a unique solution; see the detailed analysis in [28].

4.2. Existence of a solution to (40)**.** To study the existence and uniqueness of a solution to (40), we use a fixed point approach and introduce the equation system

$$\begin{cases} \dot{\mathbf{m}} = (A+G)\mathbf{m} + h + \gamma\mathbf{p}, \\ \dot{\mathbf{p}} = -(A+G)^T\mathbf{p} - \widehat{Q}\mathbf{m} + (I-\Gamma)^TQ\eta, \\ \dot{\mathbf{y}} = -A^T\mathbf{y} + Q[\mathbf{m} - (\Gamma\mathbf{m} + \eta)], \end{cases} \tag{42}$$

where $h \in C([0,T]; \mathbb{R}^n)$, $m(0) = m_0$, $\mathbf{p}(T) = H\mathbf{m}(T)$, and $\mathbf{y}(T) = -H\mathbf{m}(T)$. The next lemma identifies a sufficient condition for (42) to have a unique solution for any $h \in C([0,T]; \mathbb{R}^n)$.

LEMMA 4.2. *Suppose that the Riccati equation*

$$\dot{K} + K(A+G) + (A+G)^TK - \gamma K^2 - \widehat{Q} = 0, \qquad K(T) = -H \tag{43}$$

has a unique solution on $[0,T]$. *Then* (42) *defines a mapping from* $C([0,T]; \mathbb{R}^n)$ *to itself:*

$$\Lambda_1 : h \longmapsto BR^{-1}B^T\mathbf{y}.$$

Proof. We write $\mathbf{p} = -K\mathbf{m} + \phi$ for (42) and obtain the ODE

$$\dot{\phi} = -(A + G - \gamma K)\phi + Kh + (I-\Gamma)^TQ\eta, \quad \phi(T) = 0.$$

It follows that

$$\dot{\mathbf{m}} = (A + G - \gamma K)\mathbf{m} + h + \gamma\phi.$$

Let the fundamental solution matrices of the two ODEs

$$\dot{\varphi} = (A + G - \gamma K)\varphi, \quad \dot{\psi} = -(A + G - \gamma K)^T \psi$$

be $\Phi(t,s)$ and $\Psi(t,s)$, respectively, with $\Phi(s,s) = \Psi(s,s) = I$. Then $\Psi(t,s) = \Phi^T(s,t)$. We obtain

$$\phi(t) = -\int_t^T \Psi(t,s_1)[K(s_1)h(s_1) + (I-\Gamma)^T Q\eta]ds_1.$$

This in turn gives

$$\begin{aligned}\mathbf{m}(t) = \Phi(t,0)m_0 + \int_0^t \Phi(t,s_1)h(s_1)ds_1 \\ - \gamma \int_0^t \Phi(t,s_2)\int_{s_2}^T \Psi(s_2,s_1)[K(s_1)h(s_1) + (I-\Gamma)^T Q\eta]ds_1 ds_2.\end{aligned}$$

We further solve

$$\mathbf{y}(t) = -\int_t^T e^{-A^T(t-s_3)}Q[(I-\Gamma)\mathbf{m}(s_3) - \eta]ds_3 - e^{-A^T(t-T)}H\mathbf{m}(T),$$

which implies $\mathbf{y} \in C([0,T];\mathbb{R}^n)$. The lemma follows. □

To simplify the existence analysis for (40) in this section, we consider the case $H = 0$. Below Υ_k denotes a continuous function of t which does not depend on h and can be easily determined. Consequently,

$$\begin{aligned}\mathbf{y}(t) &= -\int_t^T e^{-A^T(t-s_3)}Q[(I-\Gamma)\mathbf{m}(s_3) - \eta]ds_3 \\ &= -\int_t^T e^{-A^T(t-s_3)}Q(I-\Gamma)\mathbf{m}(s_3)ds_3 + \Upsilon_1(t) \\ &= -\int_t^T e^{-A^T(t-s_2)}Q(I-\Gamma)\int_0^{s_2}\Phi(s_2,s_1)h(s_1)ds_1 ds_2 \\ &\quad + \gamma\int_t^T e^{-A^T(t-s_3)}Q(I-\Gamma)\int_0^{s_3}\Phi(s_3,s_2)\int_{s_2}^T \Psi(s_2,s_1)K(s_1)h(s_1)ds_1 ds_2 ds_3 \\ &\quad + \Upsilon_2(t).\end{aligned}$$

Now we have

$$\begin{aligned}\Lambda_1(h)(t) &= BR^{-1}B^T\mathbf{y}(t) \\ &= -BR^{-1}B^T\int_t^T e^{-A^T(t-s_2)}Q(I-\Gamma)\int_0^{s_2}\Phi(s_2,s_1)h(s_1)ds_1 ds_2 \\ &\quad + \gamma BR^{-1}B^T\int_t^T e^{-A^T(t-s_3)}Q(I-\Gamma)\int_0^{s_3}\Phi(s_3,s_2) \\ &\quad \cdot\int_{s_2}^T \Psi(s_2,s_1)K(s_1)h(s_1)ds_1 ds_2 ds_3 + BR^{-1}B^T\Upsilon_2(t) \\ &=: \Lambda_0(h)(t) + BR^{-1}B^T\Upsilon_2(t).\end{aligned}$$

It is clear that Λ_0 and subsequently Λ_1 are from $C([0,T];\mathbb{R}^n)$ to itself. □

Define the constants

$$c_1 = \max_{t\in[0,T]} |K(t)|, \quad c_2 = \max_{0\le t,s\le T} |\Phi(t,s)|,$$
$$c_3 = \max_{t\in[0,T]} \int_t^T |e^{A(s-t)}| s ds, \quad c_4 = \max_{t\in[0,T]} \int_t^T |e^{A(s-t)}| (Ts - \frac{s^2}{2}) ds.$$

Note that $Ts - \frac{s^2}{2} \ge 0$ for $s \in [0,T]$. Denote $|h| = \max_{t\in[0,T]} |h(t)|$.

THEOREM 4.3. *Assume $H = 0$. If*

$$c_2 |BR^{-1}B^T| \cdot |Q(I-\Gamma)| \, (c_3 + \gamma c_1 c_2 c_4) < 1, \tag{44}$$

then (40) has a unique solution.

