LINEAR-QUADRATIC-GAUSSIAN MIXED GAMES WITH CONTINUUM-PARAMETRIZED MINOR PLAYERS

SON LUU NGUYEN† AND MINYI HUANG†

Abstract. We consider a mean field linear-quadratic-Gaussian game with a major player and a large number of minor players parametrized by a continuum set. The mean field generated by the minor players is approximated by a random process depending only on the initial state and the Brownian motion of the major player, and this leads to two limiting optimal control problems with random coefficients, which are solved subject to a consistency requirement on the mean field approximation. The set of decentralized strategies constructed from the limiting control problems has an ε-Nash equilibrium property when applied to the large but finite population model.

Key words. mean field models, mixed players, differential games, decentralized control, Nash equilibria

AMS subject classifications. 91A10, 91A23, 91A25, 93E20

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1. Introduction. Stochastic dynamic games with mean field coupling have experienced intense investigation in recent years; see, e.g., [2, 11, 16, 18, 20, 21, 28, 33, 34]. The players in these models are individually insignificant but they collectively have a significant impact on a particular player. Since all players are comparably small, they may be called peers. To design low-complexity strategies, consistent mean field approximations provide a powerful approach. The fundamental idea is that in the population limit each agent optimally responds to a certain mean field which in turn is replicated by the closed-loop behaviors of the agents. Based on this procedure, the mean field may be determined by a fixed point analysis [16]. In the resulting solution, each agent only needs to know its own state information and the mass effect in the population limit which may be computed offline. This methodology has been applied to decentralized control of coupled nonlinear oscillators subject to random disturbances [34], human crowd motion [11], and decentralized charging control for large populations of plug-in electric vehicles [23]. Risk-sensitive costs and robustness issues were considered in [32]. Numerical solutions for coupled Hamilton–Jacobi–Bellman and Fokker–Planck equations in mean field games have been developed in [1]. The survey [6] on differential games presents recent progress in mean field game theory. The technique of consistent mean field approximations is also applicable to mean field social optimization [17], where each agent does not simply minimize its own cost but solves a modified optimal control problem by taking into account its social impact across the population. The mean field approach has also appeared in anonymous sequential games [19] with a continuum of players individually optimally responding to the mean field. However, the modeling of a continuum of independent processes
leads to measurability difficulties and the empirical frequency of the realizations of the continuum-indexed individual states cannot be meaningfully defined [3].

1.1. Mean field games with major and minor players. This paper considers the linear-quadratic-Gaussian (LQG) game with a major player $A_0$ and a population of minor players $\{A_i, 1 \leq i \leq N\}$. At time $t \geq 0$, the states of the players $A_0$ and $A_i$ are, respectively, denoted by $x_0(t)$ and $x_i(t)$, $1 \leq i \leq N$. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)$ be the underlying filtered probability space with the filtration $\{\mathcal{F}_t\}_{t \geq 0}$. The dynamics of the $N+1$ players are given by a system of linear stochastic differential equations (SDEs)

\begin{align}
(1.1) \quad & dx_0(t) = \left[A_0x_0(t) + B_0u_0(t) + F_0x^{(N)}(t)\right]dt + D_0dW_0(t), \\
(1.2) \quad & dx_i(t) = \left[A(\theta_i)x_i(t) + B(\theta_i)u_i(t) + F(\theta_i)x^{(N)}(t)\right]dt + D(\theta_i)dW_i(t),
\end{align}

where $x^{(N)} = (1/N)\sum_{i=1}^N x_i$ is the mean field term. The initial states $\{x_j(0), 0 \leq j \leq N\}$ are measurable with respect to $\mathcal{F}_0$ and have finite second moments. The states $x_0, x_i$ and controls $u_0, u_i$ are, respectively, $n$ and $n_1$ dimensional vectors. The noise processes $W_0$ and $W_i$ are $n_2$ dimensional independent standard Brownian motions adapted to $\{\mathcal{F}_t\}_{t \geq 0}$, which are also independent of $\{x_j(0), 0 \leq j \leq N\}$. For simplicity, we may take $\mathcal{F}_i = \sigma\{x_j(0), W_j(s), 0 \leq j \leq N, s \leq t\}$, which is the $\sigma$-algebra generated by the initial states and the Brownian motions up to time $t$. The vector $\theta_i \in \mathbb{R}^d$ is a parameter in the dynamics of player $A_i$.

For $0 \leq j \leq N$, denote $u_{-j} = (u_0, \ldots, u_{j-1}, u_{j+1}, \ldots, u_N)$. For positive semi-definite matrix $M \geq 0$, we may write the quadratic form $z^T M z$ as $|z|^2_M$. The cost function for $A_0$ is given by

\begin{align}
(1.3) \quad & J_0(u_0, u_{-0}) = \mathbb{E} \int_0^T \left[|x_0(t) - \chi_0(x^{(N)}(t))|^2_{Q_0} + u_0^T(t)R_0u_0(t)\right]dt,
\end{align}

where $\chi_0(x^{(N)}(t)) = H_0x^{(N)}(t) + \eta_0, Q_0 \geq 0$, and $R_0 > 0$. The cost function for $A_i$, $1 \leq i \leq N$, is

\begin{align}
(1.4) \quad & J_i(u_i, u_{-i}) = \mathbb{E} \int_0^T \left[|x_i(t) - \chi(x_0(t), x^{(N)}(t))|^2_Q + u_i^T(t)Ru_i(t)\right]dt,
\end{align}

where $\chi(x_0(t), x^{(N)}(t)) = Hx_0(t) + \hat{H}x^{(N)}(t) + \eta_i, Q \geq 0$, and $R > 0$. The component $Hx_0$ in the coupling term $\chi$ indicates the strong influence of the major player on each minor player. By contrast, the mutual impact of two minor players is insignificant. We assume that all matrix or vector parameters $(A_0, B_0, \ldots, \eta_i, \ldots)$ in (1.1)–(1.4) are deterministic and have compatible dimensions.

Similar interaction patterns are often seen in economic settings, a simple example being a few large corporations and many much smaller competitors. They have been well studied in cooperative game theory, usually based on static models [12, 13, 14], and these games are called mixed games due to the vastly different influences of the players.

The above type of mean field LQG game modeling was initially introduced in [15], where the presence of the major player causes an interesting phenomenon called the lack of sufficient statistics. More specifically, in order to obtain asymptotic equilibrium strategies, $A_0$ cannot simply use a strategy as a function of $(t, x_0(t))$ and $A_i$ cannot only use $(t, x_0(t), x_i(t)), i \geq 1$; it is necessary to augment the system dynamics with a new state to Markovianize the local limiting decision problems.
The work [15] analyzed the case where the minor players are from a total of $K$ classes, i.e., $\theta_i \in \{1, \ldots, K\}$. By aggregating all states in the same class, a $Kn$ dimensional process $\bar{z}(t) = [\bar{z}_1^T, \ldots, \bar{z}_K^T]^T$ is introduced to approximate the mean field effects produced by the $K$ classes of players, and the evolution of $\bar{z}$ is specified by a linear stochastic ordinary differential equation driven by the state of the major player. Subsequently, the fundamental consistent mean field approximation approach [16] is applied to determine the dynamics of $\bar{z}$. This procedure gives a set of decentralized Markov strategies in an augmented state space.

This paper considers a population of minor players parametrized by an infinite set such as a continuum set. The procedure in [15] cannot be applied to obtain a finite dimensional Markov model as seen by a given player in the population limit since the method of specifying the sub-mean field of each type of minor players will generate an infinite dimensional state space. On the other hand, from the point of view of decentralized decision making of any given player, it is even unnecessary to specify sub-mean fields in such fine scales. By contrast, in a mean field game of peers, it is adequate to approximate the aggregate effect of all agents by a single deterministic process and a continuum parameter set does not cause any difficulty in the aggregation procedure [16].

To tackle an infinite number of types of minor players, we adopt a different approach by directly treating the mean field $z$ in the population limit as a random process. A crucial issue is to find an appropriate representation of $z$ so that the consistent mean field approximation translates into a tractable problem. Intuitively, when $N$ tends to infinity, the independent Brownian motions and initial states of the minor players will altogether contribute to some deterministic effect in the formation of the mean field process, and only $x_0(0)$ and $W_0$ cause the random fluctuation of the mean field. Furthermore, due to the linear quadratic structure of the game, it is plausible to restrict $z$ to be linearly dependent on $x_0(0)$ and $W_0$. The feasibility of considering $z$ from such a class will become clear later on when the consistency analysis is carried out in the paper. Given the approximation of $x^{(N)}$, we approximate the original problems of the major player and a representative minor player, respectively, by stochastic control problems with random coefficients in the dynamics and costs [5, 31, 36]. This further enables the explicit computation of the optimal strategies by using the powerful machinery of backward stochastic differential equations [5, 22] and Riccati equations. The approach of solving stochastic control problems with random coefficients has been used in a leader-follower hierarchical game with open-loop information [35]. Also, we exploit the linear nature of the system dynamics to reduce the analysis to function spaces when the consistent mean field is identified. Due to the treatment of random coefficients, the strategies obtained are not Markov strategies, which is different from [15]. We note that non-Markov control has arisen frequently in linear quadratic stochastic control problems with random coefficients [5, 8, 31, 36]. For applications of backward stochastic differential equations in mean field optimal control, the reader is referred to [4]. In mathematical physics, random mean field limits have been studied in interacting particle systems with correlated individual driving Brownian motions [10].

1.2. Assumptions, notation and organization. We introduce the following assumptions:

(A1) The initial states $\{x_j(0), 0 \leq j \leq N\}$ are independent, and there exists a constant $C$ independent of $N$ such that $\sup_{0 \leq j \leq N} E|x_j(0)|^2 \leq C$.

(A2) There exists a distribution function $F(\theta, x)$ on $\mathbb{R}^{d+n}$ such that the sequence of empirical distribution functions $F_N(\theta, x) = \frac{1}{N} \sum_{i=1}^{N} 1_{\{\theta_i \leq \theta, E x_i(0) \leq x\}}, N \geq 1$. 

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where each inequality holds componentwise, converges to $F(\theta, x)$ weakly, i.e., for any bounded and continuous function $h(\theta, x)$ on $\mathbb{R}^{d+n}$,

$$
\lim_{N \to \infty} \int_{\mathbb{R}^{d+n}} h(\theta, x) dF_N(\theta, x) = \int_{\mathbb{R}^{d+n}} h(\theta, x) dF(\theta, x).
$$

(A3) $A(\cdot), B(\cdot), F(\cdot)$, and $D(\cdot)$ are continuous matrix-valued functions of $\theta \in \Theta$, where $\Theta$ is a compact subset of $\mathbb{R}^d$.

Remark 1.1. It follows from (A1) that $Ex_t(0)$ is from a fixed compact set independent of $N$.

Note that $\{\theta_i, i \geq 1\}$ in (A2) is treated as a deterministic sequence which jointly with $\{Ex_t(0), i \geq 1\}$ exhibits orderly statistical behavior. Let $F_N(\theta)$ and $F(\theta)$ denote the marginal distribution functions of $F_N(\theta, x)$ and $F(\theta, x)$, respectively, with respect to $\theta$. By (A2), it is clear that $\{F_N(\theta), N \geq 1\}$ converges to $F(\theta)$ weakly.

For the reader’s convenience, we note some conventions on notation. We use $C, C_0$, etc., to denote a generic constant which may change from place to place but does not depend on the population size $N$. For a vector or matrix $V$, $|V|$ denotes the Frobenius norm; $\Delta$ is the region $\{(t, s) | 0 \leq s \leq t \leq T\}$; $[Z_1; \ldots; Z_l]$ denotes the vertical concatenation of $l$ column vectors; and $L^2_\gamma([0, T]; \mathbb{R}^{k_1 \times k_2})$ denotes all $\mathbb{R}^{k_1 \times k_2}$-valued processes $\{X_t, 0 \leq t \leq T\}$ adapted to $\{\mathcal{F}_t\}_{t \geq 0}$ and satisfying $E \int_0^T |X_t|^2 dt < \infty$. The product of two spaces $L^2_\gamma([0, T]; S_1)$ and $L^2_\gamma([0, T]; S_2)$ may be written as $L^2_\gamma([0, T]; S_1 \times S_2)$, and this notation may be extended to more than two spaces.

The organization of the paper is as follows. Section 1 presents the introduction and formulates the mean field game problem. Section 2 solves two auxiliary stochastic control problems in the mean field limit. The consistency condition for mean field approximations is introduced in section 3, and section 4 shows an asymptotic Nash equilibrium property. Section 5 analyzes a scalar model with numerical solutions. Section 6 concludes the paper.

2. The limiting control problems with random coefficients. We formulate the auxiliary control problems within the population limit via the approximation of $x^{(N)}(t)$ by a process $z$ depending linearly on the initial state and the driving noise of the major player. It will be convenient to still refer to them as the control problems of the major and minor players, respectively, although $x^{(N)}(t)$ in the original dynamics is now replaced by $z$.

2.1. Two auxiliary optimal control problems.

Problem (I): Optimal control of the major player. The dynamics are given by

$$
\begin{align*}
\dot{z}(t) &= f_1(t) + f_2(t)x_0(0) + \int_0^t g(t, s) dW_0(s), \\
\dot{x}_0(t) &= [A_0x_0(t) + B_0u_0(t) + F_0z(t)]dt + D_0dW_0(t),
\end{align*}
$$

where the second equation is obtained by replacing $x^{(N)}(t)$ in (1.1) by $z(t)$. For the proposed mean field approximation, we consider $f_1 \in C([0, T], \mathbb{R}^n)$, $f_2 \in C([0, T], \mathbb{R}^{n \times n})$ and $g \in C(\Delta, \mathbb{R}^{n \times n_2})$, where $\Delta = \{(t, s) | 0 \leq s \leq t \leq T\}$. The cost function is given by

$$
J_0(u_0) = E \int_0^T \left[ |x_0(t) - H_0z(t) - \eta_0|_{Q_0}^2 + u_0^T(t)R_0u_0(t) \right] dt.
$$

Problem (II): Optimal control of the minor player. After solving Problem (I), we represent the state $x_0$ of the major player by its initial condition and its
Brownian motion and further denote the state process by \( \bar{x}_0 \). By combining \( z, \bar{x}_0 \) with the limiting dynamics of the minor player, we introduce the equation system

\[
\begin{aligned}
\dot{z}(t) &= f_1(t) + f_2(t)x_0(0) + \int_0^t g(t,s)dW_0(s), \\
\dot{x}_0(t) &= f_{x_0,1}(t) + f_{x_0,2}(t)x_0(0) + \int_0^t g_{x_0}(t,s)dW_0(s), \\
x_0(t) &= \left[A(\theta_0)x_0(t) + B(\theta_0)u_0(t) + F(\theta_0)z(t)\right]dt + D(\theta_0)dW_t(t),
\end{aligned}
\]

(2.3)

where \((f_{x_0,1}, f_{x_0,2}, g_{x_0})\) is determined from the solution of Problem (I). The cost function is given by

\[
J_\delta(z) = E \int_0^T \left[ \|x_i(t) - H\bar{x}_0(t) - \hat{H}z(t) - \eta_t^i\|^2_Q + u_i^T(t)R u_i(t) \right] dt.
\]

(2.4)

### 2.2. The analysis of Problem (I).

By viewing Problem (I) as an optimal control problem with a random coefficient process \( z \), the optimal control may be determined by the standard backward SDE approach [5]. We have the following lemma.

