SIAM J. Control Optim.
Vol. 50, No. 5, pp. 2907–2937

LINEAR-QUADRATIC-GAUSSIAN MIXED GAMES WITH CONTINUUM-PARAMETRIZED MINOR PLAYERS*

SON LUU NGUYEN† AND MINYI HUANG†

Abstract. We consider a mean field linear-quadratic-Gaussian game with a major player and a large number of minor players parametrized by a continuum set. The mean field generated by the minor players is approximated by a random process depending only on the initial state and the Brownian motion of the major player, and this leads to two limiting optimal control problems with random coefficients, which are solved subject to a consistency requirement on the mean field approximation. The set of decentralized strategies constructed from the limiting control problems has an ε-Nash equilibrium property when applied to the large but finite population model.

Key words. mean field models, mixed players, differential games, decentralized control, Nash equilibria

AMS subject classifications. 91A10, 91A23, 91A25, 93E20

DOI. 10.1137/110841217

1. Introduction.

Stochastic dynamic games with mean field coupling have experienced intense investigation in recent years; see, e.g., [2, 11, 16, 18, 20, 21, 28, 33, 34]. The players in these models are individually insignificant but they collectively have a significant impact on a particular player. Since all players are comparably small, they may be called peers. To design low-complexity strategies, consistent mean field approximations provide a powerful approach. The fundamental idea is that in the population limit each agent optimally responds to a certain mean field which in turn is replicated by the closed-loop behaviors of the agents. Based on this procedure, the mean field may be determined by a fixed point analysis [16]. In the resulting solution, each agent only needs to know its own state information and the mass effect in the population limit which may be computed offline. This methodology has been applied to decentralized control of coupled nonlinear oscillators subject to random disturbances [34], human crowd motion [11], and decentralized charging control for large populations of plug-in electric vehicles [23]. Risk-sensitive costs and robustness issues were considered in [32]. Numerical solutions for coupled Hamilton–Jacobi–Bellman and Fokker–Planck equations in mean field games have been developed in [1]. The survey [6] on differential games presents recent progress in mean field game theory. The technique of consistent mean field approximations is also applicable to mean field social optimization [17], where each agent does not simply minimize its own cost but solves a modified optimal control problem by taking into account its social impact across the population. The mean field approach has also appeared in anonymous sequential games [19] with a continuum of players individually optimally responding to the mean field. However, the modeling of a continuum of independent processes

*Received by the editors July 18, 2011; accepted for publication (in revised form) June 19, 2012; published electronically September 25, 2012. This work was supported by a Fields Institute postdoctoral fellowship and a discovery grant from the Natural Sciences and Engineering Research Council (NSERC) of Canada. This paper was presented in part at Optimization Days, Montreal, QC, Canada, May 2011, and the 50th IEEE Conference on Decision and Control, Orlando, FL, December 2011.
http://www.siam.org/journals/sicon/50-5/84121.html
†School of Mathematics and Statistics, Carleton University, Ottawa, ON K1S 5B6, Canada (snguyen@math.carleton.ca, mhuang@math.carleton.ca).

leads to measurability difficulties and the empirical frequency of the realizations of the continuum-indexed individual states cannot be meaningfully defined [3].

1.1. Mean field games with major and minor players. This paper considers the linear-quadratic-Gaussian (LQG) game with a major player $\mathcal{A}_0$ and a population of minor players $\{\mathcal{A}_i, 1 \leq i \leq N\}$. At time $t \geq 0$, the states of the players $\mathcal{A}_0$ and $\mathcal{A}_i$ are, respectively, denoted by $x_0(t)$ and $x_i(t)$, $1 \leq i \leq N$. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ be the underlying filtered probability space with the filtration $\{\mathcal{F}_t\}_{t\geq 0}$. The dynamics of the $N+1$ players are given by a system of linear stochastic differential equations (SDEs)

$$dx_0(t) = \big[A_0 x_0(t) + B_0 u_0(t) + F_0 x^{(N)}(t)\big]dt + D_0 dW_0(t), \tag{1.1}$$

$$dx_i(t) = \big[A(\theta_i)x_i(t) + B(\theta_i)u_i(t) + F(\theta_i)x^{(N)}(t)\big]dt + D(\theta_i)dW_i(t), \tag{1.2}$$

where $x^{(N)} = (1/N)\sum_{i=1}^N x_i$ is the mean field term. The initial states $\{x_j(0), 0 \leq j \leq N\}$ are measurable with respect to $\mathcal{F}_0$ and have finite second moments. The states x_0, x_i and controls u_0, u_i are, respectively, n and n_1 dimensional vectors. The noise processes W_0 and W_i are n_2 dimensional independent standard Brownian motions adapted to $\{\mathcal{F}_t\}_{t\geq 0}$, which are also independent of $\{x_j(0), 0 \leq j \leq N\}$. For simplicity, we may take $\mathcal{F}_t = \sigma\{x_j(0), W_j(s), 0 \leq j \leq N, s \leq t\}$, which is the σ-algebra generated by the initial states and the Brownian motions up to time t. The vector $\theta_i \in \mathbb{R}^d$ is a parameter in the dynamics of player $\mathcal{A}_i$.

For $0 \leq j \leq N$, denote $u_{-j} = (u_0, \ldots, u_{j-1}, u_{j+1}, \ldots, u_N)$. For positive semi-definite matrix $M \geq 0$, we may write the quadratic form $z^T M z$ as $|z|_M^2$. The cost function for $\mathcal{A}_0$ is given by

$$J_0(u_0, u_{-0}) = E\int_0^T \big[\big|x_0(t) - \chi_0(x^{(N)}(t))\big|_{Q_0}^2 + u_0^T(t)R_0u_0(t)\big]dt, \tag{1.3}$$

where $\chi_0(x^{(N)}(t)) = H_0x^{(N)}(t) + \eta_0$, $Q_0 \geq 0$, and $R_0 > 0$. The cost function for $\mathcal{A}_i$, $1 \leq i \leq N$, is

$$J_i(u_i, u_{-i}) = E\int_0^T \big[\big|x_i(t) - \chi(x_0(t), x^{(N)}(t))\big|_Q^2 + u_i^T(t)Ru_i(t)\big]dt, \tag{1.4}$$

where $\chi(x_0(t), x^{(N)}(t)) = Hx_0(t) + \hat{H}x^{(N)}(t) + \eta$, $Q \geq 0$, and $R > 0$. The component Hx_0 in the coupling term χ indicates the strong influence of the major player on each minor player. By contrast, the mutual impact of two minor players is insignificant. We assume that all matrix or vector parameters $(A_0, B_0, \ldots, \eta, \ldots)$ in (1.1)–(1.4) are deterministic and have compatible dimensions.

Similar interaction patterns are often seen in economic settings, a simple example being a few large corporations and many much smaller competitors. They have been well studied in cooperative game theory, usually based on static models [12, 13, 14], and these games are called mixed games due to the vastly different influences of the players.

The above type of mean field LQG game modeling was initially introduced in [15], where the presence of the major player causes an interesting phenomenon called the lack of sufficient statistics. More specifically, in order to obtain asymptotic equilibrium strategies, $\mathcal{A}_0$ cannot simply use a strategy as a function of $(t, x_0(t))$ and $\mathcal{A}_i$ cannot only use $(t, x_0(t), x_i(t))$, $i \geq 1$; it is necessary to augment the system dynamics with a new state to Markovianize the local limiting decision problems.