Proof. For each t,

$$\begin{aligned}
|\Lambda_0(h)(t)| &\le c_2|h| \cdot |BR^{-1}B^T| \cdot |Q(I-\Gamma)| \int_t^T |e^{A^T(s_2-t)}| s_2 ds_2 \\
&\quad + \gamma c_1 c_2^2 |h| \cdot |BR^{-1}B^T| \cdot |Q(I-\Gamma)| \int_t^T |e^{A^T(s_3-t)}| \int_0^{s_3} \int_{s_2}^T ds_1 ds_2 ds_3 \\
&= c_2|h| \cdot |BR^{-1}B^T| \cdot |Q(I-\Gamma)| \int_t^T |e^{A(s_2-t)}| s_2 ds_2 \\
&\quad + \gamma c_1 c_2^2 |h| \cdot |BR^{-1}B^T| \cdot |Q(I-\Gamma)| \int_t^T |e^{A(s_3-t)}| (Ts_3 - \frac{s_3^2}{2}) ds_3 \\
&= c_2 |BR^{-1}B^T| \cdot |Q(I-\Gamma)| \, (c_3 + \gamma c_1 c_2 c_4) \, |h|.
\end{aligned}$$

Hence, Λ_1 is a contraction and has a unique fixed point. So (40) has a unique solution. □

The constants $c_1, \dots, c_4$ in (44) do not depend on $BR^{-1}B^T$. If $BR^{-1}B^T$ is suitably small, (44) can be ensured.

EXAMPLE 4.4. *Consider the system with parameters given by Example 3.4. Take $T = 1.3$. Similar to (12), we can solve $K(t)$ on $[0,T]$ for (43). It can be shown that $K(t) \le 0$ for $t \in [0,T]$ and $|K(t)|$ attains its maximum on $[0,T]$ at $t = 0$. We have $K(0) = -0.171417$ which gives $c_1 = 0.171417$. So $c_2 \le e^{(A+G+|K(0)|)T} = 3.312961$. Furthermore,*

$$c_3 \le \int_0^T e^{As} s ds = 1.318243, \quad c_4 \le \int_0^T e^{As} (Ts - \frac{s^2}{2}) ds = 1.112937.$$

Subsequently,

$$c_2 |BR^{-1}B^T| \cdot |Q(I-\Gamma)| \, (c_3 + \gamma c_1 c_2 c_4) \le 0.861493.$$

So (44) holds.

REMARK 2. For the two-point boundary value problem, the contraction estimate in the fixed point method may be conservative and typically works on small time intervals for the solvability of (40) (see, e.g., Ch.1, Sec. 5, [44]).

We continue to derive another condition under which (40) is solvable without restriction to a small time horizon. To this end, we first rewrite (40) in the following

form:

$$\begin{cases} \begin{pmatrix} \dot{m} \\ \dot{p} \\ \dot{y} \end{pmatrix} = \widetilde{A} \begin{pmatrix} m \\ p \\ y \end{pmatrix} + \widetilde{\eta}, \\ m(0) = m_0, \quad p(T) = Hm(T), \quad y(T) = -Hm(T), \end{cases} \tag{45}$$

where

$$\widetilde{A} = \begin{pmatrix} A+G & \gamma & BR^{-1}B^T \\ -(I-\Gamma)^T Q(I-\Gamma) & -(A+G)^T & 0 \\ Q(I-\Gamma) & 0 & -A^T \end{pmatrix}, \quad \widetilde{\eta} = \begin{pmatrix} 0 \\ (I-\Gamma)^T Q\eta \\ -Q\eta \end{pmatrix}.$$

Then, by the variation of constant formula, we have

$$\begin{pmatrix} m(t) \\ p(t) \\ y(t) \end{pmatrix} = \Theta(t) \begin{pmatrix} m_0 \\ \mu \\ \nu \end{pmatrix} + \Theta(t) \int_0^t \Theta^{-1}(s)\widetilde{\eta} ds, \tag{46}$$

where $\Theta(t) = e^{\widetilde{A}t}$ and p, y have the initial conditions $p(0) = \mu$, $y(0) = \nu$. Noting the terminal condition in (45), now we present the following result.

PROPOSITION 4.5. [44, Ch.2, Sec. 3] *If for given $T > 0$, $\det(\widetilde{\Theta}(T)) \neq 0$, where*

$$\widetilde{\Theta}(T) = \begin{pmatrix} -H & I & 0 \\ H & 0 & I \end{pmatrix} \Theta(T) \begin{pmatrix} 0 & 0 \\ I & 0 \\ 0 & I \end{pmatrix},$$

then (40) has a unique solution on $[0, T]$ for any initial value m_0.

For illustration, we give the following example.

EXAMPLE 4.6. *Consider system (3)-(4) with all parameters being scalar-valued and $\Gamma = 1$, $H = 0$. We calculate*

$$\mathcal{A} = \begin{pmatrix} A+G & -\gamma \\ 0 & -(A+G) \end{pmatrix}, \quad \widetilde{A} = \begin{pmatrix} A+G & \gamma & R^{-1}B^2 \\ 0 & -(A+G) & 0 \\ 0 & 0 & -A \end{pmatrix},$$

where $\mathcal{A}$ is defined in Proposition 3.3. By direct computations, we obtain

$$\det\{[(0, I)e^{\mathcal{A}t}(0, I)^T]\} = e^{-(A+G)t} > 0$$

for all $t \in [0, T]$, which ensures (H1) *by Proposition 3.3. Moreover, $b_3 = 0$ gives $C_q = 0$ in (27) so that* (H2) *always holds true for $R > 0$. Finally, $\det(\widetilde{\Theta}(t)) = e^{-(2A+G)t} > 0$, and subsequently, (40) has a unique solution on any interval $[0, T]$. To summarize,* (H1), (H2) *and the solvability of (40) are all satisfied by the system.*

Note that the solvability of (40) in Example 4.6 does not depend on the value of $R^{-1}B^2$, which is different from the condition in Theorem 4.3.