**Lemma 2.1.** (i) There exists a unique optimal control \( \bar{u}_0 \in L^2_T(0,T;\mathbb{R}^n) \) to Problem (I).

(ii) The pair \((\bar{x}_0, \bar{u}_0) \in L^2_T(0,T;\mathbb{R}^{n+1})\) is the optimal solution to Problem (I) if and only if \( \bar{u}_0(t) = R_0^{-1}B_0^T p_0(t) \), where \((x_0, p_0, q_0) \in L^2_T(0,T;\mathbb{R}^{2n} \times \mathbb{R}^{n \times n^2})\) is the solution of the forward-backward SDE

\[
\begin{aligned}
\dot{x}_0(t) &= \left[A_0 \bar{x}_0(t) + B_0 R^{-1}_0 B^T_0 p_0(t) + F_0 z(t)\right]dt + D_0 dW_0(t), \\
\dot{p}_0(t) &= \left[R_0(\bar{x}_0(t) - H_0 z(t) - \eta_t) - A^T_0 p_0(t)\right]dt + q_0(t)dW_0(t), \\
x_0(0) &= x_0(0), \quad p_0(T) = 0.
\end{aligned}
\]

(2.5)

(iii) Equation (2.5) has a unique solution \((x_0, p_0, q_0) \in L^2_T(0,T;\mathbb{R}^{2n} \times \mathbb{R}^{n \times n^2})\).

**Proof.** Lemma 2.1(i) follows from [5, Theorem 3.1].

(ii) Let \((\bar{u}_0, \bar{x}_0)\) be the optimal pair to Problem (I) and \((p_0, q_0)\) the solution of the above backward SDE. Let \( \delta \bar{u}_0 \) and \( \delta \bar{x}_0 \) be the variations of \( \bar{u}_0 \) and \( \bar{x}_0 \), respectively, where \( \delta \bar{u}_0 \in L^2_T(0,T;\mathbb{R}^n) \). Since \( J_\delta(\cdot) \) is convex in \((x_0, u_0)\) and \( R_0 > 0 \), \( \bar{u}_0 \) is a solution to Problem (I) if and only if the first order variation \( \delta J_\delta \) of \( J_\delta \) satisfies

\[
\frac{\delta J_\delta(\bar{u}_0)}{2} = E \int_0^T \left\{ \delta \bar{x}_0^T(t) Q_0(\bar{x}_0(t) - H_0 z(t) - \eta_0) + \delta \bar{u}_0^T(t) R_0 \bar{u}_0(t) \right\} dt = 0.
\]

(2.6)

Note that \( d\delta \bar{x}_0(t) = (A_0 \delta \bar{x}_0 + B_0 \delta \bar{u}_0) dt \). By virtue of Itô’s formula,

\[
d[\delta \bar{x}_0^T(t)p_0(t)] = (A_0 \delta \bar{x}_0(t) + B_0 \delta \bar{u}_0(t))^T p_0(t) dt + \delta \bar{x}_0^T(t) \left\{ Q_0(\bar{x}_0(t) - H_0 z(t) - \eta_0) - A^T_0 p_0(t) \right\} dt + q_0(t)dW_0(t)
\]

\[
= \left\{ \delta \bar{x}_0^T(t) Q_0(\bar{x}_0(t) - H_0 z(t) - \eta_0) + \delta \bar{u}_0^T(t) B_0^T p_0(t) \right\} dt + \delta \bar{x}_0^T(t)q_0(t)dW_0(t).
\]

Combining this identity with \( p(T) = 0 \) and \( \delta \bar{x}_0(0) = 0 \) yields

\[
0 = E \delta \bar{x}_0^T(T)p_0(T) - E \delta \bar{x}_0^T(0)p_0(0)
\]

(2.7)

\[
= E \int_0^T \left\{ \delta \bar{x}_0^T(t) Q_0[\bar{x}_0(t) - H_0 z(t) - \eta_0] + \delta \bar{u}_0^T(t) B_0^T p_0(t) \right\} dt.
\]
It follows from (2.6)–(2.7) that for any \( \delta \bar{u}_0(t) \in L^2_T(0,T;\mathbb{R}^{n_1}) \),

\[
E \int_0^T (\delta \bar{u}_0^T(t)B_0^T p_0(t) - \delta \bar{u}_0^T(t)R_0 \bar{u}_0(t)) dt = 0.
\]

This implies that \( \bar{u}_0 \) is optimal if and only if \( \bar{u}_0(t) = R_0^{-1}B_0^T p_0(t) \) as desired.

(iii) Let \( P_0(t) \geq 0 \) be the unique solution of the Riccati equation (see [30])

\[
\begin{align*}
\dot{P}_0(t) + P_0(t)A_0 + A_0^T P_0(t) - P_0(t)B_0 R_0^{-1} B_0^T P_0(t) &+ Q_0 = 0, \\
P_0(T) &= 0.
\end{align*}
\]

To analyze the unique solvability of (2.5), write \( p_0(t) = -P_0(t)\bar{x}_0(t) + \nu_0(t) \), where \( \nu_0(t) \) satisfies the terminal condition \( \nu_0(T) = 0 \). Denote

\[
\kappa_0(t) = A_0 - B_0 R_0^{-1} B_0^T P_0(t).
\]

By Ito’s formula, it can be shown that (2.5) is equivalent to the forward-backward SDE

\[
\begin{align*}
\dot{x}_0(t) &= \left[ \kappa_0(t) x_0(t) + B_0 R_0^{-1} B_0^T \nu_0(t) + F_0 z(t) \right] dt + D_0 dW_0(t), \\
\dot{\nu}_0(t) &= \left\{ - \kappa_0^T(t) \nu_0(t) + \left[ (P_0(t) F_0 - Q_0 H_0) z(t) - Q_0 \eta_0 \right] \right\} dt \\
&\quad + \left[ g_0(t) + P_0(t) D_0 \right] dW_0(t), \\
x_0(0) &= x_0(0), \quad \nu_0(T) = 0,
\end{align*}
\]

where \( \nu_0 \) is now decoupled from \( \bar{x}_0 \). By Lemma A.1(ii), we can solve the second equation in (2.10) with a unique solution \( \nu_0 \in L^2_T(0,T;\mathbb{R}^{n}) \), which in turn determines a unique solution \( \bar{x}_0 \in L^2_T(0,T;\mathbb{R}^{n}) \) to the first equation. Therefore, (iii) follows from the fact that (2.5) is equivalent to (2.10).

The optimal control law is given in the form

\[
\bar{u}_0(t) = R_0^{-1} B_0^T \left[ -P_0(t) \bar{x}_0(t) + \nu_0(t) \right].
\]

2.3. An explicit solution of the optimal state process. We intend to find a representation of \( \bar{x}_0 \) in (2.10) in the form

\[
\bar{x}_0(t) = f_{\bar{x}_0,1}(t) + f_{\bar{x}_0,2}(t)x_0(0) + \int_0^t g_{\bar{x}_0}(t,s)dW_0(s),
\]

where \( f_{\bar{x}_0,1} \in C([0,T],\mathbb{R}^n), f_{\bar{x}_0,2} \in C([0,T],\mathbb{R}^{n\times n_2}), \) and \( g_{\bar{x}_0} \in C(\Delta,\mathbb{R}^{n\times n_2}) \) are to be determined.

To solve the second equation in (2.10), we denote \( \zeta_0(t) = [P_0(t) F_0 - Q_0 H_0] z(t) - Q_0 \eta_0 \) and \( \mu_0(t) = q_0(t) + P_0(t) D_0 \). Since \( z(t) = f_1(t) + f_2(t)x_0(0) + \int_0^t g(t,s)dW_0(s) \), we have

\[
\zeta_0(t) = f_{\zeta_0,1}(t) + f_{\zeta_0,2}(t)x_0(0) + \int_0^t g_{\zeta_0}(t,s)dW_0(s),
\]

where \( f_{\zeta_0,1} = [P_0(t) F_0 - Q_0 H_0] f_1(t) - Q_0 \eta_0, f_{\zeta_0,2} = [P_0(t) F_0 - Q_0 H_0] f_2(t), g_{\zeta_0}(t,s) = [P_0(t) F_0 - Q_0 H_0] g(t,s) \). The equation of \( \nu_0 \) in (2.10) may be rewritten as

\[
d\nu_0(t) = [\zeta_0(t) - \kappa_0^T(t) \nu_0(t)] dt + \mu_0(t) dW_0(t).
\]
Take $M_0(t) = \kappa_0(t)$ in (A.2) and denote the resulting solution by $\Phi_0(t,s)$. The application of Lemma A.1(ii) to (2.13) yields

$$
\nu_0(t) = f_{\nu_0,1}(t) + f_{\nu_0,2}(t)x_0(0) + \int_0^t g_{\nu_0}(t,s)dW_0(s),
$$

where

$$
f_{\nu_0,1}(t) = \int_t^T \Phi_0^T(s_1,t) \left\{ [Q_0 H_0 - P_0(s_1) F_0] f_1(s_1) + Q_0 \nu_0 \right\} ds_1,
$$

$$
f_{\nu_0,2}(t) = \int_t^T \Phi_0^T(s_1,t) [Q_0 H_0 - P_0(s_1) F_0] f_2(s_1) ds_1,
$$

$$
g_{\nu_0}(t,s) = \int_t^T \Phi_0^T(s_1,t) [Q_0 H_0 - P_0(s_1) F_0] g(s_1,s) ds_1.
$$

We continue to solve the first equation in (2.10). Let $\xi_0(t) = B_0 R_0^{-1} B_0^T \nu_0(t) + F_0 z(t)$. Then

$$
d\bar{x}_0(t) = [\xi_0(t) + \kappa_0(t) \bar{x}_0(t)] dt + D_0 dW_0(t).
$$

By (2.14) and the representation of $z(t)$,

$$
\xi_0(t) = f_{\xi_0,1}(t) + f_{\xi_0,2}(t)x_0(0) + \int_0^t g_{\xi_0}(t,s)dW_0(s),
$$

where $f_{\xi_0,j}(t) = B_0 R_0^{-1} B_0^T f_{\nu_0,j}(t) + F_0 f_j(t)$, $j = 1, 2$, and $g_{\xi_0}(t,s) = B_0 R_0^{-1} B_0^T g_{\nu_0}(t,s) + F_0 g(t,s)$. Therefore, by (2.18) and Lemma A.1(i) we obtain (2.12), where

$$
f_{\bar{x}_0,1}(t) = \int_0^t \Phi_0(t,s_1) f_{\xi_0,1}(s_1) ds_1
$$

$$
= \int_0^t \int_{s_1}^t \Phi_0(t,s_1) B_0 R_0^{-1} B_0^T \Phi_0^T(s_2,s_1) \left\{ [Q_0 H_0 - P_0(s_2) F_0] f_1(s_2) + Q_0 \nu_0 \right\} ds_2 ds_1
$$

$$
+ \int_0^t \Phi_0(t,s_1) F_0 f_1(s_1) ds_1
$$

$$
= [\Gamma_{0,1} f_1](t),
$$

$$
f_{\bar{x}_0,2}(t) = \Phi_0(t,0) + \int_0^t \Phi_0(t,s_1) f_{\xi_0,2}(s_1) ds_1
$$

$$
= \int_0^t \int_{s_1}^t \Phi_0(t,s_1) B_0 R_0^{-1} B_0^T \Phi_0^T(s_2,s_1) [Q_0 H_0 - P_0(s_2) F_0] f_2(s_2) ds_2 ds_1
$$

$$
+ \int_0^t \Phi_0(t,s_1) F_0 f_2(s_1) ds_1 + \Phi_0(t,0)
$$

$$
= [\Gamma_{0,2} f_2](t),
$$

$$
g_{\bar{x}_0}(t,s) = \int_s^t \Phi_0(t,s_1) g_{\xi_0}(s_1,s) ds_1 + \Phi_0(t,s) D_0
$$

$$
= \int_s^t \int_{s_1}^t \Phi_0(t,s_1) B_0 R_0^{-1} B_0^T \Phi_0^T(s_2,s_1) [Q_0 H_0 - P_0(s_2) F_0] g(s_2,s) ds_2 ds_1
$$

$$
+ \int_s^t \Phi_0(t,s_1) F_0 g(s_1,s) ds_1 + \Phi_0(t,s) D_0
$$

$$
= [\Lambda_0 g](t,s).
$$
2.4. The analysis of Problem (II).

**Lemma 2.2.** (i) There exists a unique optimal control \( \bar{u}_i \in L^2_{\mathbb{F}}(0, T; \mathbb{R}^{n_1}) \) to Problem (II).