The work [15] analyzed the case where the minor players are from a total of K classes, i.e., $\theta_i \in \{1, \dots, K\}$. By aggregating all states in the same class, a Kn dimensional process $\bar{z}(t) = [\bar{z}_1^T, \dots, \bar{z}_K^T]^T$ is introduced to approximate the mean field effects produced by the K classes of players, and the evolution of $\bar{z}$ is specified by a linear stochastic ordinary differential equation driven by the state of the major player. Subsequently, the fundamental consistent mean field approximation approach [16] is applied to determine the dynamics of $\bar{z}$. This procedure gives a set of decentralized Markov strategies in an augmented state space.

This paper considers a population of minor players parametrized by an infinite set such as a continuum set. The procedure in [15] cannot be applied to obtain a finite dimensional Markov model as seen by a given player in the population limit since the method of specifying the sub-mean field of each type of minor players will generate an infinite dimensional state space. On the other hand, from the point of view of decentralized decision making of any given player, it is even unnecessary to specify sub-mean fields in such fine scales. By contrast, in a mean field game of peers, it is adequate to approximate the aggregate effect of all agents by a single deterministic process and a continuum parameter set does not cause any difficulty in the aggregation procedure [16].

To tackle an infinite number of types of minor players, we adopt a different approach by directly treating the mean field z in the population limit as a random process. A crucial issue is to find an appropriate representation of z so that the consistent mean field approximation translates into a tractable problem. Intuitively, when N tends to infinity, the independent Brownian motions and initial states of the minor players will altogether contribute to some deterministic effect in the formation of the mean field process, and only $x_0(0)$ and W_0 cause the random fluctuation of the mean field. Furthermore, due to the linear quadratic structure of the game, it is plausible to restrict z to be linearly dependent on $x_0(0)$ and W_0. The feasibility of considering z from such a class will become clear later on when the consistency analysis is carried out in the paper. Given the approximation of $x^{(N)}$, we approximate the original problems of the major player and a representative minor player, respectively, by stochastic control problems with random coefficients in the dynamics and costs [5, 31, 36]. This further enables the explicit computation of the optimal strategies by using the powerful machinery of backward stochastic differential equations [5, 22] and Riccati equations. The approach of solving stochastic control problems with random coefficients has been used in a leader-follower hierarchical game with open-loop information [35]. Also, we exploit the linear nature of the system dynamics to reduce the analysis to function spaces when the consistent mean field is identified. Due to the treatment of random coefficients, the strategies obtained are not Markov strategies, which is different from [15]. We note that non-Markov control has arisen frequently in linear quadratic stochastic control problems with random coefficients [5, 8, 31, 36]. For applications of backward stochastic differential equations in mean field optimal control, the reader is referred to [4]. In mathematical physics, random mean field limits have been studied in interacting particle systems with correlated individual driving Brownian motions [10].

1.2. Assumptions, notation and organization. We introduce the following assumptions:

(A1) The initial states $\{x_j(0), 0 \le j \le N\}$ are independent, and there exists a constant C independent of N such that $\sup_{0 \le j \le N} E|x_j(0)|^2 \le C$.

(A2) There exists a distribution function $\mathbf{F}(\theta, x)$ on $\mathbb{R}^{d+n}$ such that the sequence of empirical distribution functions $\mathbf{F}_N(\theta, x) = \frac{1}{N}\sum_{i=1}^N 1_{\{\theta_i \le \theta, Ex_i(0) \le x\}}$, $N \ge$

1, where each inequality holds componentwise, converges to $\mathbf{F}(\theta, x)$ weakly, i.e., for any bounded and continuous function $h(\theta, x)$ on $\mathbb{R}^{d+n}$,

$$\lim_{N\to\infty}\int_{\mathbb{R}^{d+n}} h(\theta,x)d\mathbf{F}_N(\theta,x) = \int_{\mathbb{R}^{d+n}} h(\theta,x)d\mathbf{F}(\theta,x).$$

(A3) $A(\cdot), B(\cdot), F(\cdot)$, and $D(\cdot)$ are continuous matrix-valued functions of $\theta \in \Theta$, where Θ is a compact subset of $\mathbb{R}^d$.

Remark 1.1. It follows from (A1) that $Ex_i(0)$ is from a fixed compact set independent of N.

Note that $\{\theta_i, i \geq 1\}$ in (A2) is treated as a deterministic sequence which jointly with $\{Ex_i(0), i \geq 1\}$ exhibits orderly statistical behavior. Let $\mathbf{F}_N(\theta)$ and $\mathbf{F}(\theta)$ denote the marginal distribution functions of $\mathbf{F}_N(\theta, x)$ and $\mathbf{F}(\theta, x)$, respectively, with respect to θ. By (A2), it is clear that $\{\mathbf{F}_N(\theta), N \geq 1\}$ converges to $\mathbf{F}(\theta)$ weakly.

For the reader's convenience, we note some conventions on notation. We use C, C_0, etc., to denote a generic constant which may change from place to place but does not depend on the population size N. For a vector or matrix V, $|V|$ denotes the Frobenius norm; Δ is the region $\{(t,s)|0 \leq s \leq t \leq T\}$; $[Z_1; \dots; Z_l]$ denotes the vertical concatenation of l column vectors; and $L^2_{\mathcal{F}}(0, T; \mathbb{R}^{k_1\times k_2})$ denotes all $\mathbb{R}^{k_1\times k_2}$-valued processes $\{X_t, 0 \leq t \leq T\}$ adapted to $\{\mathcal{F}_t\}_{t\geq 0}$ and satisfying $E\int_0^T |X_t|^2 dt < \infty$. The product of two spaces $L^2_{\mathcal{F}}(0, T; S_1)$ and $L^2_{\mathcal{F}}(0, T; S_2)$ may be written as $L^2_{\mathcal{F}}(0, T; S_1 \times S_2)$, and this notation may be extended to more than two spaces.

The organization of the paper is as follows. Section 1 presents the introduction and formulates the mean field game problem. Section 2 solves two auxiliary stochastic control problems in the mean field limit. The consistency condition for mean field approximations is introduced in section 3, and section 4 shows an asymptotic Nash equilibrium property. Section 5 analyzes a scalar model with numerical solutions. Section 6 concludes the paper.

2. The limiting control problems with random coefficients.

We formulate the auxiliary control problems within the population limit via the approximation of $x^{(N)}$ by a process z depending linearly on the initial state and the driving noise of the major player. It will be convenient to still refer to them as the control problems of the major and minor players, respectively, although $x^{(N)}$ in the original dynamics is now replaced by z.

2.1. Two auxiliary optimal control problems.

Problem (I): Optimal control of the major player. The dynamics are given by

$$\begin{cases} z(t) = f_1(t) + f_2(t)x_0(0) + \int_0^t g(t,s)dW_0(s), \\ dx_0(t) = [A_0x_0(t) + B_0u_0(t) + F_0z(t)]dt + D_0dW_0(t), \end{cases} \tag{2.1}$$

where the second equation is obtained by replacing $x^{(N)}(t)$ in (1.1) by $z(t)$. For the proposed mean field approximation, we consider $f_1 \in C([0,T], \mathbb{R}^n)$, $f_2 \in C([0,T], \mathbb{R}^{n\times n})$ and $g \in C(\Delta, \mathbb{R}^{n\times n_2})$, where $\Delta = \{(t,s)|0 \leq s \leq t \leq T\}$. The cost function is given by

$$\bar{J}_0(u_0) = E\int_0^T \left[\left|x_0(t) - H_0z(t) - \eta_0\right|^2_{Q_0} + u_0^T(t)R_0u_0(t)\right]dt. \tag{2.2}$$

Problem (II): Optimal control of the minor player. After solving Problem (I), we represent the state x_0 of the major player by its initial condition and its