5. Error Estimate of the Mean Field Approximation. We suppose that (40) has a unique solution $(\mathbf{m}, \mathbf{p}, \mathbf{y})$ and accordingly take $\bar{u}$ in (34) as

$$\bar{u}^* = R^{-1}B^T\mathbf{y}. \tag{47}$$

The FBSDE system (34) now becomes

$$\begin{cases} dx_i = (Ax_i + BR^{-1}B^T y_i + Gm_i + \gamma p_i)dt + DdW_i, \\ \dot{m}_i = (A+G)m_i + B\bar{u}^* + \gamma p_i, \\ \dot{p}_i = -(A+G)^T p_i - (I-\Gamma)^T Q[\mathbb{E}x_i - (\Gamma m_i + \eta)], \\ dy_i = \left\{-A^T y_i + Q[x_i - (\Gamma m_i + \eta)]\right\} dt + \zeta_i dW_i, \end{cases} \tag{48}$$

where $x_i(0)$ is given, $m_i(0) = m_0$, $p_i(T) = H\mathbb{E}x_i(T)$, and $y_i(T) = -Hx_i(T)$. By Corollary 4.1, this FBSDE has a unique solution. In the game of N players, let y_i be solved from (48) and denote the control for $\mathcal{A}_i$ by

$$\hat{u}_i = R^{-1}By_i, \qquad 1 \le i \le N, \tag{49}$$

which is a well defined process in $L^2_{\mathcal{F}}(0,T;\mathbb{R}^{n_1})$.

For $\hat{u}^{(N)} = (1/N)\sum_{i=1}^N \hat{u}_i$, we aim to estimate

$$\mathbb{E}|\hat{u}^{(N)}(t) - \bar{u}^*(t)|^2.$$

Note that $\hat{u}_1, \dots, \hat{u}_N$ are independent, but they are not necessarily with the same distribution due to possibly different initial states of the agents. This fact will somehow complicate our error estimate. The key result of this section is the following theorem.

THEOREM 5.1. *Assume that* (H1)-(H2) *hold and that* (40) *has a unique solution. We have*

$$\sup_{0\le t\le T} \mathbb{E}|\hat{u}^{(N)} - \bar{u}^*|^2 = O(1/N) + O(|x^{(N)}(0) - m_0|^2),$$

where $x^{(N)}(0) = (1/N)\sum_{i=1}^N x_i(0)$.

The proof of Theorem 5.1 is provided in the remaining part of this section. To do this, we need to prove some lemmas under the assumptions of the theorem. Recalling (36), we take $\bar{u} = \bar{u}^*$ in (37) to write

$$\begin{cases} dx^{(N)} = \left(Ax^{(N)} + BR^{-1}B^T y^{(N)} + Gm^{(N)} + \gamma p^{(N)}\right) dt + \frac{D}{N}\sum_{i=1}^N dW_i, \\ \frac{dm^{(N)}}{dt} = (A+G)m^{(N)} + B\bar{u}^* + \gamma p^{(N)}, \\ \frac{dp^{(N)}}{dt} = -(A+G)^T p^{(N)} - (I-\Gamma)^T Q\left[\mathbb{E}x^{(N)} - (\Gamma m^{(N)} + \eta)\right], \\ dy^{(N)} = \left\{-A^T y^{(N)} + Q[x^{(N)} - (\Gamma m^{(N)} + \eta)]\right\} dt + \frac{1}{N}\sum_{i=1}^N \zeta_i dW_i, \end{cases} \tag{50}$$

where $x^{(N)}(0) = (1/N)\sum_{i=1}^N x_i(0)$, $m^{(N)}(0) = m_0$, $p^{(N)}(T) = H\mathbb{E}x^{(N)}(T)$, and $y^{(N)}(T) = -Hx^{(N)}(T)$.

Denote the ODE system

$$\begin{cases} \dot{\mathbf{x}}_N = A\mathbf{x}_N + BR^{-1}B^T\mathbf{y}_N + G\mathbf{m}_N + \gamma\mathbf{p}_N, \\ \dot{\mathbf{m}}_N = (A+G)\mathbf{m}_N + B\bar{u}^* + \gamma\mathbf{p}_N, \\ \dot{\mathbf{p}}_N = -(A+G)^T\mathbf{p}_N - (I-\Gamma)^T Q[\mathbf{x}_N - (\Gamma\mathbf{m}_N + \eta)], \\ \dot{\mathbf{y}}_N = -A^T\mathbf{y}_N + Q[\mathbf{x}_N - (\Gamma\mathbf{m}_N + \eta)], \end{cases} \tag{51}$$

where $\mathbf{x}_N(0) = (1/N)\sum_{i=1}^N x_i(0)$, $\mathbf{m}_N(0) = m_0$, $\mathbf{p}_N(T) = H\mathbf{x}_N(T)$, and $\mathbf{y}_N(T) = -H\mathbf{x}_N(T)$. The initial condition $\mathbf{x}_N(0)$ is different from that of (38).

LEMMA 5.2. (51) *has a unique solution which can be denoted as*

$$(\mathbf{x}_N, \mathbf{m}_N, \mathbf{p}_N, \mathbf{y}_N) = (\mathbb{E}x^{(N)}, m^{(N)}, p^{(N)}, \mathbb{E}y^{(N)}).$$

Proof. Existence follows by taking expectation in (50). To show uniqueness, suppose that (51) has two different solutions $(\mathbf{x}_N, \mathbf{m}_N, \mathbf{p}_N, \mathbf{y}_N)$ and $(\mathbf{x}'_N, \mathbf{m}'_N, \mathbf{p}'_N, \mathbf{y}'_N)$. Then for any $\lambda \in \mathbb{R}$,

$$(x_i, m_i, p_i, y_i, \zeta_i) + \lambda(\mathbf{x}_N - \mathbf{x}'_N, \mathbf{m}_N - \mathbf{m}'_N, \mathbf{p}_N - \mathbf{p}'_N, \mathbf{y}_N - \mathbf{y}'_N, 0)$$

satisfies (34), which is a contradiction to Theorem 3.13. Uniqueness follows. □

LEMMA 5.3. *We have*

$$\sup_{0\le t\le T}\Big(\mathbb{E}|x^{(N)} - \mathbb{E}x^{(N)}|^2 + \mathbb{E}|y^{(N)} - \mathbb{E}y^{(N)}|^2\Big) = O(1/N).$$

Proof. Define

$$(\theta_1, \theta_2) = (x^{(N)} - \mathbb{E}x^{(N)}, y^{(N)} - \mathbb{E}y^{(N)}).$$

By (50), (51) and Lemma 5.2,

$$\begin{cases} d\theta_1 = (A\theta_1 + BR^{-1}B^T\theta_2)dt + \frac{D}{N}\sum_{i=1}^N dW_i, \\ d\theta_2 = (-A^T\theta_2 + Q\theta_1)dt + \frac{1}{N}\sum_{i=1}^N \zeta_i dW_i, \end{cases}$$

where $\theta_1(0) = 0$ and $\theta_2(T) = -H\theta_1(T)$.