(ii) The pair \((\bar{x}_i, \bar{u}_i) \in L^2_{\mathbb{F}}(0, T; \mathbb{R}^{n+n_1})\) is the optimal solution to Problem (II) if and only if \(\bar{u}_i(t) = R^{-1}B^T(\theta_i)p_i(t)\), where \((\bar{x}_i, p_i, q_i, r_i) \in L^2_{\mathbb{F}}(0, T; \mathbb{R}^{2n} \times \mathbb{R}^{n \times n_2} \times \mathbb{R}^{n \times n_2})\) is the solution of the forward-backward SDE

\[
\begin{align*}
    d\bar{x}_i &= \left[A(\theta_i)\bar{x}_i + B(\theta_i)R^{-1}B^T(\theta_i)p_i + F(\theta_i)z\right]dt + D(\theta_i)d\tilde{W}_i, \\
    dp_i &= \left[Q(\bar{x}_i - H\bar{x}_0 - \tilde{H}z - \eta) - A^T(\theta_i)p_i\right]dt + q_id\tilde{W}_i + r_id\tilde{W}_0, \\
    \bar{x}_i(0) &= x_i(0), \quad \nu_i(T) = 0.
\end{align*}
\]

(iii) Equation (2.22) has a unique solution \((\bar{x}_i, p_i, q_i, r_i) \in L^2_{\mathbb{F}}(0, T; \mathbb{R}^{2n} \times \mathbb{R}^{n \times n_2} \times \mathbb{R}^{n \times n_2}).\)

**Proof.** Using Lemma A.2 instead of Lemma A.1, the proof is similar to that of Lemma 2.1. \(\square\)

Let \(P_{\theta_i}(t) \geq 0\) be the unique solution of the Riccati equation

\[
\begin{align*}
    \dot{P}_{\theta_i} + P_{\theta_i}A(\theta_i) + A^T(\theta_i)P_{\theta_i} - P_{\theta_i}B(\theta_i)R_0^{-1}B^T(\theta_i)P_{\theta_i} + Q &= 0, \\
    P_{\theta_i}(T) &= 0.
\end{align*}
\]

Write \(p_i(t) = -P_{\theta_i}(t)\bar{x}_i(t) + \nu_i(t)\), where \(\nu_{\theta_i}(t)\) satisfies \(\nu_i(T) = 0\). Denote

\[
\lambda_{\theta_i}(t) = A(\theta_i) - B(\theta_i)R^{-1}B^T(\theta_i)P_{\theta_i}(t).
\]

By Ito's formula, it is straightforward to show that (2.22) is equivalent to the forward-backward SDE

\[
\begin{align*}
    d\bar{x}_i &= \left[\lambda_{\theta_i}\bar{x}_i + B(\theta_i)R^{-1}B^T(\theta_i)\nu_{\theta_i} + F(\theta_i)z\right]dt + D(\theta_i)d\tilde{W}_i, \\
    d\nu_{\theta_i} &= \left[-\lambda_{\theta_i}\nu_{\theta_i} + (P_{\theta_i}F(\theta_i) - Q\tilde{H})z - QH\bar{x}_0 - Q\eta\right]dt \\
    &\quad + [q_i + P_{\theta_i}D(\theta_i)]d\tilde{W}_i + r_id\tilde{W}_0, \\
    \bar{x}_i(0) &= x_i(0), \quad \nu_{\theta_i}(T) = 0.
\end{align*}
\]

The optimal control law is given in the form

\[
\bar{u}_i(t) = R^{-1}B^T(\theta_i)\left[-P_{\theta_i}(t)\bar{x}_i(t) + \nu_{\theta_i}(t)\right].
\]

We will apply Lemma A.2 to represent \(\bar{x}_i(t)\) in the form

\[
\bar{x}_i(t) = f_{\bar{x}_i,1}(t) + f_{\bar{x}_i,2}(t)x_0(0) + f_{\bar{x}_i,3}(t)x_i(0) + \int_0^t g_{\bar{x}_i}(t, s)d\tilde{W}_0(s) + \int_0^t h_{\bar{x}_i}(t, s)d\tilde{W}_i(s),
\]

where \(f_{\bar{x}_i,1} \in C([0, T], \mathbb{R}^n), f_{\bar{x}_i,2}, f_{\bar{x}_i,3} \in C([0, T], \mathbb{R}^{n \times n}), \) and \(g_{\bar{x}_i}, h_{\bar{x}_i} \in C(\Delta, \mathbb{R}^{n \times n_2})\) are to be determined.

Let \(\zeta_i(t) = [P_{\theta_i}(t)F(\theta_i) - Q\tilde{H}]z(t) - QH\bar{x}_0(t) - Q\eta\) and \(\lambda_i(t) = q_i(t) + P_{\theta_i}(t)D(\theta_i).\)

By (2.12),

\[
\zeta_i(t) = f_{\zeta_i,1}(t) + f_{\zeta_i,2}(t)x_0(0) + \int_0^t g_{\zeta_i}(t, s)d\tilde{W}_0(s),
\]
where \( f_{\xi,1}(t) = [P_0(t)F(\theta_i) - Q\hat{H}]f_1(t) - QH f_{x,0,1}(t) - Q\eta, f_{\xi,2}(t) = [P_0(t)F(\theta_i) - Q\hat{H}]f_2(t) - QH f_{x,0,2}(t), \) and \( g_{\xi}(t, s) = [P_0(t)F(\theta_i) - Q\hat{H}]g(t, s) - QH g_{x,0}(t, s). \) Note that \( f_{x,0,1}, f_{x,0,2}, \) and \( g_{x,0} \) are determined in (2.19)–(2.21), and \( f_1, f_2, \) and \( g \) appear in the equation of \( z. \) By the above choices of \( \zeta(t) \) and \( \lambda_i(t), \) the second equation in (2.25) may be written as

\[
d\nu_0(t) = [\zeta(t) - \lambda_0(t)\nu_0(t)]dt + r_i(t)dW_0(t) + \lambda_i(t)dW_i(t).
\]

Take \( M_i(t) = \lambda_0(t) \) in (A.8) and denote the resulting solution by \( \Phi_{\theta_i}(t, s). \) Then by Lemma A.2(ii), we have

\[
(2.28) \quad \nu_0(t) = f_{v_{0,1}}(t) + f_{v_{0,2}}(t)x_0(0) + \int_0^t g_{v_0}(t, s)dW_0(s),
\]

where

\[
(2.29) \quad f_{v_{0,1}}(t) = \int_t^T \Phi_{\theta_i}(s, t) \left\{ [Q\hat{H} - P_0(s_1)F(\theta_i)]f_1(s_1) + QH f_{x,0,1}(s_1) + Q\eta \right\} ds_1,
\]

\[
(2.30) \quad f_{v_{0,2}}(t) = \int_t^T \Phi_{\theta_i}(s, t) \left\{ [Q\hat{H} - P_0(s_1)F(\theta_i)]f_2(s_1) + QH f_{x,0,2}(s_1) \right\} ds_1,
\]

\[
(2.31) \quad g_{v_0}(t, s) = \int_t^T \Phi_{\theta_i}(s, t) \left\{ [Q\hat{H} - P_0(s_1)F(\theta_i)]g(s_1, s) + QH g_{x,0}(s_1, s) \right\} ds_1.
\]

Next let \( \xi_i(t) = B(\theta_i)R^{-1}B^T(\theta_i)\nu_0(t) + F(\theta_i)z(t). \) By (2.28) and the equation of \( z \) in (2.1), \( \xi_i(t) = f_{\xi,1}(t) + f_{\xi,2}(t)x_0(0) + \int_0^t g_{\xi}(t, s)dW_0(s), \) where

\[
(2.32) \quad f_{\xi,1}(t) = B(\theta_i)R^{-1}B^T(\theta_i)f_{v_{0,1}}(t) + F(\theta_i)f_1(t), \quad j = 1, 2,
\]

\[
(2.33) \quad g_{\xi}(t, s) = B(\theta_i)R^{-1}B^T(\theta_i)g_{v_{0,1}}(t, s) + F(\theta_i)g(t, s).
\]

Here, \( f_{v_{0,1}}, f_{v_{0,2}}, \) and \( g_{v_0} \) are determined in (2.29)–(2.31). So the first equation in (2.25) may be written as

\[
d\bar{x}_i(t) = [\xi_i(t) + \lambda_0(t)\bar{x}_i(t)]dt + D(\theta_i)dW_i(t).
\]

By Lemma A.2(i) we obtain (2.27), where

\[
(2.34) \quad f_{\bar{x},1}(t) = \int_0^t \Phi_{\theta_i}(t, s_1)f_{\xi,1}(s_1)ds_1
\]

\[
= \int_0^t \int_{s_1}^T \Phi_{\theta_i}(t, s_1)B(\theta_i)R^{-1}B^T(\theta_i)\Phi_{\theta_i}^T(s_2, s_1)
\]

\[
\times \left( [Q\hat{H} - P_0(s_2)F(\theta_i)]f_1(s_2) + QH f_{x,0,1}(s_2) + Q\eta \right) ds_2 ds_1
\]

\[
+ \int_0^t \Phi_{\theta_i}(t, s_1)F(\theta_i)f_1(s_1)ds_1
\]

\[
=: [\Gamma_{\theta_i,1}f_1](t),
\]
\[ f_{\xi,2}(t) = \int_0^t \Phi_{\theta_i}(t, s_1) f_{\xi,2}(s_1) ds_1 \]
\[ = \int_0^t \int_s^t \Phi_{\theta_i}(t, s_1) B(\theta_i) R^{-1} B^T(\theta_i) \Phi_{\theta_i}^T(s_2, s_1) \]
\[ \times \left( (Q \dot{H} - P_{\theta_i}(s_2) F(\theta_i)) f_2(s_2) + Q H f_{\xi,2}(s_2) \right) ds_2 ds_1 \]
\[ + \int_0^t \Phi_{\theta_i}(t, s_1) F(\theta_i) f_2(s_1) ds_1 \]
\[ (2.35) =: [\Gamma_{\theta_i,2} f_2](t), \]
\[ f_{\xi,3}(t) = \Phi_{\theta_i}(t, 0), \]
\[ g_{\xi}(t, s) = \int_s^t \Phi_{\theta_i}(t, s_1) g_{\xi}(s_1, s) ds_1 \]
\[ = \int_s^t \Phi_{\theta_i}(t, s_1) B(\theta_i) R^{-1} B^T(\theta_i) \Phi_{\theta_i}^T(s_2, s_1) \]
\[ \times \left( (Q \dot{H} - P_{\theta_i}(s_2) F(\theta_i)) g(s_2, s) + Q H g_{\xi_0}(s_2, s) \right) ds_2 ds_1 \]
\[ + \int_s^t \Phi_{\theta_i}(t, s_1) F(\theta_i) g(s_1, s) ds_1 \]
\[ (2.37) =: [\Lambda_{\theta_i} g](t, s), \]
and
\[ (2.38) h_{\xi}(t, s) = \Phi_{\theta_i}(t, s) D(\theta_i). \]

3. The consistency condition. This section introduces the consistency requirement for the mean field approximation, i.e., when the control strategies obtained in section 2 are applied, the mean field replicated by the closed loop in the population limit should coincide with the process \( z \) assumed at the beginning. Based on averaging with (2.27), denote
\[ \Gamma_{1f_1}(t) = \int_\Theta [\Gamma_{\theta,1f_1}(t)] d\Phi(\theta) + \int_\Theta \Phi(0) x d\Phi(\theta, x), \quad 0 \leq t \leq T, \]
\[ \Gamma_{2f_2}(t) = \int_\Theta [\Gamma_{\theta,2f_2}(t)] d\Phi(\theta), \quad 0 \leq t \leq T, \]
\[ [\Lambda g](t, s) = \int_\Theta [\Lambda_{\theta} g](t, s) d\Phi(\theta), \quad 0 \leq s \leq t \leq T, \]
for \( f_1 \in C([0, T], \mathbb{R}^n) \), \( f_2 \in C([0, T], \mathbb{R}^{n \times n}) \), and \( g \in C(\Delta, \mathbb{R}^{n \times n_2}) \). Here, \( \Gamma_{\theta,j} f_j \), \( j = 1, 2 \), and \( \Lambda_{\theta} g \) are, respectively, defined as in (2.34), (2.35), and (2.37) with \( \theta_i = \theta \). The consistent mean field approximation reduces to analyzing the fixed point equation system
\[ \begin{cases} f_j(t) = [\Gamma_{j} f_j](t), & 0 \leq t \leq T, \quad j = 1, 2, \\ g(t, s) = [\Lambda g](t, s), & 0 \leq s \leq t \leq T, \end{cases} \]
which is called the Nash certainty equivalence (NCE) equation system. Denote the product space
\[ C_{\text{NCE}} = C([0, T], \mathbb{R}^n) \times C([0, T], \mathbb{R}^{n \times n}) \times C(\Delta, \mathbb{R}^{n \times n_2}). \]

Definition 3.1. A triple \((f_1, f_2, g) \in C_{\text{NCE}}\) satisfying (3.4) is called a (consistent) solution to the NCE equation system.
LEMMA 3.2. Assume (A1)–(A3). Then
(i) $\Gamma_1$ is a mapping from $C([0, T], \mathbb{R}^n)$ to $C([0, T], \mathbb{R}^n)$;
(ii) $\Gamma_2$ is a mapping from $C([0, T], \mathbb{R}^{n \times n})$ to $C([0, T], \mathbb{R}^{n \times n})$;
(iii) $\Lambda$ is a mapping from $C(\Delta, \mathbb{R}^{n \times n})$ to $C(\Delta, \mathbb{R}^{n \times n})$.

Proof. We only prove (iii) while (i) and (ii) may be proved by a similar argument.
To analyze $\Lambda$, we define some auxiliary operators. For $g \in C(\Delta, \mathbb{R}^{n \times n})$ and $\theta \in \Theta$, denote
\begin{equation}
[\Lambda_{\theta,1}g](t, s) = \int_s^t \Phi_\theta(t, s_1)B(\theta)R^{-1}B^T(\theta)\int_{s_1}^T \Phi_\theta^T(s_2, s_1)g(s_2, s)ds_2ds_1,
\end{equation}
\begin{equation}
[\Lambda_{\theta,2}g](t, s) = \int_s^t \Phi_\theta(t, s_1)B(\theta)R^{-1}B^T(\theta)\int_{s_1}^T \Phi_\theta^T(s_2, s_1)P_\theta(s_2)F(\theta)g(s_2, s)ds_2ds_1,
\end{equation}
\begin{equation}
[\Lambda_{\theta,3}g](t, s) = \int_s^t \Phi_\theta(t, s_1)F(\theta)g(s_1, s)ds_1,
\end{equation}
and
\begin{equation}
[\Lambda,g](t, s) = \int_\Theta [\Lambda_{\theta,i}g](t, s)d\Phi(\theta), \quad i = 1, 2, 3.
\end{equation}

In view of (A3) and Remark A.4 (see Appendix A),
\begin{equation}
\sup_{\theta \in \Theta} \sup_{0 \leq t \leq T} \max \left\{ \left| \frac{d\Phi_\theta(t, s)}{dt} \right|, |\Phi_\theta(t, s)| \right\} \leq C, \quad \sup_{\theta \in \Theta} |B(\theta)| \leq C.
\end{equation}
Consider $(t, s) \in \Delta, (t, s') \in \Delta, s \leq s'$. Then
\begin{align}
\sup_{\theta \in \Theta} |[\Lambda_{\theta,1}g](t, s') - [\Lambda_{\theta,1}g](t, s)| &\leq \sup_{\theta \in \Theta} \left| \int_{s'}^t \Phi_\theta(t, s_1)B(\theta)R^{-1}B^T(\theta)\int_{s_1}^T \Phi_\theta^T(s_2, s_1)(g(s_2, s') - g(s_2, s))ds_2ds_1 \right| \\
&+ \sup_{\theta \in \Theta} \left| \int_s^{s'} \Phi_\theta(t, s_1)B(\theta)R^{-1}B^T(\theta)\int_{s_1}^T \Phi_\theta^T(s_2, s_1)g(s_2, s)ds_2ds_1 \right| \\
&\leq C \sup_{s' \leq s \leq T} |g(s_2, s') - g(s_2, s)| + C|s' - s| \sup_{(t, s) \in \Delta} |g(t, s)|.
\end{align}
Here $C$ does not depend on $g$.
Note that
\begin{align}
\sup_{\theta \in \Theta} \sup_{(t, s), (t', s') \in \Delta} |\Phi_\theta(t', s) - \Phi_\theta(t, s)| &\leq \sup_{\theta \in \Theta} \sup_{0 \leq t \leq T} \left| \frac{d\Phi_\theta(t, s)}{dt} \right||t - t'| \leq C|t - t'|.
\end{align}
Next consider any $(t, s) \in \Delta, (t', s) \in \Delta, t \leq t'$. For some constant $C$,
\begin{align}
\sup_{\theta \in \Theta} |[\Lambda_{\theta,1}g](t', s) - [\Lambda_{\theta,1}g](t, s)| &\leq \sup_{\theta \in \Theta} \left| \int_{s'}^{t'} \Phi_\theta(t', s_1)B(\theta)R^{-1}B^T(\theta)\Phi_\theta^T(s_2, s_1)g(s_2, s)ds_2ds_1 \right| \\
&+ \sup_{\theta \in \Theta} \left| \int_s^{t'} \Phi_\theta(t', s_1) - \Phi_\theta(t, s_1) \right|B(\theta)R^{-1}B^T(\theta)\Phi_\theta^T(s_2, s_1)g(s_2, s)ds_2ds_1 \\
&\leq C|t' - t|.
\end{align}
Let \((t, s) \in \Delta\) and \(\delta t, \delta s\) be such that \((t+\delta t, s+\delta s) \in \Delta\). By the uniform continuity of \(g\) on \(\Delta\) and (3.6)–(3.7), \(|\Lambda_1 g(t+\delta t, s+\delta s) - [\Lambda_1 g](t, s)| \to 0\) uniformly with respect to \((t, s)\), as \(\max\{|\delta s|, |\delta t|\} \to 0\). Hence, \(\Lambda_1\) is a mapping from \(C(\Delta, \mathbb{R}^{n \times n_2})\) to \(C(\Delta, \mathbb{R}^{n \times n_z})\). Similarly, we can show that \(\Lambda_2\) and \(\Lambda_3\) are mappings from \(C(\Delta, \mathbb{R}^{n \times n_z})\) to \(C(\Delta, \mathbb{R}^{n \times n_z})\).