Brownian motion and further denote the state process by $\bar{x}_0$. By combining z, $\bar{x}_0$ with the limiting dynamics of the minor player, we introduce the equation system

$$(2.3)\qquad \begin{cases} z(t) = f_1(t) + f_2(t)x_0(0) + \int_0^t g(t,s)dW_0(s), \\ \bar{x}_0(t) = f_{\bar{x}_0,1}(t) + f_{\bar{x}_0,2}(t)x_0(0) + \int_0^t g_{\bar{x}_0}(t,s)dW_0(s), \\ dx_i(t) = \big[A(\theta_i)x_i(t) + B(\theta_i)u_i(t) + F(\theta_i)z(t)\big]dt + D(\theta_i)dW_i(t), \end{cases}$$

where $(f_{\bar{x}_0,1}, f_{\bar{x}_0,2}, g_{\bar{x}_0})$ is determined from the solution of Problem (I). The cost function is given by

$$(2.4)\qquad \bar{J}_i(u_i) = E\int_0^T \big[\big|x_i(t) - H\bar{x}_0(t) - \hat{H}z(t) - \eta\big|_Q^2 + u_i^T(t)Ru_i(t)\big]dt.$$

2.2. The analysis of Problem (I). By viewing Problem (I) as an optimal control problem with a random coefficient process z, the optimal control may be determined by the standard backward SDE approach [5]. We have the following lemma.

LEMMA 2.1. (i) *There exists a unique optimal control* $\bar{u}_0 \in L^2_{\mathcal{F}}(0,T;\mathbb{R}^{n_1})$ *to Problem* (I).

(ii) *The pair* $(\bar{x}_0, \bar{u}_0) \in L^2_{\mathcal{F}}(0,T;\mathbb{R}^{n+n_1})$ *is the optimal solution to Problem* (I) *if and only if* $\bar{u}_0(t) = R_0^{-1}B_0^Tp_0(t)$, *where* $(\bar{x}_0, p_0, q_0) \in L^2_{\mathcal{F}}(0,T;\mathbb{R}^{2n}\times\mathbb{R}^{n\times n_2})$ *is the solution of the forward-backward SDE*

$$(2.5)\qquad \begin{cases} d\bar{x}_0(t) = \big[A_0\bar{x}_0(t) + B_0R_0^{-1}B_0^Tp_0(t) + F_0z(t)\big]dt + D_0dW_0(t), \\ dp_0(t) = \big[Q_0\big(\bar{x}_0(t) - H_0z(t) - \eta_0\big) - A_0^Tp_0(t)\big]dt + q_0(t)dW_0(t), \\ \bar{x}_0(0) = x_0(0), \quad p_0(T) = 0. \end{cases}$$

(iii) *Equation* (2.5) *has a unique solution* $(\bar{x}_0, p_0, q_0) \in L^2_{\mathcal{F}}(0,T;\mathbb{R}^{2n}\times\mathbb{R}^{n\times n_2})$.

Proof. Lemma 2.1(i) follows from [5, Theorem 3.1].

(ii) Let $(\bar{u}_0, \bar{x}_0)$ be the optimal pair to Problem (I) and (p_0, q_0) the solution of the above backward SDE. Let $\delta\bar{u}_0$ and $\delta\bar{x}_0$ be the variations of $\bar{u}_0$ and $\bar{x}_0$, respectively, where $\delta\bar{u}_0 \in L^2_{\mathcal{F}}(0,T;\mathbb{R}^{n_1})$. Since $\bar{J}_0(\cdot)$ is convex in (x_0, u_0) and $R_0 > 0$, $\bar{u}_0$ is a solution to Problem (I) if and only if the first order variation $\delta\bar{J}_0$ of $\bar{J}_0$ satisfies

$$(2.6)\qquad \frac{\delta\bar{J}_0(\bar{u}_0)}{2} = E\int_0^T \Big\{\delta\bar{x}_0^T(t)Q_0\big(\bar{x}_0(t) - H_0z(t) - \eta_0\big) + \delta\bar{u}_0^T(t)R_0\bar{u}_0(t)\Big\}dt = 0.$$

Note that $d\delta\bar{x}_0(t) = (A_0\delta\bar{x}_0 + B_0\delta\bar{u}_0)dt$. By virtue of Ito's formula,

$$\begin{aligned} d[\delta\bar{x}_0^T(t)p_0(t)] &= \big(A_0\delta\bar{x}_0(t) + B_0\delta\bar{u}_0(t)\big)^Tp_0(t)dt \\ &\quad + \delta\bar{x}_0^T(t)\Big\{\Big[Q_0\big(\bar{x}_0(t) - H_0z(t) - \eta_0\big) - A_0^Tp_0(t)\Big]dt + q_0(t)dW_0(t)\Big\} \\ &= \Big\{\delta\bar{x}_0^T(t)Q_0\big(\bar{x}_0(t) - H_0z(t) - \eta_0\big) + \delta\bar{u}_0^T(t)B_0^Tp_0(t)\Big\}dt \\ &\quad + \delta\bar{x}_0^T(t)q_0(t)dW_0(t). \end{aligned}$$

Combining this identity with $p(T) = 0$ and $\delta\bar{x}_0(0) = 0$ yields

$$\begin{aligned} 0 &= E\delta\bar{x}_0^T(T)p_0(T) - E\delta\bar{x}_0^T(0)p_0(0) \\ (2.7)\qquad &= E\int_0^T \Big\{\delta\bar{x}_0^T(t)Q_0\big[\bar{x}_0(t) - H_0z(t) - \eta_0\big] + \delta\bar{u}_0^T(t)B_0^Tp_0(t)\Big\}dt. \end{aligned}$$

It follows from (2.6)–(2.7) that for any $\delta\bar{u}_0(t) \in L^2_{\mathcal{F}}(0,T;\mathbb{R}^{n_1})$,

$$E\int_0^T \big(\delta\bar{u}_0^T(t)B_0^T p_0(t) - \delta\bar{u}_0^T(t)R_0\bar{u}_0(t)\big)dt = 0.$$

This implies that $\bar{u}_0$ is optimal if and only if $\bar{u}_0(t) = R_0^{-1}B_0^T p_0(t)$ as desired.

(iii) Let $P_0(t) \geq 0$ be the unique solution of the Riccati equation (see [30])

$$\begin{cases} \dot{P}_0(t) + P_0(t)A_0 + A_0^T P_0(t) - P_0(t)B_0R_0^{-1}B_0^T P_0(t) + Q_0 = 0, \\ P_0(T) = 0. \end{cases} \tag{2.8}$$

To analyze the unique solvability of (2.5), write $p_0(t) = -P_0(t)\bar{x}_0(t) + \nu_0(t)$, where $\nu_0(t)$ satisfies the terminal condition $\nu_0(T) = 0$. Denote

$$\mathbb{A}_0(t) = A_0 - B_0R_0^{-1}B_0^T P_0(t). \tag{2.9}$$

By Ito's formula, it can be shown that (2.5) is equivalent to the forward-backward SDE

$$\begin{cases} d\bar{x}_0(t) = \Big[\mathbb{A}_0(t)\bar{x}_0(t) + B_0R_0^{-1}B_0^T\nu_0(t) + F_0z(t)\Big]dt + D_0dW_0(t), \\ d\nu_0(t) = \Big\{ - \mathbb{A}_0^T(t)\nu_0(t) + \big[\big(P_0(t)F_0 - Q_0H_0\big)z(t) - Q_0\eta_0\big]\Big\}dt \\ \qquad\qquad + \big[q_0(t) + P_0(t)D_0\big]dW_0(t), \\ \bar{x}_0(0) = x_0(0), \quad \nu_0(T) = 0, \end{cases} \tag{2.10}$$

where ν_0 is now decoupled from $\bar{x}_0$. By Lemma A.1(ii), we can solve the second equation in (2.10) with a unique solution $\nu_0 \in L^2_{\mathcal{F}}(0,T;\mathbb{R}^n)$, which in turn determines a unique solution $\bar{x}_0 \in L^2_{\mathcal{F}}(0,T;\mathbb{R}^n)$ to the first equation. Therefore, (iii) follows from the fact that (2.5) is equivalent to (2.10). ∎