Let P be the solution of the Riccati equation

$$\dot P + A^TP + PA - PBR^{-1}B^TP + Q = 0, \quad P(T) = H.$$

Denote $\theta_2 = -P\theta_1 + \psi$, where $\psi(T) = 0$. This gives the equation

$$d\psi = -(A - BR^{-1}B^TP)^T\psi dt + \frac{1}{N}\sum_{i=1}^N (PD + \zeta_i)dW_i,$$

where $\psi(T) = 0$. There is a unique solution $\psi = 0$ for $t \in [0, T]$. This implies

$$d\theta_1 = (A - BR^{-1}B^TP)\theta_1 dt + \frac{D}{N}\sum_{i=1}^N dW_i.$$

Hence, $\sup_{0\le t\le T}\mathbb{E}|\theta_1(t)|^2 = O(1/N)$. The lemma follows since $\theta_2 = -P\theta_1$. □

When $(\mathbf{m}, \mathbf{p}, \mathbf{y})$ is a unique solution of (40), it can be shown that $(\mathbf{x}, \mathbf{m}, \mathbf{y}, \mathbf{p}) := (\mathbf{m}, \mathbf{m}, \mathbf{y}, \mathbf{p})$ is a unique solution of (38) under the condition (39).

LEMMA 5.4. *We have*

$$\sup_{0\le t\le T}[|\mathbf{x}_N - \mathbf{x}| + |\mathbf{m}_N - \mathbf{m}| + |\mathbf{p}_N - \mathbf{p}| + |\mathbf{y}_N - \mathbf{y}|] = O(|x^{(N)}(0) - m_0|).$$

Proof. Consider

$$\begin{cases} \dot h_1 = Ah_1 + BR^{-1}B^Th_4 + Gh_2 + \gamma h_3, \\ \dot h_2 = (A + G)h_2 + \gamma h_3, \\ \dot h_3 = -(A + G)^Th_3 - (I - \Gamma)^TQ(h_1 - \Gamma h_2), \\ \dot h_4 = -A^Th_4 + Q(h_1 - \Gamma h_2), \end{cases} \tag{52}$$

where $h_1(0)$ is given, $h_2(0) = 0$, $h_3(T) = Hh_1(T)$, and $h_4(T) = -Hh_1(T)$. It is constructed as a homogeneous version of (51). We claim that (52) has a unique

solution for any given value of $h_1(0)$. If this were not true, there would exist $h_1(0)$ such that (52) has multiple solutions which, in turn, can be used to construct multiple solutions to (51). This would give a contradiction to Lemma 5.2.

It is clear that

$$(\mathbf{x}_N - \mathbf{x}, \mathbf{m}_N - \mathbf{m}, \mathbf{p}_N - \mathbf{p}, \mathbf{y}_N - \mathbf{y}) =: (h_1, h_2, h_3, h_4),$$

is a solution of (52) with $h_1(0) = \mathbf{x}_N(0) - m_0$.

Let $e_1, \dots e_n$ be a canonical basis of $\mathbb{R}^n$. For $h_1(0) = e_k$, we obtain a solution of (52), denoted by $h^k = (h_1^k, h_2^k, h_3^k, h_4^k)$. Let $(z)_k$ be the kth component of a vector z. We may uniquely denote $(\mathbf{x}_N - \mathbf{x}, \mathbf{m}_N - \mathbf{m}, \mathbf{p}_N - \mathbf{p}, \mathbf{y}_N - \mathbf{y})$ as a linear combination of $h^1, \dots, h^n$:

$$(\mathbf{x}_N - \mathbf{x}, \mathbf{m}_N - \mathbf{m}, \mathbf{p}_N - \mathbf{p}, \mathbf{y}_N - \mathbf{y}) = \sum_{k=1}^{n} (x_N(0) - m_0)_k (h_1^k, h_2^k, h_3^k, h_4^k).$$

The lemma follows readily. □

Proof of Theorem 5.1. For $\bar{u} = \bar{u}^*$, we write $\hat{u}^{(N)} = R^{-1}B^T(1/N)\sum_{i=1}^N y_i = R^{-1}B^T y^{(N)}$. We have

$$\begin{aligned}
|\hat{u}^{(N)} - \bar{u}^*|^2 &= \mathbb{E}|R^{-1}B^T(y^{(N)} - \mathbf{y})|^2 \\
&\le C\mathbb{E}|y^{(N)} - \mathbf{y}|^2 \\
&= CE|y^{(N)} - \mathbb{E}y^{(N)} + \mathbb{E}y^{(N)} - \mathbf{y}|^2 \\
&\le C(1/N) + C|\mathbf{y}_N - \mathbf{y}|^2 \\
&= O(1/N) + O(|x^{(N)}(0) - m_0|^2).
\end{aligned}$$

The second inequality follows from Lemmas 5.2 and 5.3, and the last step follows from Lemma 5.4. □

6. Robust Nash Equilibrium. Throughout this section, we assume that (40) has a unique solution and take $\bar{u} = \bar{u}^*$ determined by (47). For $f \in L^2(0,T;\mathbb{R}^n)$ and $u_i \in L^2_{\mathcal{F}}(0,T;\mathbb{R}^{n_1})$, $1 \le i \le N$, recall the worst case cost

$$J_i^{\text{wo}}(u_i, u_{-i}) = \sup_{f \in L^2(0,T;\mathbb{R}^n)} J_i(u_i, u_{-i}, f).$$

It is clear that for each i and any (u_i, u_{-i}), $\sup_f J_i(u_i, u_{-i}, f) \ge 0$.