By replacing the subscript \(\theta\) by 0, \(B(\theta)\) by \(B_0\), and \(F(\theta)\) by \(F_0\) in (3.5), we define \(\Lambda_{0,i}\) for \(i = 1, 2, 3\) in a similar manner. It can also be shown that each \(\Lambda_{0,i}\) is a mapping from \(C(\Delta, \mathbb{R}^{n \times n_z})\) to \(C(\Delta, \mathbb{R}^{n \times n_z})\). Note that

\[
[\Lambda_0 g](t, s) = [\Lambda_{0,1}(Q_0 H_0 g)](t, s) - [\Lambda_{0,2} g](t, s) + [\Lambda_{0,3} g](t, s) + \Phi_0(t, s) D_0,
\]

so \(\Lambda_0\) is also a mapping from \(C(\Delta, \mathbb{R}^{n \times n_z})\) to \(C(\Delta, \mathbb{R}^{n \times n_z})\). Finally, we have

\[
[\Lambda g](t, s) = [\Lambda_1(Q\dot{H} g + Q H \Lambda_0 g)](t, s) - [\Lambda_2 g](t, s) + [\Lambda_3 g](t, s).
\]

This proves (iii).

By replacing \((F(\theta, x), F(\theta))\) by \((F_N(\theta, x), F_N(\theta))\) in (3.1)–(3.3), we define

\[
\begin{align*}
[\Gamma_1^{(N)} f_1](t) &= \int_0^t \int_\Theta \int_{\mathbb{R}^n} \Phi_N(t, x) dF_N(x) + \int_\Theta \int_{\mathbb{R}^n} \Phi_N(t, x) dF_N(x), \\
&\quad 0 \leq t \leq T, \\
[\Gamma_2^{(N)} f_2](t) &= \int_0^t \int_\Theta \int_{\mathbb{R}^n} \Phi_N(t, x) dF_N(x), \\
&\quad 0 \leq t \leq T, \\
[\Lambda^{(N)} g](t, s) &= \int_\Theta [\Lambda_N g](t, s) dF_N(x), \\
&\quad 0 \leq s \leq t \leq T,
\end{align*}
\]

where \(f_1 \in C([0, T], \mathbb{R}^n), f_2 \in C([0, T], \mathbb{R}^{n \times n}),\) and \(g \in C(\Delta, \mathbb{R}^{n \times n_z}).\)

By the same argument as in the proof of Lemma 3.2, we can show that \(\Gamma_1^{(N)}\) is a mapping from \(C([0, T], \mathbb{R}^n)\) to \(C([0, T], \mathbb{R}^{n \times n})\), \(\Gamma_2^{(N)}\) is a mapping from \(C([0, T], \mathbb{R}^{n \times n})\) to \(C([0, T], \mathbb{R}^{n \times n_z})\), and \(\Lambda^{(N)}\) is a mapping from \(C(\Delta, \mathbb{R}^{n \times n_z})\) to \(C(\Delta, \mathbb{R}^{n \times n_z})\).

**Remark 3.** In view of Lemma A.3 and Remark A.4, for fixed \(t \in [0, T]\) and \((t, s) \in \Delta\) and fixed \(f_1 \in C([0, T], \mathbb{R}^n), f_2 \in C([0, T], \mathbb{R}^{n \times n}),\) and \(g \in C(\Delta, \mathbb{R}^{n \times n_z}),\)

\[
[\Gamma_1^{(N)} f_1](t), [\Gamma_2^{(N)} f_2](t) \text{ and } [\Lambda^{(N)} g](t, s)
\]

are bounded and continuous vector- or matrix-valued functions of \(\theta\). By Assumption (A2), \([\Gamma_1^{(N)} f_1](t) \to [\Gamma_1^{(N)} f_1](t)\) as \(N \to \infty\).

**Remark 3.** It follows from (3.6)–(3.7) that for each \(g \in C(\Delta, \mathbb{R}^{n \times n_z}),\) the sequence of matrix-valued functions \(\{\Lambda^{(N)} g, N \geq 1\}\) is uniformly equicontinuous on \(\Delta\). Similarly, we may show that for fixed \(f_1 \in C([0, T], \mathbb{R}^n)\) and \(f_2 \in C([0, T], \mathbb{R}^{n \times n}),\) the sequences of vector- or matrix-valued functions \(\{\Gamma_2^{(N)} f_2, N \geq 1\}\) and \(\{\Gamma_3^{(N)} f_3, N \geq 1\}\) are both uniformly equicontinuous on \([0, T]\).

Denote the linear operators \(\Gamma_{0,j}, \Gamma_{0,2},\) and \(\Lambda_0\) on \(C([0, T], \mathbb{R}^n), C([0, T], \mathbb{R}^{n \times n}),\) and \(C(\Delta, \mathbb{R}^{n \times n_z})\), respectively, as follows:

\[
[\Gamma_{0,j} f_j](t) = \int_0^t \int_{s_1}^T \Phi(t, s_1) B_0 R_0^{-1} B_0^T \Phi_0(s_2, s_1) (Q_0 H_0 - P_0(s_2) F_0) f_j(s_2) ds_2 ds_1
\]

\[
+ \int_0^t \Phi(t, s_1) F_0 f_j(s_1) ds_1, \quad j = 1, 2,
\]

where \(\Phi(t, s)\) and \(\Phi_0(t, s)\) are defined by (3.1) and (3.2), respectively.
\[
[\bar{\Lambda}g](t, s) = \int_s^t \int_{s_1}^T \Phi_0(t, s_1)B_0R_0^{-1}B_0^T \Phi_0^T(s_2, s_1)(Q_0H_0 - P_0(s_2)F_0)g(s_2, s)ds_2ds_1 + \int_s^t \Phi_0(t, s_1)F_0g(s_1, s)ds_1,
\]

which are obtained by retaining the linear term of the affine operators \(\Gamma_{0,j}\) and \(\Lambda_0\), respectively.

Corresponding to \(\Gamma_{\theta,1}\), \(\Gamma_{\theta,2}\), and \(\Lambda_\theta\), define the linear operators \(\tilde{\Gamma}_{\theta,1}\), \(\tilde{\Gamma}_{\theta,2}\), and \(\tilde{\Lambda}_\theta\) on \(C([0, T], \mathbb{R}^n)\), \(C([0, T], \mathbb{R}^{n\times n})\), and \(C(\Delta, \mathbb{R}^{n\times n_2})\), respectively, as follows:

\[
[\tilde{\Gamma}_{\theta,j}f_j](t) = \int_0^t \Phi_0(t, s_1)B(t)R^{-1}B^T(\theta)\Phi_0^T(s_2, s_1)
\times \left( (Q\hat{H} - P_0(s_2)F(\theta))f_j(s_2) + QH[\tilde{\Gamma}_{\theta,j}f_j](s_2) \right)ds_2ds_1
+ \int_0^t \Phi_0(t, s_1)F(t)f_j(s_1)ds_1, \quad j = 1, 2,
\]

\[
[\tilde{\Lambda}_\theta g](t, s) = \int_s^t \int_{s_1}^T \Phi_0(t, s_1)B(t)R^{-1}B^T(\theta)\Phi_0^T(s_2, s_1)
\times \left( (Q\hat{H} - P_0(s_2)F(\theta))g(s_2, s) + QH[\tilde{\Lambda}_\theta g](s_2, s) \right)ds_2ds_1
+ \int_s^t \Phi_0(t, s_1)F(t)g(s_1, s)ds_1.
\]

Let \(C([0, T], \mathbb{R}^n)\), \(C([0, T], \mathbb{R}^{n\times n})\), and \(C(\Delta, \mathbb{R}^{n\times n_2})\) be endowed with the usual sup-norms \(\| \cdot \|_\infty\) so that they are all Banach spaces. Define the linear operators

\[(3.12) \quad [\tilde{\Gamma}_j f_j](t) = \int_{\theta} [\tilde{\Gamma}_{\theta,j}f_j](t) d\mathbf{F}(\theta), \quad [\tilde{\Lambda}g](t, s) = \int_{\theta} [\tilde{\Lambda}_\theta g](t, s) d\mathbf{F}(\theta).\]

By Lemma 3.2, we see that \(\tilde{\Gamma}_1\) (resp., \(\tilde{\Gamma}_2\) and \(\Lambda\)) is a mapping from \(C([0, T], \mathbb{R}^n)\) (resp., \(C([0, T], \mathbb{R}^{n\times n})\) and \(C(\Delta, \mathbb{R}^{n\times n_2})\)) to itself.

**Remark 3.5.** For \(f_2 \in C([0, T], \mathbb{R}^{n\times n})\), we write \(f_2 = [f_{2,1}, \ldots, f_{2,n}]\), where \(f_{2,k} \in C([0, T], \mathbb{R}^n)\) for \(k = 1, \ldots, n\). By (3.12), \([\tilde{\Gamma}_2 f_2](t) = [ [\tilde{\Gamma}_1 f_{2,1}](t), \ldots, [\tilde{\Gamma}_1 f_{2,n}](t) ] \).

We have the following solvability result for the NCE equation system.

**Theorem 3.6.** Under Assumptions (A1)–(A3), if the norms of the linear operators \(\Gamma_1\) and \(\Lambda\) satisfy \(\|\Gamma_1\| < 1\) and \(\|\Lambda\| < 1\), there exists a unique solution \((f_1, f_2, g) \in C_{\text{NCE}}\) to (3.4).

**Proof.** For any \(f_1, f_1' \in C([0, T], \mathbb{R}^n)\) and \(g, g' \in C(\Delta, \mathbb{R}^{n\times n_2})\), we have

\[
\|\Gamma_1 f_1 - \Gamma_1 f_1'\|_\infty = \|\tilde{\Gamma}_1(f_1 - f_1')\|_\infty \leq \|\tilde{\Gamma}_1\| \|f_1 - f_1'\|_\infty \leq \alpha \|f_1 - f_1'\|_\infty,
\]

\[
\|\Lambda g - \Lambda g'\|_\infty = \|\tilde{\Lambda}(g - g')\|_\infty \leq \|\tilde{\Lambda}\| \|g - g'\|_\infty \leq \alpha \|g - g'\|_\infty
\]

for some \(\alpha \in (0, 1)\). By Banach fixed point theorem, \(\Gamma_1 f_1 = f_1\) and \(\Lambda g = g\) have a unique solution. Next, we write \(\Gamma_2 f_2 = f_2\) in the equivalent form

\[
\tilde{\Gamma}_2 f_2 + f_2' = f_2,
\]

where \(f_2' \in C([0, T], \mathbb{R}^{n\times n})\) is known. Denote \(f_2 = [f_{2,1}, \ldots, f_{2,n}]\), \(f_2' = [f'_{2,1}, \ldots, f'_{2,n}]\).

By Remark 3.5, we may further write an equivalent set of \(n\) decoupled equations of the form

\[
\tilde{\Gamma}_1 f_{2,k} + f'_{2,k} = f_{2,k}, \quad 1 \leq k \leq n,
\]
4. Decentralized strategies. Throughout this section we assume that there exists a solution \((f_1, f_2, g) \in C_{NCE}\) to the NCE equation system (3.4). Consider (1.1)–(1.2). Let the control laws of \(\mathcal{A}_0\) and \(\mathcal{A}_i\), \(1 \leq i \leq N\), be given by

\[
\begin{align*}
\dot{u}_0(t) &= R_0^{-1}B_0^T[-P_0(t)x_0(t) + \nu_0(t)], \\
\dot{u}_i(t) &= R^{-1}B^T(\theta_i)[-P_0(t)x_i(t) + \nu_0(t)], \quad 1 \leq i \leq N,
\end{align*}
\]

where \(P_0(t)\) and \(P_0(t)\) are determined by (2.8) and (2.23), and \(\nu_0(t)\) and \(\nu_0(t)\) are determined by (2.10) and (2.25) with \((f_1, f_2, g)\) being a solution to (3.4). Their explicit solutions are given by (2.14) and (2.28). The control laws (4.1)–(4.2) use the states of the system of \(N + 1\) players and are different from (2.11) and (2.26). Since each strategy depends on the current state information and the major player’s Brownian motion via the solution of the backward SDE, we call it a partial state feedback strategy.