The optimal control law is given in the form

$$\bar{u}_0(t) = R_0^{-1}B_0^T\big[- P_0(t)\bar{x}_0(t) + \nu_0(t)\big]. \tag{2.11}$$

2.3. An explicit solution of the optimal state process. We intend to find a representation of $\bar{x}_0$ in (2.10) in the form

$$\bar{x}_0(t) = f_{\bar{x}_0,1}(t) + f_{\bar{x}_0,2}(t)x_0(0) + \int_0^t g_{\bar{x}_0}(t,s)dW_0(s), \tag{2.12}$$

where $f_{\bar{x}_0,1} \in C([0,T],\mathbb{R}^n)$, $f_{\bar{x}_0,2} \in C([0,T],\mathbb{R}^{n\times n})$, and $g_{\bar{x}_0} \in C(\Delta,\mathbb{R}^{n\times n_2})$ are to be determined.

To solve the second equation in (2.10), we denote $\zeta_0(t) = [P_0(t)F_0 - Q_0H_0]z(t) - Q_0\eta_0$ and $\mu_0(t) = q_0(t) + P_0(t)D_0$. Since $z(t) = f_1(t) + f_2(t)x_0(0) + \int_0^t g(t,s)dW_0(s)$, we have

$$\zeta_0(t) = f_{\zeta_0,1}(t) + f_{\zeta_0,2}(t)x_0(0) + \int_0^t g_{\zeta_0}(t,s)dW_0(s),$$

where $f_{\zeta_0,1}(t) = [P_0(t)F_0 - Q_0H_0]f_1(t) - Q_0\eta_0$, $f_{\zeta_0,2}(t) = [P_0(t)F_0 - Q_0H_0]f_2(t)$, $g_{\zeta_0}(t,s) = [P_0(t)F_0 - Q_0H_0]g(t,s)$. The equation of ν_0 in (2.10) may be rewritten as

$$d\nu_0(t) = [\zeta_0(t) - \mathbb{A}_0^T(t)\nu_0(t)]dt + \mu_0(t)dW_0(t). \tag{2.13}$$

Take $M_0(t) = \mathbb{A}_0(t)$ in (A.2) and denote the resulting solution by $\Phi_0(t,s)$. The application of Lemma A.1(ii) to (2.13) yields

$$\nu_0(t) = f_{\nu_0,1}(t) + f_{\nu_0,2}(t)x_0(0) + \int_0^t g_{\nu_0}(t,s)dW_0(s), \tag{2.14}$$

where

$$f_{\nu_0,1}(t) = \int_t^T \Phi_0^T(s_1,t)\Big\{\big[Q_0H_0 - P_0(s_1)F_0\big]f_1(s_1) + Q_0\eta_0\Big\}ds_1, \tag{2.15}$$

$$f_{\nu_0,2}(t) = \int_t^T \Phi_0^T(s_1,t)\big[Q_0H_0 - P_0(s_1)F_0\big]f_2(s_1)ds_1, \tag{2.16}$$

$$g_{\nu_0}(t,s) = \int_t^T \Phi_0^T(s_1,t)\big[Q_0H_0 - P_0(s_1)F_0\big]g(s_1,s)ds_1. \tag{2.17}$$

We continue to solve the first equation in (2.10). Let $\xi_0(t) = B_0R_0^{-1}B_0^T\nu_0(t) + F_0z(t)$. Then

$$d\bar{x}_0(t) = [\xi_0(t) + \mathbb{A}_0(t)\bar{x}_0(t)]dt + D_0dW_0(t). \tag{2.18}$$

By (2.14) and the representation of $z(t)$,

$$\xi_0(t) = f_{\xi_0,1}(t) + f_{\xi_0,2}(t)x_0(0) + \int_0^t g_{\xi_0}(t,s)dW_0(s),$$

where $f_{\xi_0,j}(t) = B_0R_0^{-1}B_0^Tf_{\nu_0,j}(t)+F_0f_j(t)$, $j=1,2$, and $g_{\xi_0}(t,s) = B_0R_0^{-1}B_0^Tg_{\nu_0}(t,s)+F_0g(t,s)$. Therefore, by (2.18) and Lemma A.1(i) we obtain (2.12), where

$$\begin{aligned}
f_{\bar{x}_0,1}(t) &= \int_0^t \Phi_0(t,s_1)f_{\xi_0,1}(s_1)ds_1\\
&= \int_0^t\int_{s_1}^T \Phi_0(t,s_1)B_0R_0^{-1}B_0^T\Phi_0^T(s_2,s_1)\Big\{\big[Q_0H_0 - P_0(s_2)F_0\big]f_1(s_2) + Q_0\eta_0\Big\}ds_2ds_1\\
&\quad + \int_0^t \Phi_0(t,s_1)F_0f_1(s_1)ds_1\\
&=: [\Gamma_{0,1}f_1](t),
\end{aligned} \tag{2.19}$$

$$\begin{aligned}
f_{\bar{x}_0,2}(t) &= \Phi_0(t,0) + \int_0^t \Phi_0(t,s_1)f_{\xi_0,2}(s_1)ds_1\\
&= \int_0^t\int_{s_1}^T \Phi_0(t,s_1)B_0R_0^{-1}B_0^T\Phi_0^T(s_2,s_1)\big[Q_0H_0 - P_0(s_2)F_0\big]f_2(s_2)ds_2ds_1\\
&\quad + \int_0^t \Phi_0(t,s_1)F_0f_2(s_1)ds_1 + \Phi_0(t,0)\\
&=: [\Gamma_{0,2}f_2](t),
\end{aligned} \tag{2.20}$$

$$\begin{aligned}
g_{\bar{x}_0}(t,s) &= \int_s^t \Phi_0(t,s_1)g_{\xi_0}(s_1,s)ds_1 + \Phi_0(t,s)D_0\\
&= \int_s^t\int_{s_1}^T \Phi_0(t,s_1)B_0R_0^{-1}B_0^T\Phi_0^T(s_2,s_1)\big[Q_0H_0 - P_0(s_2)F_0\big]g(s_2,s)ds_2ds_1\\
&\quad + \int_s^t \Phi_0(t,s_1)F_0g(s_1,s)ds_1 + \Phi_0(t,s)D_0\\
&=: [\Lambda_0g](t,s).
\end{aligned} \tag{2.21}$$

2.4. The analysis of Problem (II).

LEMMA 2.2. (i) *There exists a unique optimal control* $\bar{u}_i \in L^2_{\mathcal{F}}(0,T;\mathbb{R}^{n_1})$ *to Problem* (II).