Consider the set of strategies $(\hat{u}_i, \hat{u}_{-i})$ given by (49) for a population of N players with dynamics (3). It should be emphasized that we only use (48)-(49) to make a well defined process $\hat{u}_i$ in $L^2_{\mathcal{F}}(0,T;\mathbb{R}^{n_1})$ which should not be understood as a feedback strategy. The main result of this section is the next theorem which characterizes the performance of this set of strategies.

THEOREM 6.1. *Assume* (i) (H1)-(H2) *hold;* (ii) $\sup_{i\ge 0}|x_i(0)| \le M_0$ *where* M_0 *does not depend on* N*;* (iii) (40) *has a unique solution. Then the set of strategies* $(\hat{u}_1, \dots, \hat{u}_N)$ *given by* (49) *is a robust* ε_N*-Nash equilibrium for the* N *players, i.e.,*

$$J_i^{\text{wo}}(\hat{u}_i, \hat{u}_{-i}) - \varepsilon_N \le \inf_{u_i \in \mathcal{U}} J_i^{\text{wo}}(u_i, \hat{u}_{-i}) \le J_i^{\text{wo}}(\hat{u}_i, \hat{u}_{-i}), \tag{53}$$

where $0 \le \varepsilon_N = O(1/\sqrt{N} + |x^{(N)}(0) - m_0|)$ *and* $x^{(N)}(0) = (1/N)\sum_{j=1}^N x_j(0)$.

The rest part of this section is devoted to the proof of Theorem 6.1. For any given $f \in L^2(0,T;\mathbb{R}^n)$, denote the state processes of (3) corresponding to $(\hat{u}_i, \hat{u}_{-i}, f)$ by $\hat{x}_j$, $1 \le j \le N$, and $\hat{x}^{(N)} = (1/N)\sum_{j=1}^N \hat{x}_j$. Denote

$$\dot{\bar{m}} = (A+G)\bar{m} + B\bar{u}^* + f, \quad \bar{m}(0) = m_0 \tag{54}$$

All subsequent lemmas are proved under the assumptions of Theorem 6.1.

LEMMA 6.2. *We have*

$$\sup_{0\le t\le T, f} \mathbb{E}|\hat{x}^{(N)} - \bar{m}|^2 \le C(1/N + |x^{(N)}(0) - m_0|^2).$$

Proof. Note that

$$d\hat{x}^{(N)} = [(A+G)\hat{x}^{(N)} + B\hat{u}^{(N)} + f]dt + (D/N)\sum_{i=1}^N dW_i.$$

Therefore,

$$d(\hat{x}^{(N)} - \bar{m}) = [(A+G)(\hat{x}^{(N)} - \bar{m}) + B(\hat{u}^{(N)} - \bar{u}^*)]dt + (D/N)\sum_{i=1}^N dW_i.$$

By linear SDE estimates,

$$\begin{aligned}\mathbb{E}|\hat{x}^{(N)}(t) - \bar{m}(t)|^2 \le{}& C|x^{(N)}(0) - m_0|^2 + C/N \\ &+ C\mathbb{E}\int_0^t |\hat{u}^{(N)}(\tau) - \bar{u}^*(\tau)|^2 d\tau.\end{aligned}$$

By Theorem 5.1, the lemma follows. □

LEMMA 6.3. *There exists a constant $\hat{C}_0$ independent of N such that*

$$\max_{1\le i\le N} \sup_f J_i(\hat{u}_i, \hat{u}_{-i}, f) \le \hat{C}_0.$$

Proof. Denote

$$dx_i' = (Ax_i' + B\hat{u}_i + G\bar{m} + f)dt + DdW_i, \tag{55}$$

where $x_i'(0) = x_i(0)$. By Lemma 6.2, it is easy to show

$$\sup_{0\le t\le T, f} \mathbb{E}|\hat{x}_i(t) - x_i'(t)|^2 \le C(1/N + |x^{(N)}(0) - m_0|^2).$$

We have

$$\begin{aligned}J_i(\hat{u}_i, \hat{u}_{-i}, f) = \bar{J}_i(\hat{u}_i, f) &+ \mathbb{E}\int_0^T |(\hat{x}_i - x_i') + \Gamma(\bar{m} - \hat{x}^{(N)})|_Q^2 dt + \mathbb{E}|\hat{x}_i(T) - x_i'(T)|_H^2 \\ &+ 2\mathbb{E}\int_0^T [x_i' - (\Gamma\bar{m} + \eta)]^T Q[(\hat{x}_i - x_i') + \Gamma(\bar{m} - \hat{x}^{(N)})]dt \\ &+ 2\mathbb{E}[x_i'^T(T)H(\hat{x}_i(T) - x_i'(T))].\end{aligned} \tag{56}$$

Combining Lemma 3.6 with condition (ii) in Theorem 6.1, we obtain

$$\bar{J}_i(\hat{u}_i, f) \le C - (\epsilon_0/2)\|f\|^2_{L^2} \tag{57}$$

for $\epsilon_0 > 0$, where C does not depend on (i, N). Since neither $\hat{x}_i - x'_i$ nor $\bar{m} - \hat{x}^{(N)}$ depend on f, there exists a constant C_1 such that

$$\begin{aligned}
&\left| \mathbb{E} \int_0^T [x'_i - (\Gamma \bar{m} + \eta)]^T Q[(\hat{x}_i - x'_i) + \Gamma(\bar{m} - \hat{x}^{(N)})] dt \right| \\
&\le C_1 \left(\mathbb{E} \int_0^T |x'_i - (\Gamma \bar{m} + \eta)|^2_Q dt \right)^{1/2} \\
&\le C_2 (1 + \|f\|^2_{L^2})^{1/2} \\
&\le C_3 + (\epsilon_0/16)\|f\|^2_{L^2}, \qquad (58)
\end{aligned}$$

where the second inequality follows from elementary estimates based on the solutions of (54) and (55). Similarly,

$$\mathbb{E}[x_i'^T(T) H(\hat{x}_i(T) - x'_i(T))] \le C_4 + (\epsilon_0/16)\|f\|^2_{L^2}. \tag{59}$$