After the control laws (4.1)–(4.2) are applied, the dynamics of \(\mathcal{A}_0\) and \(\mathcal{A}_i\) may be written as

\[
\begin{align*}
\dot{x}_0(t) &= \left[\lambda_0(t)x_0(t) + B_0R_0^{-1}B_0^TP_0(t)\nu_0(t) + F_0x^{(N)}(t)\right]dt + \nu_0(t)dW_0(t), \\
\dot{x}_i(t) &= \left[\lambda_0(t)x_i(t) + B(\theta_i)R^{-1}B^T(\theta_i)P_0(t)\nu_0(t) + F(\theta_i)x^{(N)}(t)\right]dt \\
&\quad + D(\theta_i)dW_i(t),
\end{align*}
\]

where \(1 \leq i \leq N\) and \(x^{(N)} = (1/N)\sum_{i=1}^{N} x_i\). For further performance estimates, we construct the auxiliary equation system for the \(N + 1\) players

\[
\begin{align*}
\dot{\bar{x}}_0(t) &= \left[\lambda_0(t)\bar{x}_0(t) + B_0R_0^{-1}B_0^TP_0(t)\nu_0(t) + F_0\bar{z}(t)\right]dt + \nu_0(t)dW_0(t), \\
\dot{\bar{x}}_i(t) &= \left[\lambda_0(t)\bar{x}_i(t) + B(\theta_i)R^{-1}B^T(\theta_i)P_0(t)\nu_0(t) + F(\theta_i)\bar{z}(t)\right]dt \\
&\quad + D(\theta_i)dW_i(t), \quad 1 \leq i \leq N,
\end{align*}
\]

where \(\bar{x}_0(0) = x_0(0)\) and \(\bar{x}_i(0) = x_i(0)\). From (2.12)–(2.21) and (2.27)–(2.38), we have

\[
\begin{align*}
\bar{x}_0(t) &= [\Gamma_{0,1}f_1](t) + [\Gamma_{0,2}f_2](t)x_0(0) + \int_0^t [\lambda_0g](t, s)dW_0(s), \\
\bar{x}_i(t) &= [\Gamma_{\theta_i1}f_1](t) + [\Gamma_{\theta_i2}f_2](t)x_0(0) + \Phi_{\theta_i}(t, 0)x_i(0) \\
&\quad + \int_0^t [\lambda_{\theta_i}g](t, s)dW_0(s) + \int_0^t \Phi_{\theta_i}(t, s)dW_i(s), \quad 1 \leq i \leq N.
\end{align*}
\]

Denote \(\bar{x}^{(N)}(t) = (1/N)\sum_{i=1}^{N} \bar{x}_i(t)\).
Define $\epsilon_N \geq 0$ by

$$
\epsilon_N^2 = \epsilon_{f_1,N}^2 + \epsilon_{f_2,N}^2 + \epsilon_{g,N}^2,
$$

where

$$
\begin{align*}
\epsilon_{f_1,N}^2 &= \int_0^T \left| \int_\Theta [\Gamma_\theta f_1](t)dF_N(\theta) - \int_\Theta \Phi_\theta(t,0)xdF_N(\theta,x) \right|^2 dt, \\
\epsilon_{f_2,N}^2 &= \int_0^T \left| \int_\Theta [\Gamma_\theta f_2](t)dF_N(\theta) - \int_\Theta \Phi_\theta(t,0)xdF(\theta,x) \right|^2 dt, \\
\epsilon_{g,N}^2 &= \int_0^T \int_0^t \left| \int_\Theta [\Lambda g](t,s)dF_N(\theta) - \int_\Theta [\Lambda g](t,s)dF(\theta) \right|^2 dsdt.
\end{align*}
$$

**Lemma 4.1.** Assume (A1)–(A3). We have $\lim_{N \to \infty} \epsilon_N = 0$.

*Proof.* Note that

$$
\begin{align*}
\epsilon_{f,j,N}^2 &= \int_0^T \left| \int_\Theta [\Gamma_\theta^{(N)} f_j](t) - [\Gamma_j f_j](t) \right|^2 dt, \quad j = 1, 2, \\
\epsilon_{g,N}^2 &= \int_0^T \int_0^t \left| \int_\Theta [\Lambda^{(N)} g](t,s) - [\Lambda g](t,s) \right|^2 dsdt,
\end{align*}
$$

where $\Gamma_\theta^{(N)} f_j$ and $\Lambda^{(N)} g$ are defined in (3.8)–(3.10), and $\Gamma_j f_j$ and $\Lambda g$ are defined in (3.1)–(3.3).

Since uniform equicontinuity and pointwise convergence imply uniform convergence, by Remarks 3.3 and 3.4 we conclude

$$
\sup_{t \in [0,T]} \left| \int_\Theta [\Gamma_\theta^{(N)} f_j](t) - [\Gamma_j f_j](t) \right| \to 0, \quad j = 1, 2,
$$

$$
\sup_{(t,s) \in \Delta} \left| \int_\Theta [\Lambda^{(N)} g](t,s) - [\Lambda g](t,s) \right| \to 0, \quad \text{as } N \to \infty.
$$

The lemma follows. \(\square\)

Let $(f_1, f_2, g) \in C_{\text{NCE}}$ be a solution to (3.4) and

$$(4.8) \quad z(t) = f_1(t) + f_2(t)x_0(0) + \int_0^t g(t,s)dW_0(s), \quad 0 \leq t \leq T.$$ 

We have the error estimate on the mean field approximation.

**Lemma 4.2.** Assume (A1)–(A3) and let $z$ be given by (4.8). Then

$$
E \int_0^T \left| z(t) - \hat{x}^{(N)}(t) \right|^2 dt = O\left( \epsilon_N^2 + \frac{1}{N} \right).
$$
Proof. We have \( \frac{1}{N} \sum_{i=1}^{N} \Phi_{\theta_i}(t,0)E x_i(0) = \int_{\Theta \times \mathbb{R}^n} \Phi_{\theta}(t,0) x dF_N(\theta, x) \). Then (3.4) and (4.7) yield

\[
Z(t) = \left[ \int_{\Theta} [\Gamma_{\theta_1} f_1(t)](t) dF(\theta) + \int_{\Theta \times \mathbb{R}^n} \Phi_{\theta}(t,0) x dF(\theta, x) \right] + \int_{\Theta} [\Gamma_{\theta_2} f_2](t) dF(\theta) x_0(0) \\
+ \int_{0}^{t} \int_{\Theta} [\Lambda_{\theta} g](t, s) dF(\theta) dW_0(s),
\]

\[
\overline{x}^{(N)}(t) = \left[ \int_{\Theta} [\Gamma_{\theta_1} f_1](t) dF_N(\theta) + \int_{\Theta \times \mathbb{R}^n} \Phi_{\theta}(t,0) x dF_N(\theta, x) \right] \\
+ \int_{\Theta} [\Gamma_{\theta_2} f_2](t) dF_N(\theta) x_0(0) + \int_{0}^{t} \int_{\Theta} [\Lambda_{\theta} g](t, s) dF_N(\theta) dW_0(s) \\
+ \frac{1}{N} \sum_{i=1}^{N} \Phi_{\theta_i}(t,0) [x_i(0) - E x_i(0)] + \frac{1}{N} \sum_{i=1}^{N} \int_{0}^{t} \Phi_{\theta_i}(t, s) D(\theta_i) dW_i(s).
\]

Hence

\[
E \int_{0}^{T} \left| Z(t) - \overline{x}^{(N)}(t) \right|^2 dt \\
\leq 5 \int_{0}^{T} \left| \int_{\Theta} [\Gamma_{\theta_1} f_1](t) dF(\theta) + \int_{\Theta \times \mathbb{R}^n} \Phi_{\theta}(t,0) x dF(\theta, x) \right|^2 dt \\
- \int_{\Theta} [\Gamma_{\theta_1} f_1](t) dF_N(\theta) - \int_{\Theta \times \mathbb{R}^n} \Phi_{\theta}(t,0) x dF_N(\theta, x) \left| \int_{0}^{t} \int_{\Theta} [\Gamma_{\theta_2} f_2](t) dF(\theta) - \int_{\Theta} [\Gamma_{\theta_2} f_2](t) dF_N(\theta) \right|^2 dt \\
+ 5E|x_0(0)|^2 \int_{0}^{T} \left| \int_{\Theta} [\Gamma_{\theta_1} f_1](t) dF(\theta) - \int_{\Theta} [\Gamma_{\theta_1} f_1](t) dF_N(\theta) \right|^2 dt \\
+ 5E \int_{0}^{T} \int_{0}^{t} \left| \int_{\Theta} \{[\Lambda_{\theta} g](t, s) dF(\theta) - [\Lambda_{\theta} g](t, s) dF_N(\theta) \} dW_0(s) \right|^2 dt \\
+ \frac{5}{N^2} \int_{0}^{T} \left| \sum_{i=1}^{N} \Phi_{\theta_i}(t,0) [x_i(0) - E x_i(0)] \right|^2 dt \\
+ \frac{5}{N^2} \int_{0}^{T} \left| \sum_{i=1}^{N} \int_{0}^{t} \Phi_{\theta_i}(t, s) D(\theta_i) dW_i(s) \right|^2 dt \\
\leq 5(1 + E|x_0(0)|^2) \left( \epsilon_{f_1,N}^2 + \epsilon_{f_2,N}^2 + \epsilon_{f_3,N}^2 \right) + \frac{5}{N^2} (NC)
\]

(4.9) \( \leq C \left( \epsilon_N^2 + \frac{1}{N} \right) \).

The second inequality follows from the independence of \( \{W_i, i \geq 1\} \), the independence of \( \{x_i(0), i \geq 1\} \) and the boundedness of \( \Phi_{\theta}(t, s) \) as implied by Remark A.4. The last inequality follows from (A1). \( \square \)

By using Lemma 4.2 we may further establish the next theorem.

**Theorem 4.3.** Assume (A1)–(A3). We have

\[
E \int_{0}^{T} \left[ |Z(t) - x^{(N)}(t)|^2 + \sup_{0 \leq j \leq N} |x_j(t) - \overline{x}_j(t)|^2 \right] dt = O \left( \epsilon_N^2 + \frac{1}{N} \right).
\]

**Proof.** See Appendix B. \( \square \)
Consider the system of $N + 1$ players described by (1.1)–(1.2). Denote the class $\mathcal{U}_W$ consisting of all $\mathbb{R}^{(N+1)n}$-valued processes $y_W$ of the form

$$y_W(t) = \int_0^t [h_0(t,s)dW_0(s); h_1(t,s)dW_1(s); \ldots; h_N(t,s)dW_N(s)],$$

where each $h_j$ is an $\mathbb{R}^{n \times n_j}$-valued bounded measurable function on $\Delta$.

Denote $\xi = (x_0, x_1, \ldots, x_N, y_W)$. For any $j = 0, \ldots, N$, the admissible control set $\mathcal{U}_j$ of player $\mathcal{A}_j$ consists of controls $u_j$ of the form $f(t, \xi_j)$ for some $y_W \in \mathcal{U}_W$ and some continuous function $f$, Lipschitz continuous in $\xi_j$. So the control of a player may not be purely in a feedback form since the noise process may be used via $y_W$; this general form of controls is necessary in order to include the decentralized strategies (4.1)–(4.2). Since the control still uses the players’ states, $(u_0, u_1, \ldots, u_n)$ will be called a set of partial state feedback strategies. Note that $\mathcal{U}_j$ is not restricted to be decentralized. Given each set of strategies in $\mathcal{U}_0 \times \cdots \times \mathcal{U}_N$, the closed-loop system has a unique strong solution. Recall $u_{-j} = (u_0, u_1, \ldots, u_{j-1}, u_{j+1}, \ldots, u_N)$ for $j = 0, \ldots, N$.

**Definition 4.4.** A set of controls $u_j \in \mathcal{U}_j$, $0 \leq j \leq N$, for the $N + 1$ players is called an $\varepsilon$-Nash equilibrium with respect to the costs $J_j$, $0 \leq j \leq N$, where $\varepsilon \geq 0$, if for any $j$, $0 \leq j \leq N$, $J_j(u_j, u_{-j}) \leq J_j(u_j', u_{-j}) + \varepsilon$, when any alternative $u_j'$ is applied by player $\mathcal{A}_j$.

**Theorem 4.5.** Assume (A1)–(A3). Let $\hat{u}_0$ and $\hat{u}_j$ be the optimal controls in Problems (I) and (II). For $0 \leq j \leq N$,

$$J_j(\hat{u}_j, \hat{u}_{-j}) - J_j(\hat{u}_j) = O\left(\varepsilon N + \frac{1}{\sqrt{N}}\right).$$

Proof. See Appendix C.

**Theorem 4.6.** Assume (A1)–(A3). Then the set of controls $\hat{u}_j$, $0 \leq j \leq N$, for the $N + 1$ players is an $\varepsilon$-Nash equilibrium, i.e., for $0 \leq j \leq N$,

$$J_j(\hat{u}_j, \hat{u}_{-j}) - \varepsilon \leq \inf_{u_j \in \mathcal{U}_j} J_j(u_j, \hat{u}_{-j}) \leq J_j(\hat{u}_j, \hat{u}_{-j}),$$

where $0 \leq \varepsilon = O(\varepsilon N + 1/\sqrt{N})$.

Proof. See Appendix C.

**5. The scalar model and numerical solutions.** The dynamics of the $N + 1$ players are given by

$$\begin{align*}
dx_0 &= [a_0x_0(t) + b_0u_0(t)]dt + D_0dw_0(t), \quad t \geq 0, \\
dx_i &= [a_ix_i(t) + b_iu_i(t)]dt + Ddw_i(t), \quad 1 \leq i \leq N,
\end{align*}$$

where $x_0$, $u_0$, $w_0$, $x_i$, $u_i$, and $w_i$ are all scalar processes. For simplicity, we only let $a_i$ be dependent on the players, and furthermore, $E x_i(0) = 0$ for $i \geq 1$. Suppose $a_i \in \Theta := [\underline{a}, \overline{a}]$, which is a compact interval.