(ii) *The pair* $(\bar{x}_i, \bar{u}_i) \in L^2_{\mathcal{F}}(0,T;\mathbb{R}^{n+n_1})$ *is the optimal solution to Problem* (II) *if and only if* $\bar{u}_i(t) = R^{-1}B^T(\theta_i)p_i(t)$, *where* $(\bar{x}_i, p_i, q_i, r_i) \in L^2_{\mathcal{F}}(0,T;\mathbb{R}^{2n}\times\mathbb{R}^{n\times n_2}\times\mathbb{R}^{n\times n_2})$ *is the solution of the forward-backward SDE*

$$\begin{cases} d\bar{x}_i = \Big[A(\theta_i)\bar{x}_i + B(\theta_i)R^{-1}B^T(\theta_i)p_i + F(\theta_i)z\Big]dt + D(\theta_i)dW_i, \\ dp_i = \Big[Q\big(\bar{x}_i - H\bar{x}_0 - \hat{H}z - \eta\big) - A^T(\theta_i)p_i\Big]dt + q_i dW_i + r_i dW_0, \\ \bar{x}_i(0) = x_i(0), \quad p_i(T) = 0. \end{cases} \tag{2.22}$$

(iii) *Equation* (2.22) *has a unique solution* $(\bar{x}_i, p_i, q_i, r_i) \in L^2_{\mathcal{F}}(0,T;\mathbb{R}^{2n}\times\mathbb{R}^{n\times n_2}\times\mathbb{R}^{n\times n_2})$.

Proof. Using Lemma A.2 instead of Lemma A.1, the proof is similar to that of Lemma 2.1. ∎

Let $P_{\theta_i}(t) \geq 0$ be the unique solution of the Riccati equation

$$\begin{cases} \dot{P}_{\theta_i} + P_{\theta_i}A(\theta_i) + A^T(\theta_i)P_{\theta_i} - P_{\theta_i}B(\theta_i)R_0^{-1}B^T(\theta_i)P_{\theta_i} + Q = 0, \\ P_{\theta_i}(T) = 0. \end{cases} \tag{2.23}$$

Write $p_i(t) = -P_{\theta_i}(t)\bar{x}_i(t) + \nu_{\theta_i}(t)$, where $\nu_{\theta_i}(t)$ satisfies $\nu_{\theta_i}(T) = 0$. Denote

$$\mathbb{A}_{\theta_i}(t) = A(\theta_i) - B(\theta_i)R^{-1}B^T(\theta_i)P_{\theta_i}(t). \tag{2.24}$$

By Ito's formula, it is straightforward to show that (2.22) is equivalent to the forward-backward SDE

$$\begin{cases} d\bar{x}_i = \Big[\mathbb{A}_{\theta_i}\bar{x}_i + B(\theta_i)R^{-1}B^T(\theta_i)\nu_{\theta_i} + F(\theta_i)z\Big]dt + D(\theta_i)dW_i, \\ d\nu_{\theta_i} = \Big[-\mathbb{A}^T_{\theta_i}\nu_{\theta_i} + (P_{\theta_i}F(\theta_i) - Q\hat{H})z - QH\bar{x}_0 - Q\eta\Big]dt \\ \qquad\qquad + [q_i + P_{\theta_i}D(\theta_i)]dW_i + r_i dW_0, \\ \bar{x}_i(0) = x_i(0), \quad \nu_{\theta_i}(T) = 0. \end{cases} \tag{2.25}$$

The optimal control law is given in the form

$$\bar{u}_i(t) = R^{-1}B^T(\theta_i)\big[-P_{\theta_i}(t)\bar{x}_i(t) + \nu_{\theta_i}(t)\big]. \tag{2.26}$$

We will apply Lemma A.2 to represent $\bar{x}_i(t)$ in the form

$$\begin{aligned} \bar{x}_i(t) &= f_{\bar{x}_i,1}(t) + f_{\bar{x}_i,2}(t)x_0(0) + f_{\bar{x}_i,3}(t)x_i(0) \\ &\quad + \int_0^t g_{\bar{x}_i}(t,s)dW_0(s) + \int_0^t h_{\bar{x}_i}(t,s)dW_i(s), \end{aligned} \tag{2.27}$$

where $f_{\bar{x}_i,1} \in C([0,T],\mathbb{R}^n)$, $f_{\bar{x}_i,2}$, $f_{\bar{x}_i,3} \in C([0,T],\mathbb{R}^{n\times n})$, and $g_{\bar{x}_i}$, $h_{\bar{x}_i} \in C(\Delta,\mathbb{R}^{n\times n_2})$ are to be determined.

Let $\zeta_i(t) = [P_{\theta_i}(t)F(\theta_i) - Q\hat{H}]z(t) - QH\bar{x}_0(t) - Q\eta$ and $\lambda_i(t) = q_i(t) + P_{\theta_i}(t)D(\theta_i)$. By (2.12),

$$\zeta_i(t) = f_{\zeta_i,1}(t) + f_{\zeta_i,2}(t)x_0(0) + \int_0^t g_{\zeta_i}(t,s)dW_0(s),$$

where $f_{\zeta_i,1}(t) = [P_{\theta_i}(t)F(\theta_i) - Q\hat{H}]f_1(t) - QHf_{\bar{x}_0,1}(t) - Q\eta$, $f_{\zeta_i,2}(t) = [P_{\theta_i}(t)F(\theta_i) - Q\hat{H}]f_2(t) - QHf_{\bar{x}_0,2}(t)$, and $g_{\zeta_i}(t,s) = [P_{\theta_i}(t)F(\theta_i) - Q\hat{H}]g(t,s) - QHg_{\bar{x}_0}(t,s)$. Note that $f_{\bar{x}_0,1}$, $f_{\bar{x}_0,2}$, and $g_{\bar{x}_0}$ are determined in (2.19)–(2.21), and f_1, f_2, and g appear in the equation of z. By the above choices of $\zeta_i(t)$ and $\lambda_i(t)$, the second equation in (2.25) may be written as

$$d\nu_{\theta_i}(t) = [\zeta_i(t) - \mathbb{A}_{\theta_i}^T \nu_{\theta_i}(t)]dt + r_i(t)dW_0(t) + \lambda_i(t)dW_i(t).$$

Take $M_i(t) = \mathbb{A}_{\theta_i}(t)$ in (A.8) and denote the resulting solution by $\Phi_{\theta_i}(t,s)$. Then by Lemma A.2(ii), we have

$$\nu_{\theta_i}(t) = f_{\nu_{\theta_i},1}(t) + f_{\nu_{\theta_i},2}(t)x_0(0) + \int_0^t g_{\nu_{\theta_i}}(t,s)dW_0(s), \tag{2.28}$$

where

$$f_{\nu_{\theta_i},1}(t) = \int_t^T \Phi_{\theta_i}^T(s_1,t)\Big\{\big[Q\hat{H} - P_{\theta_i}(s_1)F(\theta_i)\big]f_1(s_1) + QHf_{\bar{x}_0,1}(s_1) + Q\eta\Big\}ds_1, \tag{2.29}$$

$$f_{\nu_{\theta_i},2}(t) = \int_t^T \Phi_{\theta_i}^T(s_1,t)\Big\{\big[Q\hat{H} - P_{\theta_i}(s_1)F(\theta_i)\big]f_2(s_1) + QHf_{\bar{x}_0,2}(s_1)\Big\}ds_1, \tag{2.30}$$

$$g_{\nu_{\theta_i}}(t,s) = \int_t^T \Phi_{\theta_i}^T(s_1,t)\Big\{\big[Q\hat{H} - P_{\theta_i}(s_1)F(\theta_i)\big]g(s_1,s) + QHg_{\bar{x}_0}(s_1,s)\Big\}ds_1. \tag{2.31}$$