Finally combining (56)-(59) with Lemma 6.2 leads to

$$J_i(\hat{u}_i, \hat{u}_{-i}, f) \le C - (\epsilon_0/4)\|f\|^2_{L^2}.$$

The lemma follows. □

Consider the set of strategies $(u_i, \hat{u}_{-i})$ and the corresponding state processes

$$dx_i = (Ax_i + Bu_i + Gx^{(N)} + f)dt + DdW_i, \tag{60}$$

$$dx_j = (Ax_j + B\hat{u}_j + Gx^{(N)} + f)dt + DdW_j, \quad 1 \le j \le N,\ j \ne i. \tag{61}$$

Lemma 6.4. *If u_i in (60) satisfies $\sup_f J_i(u_i, \hat{u}_{-i}, f) \le \hat{C}_0$, there exists $\hat{C}_1$ independent of N such that*

$$\mathbb{E} \int_0^T |u_i(t)|^2 dt \le \hat{C}_1. \tag{62}$$

Proof. Suppose $\sup_f J_i(u_i, \hat{u}_{-i}, f) \le \hat{C}_0$. Then for any f,

$$\mathbb{E} \int_0^T \left(|x_i - (\Gamma x^{(N)} + \eta)|^2_Q + u_i^T R u_i - \frac{1}{\gamma}|f(t)|^2 \right) dt + \mathbb{E}[x_i^T(T) H x_i(T)] \le \hat{C}_0,$$

where $(x_1, \cdots, x_N)$ is generated by $(u_i, \hat{u}_{-i})$ and f. Taking $f = 0$, we obtain

$$\mathbb{E} \int_0^T \left(|x_i - (\Gamma x^{(N)} + \eta)|^2_Q + u_i^T R u_i \right) dt \le \hat{C}_0.$$

Therefore, (62) holds. □

Let $\mathcal{U}_{\hat{C}_1}$ denote the set of processes $u_i \in L^2_{\mathcal{F}}(0, T; \mathbb{R}^{n_1})$ which satisfy (62). For (60)-(61), denote $x^{(N)} = (1/N)\sum_{j=1}^N x_j$.

LEMMA 6.5. *Suppose* $u_i \in \mathcal{U}_{\hat{C}_1}$ *in* (60). *Then*

$$\sup_{0\le t\le T, f, u_i\in\mathcal{U}_{\hat{C}_1}} \mathbb{E}|x^{(N)}(t)-\bar{m}(t)|^2 = O(1/N + |x^{(N)}(0)-m_0|^2).$$

Proof. Rewrite (60) in the form

$$dx_i = [Ax_i + B\hat{u}_i + Gx^{(N)} + f]dt + B(u_i-\hat{u}_i)dt + DdW_i. \tag{63}$$

By (61) and (63),

$$dx^{(N)} = [(A+G)x^{(N)} + B\hat{u}^{(N)} + f]dt + \frac{B}{N}(u_i-\hat{u}_i)dt + \frac{D}{N}\sum_{j=1}^N dW_j,$$

which combined with (54) gives

$$\begin{aligned} d(x^{(N)}-\bar{m}) = {} & [(A+G)(x^{(N)}-\bar{m}) + B(\hat{u}^{(N)}-\bar{u}^*)]dt \\ & + \frac{B}{N}(u_i-\hat{u}_i)dt + \frac{D}{N}\sum_{j=1}^N dW_j. \end{aligned}$$

By Theorem 5.1 and the fact $\mathbb{E}\int_0^T |u_i-\hat{u}_i|^2dt \le C$ for all $u_i \in \mathcal{U}_{\hat{C}_1}$, where the constant C does not depend on (f, u_i), elementary SDE estimates lead to

$$\sup_{0\le t\le T, f} \mathbb{E}|x^{(N)}(t)-\bar{m}(t)|^2 \le C(1/N + |x^{(N)}(0)-m_0|^2),$$

where C does not depend on u_i. The lemma follows. □

LEMMA 6.6. *For each* $u_i \in \mathcal{U}_{\hat{C}_1}$, $\sup_f J_i(u_i, \hat{u}_{-i}, f)$ *is finite and attained by some* f *depending on* u_i *and so denoted as* f_{u_i}. *Moreover,*

$$\sup_{u_i\in\mathcal{U}_{\hat{C}_1}} |\sup_f J_i(u_i, \hat{u}_{-i}, f) - \bar{J}_i(u_i, \hat{f}_{u_i})| = O(1/\sqrt{N} + |x^{(N)}(0)-m(0)|),$$

where $\hat{f}_{u_i}$ *is determined by Theorem 3.7 for the given* u_i.

Proof. Note that we have

$$\begin{aligned} dx_i &= [Ax_i + Bu_i + G\bar{m} + G(x^{(N)}-\bar{m}) + f]dt + DdW_i, \\ \dot{\bar{m}} &= (A+G)\bar{m} + B\bar{u}^* + f, \end{aligned} \tag{64}$$

where $\bar{m}(0) = m_0$. Define the auxiliary process

$$dx_i^\dagger = (Ax_i^\dagger + Bu_i + G\bar{m} + f)dt + DdW_i,$$

where $x_i^\dagger(0) = x_i(0)$ and (u_i, f, W_i) is the same as in (64). By Lemma 6.5, it is easy to show

$$\sup_{0\le t\le T, f} \mathbb{E}|x_i(t)-x_i^\dagger(t)|^2 = O(1/N + |x^{(N)}(0)-m(0)|^2). \tag{65}$$

We have the relation

$$\begin{aligned} |x_i - (\Gamma x^{(N)} + \eta)|_Q^2 = {} & |x_i^\dagger - (\Gamma\bar{m} + \eta)|_Q^2 + |(x_i - x_i^\dagger) + \Gamma(\bar{m} - x^{(N)})|_Q^2 \\ & + 2[x_i^\dagger - (\Gamma\bar{m}+\eta)]^T Q[(x_i - x_i^\dagger) + \Gamma(\bar{m} - x^{(N)})]. \end{aligned}$$