The cost functions for $\mathcal{A}_0$ and $\mathcal{A}_i$, $1 \leq i \leq N$, are given by

$$\begin{align*}
J_0(u_0, u_{-0}) &= E \int_0^T \left\{ q_0(x_0(t) - h_0x^{(N)}(t) - \eta_0)^2 + u_0^2(t) \right\} dt, \\
J_i(u_i, u_{-i}) &= E \int_0^T \left\{ q(x_i(t) - h_ix^{(N)}(t) - \eta_i)^2 + u_i^2(t) \right\} dt.
\end{align*}$$

We introduce the following assumption to replace (A2).
Then
\[ a(5.4) \]

Then, \( \exp \) where \( a \) weakly.

\[ \tilde{\Lambda}_0 \triangleright \Gamma_0 \quad \tilde{\Gamma}_0 \]

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Denote

\[ \varphi_0(t) = q_0b_0^2 h_0 \int_0^t \int_s^T e^{\int_s^t a(s_2)ds_2} e^{\int_s^t a(s_1)ds_1} ds_2 ds_1, \]

\[ \varphi_1(t) = q_0^2 h_0 \int_0^t \int_s^T \frac{\|a(s_2)ds_2\|}{2} e^{\int_s^t a(s_1)ds_1} \big( h\varphi_0(s_2) + \eta \big) dF(a) ds_2 ds_1, \]

\[ \varphi_2(t) = q_0^2 h \int_0^t \int_s^T \frac{\|a(s_2)ds_2\|}{2} e^{\int_s^t a(s_1)ds_1} \big( h\varphi_0(s_2) + \eta \big) dF(a) ds_2 ds_1, \]

\[ \psi(t) = q_0^2 h D_0 \int_0^t \int_s^T \frac{\|a(s_2)ds_2\|}{2} e^{\int_s^t a(s_1)ds_1} \big( h\varphi_0(s_2) + \eta \big) dF(a) ds_2 ds_1. \]

Taking into account \( E x_i(0) = 0, 1 \leq i \leq N \), we obtain the NCE equation system

(5.5)

\[ \begin{cases} f_j(t) = [\hat{\Gamma} f_j](t) + \varphi_j(t), & j = 1, 2, \\ g(t, s) = [\hat{\Lambda} g](t, s) + \psi(t, s). \end{cases} \]

**Corollary 5.1.** The NCE equation system (5.5) has a unique solution if

\[ \frac{q_0^2 T^2}{2} \left( \|\hat{\Gamma}\| + \frac{h_0 h_1 q_0 b_0^2 T^2}{2} e^{2T a_0 \psi_0} \right) \int_\mathbb{R} e^{2T a_0 \psi_0} dF(a) < 1. \]

**Proof.** By Theorem 3.6, it suffices to show \( \|\hat{\Gamma}\| < 1 \) and \( \|\hat{\Lambda}\| < 1 \). Since \( a_0(t) \leq a_0 \) for \( 0 \leq t \leq T \),

\[ \|\hat{\Gamma}\| \leq q_0 b_0^2 |h_0| \int_0^t \int_s^T e^{(t-s_1) a_0} e^{(s_2-s_1) a_0} ds_2 ds_1 \leq q_0 b_0^2 |h_0| \frac{T^2}{2} e^{2T a_0 \psi_0}. \]

Since \( a(t) \leq a \) for \( 0 \leq t \leq T \), we have

\[ \|\hat{\Gamma}\| \leq q_0^2 (|h_0| \|\hat{\Gamma}\|_0 + |\hat{h}|) \int_0^T \int_s^T e^{(t-s_1) a} e^{(s_2-s_1) a} ds_2 ds_1 dF(a) \]

\[ \leq \frac{q_0^2 T^2}{2} \left( |\hat{h}| + \frac{h_0 h_1 q_0 b_0^2 T^2}{2} e^{2T a_0 \psi_0} \right) \int_\mathbb{R} e^{2T a_0 \psi_0} dF(a) < 1. \]

Similarly, \( \|\hat{\Lambda}\| < 1 \). This completes the proof. \( \square \)

In the numerical solution for (5.5), we take the parameters \( T = 1 \),

\[ [a_0, b_0, D_0, q_0, h_0, \eta_0] = [0.5, 1, 1, 0.6, 1.5], \]

\[ [\underline{a}, \underline{a}, b, D, q, h, \hat{h}, \eta] = [0.1, 0.4, 1, 1, 1.2, 0.5, 0.4, 0.5], \]

and \( F(a) \) is a uniform distribution on \( [\underline{a}, \underline{a}] \). It can be verified that the condition of Corollary 5.1 holds for this example. For the iteration of the fixed point equations in (5.5), we take a step size of 0.05 to discretize \( t \) and \( a \), and the solutions of \( f_1, f_2, \) and \( g \) are shown in Figure 1.
6. Concluding remarks. This paper considers mean field LQG games with a major player and continuum-parametrized minor players. We introduce random mean field approximations, solve the resulting limiting problems as stochastic optimal control with random coefficients, and further obtain decentralized strategies for the players.

For the model analyzed in the paper, the minor players are affected by the major player only via their cost functions. The mean field in the closed loop is nonresponsive to the strategy change of the major player when the minor players implement their $\varepsilon$-Nash strategies.

If the state of the major player appears in the dynamics of each minor player, the mean field will become responsive to the major player’s control. More specifically, the strategy change of the major player causes a change in the major player’s state and consequently a change in each minor player’s state, leading to a change of the mean field. Thus, there is a strong coupling between the major player’s strategy change and the mean field evolution. In this case, the design of the major player’s strategy should address its ability in simultaneously perturbing its own state process and the mean field; some initial progress has been made in the recent work [27] by introducing a procedure called anticipative variational calculations for the major player’s limiting control problem. The analysis in [27] treated homogeneous minor players and it is of interest to generalize it to continuum-parametrized minor players.

**Fig. 1.** The numerical solution of $f_1(t)$, $f_2(t)$, and $g(t,s)$. 

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Appendix A. Auxiliary lemmas. To solve the forward-backward SDE systems in section 2, we introduce two general lemmas.

Lemma A.1. (i) Let \( M_0 \in C([0,T],\mathbb{R}^{n \times n}) \), \( D_0 \in \mathbb{R}^{n \times n_2} \), and

\[
\xi_0(t) = f_{\xi_0,1}(t) + f_{\xi_0,2}(t)x_0(0) + \int_0^t g_{\xi_0}(t,s) \, dW_0(s),
\]

where \( f_{\xi_0,1} \in C([0,T],\mathbb{R}^n) \), \( f_{\xi_0,2} \in C([0,T],\mathbb{R}^{n \times n}) \), \( g_{\xi_0} \in C(\Delta,\mathbb{R}^{n \times n_2}) \) are deterministic functions and \( W_0 \) is an \( n_2 \)-dimensional standard Brownian motion. Let \( x_0 \) be the solution of the SDE

\[
\begin{aligned}
\frac{dx_0(t)}{dt} &= (\xi_0(t) + M_0(t)x_0(t)) \, dt + D_0 \, dW_0(t) \\
\end{aligned}
\]

with the initial condition \( x_0(0) \) independent of \( W_0 \). Then

\[
x_0(t) = f_{x_0,1}(t) + f_{x_0,2}(t)x_0(0) + \int_0^t g_{x_0}(t,s) \, dW_0(s),
\]

where

\[
\begin{align*}
    f_{x_0,1}(t) &= \int_0^t \Phi_0(t,s_1)f_{\xi_0,1}(s_1) \, ds_1, \\
    f_{x_0,2}(t) &= \Phi_0(t,0) + \int_0^t \Phi_0(t,s_1)f_{\xi_0,2}(s_1) \, ds_1, \\
    g_{x_0}(t,s) &= \Phi_0(t,s)D_0 + \int_s^t \Phi_0(t,s_1)g_{\xi_0}(s_1,s) \, ds_1,
\end{align*}
\]

and \( \Phi_0(\cdot,\cdot) \) is the unique solution of the system

\[
\begin{aligned}
    \left\{ 
    d\Phi_0(t,s) &= M_0(t)\Phi_0(t,s) \, dt, \\
    &\Phi_0(s,s) = I, \quad 0 \leq s, t \leq T.
    \right. 
\end{aligned}
\]

(ii) Let \( M_0, W_0, x_0(0), \) and \( \Phi_0 \) be the same as in (i), and \( \zeta_0(t) = f_{\zeta_0,1}(t) + f_{\zeta_0,2}(t)x_0(0) + \int_0^t g_{\zeta_0}(t,s) \, dW_0(s) \), where \( f_{\zeta_0,1} \in C([0,T],\mathbb{R}^n) \), \( f_{\zeta_0,2} \in C([0,T],\mathbb{R}^{n \times n}) \), \( g_{\zeta_0} \in C(\Delta,\mathbb{R}^{n \times n_2}) \) and \( x_0(0) \) is independent of \( W_0 \). The backward SDE

\[
\begin{aligned}
    \left\{ 
    dv_0(t) &= (\zeta_0(t) - M_0^T(t)v_0(t)) \, dt + \mu_0(t) \, dW_0(t), \\
    v_0(T) &= 0
    \right. 
\end{aligned}
\]

has a unique solution \( (v_0,\mu_0) \) and \( v_0(t) = f_{v_0,1}(t) + f_{v_0,2}(t)x_0(0) + \int_0^t g_{v_0}(t,s) \, dW_0(s) \), where

\[
\begin{align*}
    f_{v_0,j}(t) &= -\int_t^T \Phi_0^T(t,s_1)f_{\zeta_0,j}(s_1) \, ds_1, \quad j = 1, 2, \\
    g_{v_0}(t,s) &= -\int_t^T \Phi_0^T(t,s_1)g_{\zeta_0}(s_1,s) \, ds_1.
\end{align*}
\]
Proof. (i) By Fubini’s theorem for stochastic integrals,
\[
x_0(t) = \int_0^t \Phi_0(t, s_1) \left( f_{\xi_0,1}(s_1) + f_{\xi_0,2}(s_1) x_0(0) + \int_0^{s_1} g_{\xi_0}(s_1, s) dW_0(s) \right) ds_1 \\
n + \Phi_0(t, 0) x_0(0) + \int_0^t \Phi_0(t, s) D_0 dW_0(s)
\]
\[
= \int_0^t \Phi_0(t, s_1) f_{\xi_0,1}(s_1) ds_1 + \left( \Phi_0(t, 0) + \int_0^t \Phi_0(t, s_1) f_{\xi_0,2}(s_1) ds_1 \right) x_0(0) \\
+ \int_0^t \left( \Phi_0(t, s) D_0 + \int_s^t \Phi_0(t, s_1) g_{\xi_0}(s_1, s) ds_1 \right) dW_0(s).
\]
This gives the desired identities.

(ii) The linear backward SDE has a unique solution [5, 36]. Let \( \Psi_0(t, s) = \Phi_O^T(t, s) \) for \( t \geq 0 \) and \( s \geq 0 \). Then by [7, Chapter 2],
\[
\begin{aligned}
\{ d\Psi_0(t, s) &= -M_0^T(t)\Psi_0(t, s) dt, \\
\Psi_0(t, s) &= I, & t \geq 0, & s \geq 0.
\end{aligned}
\]
Denote \( \hat{\zeta}_0(t) = \Psi_0(0, t) \zeta_0(0), \nu_0(t) = \Psi_0(0, t) \nu_0(t), \) and \( \hat{\mu}_0(t) = \Psi_0(0, t) \mu_0(t) \). Then
\[
\dot{\zeta}_0(t) = f_{\zeta_0,1}(t) + f_{\zeta_0,2}(t)x_0(0) + \int_0^t g_{\zeta_0}(t, s) dW_0(s),
\]
where
\[
\begin{aligned}
f_{\zeta_0,1}(t) &= \Psi_0(0, t) f_{\zeta_0,1}(t), & f_{\zeta_0,2}(t) &= \Psi_0(0, t) f_{\zeta_0,2}(t), & g_{\zeta_0}(t, s) &= \Psi_0(0, t) g_{\zeta_0}(t, s).
\end{aligned}
\]
By virtue of (A.4), \( d\Psi_0(0, t) = \Psi_0(0, t) M_0^T(t) dt \). Thus, by Ito’s formula,
\[
\begin{aligned}
d\nu_0(t) &= \dot{\zeta}_0(t) dt + \dot{\mu}_0(t) dW_0(t), \\
\nu_0(T) &= 0.
\end{aligned}
\]
By (A.6) and Fubini’s theorem,
\[
\begin{aligned}
\dot{\nu}_0(T) - \nu_0(0) &= \int_t^T f_{\zeta_0,1}(s) ds + \int_t^T f_{\zeta_0,2}(s) dW_0(x_0(0) + \int_0^t \int_t^T g_{\zeta_0}(s, s_1) dsdW_0(s_1) \\
&+ \int_t^T \int_s^T g_{\zeta_0}(s, s_1) dsdW_0(s_1) + \int_t^T \dot{\mu}_0(s) dW_0(s).
\end{aligned}
\]
Taking conditional expectation with respect to \( \sigma \{ x_0(0), W_0(s), s \leq t \} \)
\[
\begin{aligned}
\dot{\nu}_0(t) &= -\int_t^T f_{\zeta_0,1}(s) ds + \int_t^T f_{\zeta_0,2}(s) dW_0(x_0(0) - \int_0^t \int_t^T g_{\zeta_0}(s, s_1) dsdW_0(s_1) \\
\end{aligned}
\]
This implies that \( \dot{\nu}_0(0) = -\int_0^T f_{\zeta_0,1}(s) ds - \int_0^T f_{\zeta_0,2}(s) dsx_0(0) \) and
\[
\begin{aligned}
\dot{\nu}_0(t) - \nu_0(0) &= \int_0^t f_{\zeta_0,1}(s) ds + \int_0^t f_{\zeta_0,2}(s) dW_0(x_0(0) - \int_0^t \int_t^T g_{\zeta_0}(s, s_1) dsdW_0(s_1) \\
&= \int_0^t \dot{\zeta}_0(s) ds + \int_0^t \dot{\mu}_0(s) dW_0(s),
\end{aligned}
\]
where the second equality follows from (A.6). By using Fubini’s theorem again we obtain \( \hat{\mu}_0(s) = -\int_s^T g_{\xi_0}(t, s)dt \) as a deterministic function. Since \( \Psi_0(t, s) = \Psi_0(t, 0)\Psi_0(0, s) \) and \( \Psi_0(t, 0) = \Psi_0^{-1}(0, t) \), we have \( \nu_0(t) = \Psi_0(t, 0)\nu_0(t) \), \( \zeta_0(t) = \Psi_0(t, 0)\zeta_0(t) \), and \( \mu_0(t) = \Psi_0(t, 0)\mu_0(t) \). Therefore, from (A.5) and (A.7) we have

\[
\nu_0(t) = -\int_t^T \Psi_0(t, s)f_{\xi_0,1}(s)ds - \int_t^T \Psi_0(t, s)f_{\xi_0,2}(s)dsx_0(0)
- \int_t^0 \int_t^T \Psi_0(t, s)g_{\xi_0}(s, s_1)dsdW_0(s_1).
\]

Since \( \Psi_0(t, s) = \Phi_0^T(s, t) \), this completes the proof. \( \square \)

**Lemma A.2.** (i) Let \( W_0, W_i \) be two independent standard Brownian motions, which are also independent of \( \mathbb{R}^n \)-valued random vectors \( x_0(0) \) and \( x_i(0) \). Let \( M_i \in C([0, T], \mathbb{R}^{n \times n}) \), \( D_i \in \mathbb{R}^{n \times n} \), and

\[
\xi_i(t) = f_{\xi_i,1}(t) + f_{\xi_i,2}(t)x_0(0) + \int_0^t g_{\xi_i}(t, s)dW_0(s),
\]

where \( f_{\xi_i,1} \in C([0, T], \mathbb{R}^n) \), \( f_{\xi_i,2} \in C([0, T], \mathbb{R}^{n \times n}) \), \( g_{\xi_i} \in C(\Delta, \mathbb{R}^{n \times n}) \). Let \( x_i \) be the solution of

\[
dx_i(t) = (\xi_i(t) + M_i(t)x_i(t))dt + D_idW_i(t).
\]