Next let $\xi_i(t) = B(\theta_i)R^{-1}B^T(\theta_i)\nu_{\theta_i}(t) + F(\theta_i)z(t)$. By (2.28) and the equation of z in (2.1), $\xi_i(t) = f_{\xi_i,1}(t) + f_{\xi_i,2}(t)x_0(0) + \int_0^t g_{\xi_i}(t,s)dW_0(s)$, where

$$f_{\xi_i,j}(t) = B(\theta_i)R^{-1}B^T(\theta_i)f_{\nu_{\theta_i},j}(t) + F(\theta_i)f_j(t), \quad j = 1,2, \tag{2.32}$$

$$g_{\xi_i}(t,s) = B(\theta_i)R^{-1}B^T(\theta_i)g_{\nu_{\theta_i}}(t,s) + F(\theta_i)g(t,s). \tag{2.33}$$

Here, $f_{\nu_{\theta_i},1}(t)$, $f_{\nu_{\theta_i},2}(t)$, and $g_{\nu_i}(t,s)$ are determined in (2.29)–(2.31). So the first equation in (2.25) may be written as

$$d\bar{x}_i(t) = [\xi_i(t) + \mathbb{A}_{\theta_i}(t)\bar{x}_i(t)]dt + D(\theta_i)dW_i(t).$$

By Lemma A.2(i) we obtain (2.27), where

$$\begin{aligned} f_{\bar{x}_i,1}(t) &= \int_0^t \Phi_{\theta_i}(t,s_1)f_{\xi_i,1}(s_1)ds_1 \\ &= \int_0^t \int_{s_1}^T \Phi_{\theta_i}(t,s_1)B(\theta_i)R^{-1}B^T(\theta_i)\Phi_{\theta_i}^T(s_2,s_1) \\ &\quad \times \Big((Q\hat{H} - P_{\theta_i}(s_2)F(\theta_i))f_1(s_2) + QHf_{\bar{x}_0,1}(s_2) + Q\eta\Big)ds_2ds_1 \\ &\quad + \int_0^t \Phi_{\theta_i}(t,s_1)F(\theta_i)f_1(s_1)ds_1 \\ &=: [\Gamma_{\theta_i,1}f_1](t), \end{aligned} \tag{2.34}$$

$$
\begin{aligned}
f_{\bar{x}_i,2}(t) &= \int_0^t \Phi_{\theta_i}(t,s_1) f_{\xi_i,2}(s_1)ds_1 \\
&= \int_0^t \int_{s_1}^T \Phi_{\theta_i}(t,s_1)B(\theta_i)R^{-1}B^T(\theta_i)\Phi_{\theta_i}^T(s_2,s_1) \\
&\quad \times \Big((Q\hat{H} - P_{\theta_i}(s_2)F(\theta_i))f_2(s_2) + QHf_{\bar{x}_0,2}(s_2)\Big) ds_2 ds_1 \\
&\quad + \int_0^t \Phi_{\theta_i}(t,s_1)F(\theta_i)f_2(s_1)ds_1 \\
&=: [\Gamma_{\theta_i,2}f_2](t), \qquad (2.35)
\end{aligned}
$$

$$
f_{\bar{x}_i,3}(t) = \Phi_{\theta_i}(t,0), \qquad (2.36)
$$

$$
\begin{aligned}
g_{\bar{x}_i}(t,s) &= \int_s^t \Phi_{\theta_i}(t,s_1) g_{\xi_i}(s_1,s)ds_1 \\
&= \int_s^t \int_{s_1}^T \Phi_{\theta_i}(t,s_1)B(\theta_i)R^{-1}B^T(\theta_i)\Phi_{\theta_i}^T(s_2,s_1) \\
&\quad \times \Big((Q\hat{H} - P_{\theta_i}(s_2)F(\theta_i))g(s_2,s) + QHg_{\bar{x}_0}(s_2,s)\Big) ds_2 ds_1 \\
&\quad + \int_s^t \Phi_{\theta_i}(t,s_1)F(\theta_i)g(s_1,s)ds_1 \\
&=: [\Lambda_{\theta_i}g](t,s), \qquad (2.37)
\end{aligned}
$$

and

$$
h_{\bar{x}_i}(t,s) = \Phi_{\theta_i}(t,s)D(\theta_i). \qquad (2.38)
$$

3. The consistency condition. This section introduces the consistency requirement for the mean field approximation, i.e., when the control strategies obtained in section 2 are applied, the mean field replicated by the closed loop in the population limit should coincide with the process z assumed at the beginning. Based on averaging with (2.27), denote

$$
[\Gamma_1 f_1](t) = \int_\Theta [\Gamma_{\theta,1}f_1](t)d\mathbf{F}(\theta) + \int_{\Theta\times\mathbb{R}^n} \Phi_\theta(t,0)x d\mathbf{F}(\theta,x), \quad 0 \le t \le T, \qquad (3.1)
$$

$$
[\Gamma_2 f_2](t) = \int_\Theta [\Gamma_{\theta,2}f_2](t)d\mathbf{F}(\theta), \quad 0 \le t \le T, \qquad (3.2)
$$

$$
[\Lambda g](t,s) = \int_\Theta [\Lambda_\theta g](t,s)d\mathbf{F}(\theta), \quad 0 \le s \le t \le T, \qquad (3.3)
$$

for $f_1 \in C([0,T],\mathbb{R}^n)$, $f_2 \in C([0,T],\mathbb{R}^{n\times n})$, and $g \in C(\Delta,\mathbb{R}^{n\times n_2})$. Here, $\Gamma_{\theta,j}f_j$, $j=1,2$, and $\Lambda_\theta g$ are, respectively, defined as in (2.34), (2.35), and (2.37) with $\theta_i = \theta$. The consistent mean field approximation reduces to analyzing the fixed point equation system

$$
\begin{cases} f_j(t) = [\Gamma_j f_j](t), \quad 0 \le t \le T,\ j=1,2, \\ g(t,s) = [\Lambda g](t,s), \quad 0 \le s \le t \le T, \end{cases} \qquad (3.4)
$$

which is called the Nash certainty equivalence (NCE) equation system. Denote the product space

$$
C_{\text{NCE}} = C([0,T],\mathbb{R}^n) \times C([0,T],\mathbb{R}^{n\times n}) \times C(\Delta,\mathbb{R}^{n\times n_2}).
$$

Definition 3.1. *A triple $(f_1,f_2,g) \in C_{\text{NCE}}$ satisfying (3.4) is called a (consistent) solution to the NCE equation system.*

LEMMA 3.2. *Assume* (A1)–(A3). *Then*
(i) Γ_1 *is a mapping from* $C([0,T],\mathbb{R}^n)$ *to* $C([0,T],\mathbb{R}^n)$*;*
(ii) Γ_2 *is a mapping from* $C([0,T],\mathbb{R}^{n\times n})$ *to* $C([0,T],\mathbb{R}^{n\times n})$*;*
(iii) Λ *is a mapping from* $C(\Delta,\mathbb{R}^{n\times n_2})$ *to* $C(\Delta,\mathbb{R}^{n\times n_2})$.