The cost can be rewritten as

$$
\begin{aligned}
J_i(u_i,\hat{u}_{-i},f) = \bar{J}_i(u_i,f) &+ \mathbb{E}\int_0^T |(x_i-x_i^\dagger)+\Gamma(\bar{m}-x^{(N)})|_Q^2 dt \\
&+ \mathbb{E}\left[|x_i(T)-x_i^\dagger(T)|_H^2\right] \\
&+ 2\mathbb{E}\int_0^T \left[x_i^\dagger-(\Gamma\bar{m}+\eta)\right]^T Q\left[(x_i-x_i^\dagger)+\Gamma(\bar{m}-x^{(N)})\right] dt \\
&+ 2\mathbb{E}\left[(x_i^\dagger(T))^T H(x_i(T)-x_i^\dagger(T))\right] \qquad (66)\\
\le \bar{J}_i(u_i,f) &+ C\left(1/N+|x^{(N)}(0)-m_0|^2\right) \\
&+ 2\mathbb{E}\int_0^T \left[x_i^\dagger-(\Gamma\bar{m}+\eta)\right]^T Q\left[(x_i-x_i^\dagger)+\Gamma(\bar{m}-x^{(N)})\right] dt \\
&+ 2\mathbb{E}\left[(x_i^\dagger(T))^T H(x_i(T)-x_i^\dagger(T))\right], \qquad (67)
\end{aligned}
$$

where the inequality follows from Lemma 6.5 and (65). Note that neither $x_i - x_i^\dagger$ nor $\bar{m} - x^{(N)}$ in (66) depend on f. The terms $x_i^\dagger$ and $x_i^\dagger - (\Gamma\bar{m}+\eta)$ are affine in f, and $-\bar{J}_i(u_i,f)$ is convex in f by Lemma 3.6. Consequently, it follows from (66) that $-J_i(u_i,\hat{u}_{-i},f)$ is convex in f. For $u_i \in \mathcal{U}_{\hat{C}_1}$, in analogue to (57), we obtain

$$\bar{J}_i(u_i,f) \le C - (\epsilon_0/2)\|f\|_{L^2}^2, \qquad (68)$$

where C doest not depend on u_i. We have

$$
\begin{aligned}
&\left|\mathbb{E}\int_0^T \left[x_i^\dagger-(\Gamma\bar{m}+\eta)\right]^T Q\left[(x_i-x_i^\dagger)+\Gamma(\bar{m}-x^{(N)})\right] dt\right| \\
\le &\left\{\mathbb{E}\int_0^T |x_i^\dagger-(\Gamma\bar{m}+\eta)|_Q^2 dt\right\}^{1/2}\cdot\left\{\mathbb{E}\int_0^T |(x_i-x_i^\dagger)+\Gamma(\bar{m}-x^{(N)})|_Q^2 dt\right\}^{1/2} \\
\le &\, C\left(1/\sqrt{N}+|x^{(N)}(0)-m_0|\right)(1+\|f\|_{L^2}^2)^{1/2} \\
\le &\, C+(\epsilon_0/16)\|f\|_{L^2}^2.
\end{aligned}
$$

Similarly,

$$\left|\mathbb{E}\left[(x_i^\dagger(T))^T H(x_i(T)-x_i^\dagger(T))\right]\right| \le C+(\epsilon_0/16)\|f\|_{L^2}^2.$$

Hence, (67) gives

$$J_i(u_i,\hat{u}_{-i},f) \le C - (\epsilon_0/4)\|f\|_{L^2}^2, \qquad (69)$$

where C does not depend on (N,u_i). So for given $u_i \in \mathcal{U}_{\hat{C}_1}$, $J_i(u_i,\hat{u}_{-i},f)$ attains a finite supremum at some f_{u_i} since it is a continuous functional of f and satisfies (69) [37, Theorem 7.3.7], and by (69) we may further find a constant $\hat{C}_2$ such that

$$\sup_{u_i\in\mathcal{U}_{\hat{C}_1}} \|f_{u_i}\|_{L^2} \le \hat{C}_2. \qquad (70)$$

By (67),

$$\begin{aligned} J_i(u_i,\hat{u}_{-i},f) \le{}& \bar{J}_i(u_i,f) + C(1/N + |x^{(N)}(0)-m_0|^2) \\ &+ C\left(1/N + |x^{(N)}(0)-m_0|^2\right)^{1/2}\left(\mathbb{E}\int_0^T |x_i^\dagger - (\Gamma\bar{m}+\eta)|_Q^2 dt\right)^{1/2} \\ &+ C\left(1/N + |x^{(N)}(0)-m_0|^2\right)^{1/2}\left(\mathbb{E}|x_i^\dagger(T)|^2\right)^{1/2}. \end{aligned} \tag{71}$$

Now for $u_i \in \mathcal{U}_{\hat{C}_1}$ and the resulting f_{u_i} satisfying (70), we further obtain

$$\mathbb{E}|x_i^\dagger(T)|^2 + \mathbb{E}\int_0^T |x_i^\dagger - (\Gamma\bar{m}+\eta)|_Q^2 dt \le C.$$

For $u_i \in \mathcal{U}_{\hat{C}_1}$, (71) gives

$$\begin{aligned} \sup_f J_i(u_i,\hat{u}_{-i},f) &\le \bar{J}_i(u_i,f_{u_i}) + C(1/\sqrt{N} + |x^{(N)}(0)-m_0|) \\ &\le \bar{J}_i(u_i,\hat{f}_{u_i}) + C(1/\sqrt{N} + |x^{(N)}(0)-m_0|), \end{aligned}$$

where $\hat{f}_{u_i}$ is determined by Theorem 3.7. Due to (68),

$$\sup_{u_i\in\mathcal{U}_{\hat{C}_1}} \|\hat{f}_{u_i}\|_{L^2} \le C \tag{72}$$

for some constant C. By (72) and the method in (66), we similarly derive

$$J_i(u_i,\hat{u}_{-i},\hat{f}_{u_i}) \ge \bar{J}_i(u_i,\hat{f}_{u_i}) - C(1/\sqrt{N} + |x^{(N)}(0)-m_0|).$$

Hence, for all $u_i \in \mathcal{U}_{\hat{C}_1}$,

$$\sup_f J_i(u_i,\hat{u}_{-i},f) \ge \bar{J}_i(u_i,\hat{f}_{u_i}) - C(1/\sqrt{N} + |x^{(N)}(0)-m_0|).$$