Then

\[
x_i(t) = f_{x_i,1}(t) + f_{x_i,2}(t)x_0(0) + f_{x_i,3}(t)x_i(t) + \int_0^t g_{x_i}(t, s)dW_0(s) + \int_0^t h_{x_i}(t, s)dW_i(s),
\]

where

\[
f_{x_i,1}(t) = \int_0^t \Phi_i(t, s)f_{\xi_i,1}(s)ds,
f_{x_i,2}(t) = \int_0^t \Phi_i(t, s)f_{\xi_i,2}(s)ds,
f_{x_i,3}(t) = \Phi_i(t, 0),
g_{x_i}(t, s) = \int_s^t \Phi_i(t, s_1)g_{\xi_i}(s_1, s)ds_1,
h_{x_i}(t, s) = \Phi_i(t, s)D_i,
\]

and \( \Phi_i(\cdot, \cdot) \) is the unique solution of

\[
\begin{cases}
d\Phi_i(t, s) = M_i(t)\Phi_i(t, s)dt, \\
\Phi_i(t, s) = I, & 0 \leq s, t \leq T.
\end{cases}
\]

(ii) Let \( (M_i, W_0, W_i, \Phi_i, x_0(0)) \) be the same as in (i), and \( \zeta_i(t) = f_{\zeta_i,1}(t) + f_{\zeta_i,2}(t)x_0(0) + \int_0^t g_{\zeta_i}(t, s)dW_0(s) \), where \( f_{\zeta_i,1} \in C([0, T], \mathbb{R}^n) \), \( f_{\zeta_i,2} \in C([0, T], \mathbb{R}^{n \times n}) \), \( g_{\zeta_i} \in C(\Delta, \mathbb{R}^{n \times n}) \). Then

\[
\begin{cases}
d\nu_i(t) = (\zeta_i(t) - M_i^T(t)\nu_i(t))dt + \mu_i(t)dW_0(t) + \lambda_i(t)dW_i(t), \\
\nu_i(T) = 0,
\end{cases}
\]

has a unique solution \( (\nu_i, \mu_i, \lambda_i) \) which satisfies

\[
\nu_i(t) = f_{\nu_i,1}(t) + f_{\nu_i,2}(t)x_0(0) + \int_0^t g_{\nu_i}(t, s)dW_0(s),
\]
where
\[
f_{\nu,j}(t) = -\int_t^T \Phi_t^T(s_1,t) f_{\zeta_i,j}(s_1) ds_1, \quad j = 1, 2,
\]
\[
g_{\nu}(t,s) = -\int_t^s \Phi_t^T(s_1,t) g_{\zeta_i}(s_1) ds_1.
\]

**Proof.** The proof of this lemma is similar to that of Lemma A.1 and we omit the details. \(\Box\)

**Lemma A.3.** Let \(A(\theta)\) and \(B(\theta)\) be continuous matrix-valued functions on a compact set \(\Theta\). For each \(\theta \in \Theta\), let \(P_\theta(t)\) be the solution to the Riccati equation
\[
\begin{aligned}
&P_\theta(t) + P_\theta(t)A(t) + A^T(t)P_\theta(t) - P_\theta(t)B(t) R^{-1}B^T(t) P_\theta(t) + Q = 0, \\
&P_\theta(T) = 0.
\end{aligned}
\]
Then \(P_\theta(t)\) is continuous in \((t, \theta)\), and there exists a constant \(C\) such that
\[
\sup_{\theta \in \Theta, 0 \leq t \leq T} |P_\theta(t)| \leq C.
\]

**Proof.** We use the method of Bernoulli substitution to transform (A.9) into linear equations [24, 29]. Let \(U_\theta(t)\) and \(V_\theta(t)\) be determined by
\[
\begin{aligned}
&\begin{cases}
\dot{U}_\theta(t) = \left[ A(\theta) - B(\theta) R^{-1}B^T(\theta) P_\theta(t) \right] U_\theta(t), \\
V_\theta(t) = P_\theta(t) U_\theta(t),
\end{cases} \quad \begin{cases}
V_\theta(t) = P_\theta(t) U_\theta(t), \\
V_\theta(T) = 0.
\end{cases}
\end{aligned}
\]
It is clear that \(U_\theta^{-1}\) exists for \((t, \theta) \in [0, T] \times \Theta\), and \((U_\theta, V_\theta)\) satisfies the ODE
\[
\begin{bmatrix}
\dot{U}_\theta(t) \\
\dot{V}_\theta(t)
\end{bmatrix} = \begin{bmatrix}
A(\theta) & -B(\theta) R^{-1}B^T(\theta) \\
-Q & -A^T(\theta)
\end{bmatrix} \begin{bmatrix}
U_\theta(t) \\
V_\theta(t)
\end{bmatrix}, \quad \begin{bmatrix}
U_\theta(T) \\
V_\theta(T)
\end{bmatrix} = \begin{bmatrix}
I \\
0
\end{bmatrix}.
\]
Since \(A(\cdot)\) and \(B(\cdot)\) are continuous functions of \(\theta\), by [9, Theorem 7.4, p. 29], \(U_\theta(t)\) and \(V_\theta(t)\) are continuous in \((t, \theta) \in [0, T] \times \Theta\). Then \(U_\theta^{-1}(t)\) is continuous in \((t, \theta)\), so is \(P_\theta(t) = V_\theta(t) U_\theta^{-1}(t)\). By the compactness of \(\Theta\), we obtain the desired bound for \(|P_\theta(t)|\). \(\Box\)

**Remark A.4.** Let \(\Theta\) be a compact set, and \(M_\theta(t)\) a continuous \(\mathbb{R}^{n \times n}\)-valued function of \((t, \theta) \in [0, T] \times \Theta\). Let \(\Phi_\theta(t,s)\) be the solution of \(d\Phi_\theta(t,s) = M_\theta(t) \Phi_\theta(t,s) dt\), \(\Phi_\theta(s,s) = I\) for \(s \leq t \leq T, \theta \in \Theta\). Then \(\Phi_\theta(t,s)\) is continuous in \((t, s, \theta)\) and hence \(\sup_{\theta \in \Theta} \sup_{0 \leq t, s \leq T} |\Phi_\theta(t,s)| < \infty\).

**Appendix B. Proof of Theorem 4.3.** Let \(A_0(t)\) and \(A_{\theta_i}(t)\) be defined as in (2.9) and (2.24), respectively. By (4.3) and (4.5),
\[
\begin{aligned}
x_0(t) &= \bar{x}_0(t) + \int_0^t \Phi_0(t, s_1) F_0(x^{(N)}(s_1) - z(s_1)) ds_1, \\
x_i(t) &= z(t) + (\bar{x}_i(t) - z(t)) + \int_0^t \Phi_{\theta_i}(t, s_1) F(\theta_i)(x^{(N)}(s_1) - z(s_1)) ds_1.
\end{aligned}
\]
By adding both sides of (B.2) for $i = 1, \ldots, N$ and normalizing by $1/N$, we obtain

$$x^{(N)}(t) - z(t) = (\bar{x}^{(N)}(t) - z(t)) + \frac{1}{N} \sum_{i=1}^{N} \int_{0}^{t} \Phi_{0i}(t, s_{1})F(\theta_{i}) (x^{(N)}(s_{1}) - z(s_{1})) ds_{1}.$$ 

By (A3) and Remark A.4,

$$|x^{(N)}(t) - z(t)| \leq |\bar{x}^{(N)}(t) - z(t)| + C \int_{0}^{t} |z(s_{1}) - x^{(N)}(s_{1})| ds_{1}. $$

Hence, for $0 \leq t \leq T$,

$$E \int_{0}^{t} |x^{(N)}(s) - z(s)|^2 ds \leq 2E \int_{0}^{t} |\bar{x}^{(N)}(s) - z(s)|^2 ds + 2C^2 E \int_{0}^{t} \left( \int_{0}^{s} |x^{(N)}(s_{1}) - z(s_{1})| ds_{1} \right)^2 ds.$$ 

Denote $\alpha(t) = E \int_{0}^{t} |x^{(N)}(s) - z(s)|^2 ds$. By Lemma 4.2, we have

$$\alpha(t) \leq C \left( \epsilon_{N}^2 + \frac{1}{N} \right) + 2C^2T \int_{0}^{t} \alpha(s) ds, \quad 0 \leq t \leq T.$$ 

By Gronwall’s inequality,

$$(B.3) \quad E \int_{0}^{t} |x^{(N)}(s) - z(s)|^2 ds \leq C \left( \epsilon_{N}^2 + \frac{1}{N} \right) e^{2C^2T^2}.$$ 

It follows from (B.1) and Remark A.4 that

$$E \int_{0}^{T} |x_{0}(t) - \bar{x}_{0}(t)|^2 dt = E \int_{0}^{T} \left| \int_{0}^{t} \Phi_{0i}(t, s_{1})F_{0i} (x^{(N)}(s_{1}) - z(s_{1})) ds_{1} \right|^2 dt 
\leq CE \int_{0}^{T} t \int_{0}^{t} |x^{(N)}(s_{1}) - z(s_{1})|^2 ds_{1} dt 
= O \left( \epsilon_{N}^2 + \frac{1}{N} \right).$$

(B.4) 

Similarly, by virtue of (B.2) and Remark A.4 we have

$$\sup_{1 \leq i \leq N} E \int_{0}^{T} |x_{i}(t) - \bar{x}_{i}(t)|^2 dt = O \left( \epsilon_{N}^2 + \frac{1}{N} \right).$$

Combining (B.3)–(B.5), we obtain (4.10). \[ \square \]

**Appendix C. Proof of Theorems 4.5–4.6.** Before proving the two theorems, we give two auxiliary lemmas. Suppose that $\xi_{a}(\cdot)$ and $\xi_{b}(\cdot)$ are random processes which are adapted to the filtration $\{\mathcal{F}_{t}\}_{t \geq 0}$ and satisfy

$$(C.1) \quad \epsilon_{a} = \left( \int_{0}^{T} E[\xi_{a}(t)]^2 dt \right)^{1/2} < \infty, \quad \epsilon_{b} = \left( \int_{0}^{T} E[\xi_{b}(t)]^2 dt \right)^{1/2} < \infty.$$
C.1. A perturbed version of Problem (I) for the major player. Let \( z(t) \) be given as in (4.8). Recall that for the optimal control problem with dynamics and cost

\[
\begin{align*}
\dot{x}(t) &= (A_0x(t) + B_0u(t) + F_0z(t))dt + D_0dW(t), \\
J_0(u_0) &= \int_0^T \left( |x(t) - H_0z(t) - \eta_0|^2_{Q_0} + u_0^T(t)R_0u_0(t) \right)dt,
\end{align*}
\]

the optimal control law is \( \bar{u}_0(t) = R_0^{-1}B_0^T(-P_0(t)x_0(t) + \nu_0(t)) \), where \( P_0(t) \) and \( \nu_0(t) \) are determined by (2.8) and (2.14)–(2.17).

Subsequently, we consider a perturbed version of (C.2)–(C.3) having the dynamics and cost

\[
\begin{align*}
\dot{x}_0(t) &= (A_0x_0(t) + B_0u_0(t) + F_0z(t) + \xi_0(t))dt + D_0dW(t), \\
J_0^\xi(u_0) &= \int_0^T \left( |x_0(t) - H_0z(t) + \xi_0(t) - \eta_0|^2_{Q_0} + u_0^T(t)R_0u_0(t) \right)dt.
\end{align*}
\]

**Lemma C.1.** Let \( u_0 \) be the optimal control of the control problem (C.2)–(C.3). For any \( u_0 \) adapted to \( \{\mathcal{F}_t\}_{t \geq 0} \), we have

\[
J_0^\xi(u_0) \geq \tilde{J}_0(\bar{u}_0) - O(\epsilon_a + \epsilon_b),
\]

provided that for a fixed constant \( C_0 \), the pair \((x_0, u_0)\) in (C.4) has the prior upper bound

\[
E \int_0^T (|x_0(t)|^2 + |u_0(t)|^2)dt \leq C_0.
\]

**Proof.** We write (C.4) in the form

\[
\begin{align*}
\dot{x}_0(t) &= (A_0x_0(t) + B_0u_0(t) + B_0R_0^{-1}B_0^T \nu_0(t) + F_0z(t) + \xi_0(t))dt + D_0dW(t), \\
\tilde{J}_0(u_0) &= \int_0^T \left( |\tilde{x}_0 - H_0z - \eta_0|^2_{Q_0} + |u_0'' - R_0^{-1}B_0^T(P_0(t)\tilde{x}_0(t) - \nu_0(t))|^2 \right)dt,
\end{align*}
\]

where \( \tilde{x}_0(0) = x_0(0) \) and \( u_0'' \) is adapted to \( \{\mathcal{F}_t\}_{t \geq 0} \). It is clear that \( \tilde{J}_0(u_0'') \) attains its minimum when \( u_0'' \equiv 0 \) and the corresponding cost is equal to \( \tilde{J}_0(\bar{u}_0) \). Hence

\[
\tilde{J}_0(u_0'') \geq \tilde{J}_0(0) = \tilde{J}_0(\bar{u}_0).
\]

Take \( u_0'' = u_0' \) in (C.8) and let \( \tilde{x}_0 \) be the associated solution. Then \( \tilde{x}_0(t) = \tilde{x}_0(0) + \int_0^t \Phi_0(t,s)u_0(s)ds \), where \( \tilde{x}_0(t) \) is the optimal state process of (C.2) as determined in (4.5). This implies \( E \int_0^T |\tilde{x}_0(t)|^2dt \leq C_2 \). It follows from (C.7)–(C.8) that

\[
E \int_0^T |x_0(t) - \tilde{x}_0(t)|^2dt \leq C_2^2,
\]

where \( C \) is independent of \( \xi_0 \) and \((x_0, u_0)\). Following the proving argument in [15,
Lemma A.3] we may use an expansion of \((x_0, u_0)\) around \((\tilde{x}_0, \tilde{u}_0)\) and (C.10) to further show \(|J_0^u(u_0) - \tilde{J}_0(u_0)| \leq C(\epsilon_a + \epsilon_b)\), where \(C\) does not change with \((\xi_a, \xi_b)\). This combined with (C.9) proves the lemma.

**C.2. A perturbed version of Problem (II) for the minor player.** Let \(z(t)\) be given as in (4.8) and \(\tilde{x}_0(t)\) given as in (2.12) and (2.19)–(2.21). Now we consider a perturbed version of Problem (II). Let the dynamics and cost be given by

\[
\begin{align*}
\text{(C.11)} & \quad dx_i(t) = \left(A(\theta_i)x_i(t) + B(\theta_i)u_i(t) + F(\theta_i)z(t) + \xi_i(t)\right)dt + D(\theta_i)dW_i(t), \quad t \geq 0, \\
\text{(C.12)} & \quad J_i^\epsilon(u_i) = E\int_0^T \left(|x_i(t) - H\tilde{x}_0(t) - \tilde{H}z(t) + \xi_i(t) - \eta_i|_Q^2 + u_i^T(t)Ru_i(t)\right)dt.
\end{align*}
\]

The proof of the next lemma is similar to that of Lemma C.1 and hence is omitted.