Proof. We only prove (iii) while (i) and (ii) may be proved by a similar argument. To analyze Λ, we define some auxiliary operators. For $g\in C(\Delta,\mathbb{R}^{n\times n_2})$ and $\theta\in\Theta$, denote

$$
\begin{aligned}
&[\Lambda_{\theta,1}g](t,s)=\int_s^t\Phi_\theta(t,s_1)B(\theta)R^{-1}B^T(\theta)\int_{s_1}^T\Phi_\theta^T(s_2,s_1)g(s_2,s)ds_2ds_1,\\
&[\Lambda_{\theta,2}g](t,s)=\int_s^t\Phi_\theta(t,s_1)B(\theta)R^{-1}B^T(\theta)\int_{s_1}^T\Phi_\theta^T(s_2,s_1)P_\theta(s_2)F(\theta)g(s_2,s)ds_2ds_1,\\
&[\Lambda_{\theta,3}g](t,s)=\int_s^t\Phi_\theta(t,s_1)F(\theta)g(s_1,s)ds_1,
\end{aligned}\tag{3.5}
$$

and

$$[\Lambda_ig](t,s)=\int_\Theta[\Lambda_{\theta,i}g](t,s)d\mathbf{F}(\theta),\quad i=1,2,3.$$

In view of (A3) and Remark A.4 (see Appendix A),

$$\sup_{\theta\in\Theta}\sup_{0\le t,s\le T}\max\Big\{\Big|\frac{d\Phi_\theta(t,s)}{dt}\Big|,|\Phi_\theta(t,s)|\Big\}\le C,\quad \sup_{\theta\in\Theta}|B(\theta)|\le C.$$

Consider $(t,s)\in\Delta,(t,s')\in\Delta$, $s\le s'$. Then

$$
\begin{aligned}
&\sup_{\theta\in\Theta}\big|[\Lambda_{\theta,1}g](t,s')-[\Lambda_{\theta,1}g](t,s)\big|\\
&\quad\le\sup_{\theta\in\Theta}\Big|\int_{s'}^t\Phi_\theta(t,s_1)B(\theta)R^{-1}B^T(\theta)\int_{s_1}^T\Phi_\theta^T(s_2,s_1)\big(g(s_2,s')-g(s_2,s)\big)ds_2ds_1\Big|\\
&\qquad+\sup_{\theta\in\Theta}\Big|\int_s^{s'}\Phi_\theta(t,s_1)B(\theta)R^{-1}B^T(\theta)\int_{s_1}^T\Phi_\theta^T(s_2,s_1)g(s_2,s)ds_2ds_1\Big|\\
&\quad\le C\sup_{s'\le s_2\le T}\big|g(s_2,s')-g(s_2,s)\big|+C|s'-s|\sup_{(t,s)\in\Delta}|g(t,s)|.
\end{aligned}\tag{3.6}
$$

Here C does not depend on g.

Note that

$$\sup_{\theta\in\Theta}\sup_{(t,s),(t',s)\in\Delta}|\Phi_\theta(t',s)-\Phi_\theta(t,s)|\le\sup_{\theta\in\Theta}\sup_{0\le t,s\le T}\Big|\frac{d\Phi_\theta(t,s)}{dt}\Big||t-t'|\le C|t-t'|.$$

Next consider any $(t,s)\in\Delta$, $(t',s)\in\Delta$, $t\le t'$. For some constant C,

$$
\begin{aligned}
&\sup_{\theta\in\Theta}\big|[\Lambda_{\theta,1}g](t',s)-[\Lambda_{\theta,1}g](t,s)\big|\\
&\quad\le\sup_{\theta\in\Theta}\Big|\int_t^{t'}\int_{s_1}^T\Phi_\theta(t',s_1)B(\theta)R^{-1}B^T(\theta)\Phi_\theta^T(s_2,s_1)g(s_2,s)ds_2ds_1\Big|\\
&\qquad+\sup_{\theta\in\Theta}\Big|\int_s^t\int_{s_1}^T\big(\Phi_\theta(t',s_1)-\Phi_\theta(t,s_1)\big)B(\theta)R^{-1}B^T(\theta)\Phi_\theta^T(s_2,s_1)g(s_2,s)ds_2ds_1\Big|\\
&\quad\le C|t'-t|.
\end{aligned}\tag{3.7}
$$

Let $(t,s)\in\Delta$ and $\delta t,\delta s$ be such that $(t+\delta t,s+\delta s)\in\Delta$. By the uniform continuity of g on Δ and (3.6)–(3.7), $\big|[\Lambda_1 g](t+\delta t,s+\delta s)-[\Lambda_1 g](t,s)\big|\to 0$ uniformly with respect to (t,s), as $\max\{|\delta s|,|\delta t|\}\to 0$. Hence, Λ_1 is a mapping from $C(\Delta,\mathbb{R}^{n\times n_2})$ to $C(\Delta,\mathbb{R}^{n\times n_2})$. Similarly, we can show that Λ_2 and Λ_3 are mappings from $C(\Delta,\mathbb{R}^{n\times n_2})$ to $C(\Delta,\mathbb{R}^{n\times n_2})$.

By replacing the subscript θ by 0, $B(\theta)$ by B_0, and $F(\theta)$ by F_0 in (3.5), we define $\Lambda_{0,i}$ for $i=1,2,3$ in a similar manner. It can also be shown that each $\Lambda_{0,i}$ is a mapping from $C(\Delta,\mathbb{R}^{n\times n_2})$ to $C(\Delta,\mathbb{R}^{n\times n_2})$. Note that

$$[\Lambda_0 g](t,s)=[\Lambda_{0,1}(Q_0H_0 g)](t,s)-[\Lambda_{0,2}g](t,s)+[\Lambda_{0,3}g](t,s)+\Phi_0(t,s)D_0,$$

so Λ_0 is also a mapping from $C(\Delta,\mathbb{R}^{n\times n_2})$ to $C(\Delta,\mathbb{R}^{n\times n_2})$. Finally, we have

$$[\Lambda g](t,s)=[\Lambda_1(Q\hat{H}g+QH\Lambda_0 g)](t,s)-[\Lambda_2 g](t,s)+[\Lambda_3 g](t,s).$$

This proves (iii). ∎

By replacing $(\mathbf{F}(\theta,x),\mathbf{F}(\theta))$ by $(\mathbf{F}_N(\theta,x),\mathbf{F}_N(\theta))$ in (3.1)–(3.3), we define

$$[\Gamma_1^{(N)}f_1](t)=\int_\Theta[\Gamma_{\theta,1}f_1](t)d\mathbf{F}_N(\theta)+\int_{\Theta\times\mathbb{R}^n}\Phi_\theta(t,0)x\,d\mathbf{F}_N(\theta,x),\quad 0\le t\le T, \tag{3.8}$$

$$[\Gamma_2^{(N)}f_2](t)=\int_\Theta[\Gamma_{\theta,2}f_2](t)d\mathbf{F}_N(\theta),\quad 0\le t\le T, \tag{3.9}$$

$$[\Lambda^{(N)}g](t,s)=\int_\Theta[\Lambda_\theta g](t,s)d\mathbf{F}_N(\theta),\quad 0\le s\le t\le T, \tag{3.10}$$

where $f_1\in C([0,T],\mathbb{R}^n)$, $f_2\in C([0,T],\mathbb{R}^{n\times n})$, and $g\in C(\Delta,\mathbb{R}^{n\times n_2})$.

By the same argument as in the proof of Lemma 3.2, we can show that $\Gamma_1^{(N)}$ is a mapping from $C([0,T],\mathbb{R}^n)$ to $C([0,T],\mathbb{R}^n)$, $\Gamma_2^{(N)}$ is a mapping from $C([0,T],\mathbb{R}^{n\times n})$ to $C([0,T],\mathbb{R}^{n\times n})$, and $\Lambda^{(N)}$ is a mapping from $C(\Delta,\mathbb{R}^{n\times n_2})$ to $C(\Delta,\mathbb{R}^{n\times n_2})$.