The constant C in various places does not depend on u_i. The lemma follows. □

Proof of Theorem 6.1. It suffices to show the first inequality by checking $u_i \in \mathcal{U}_{\hat{C}_1}$. By Lemma 6.6, we have

$$\begin{aligned} \sup_f J_i(u_i,\hat{u}_{-i},f) &\ge \bar{J}_i(u_i,\hat{f}_{u_i}) - C_1(1/\sqrt{N} + |x^{(N)}(0)-m_0|) \\ &\ge \bar{J}_i(\hat{u}_i,\hat{f}_{\hat{u}_i}) - C_1(1/\sqrt{N} + |x^{(N)}(0)-m_0|). \end{aligned} \tag{73}$$

On the other hand, by taking the particular control $\hat{u}_i$ in Lemma 6.6,

$$\sup_f J_i(\hat{u}_i,\hat{u}_{-i},f) \le \bar{J}_i(\hat{u}_i,\hat{f}_{\hat{u}_i}) + C_2(1/\sqrt{N} + |x^{(N)}(0)-m_0|). \tag{74}$$

Subsequently, (73) and (74) imply

$$\sup_f J_i(u_i,\hat{u}_{-i},f) \ge \sup_f J_i(\hat{u}_i,\hat{u}_{-i},f) - (C_1+C_2)(1/\sqrt{N} + |x^{(N)}(0)-m_0|).$$

This completes the proof. □

7. Conclusion. This paper introduces a class of mean field LQG games with drift uncertainty. By using the idea of robust optimization, the individual agent's strategy is designed by minimizing the worst case cost. When the resulting decentralized strategies are implemented in a finite population, their performance is characterized as a robust ε-Nash equilibrium.

In this paper we only deal with drift uncertainty. If the Brownian motions are also subject to an uncertain coefficient process to model volatility uncertainty [19, 42], the resulting optimal control problems will give a set of more complicated FBSDE. It is also of potential interest to address model uncertainty of the mean field game in a different setup by considering measure uncertainty [17, 40, 53] in the robust optimization problem. This will necessitate the use of different techniques for analysis. DE.

Appendix A. For proving Lemma 3.9, we give another lemma first. Consider an auxiliary optimal control problem with dynamics

$$\text{(A.1)} \qquad \begin{cases} \dot{z}_i = Az_i + Bv_i + Gz + \gamma q, \\ \dot{z} = (A+G)z + \gamma q, \\ \dot{q} = -(A+G)^T q - (I-\Gamma)^T Q(\mathbb{E}z_i - \Gamma z), \end{cases}$$

where $z_i(0) = z(0) = 0$, $q(T) = H\mathbb{E}z_i(T)$ and $v_i \in L^2_{\mathcal{F}}(0,T;\mathbb{R}^{n_1})$. Following the argument in the proof of Lemma 3.8, under (H1) we can show the existence and uniqueness of a solution to (A.1). The optimal control problem is

$$\textbf{(P2b)} \quad \text{minimize} \quad \bar{J}_i^b(v_i) = \mathbb{E}\int_0^T \left\{ |z_i - \Gamma z|_Q^2 + v_i^T R v_i - \gamma |q(t)|^2 \right\} dt + \mathbb{E}|z_i(T)|_H^2.$$

Similarly, we may define positive definiteness of $\bar{J}_i^b$ as in Section 3.

LEMMA A.1. *$\bar{J}_i^a$ is positive semi-definite (resp., positive definite) if and only if $\bar{J}_i^b$ is positive semi-definite (resp., positive definite).*

Proof. If suffices to show the "only if" part.

Suppose that $\bar{J}_i^a$ is positive semi-definite. Consider any control $v_i \in L^2_{\mathcal{F}}(0,T;\mathbb{R}^{n_1})$ for $\bar{J}_i^b$, and this gives a unique solution (z_i, z, q). We take expectation in (A.1) to obtain

$$\begin{cases} \dot{\bar{z}}_i = A\bar{z}_i + B\bar{v}_i + Gz + \gamma q, \\ \dot{z} = (A+G)z + \gamma q, \\ \dot{q} = -(A+G)^T q - (I-\Gamma)^T Q(\bar{z}_i - \Gamma z), \end{cases}$$

where $\bar{z}_i = \mathbb{E}z_i$ and $\bar{v}_i = \mathbb{E}v_i$.

It follows that

$$\begin{aligned} \bar{J}_i^b(v_i) &= \bar{J}_i^a(\bar{v}_i) + \mathbb{E}\int_0^T \left[|z_i - \mathbb{E}z_i|_Q^2 + |v_i - \mathbb{E}v_i|_R^2 \right] dt + \mathbb{E}|z_i(T) - \mathbb{E}z_i(T)|_H^2 \\ &\geq \bar{J}_i^a(\bar{v}_i) \geq 0. \end{aligned}$$

On the other hand, $\bar{J}_i^a(0) = 0$. This shows that $\bar{J}_i^b$ is positive semi-definite. The above reasoning is also valid for the positive definite case. This proves the "only if" part. $\square$

Proof of Lemma 3.9. Let (x_i, m_i, p_i) and (x_i', m_i', p_i') be two state processes in (P2) corresponding to the controls u_i and u_i', respectively. Assume $\lambda_1 \in [0,1]$ and $\lambda_1 + \lambda_2 = 1$. We have

$$\begin{aligned}
&\lambda_1 \bar{J}_i(u_i) + \lambda_2 \bar{J}_i(u_i') - \bar{J}_i(\lambda_1 u_i + \lambda_2 u_i') \\
&= \lambda_1\lambda_2 \mathbb{E} \int_0^T \left\{ |x_i - x_i' - \Gamma(m_i - m_i')|_Q^2 + |u_i - u_i'|_R^2 - \gamma |p_i(t) - p_i'(t)|^2 \right\} dt \\
&\quad + \lambda_1\lambda_2 \mathbb{E} |x_i(T) - x_i'(T)|_H^2.
\end{aligned}$$

Denote $z_i = x_i - x_i'$, $z = m_i - m_i'$, $q = p_i - p_i'$ and $v_i = u_i - u_i'$. It is obvious

$$\lambda_1 \bar{J}_i(u_i) + \lambda_2 \bar{J}_i(u_i') - \bar{J}_i(\lambda_1 u_i + \lambda_2 u_i') = \lambda_1\lambda_2 \bar{J}_i^b(v_i).$$

Recalling Lemma A.1, this completes the proof. □

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