**Lemma C.2.** Let \(\tilde{u}_i\) be the optimal control of Problem (II) with \(z\) given by (4.8) and \(\tilde{x}_0\) given by (2.12) and (2.19)–(2.21). For any \(u_i\) adapted to \(\{F_t\}_{t \geq 0}\), we have \(J_i^\epsilon(u_i) \geq J_i(\tilde{u}_i) - O(\epsilon_a + \epsilon_b)\), provided that for a fixed constant \(C_0\), the pair \((x_i, u_i)\) in (C.11) satisfies \(E\int_0^T(|x_i(t)|^2 + |u_i(t)|^2)dt \leq C_0\).

**C.3. Proof of Theorem 4.5.** Let \(x_j\), \(0 \leq j \leq N\), be determined by (4.3)–(4.4), which is associated with the controls \(\tilde{u}_j\), \(0 \leq j \leq N\), given by (4.1)–(4.2). We first show \(|J_0(\tilde{u}_0, \tilde{u}_{-0}) - \tilde{J}_0(\tilde{u}_0)| = O(\epsilon_N + 1/\sqrt{N})\). We have

\[
\begin{align*}
\text{(C.13)} & \quad |J_0(\tilde{u}_0, \tilde{u}_{-0}) - \tilde{J}_0(\tilde{u}_0)| \\
& \quad \leq E\int_0^T \left(|x_0 - H_0x^{(N)} - \eta_0|_Q^2 - |\tilde{x}_0 - H_0\tilde{z} - \eta_0|_Q^2\right)dt \\
& \quad + E\int_0^T \left(|\tilde{u}_0|^2_{R_0} - |\tilde{u}_0|^2_{R_0}\right)dt \\
& \quad \leq E\int_0^T |Q_0|\left\{\left(|x_0 - \tilde{x}_0| - H_0(x^{(N)} - z)\right)^2 + 2|\tilde{x}_0 - H_0\tilde{z} - \eta_0||x_0 - \tilde{x}_0| - H_0(x^{(N)} - z)\right\}dt \\
& \quad + E\int_0^T |R_0|\left\{\left|\tilde{u}_0 - \tilde{u}_0\right|^2 + 2\tilde{u}_0|\tilde{u}_0 - \tilde{u}_0\right\}dt.
\end{align*}
\]

It follows from Theorem 4.3 that

\[
\text{(C.14)} \quad E\int_0^T |(x_0 - \tilde{x}_0) - H_0(x^{(N)} - z)|^2 dt = O\left(\epsilon_N^2 + \frac{1}{N}\right).
\]

Next, by Schwarz’s inequality,

\[
\begin{align*}
\text{(C.15)} & \quad E\int_0^T |\tilde{x}_0 - H_0\tilde{z} - \eta_0|(x_0 - \tilde{x}_0) + H_0(x^{(N)} - z)|dt \\
& \quad \leq \left(\int_0^T E|x_0 - \tilde{x}_0|^2 dt\right)^{1/2} \left(\int_0^T E\left|(x_0 - \tilde{x}_0) - H_0(x^{(N)} - z)\right|^2 dt\right)^{1/2} \\
& \quad = O\left(\epsilon_N + \frac{1}{\sqrt{N}}\right).
\end{align*}
\]
We use (2.11), (4.1), and (4.10) to obtain
\[ E \int_0^T |\dot{u}_0 - \bar{u}_0|^2 dt = E \int_0^T |R_0^{-1} B_0^T P_0(t)(x_0 - \bar{x}_0)|^2 dt = O\left(\epsilon_N^2 + \frac{1}{N}\right), \]
and subsequently,
\[ E \int_0^T |R_0| \left\{ |\dot{u}_0 - \bar{u}_0|^2 + 2|\bar{u}_0||\dot{u}_0 - \bar{u}_0| \right\} dt = O\left(\epsilon_N + \frac{1}{\sqrt{N}}\right). \]
By (C.13)–(C.16), it follows that \( |J_0(\bar{u}_0, \bar{u}_--0) - \bar{J}_0(\bar{u}_0)| = O\left(\epsilon_N + 1/\sqrt{N}\right). \)

Similarly, we can prove that \( |J_i(\hat{u}_i, \bar{u}_{-i}) - J_i(\bar{u}_i)| = O\left(\epsilon_N + 1/\sqrt{N}\right) \) for \( i = 1, \ldots, N. \)

\[ \square \]

**C.4. Proof of Theorem 4.6.** It suffices to show the first inequality, and the second is evident.

**Step 1. The case for the major player \( A_0 \) to use an alternative strategy \( u_0 \).** Each minor player takes the control law \( \hat{u}_i \) given in (4.2). Denote \( \xi_a(t) = F_0(x^{(N)}(t) - z(t)) \) and \( \xi_b(t) = H_0(z(t) - x^{(N)}(t)) \). The dynamics and cost of \( A_0 \) may be written as
\[ dx_0(t) = (A_0 x_0(t) + B_0 u_0(t) + F_0 z(t) + \xi_a(t)) dt + D_0 dW_0(t) \]
and
\[ J_0(u_0, \bar{u}_-) = E \int_0^T \left\{ |x_0(t) - H_0 z(t) + \xi_b(t) - \eta_0|^2 + u_0^T(t) R_0 u_0(t) \right\} dt. \]

We follow the notation in (C.1) to define \( \epsilon_a \) and \( \epsilon_b \). Note that when the set of strategies \( (u_0, \hat{u}_1, \ldots, \hat{u}_N) \) is applied, \( z(t) - x^{(N)}(t) \) is the same as in Theorem 4.3 since \( (x_0, u_0) \) does not appear in the dynamics of \( x_i, i \geq 1 \). Hence,
\[ \epsilon_a + \epsilon_b = \left( \int_0^T E|\xi_a(t)|^2 dt \right)^{1/2} + \left( \int_0^T E|\xi_b(t)|^2 dt \right)^{1/2} = O\left(\epsilon_N + \frac{1}{\sqrt{N}}\right). \]

If \( u_0 \) is simply taken as \( \bar{u}_0 \), we may apply Theorem 4.5 to show that \( J_0(\bar{u}_0, \bar{u}_-) \) is upper bounded by a constant \( C_0 \) independent of \( N \). So, it suffices to restrict our attention to all \( u_0 \) such that \( J_0(u_0, \bar{u}_-) \leq C_0 \), which further ensures that \( E \int_0^T (|x_0(t)|^2 + |u_0(t)|^2) dt \leq C_1 \) for some constant \( C_1 \) independent of \( N \). In view of Lemma C.1, \( J_0(u_0, \bar{u}_-) \geq J_0(\bar{u}_0) - O(\epsilon_a + \epsilon_b) \). Therefore, by (C.17) and Theorem 4.5,
\[ J_0(u_0, \bar{u}_-) \geq \bar{J}_0(\bar{u}_0) - O\left(\epsilon_N + \frac{1}{\sqrt{N}}\right) \geq J_0(\bar{u}_0, \bar{u}_-) - O\left(\epsilon_N + \frac{1}{\sqrt{N}}\right). \]
This proves the lower bound in (4.12).

**Step 2. The case for any given minor player \( A_i \), to use an alternative strategy \( u_i \).** After all players, except \( A_i \), apply the control laws given in (4.1)–(4.2), the dynamics of \( A_0 \) and \( A_j, j \neq i \), may be written in the form
\[ dx_0(t) = \left( \hat{\kappa}_0(t)x_0(t) + B_0 R_0^{-1} B_0^T v_0(t) + F_0 z(t) \right) dt + D_0 dW_0(t) \]
\[ dx_j(t) = \left( \hat{\kappa}_j(t)x_j(t) + B(\theta_j) R^{-1} B^T(\theta_j) v_0(t) + F(\theta_j) z(t) \right) dt + D(\theta_j) dW_j(t) \]
\[ 1 \leq j \leq N, j \neq i. \]
We write the dynamics of $\mathcal{A}_t$ as follows:
\[
d x_i(t) = \left( \dot{h}_i(t)x_i(t) + B(\theta_i)R^{-1}B^T(\theta_i)\nu_i(t) + F(\theta_i)z(t) \right) dt + D(\theta_i)dW_i(t)
\]
(C.20)
\[
+ F(\theta_i)(x_1(N))(t) - z(t)) dt + B(\theta_i)(u_i(t) - \hat{u}_i(t)) dt,
\]
where $\hat{u}_i(t) = R^{-1}B^T(\theta_i)[-P_0(\theta_i)x_1(t) + \nu_0(t)]$ depends on $x_i$ and hence is affected by $u_i$. Since $J_i(\hat{u}_i, \hat{u}_{-i}) \leq C_0$ for some fixed $C_0$ independent of $N$, it suffices to consider $u_i$ such that $J_i(u_i, \hat{u}_{-i}) \leq C_0$. This implies that $E \int_0^T |u_i(t)|^2 dt \leq C_1$ for some fixed $C_1$. So by (4.2) there exists $C_2$ such that for all $t \in [0, T]$,
\[
E \int_0^t |u_i(s) - \hat{u}_i(s)|^2 ds \leq C_2 + C_2 E \int_0^t |x_i(s)|^2 ds.
\]
(C.21)
In parallel with (B.1),
\[
x_0(t) = \bar{x}_0(t) + \int_0^t \Phi_0(t, s_1)F_0(x_1(N)(s_1) - z(s_1))ds_1,
\]
(C.22)
\[
x_j(t) = \bar{x}_j(t) + \int_0^t \Phi_j(t, s_1)F(\theta_j)(x_1(N)(s_1) - z(s_1))ds_1, \quad 1 \leq j \leq N, j \neq i,
\]
(C.23)
\[
x_i(t) = \bar{x}_i(t) + \int_0^t \Phi_i(t, s_1)F(\theta_i)(x_1(N)(s_1) - z(s_1))ds
\]
\[
+ \int_0^t \Phi_i(t, s_1)B(\theta_i)(u_i(s_1) - \hat{u}_i(s_1))ds_1.
\]
(C.24)
Adding up the $N$ equations in (C.23)–(C.24), we have
\[
x^{(N)}(t) - z(t) = \bar{x}^{(N)} - z(t) + \frac{1}{N} \sum_{j=1}^N \int_0^t \Phi_j(t, s_1)F(\theta_j)(x^{(N)}(s_1) - z(s_1))ds_1
\]
\[
+ \frac{1}{N} \int_0^t \Phi_i(t, s_1)B(\theta_i)(u_i(s_1) - \hat{u}_i(s_1))ds_1.
\]
Therefore, for $0 \leq t \leq T$,
\[
|x^{(N)}(t) - z(t)| \leq |\bar{x}^{(N)} - z(t)| + C \int_0^t |x^{(N)}(s) - z(s)|ds + \frac{C}{N} \int_0^t |u_i(s) - \hat{u}_i(s)|ds.
\]
By Schwarz’s inequality,
\[
E \int_0^t |x^{(N)}(s) - z(s)|^2 ds
\]
\[
\leq 3E \int_0^t |\bar{x}^{(N)}(s) - z(s)|^2 ds + 3CE \int_0^t \left( \int_0^s |x^{(N)}(s_1) - z(s_1)|ds_1 \right)^2 ds
\]
\[
+ \frac{3C}{N} E \int_0^t \left( \int_0^s |u_i(s_1) - \hat{u}_i(s_1)|ds_1 \right)^2 ds
\]
(C.25)
\[
\leq C \left( \epsilon_N^2 + \frac{1}{N} + \frac{1}{N} \int_0^t E|x_i(s)|^2 ds \right) + CE \int_0^t \int_0^s |x^{(N)}(s_1) - z(s_1)|^2 ds_1 ds.
\]
By Gronwall’s inequality,
\[
E \int_0^t |x^{(N)}(s) - z(s)|^2 ds \leq C \left( \epsilon_N^2 + \frac{1}{N} + \frac{1}{N} \int_0^t E|x_i(s)|^2 ds \right).
\]
(C.26)
Combining (C.20), (C.21), and (C.26), we may use Gronwall’s inequality to show
\[
\sup_{0 \leq t \leq T} E|x_i(t)|^2 \leq C_3 \text{ for some constant } C_3.
\]
Therefore,
\[
E \int_0^T |x^{(N)}(s) - z(s)|^2 ds \leq C \left( \epsilon_N^2 + \frac{1}{N} \right).
\]

It follows from (C.22) and (C.27) that
\[
E \int_0^T |x_0(s) - \bar{x}_0(s)|^2 ds \leq C \left( \epsilon_N^2 + \frac{1}{N} \right).
\]

Next, denote \( \xi_1(t) = F(\theta_i)(x^{(N)}(t) - z(t)) \) and \( \xi_2(t) = H(\bar{x}_0(t) - x_0(t)) + \bar{H}(z(t) - x^{(N)}(t)) \). Then the dynamics and cost of player \( \mathcal{A}_i \) may be written as
\[
dx_i(t) = \left( A(\theta_i)x_i(t) + B(\theta_i)u_i(t) + F(\theta_i)z(t) + \xi_1(t) \right) dt + D(\theta_i)dW_i(t), \quad t \geq 0,
J_i(u_i) = E \int_0^T \left( |x_i(t) - H\bar{x}_0(t) - \bar{H}z(t) + \xi_2(t)|^2 + u_i^T(t)Ru_i(t) \right) dt.
\]

It is clear that \( J_i(u_i) = J_i^2(\bar{u}_i) \) which is defined in (C.12). In view of (C.27)–(C.28),
\[
\epsilon_a + \epsilon_b = \left\{ \int_0^T E|\xi_a(t)|^2 dt \right\}^{1/2} + \left\{ \int_0^T E|\xi_b(t)|^2 dt \right\}^{1/2} = O \left( \epsilon_N + \frac{1}{\sqrt{N}} \right).
\]

Since \( E \int_0^T (|x_i(t)|^2 + |u_i(t)|^2) dt \leq C \) for some constant \( C \) independent of \( N \), by Lemma C.2, \( J_i(u_i, \bar{u}_{-i}) \geq J_i(\bar{u}_i) - O(\epsilon_a + \epsilon_b) \). By (C.29) and Theorem 4.5,
\[
J_i(u_i, \bar{u}_{-i}) \geq J_i(\bar{u}_i) - O \left( \epsilon_N + \frac{1}{\sqrt{N}} \right) \geq J_i(u_i, \bar{u}_{-i}) - O \left( \epsilon_N + \frac{1}{\sqrt{N}} \right).
\]

This proves the lower bound in (4.12) for player \( \mathcal{A}_i \).

REFERENCES

[27] S. L. Nguyen and M. Huang, Mean field LQG games with mass behavior responsive to a major player, to be presented at the 51st IEEE Conference on Decision and Control, Maui, HI, 2012.