Remark 3.3. In view of Lemma A.3 and Remark A.4, for fixed $t\in[0,T]$ and $(t,s)\in\Delta$ and fixed $f_1\in C([0,T],\mathbb{R}^n)$, $f_2\in C([0,T],\mathbb{R}^{n\times n})$, and $g\in C(\Delta,\mathbb{R}^{n\times n_2})$, $[\Gamma_{\theta,1}f_1](t),[\Gamma_{\theta,2}f_2](t)$ and $[\Lambda_\theta g](t,s)$ are bounded and continuous vector- or matrix-valued functions of θ. By Assumption (A2), $[\Gamma_j^{(N)}f_j](t)\to[\Gamma_j f_j](t)$, $j=1,2$, and $[\Lambda^{(N)}g](t,s)\to[\Lambda g](t,s)$, as $N\to\infty$.

Remark 3.4. It follows from (3.6)–(3.7) that for each $g\in C(\Delta,\mathbb{R}^{n\times n_2})$, the sequence of matrix-valued functions $\{\Lambda^{(N)}g, N\ge 1\}$ is uniformly equicontinuous on Δ. Similarly, we may show that for fixed $f_1\in C([0,T],\mathbb{R}^n)$ and $f_2\in C([0,T],\mathbb{R}^{n\times n})$, the sequences of vector- or matrix-valued functions $\{\Gamma_j^{(N)}f_j, N\ge 1\}$, $j=1,2$, are both uniformly equicontinuous on $[0,T]$.

Denote the linear operators $\bar\Gamma_{0,1}$, $\bar\Gamma_{0,2}$, and $\bar\Lambda_0$ on $C([0,T],\mathbb{R}^n)$, $C([0,T],\mathbb{R}^{n\times n})$, and $C(\Delta,\mathbb{R}^{n\times n_2})$, respectively, as follows:

$$[\bar\Gamma_{0,j}f_j](t)=\int_0^t\int_{s_1}^T\Phi_0(t,s_1)B_0R_0^{-1}B_0^T\Phi_0^T(s_2,s_1)\big(Q_0H_0-P_0(s_2)F_0\big)f_j(s_2)ds_2ds_1$$
$$+\int_0^t\Phi_0(t,s_1)F_0f_j(s_1)ds_1,\quad j=1,2, \tag{3.11}$$

$$[\bar{\Lambda}_0 g](t,s) = \int_s^t \int_{s_1}^T \Phi_0(t,s_1) B_0 R_0^{-1} B_0^T \Phi_0^T(s_2,s_1)\big(Q_0 H_0 - P_0(s_2) F_0\big) g(s_2,s) ds_2 ds_1$$
$$+ \int_s^t \Phi_0(t,s_1) F_0 g(s_1,s) ds_1,$$

which are obtained by retaining the linear term of the affine operators $\Gamma_{0,j}$ and Λ_0, respectively.

Corresponding to $\Gamma_{\theta,1}$, $\Gamma_{\theta,2}$, and Λ_θ, define the linear operators $\bar{\Gamma}_{\theta,1}$, $\bar{\Gamma}_{\theta,2}$, and $\bar{\Lambda}_\theta$ on $C([0,T],\mathbb{R}^n)$, $C([0,T],\mathbb{R}^{n\times n})$, and $C(\Delta,\mathbb{R}^{n\times n_2})$, respectively, as follows:

$$\begin{aligned}
[\bar{\Gamma}_{\theta,j} f_j](t) = {} & \int_0^t \int_{s_1}^T \Phi_\theta(t,s_1) B(\theta) R^{-1} B^T(\theta) \Phi_\theta^T(s_2,s_1) \\
& \times \Big((Q\hat{H} - P_\theta(s_2) F(\theta)) f_j(s_2) + QH[\bar{\Gamma}_{0,j} f_j](s_2) \Big) ds_2 ds_1 \\
& + \int_0^t \Phi_\theta(t,s_1) F(\theta) f_j(s_1) ds_1, \quad j = 1,2, \\
[\bar{\Lambda}_\theta g](t,s) = {} & \int_s^t \int_{s_1}^T \Phi_\theta(t,s_1) B(\theta) R^{-1} B^T(\theta) \Phi_\theta^T(s_2,s_1) \\
& \times \Big((Q\hat{H} - P_\theta(s_2) F(\theta)) g(s_2,s) + QH[\bar{\Lambda}_0 g](s_2,s) \Big) ds_2 ds_1 \\
& + \int_s^t \Phi_\theta(t,s_1) F(\theta) g(s_1,s) ds_1.
\end{aligned}$$

Let $C([0,T],\mathbb{R}^n)$, $C([0,T],\mathbb{R}^{n\times n})$, and $C(\Delta,\mathbb{R}^{n\times n_2})$ be endowed with the usual sup-norms $\|\cdot\|_\infty$ so that they are all Banach spaces. Define the linear operators

$$[\bar{\Gamma}_j f_j](t) = \int_\Theta [\bar{\Gamma}_{\theta,j} f_j](t) d\mathbf{F}(\theta), \quad [\bar{\Lambda} g](t,s) = \int_\Theta [\bar{\Lambda}_\theta g](t,s) d\mathbf{F}(\theta). \tag{3.12}$$

By Lemma 3.2, we see that $\bar{\Gamma}_1$ (resp., $\bar{\Gamma}_2$ and $\bar{\Lambda}$) is a mapping from $C([0,T],\mathbb{R}^n)$ (resp., $C([0,T],\mathbb{R}^{n\times n})$ and $C(\Delta,\mathbb{R}^{n\times n_2})$) to itself.

Remark 3.5. For $f_2 \in C([0,T],\mathbb{R}^{n\times n})$, we write $f_2 = [f_{2,1},\ldots,f_{2,n}]$, where $f_{2,k} \in C([0,T],\mathbb{R}^n)$ for $k = 1,\ldots,n$. By (3.12), $[\bar{\Gamma}_2 f_2](t) = \big[[\bar{\Gamma}_1 f_{2,1}](t),\ldots,[\bar{\Gamma}_1 f_{2,n}](t)\big]$.

We have the following solvability result for the NCE equation system.

THEOREM 3.6. *Under Assumptions* (A1)–(A3)*, if the norms of the linear operators* $\bar{\Gamma}_1$ *and* $\bar{\Lambda}$ *satisfy* $\|\bar{\Gamma}_1\| < 1$ *and* $\|\bar{\Lambda}\| < 1$*, there exists a unique solution* $(f_1, f_2, g) \in C_{\mathrm{NCE}}$ *to* (3.4).

Proof. For any $f_1, f_1' \in C([0,T],\mathbb{R}^n)$ and $g, g' \in C(\Delta,\mathbb{R}^{n\times n_2})$, we have

$$\|\Gamma_1 f_1 - \Gamma_1 f_1'\|_\infty = \|\bar{\Gamma}_1(f_1 - f_1')\|_\infty \le \|\bar{\Gamma}_1\| \|f_1 - f_1'\|_\infty \le \alpha \|f_1 - f_1'\|_\infty,$$
$$\|\Lambda g - \Lambda g'\|_\infty = \|\bar{\Lambda}(g - g')\|_\infty \le \|\bar{\Lambda}\| \|g - g'\|_\infty \le \alpha \|g - g'\|_\infty$$

for some $\alpha \in (0,1)$. By Banach fixed point theorem, $\Gamma_1 f_1 = f_1$ and $\Lambda g = g$ have a unique solution. Next, we write $\Gamma_2 f_2 = f_2$ in the equivalent form

$$\bar{\Gamma}_2 f_2 + f_2' = f_2,$$

where $f_2' \in C([0,T],\mathbb{R}^{n\times n})$ is known. Denote $f_2 = [f_{2,1},\ldots,f_{2,n}]$, $f_2' = [f_{2,1}',\ldots,f_{2,n}']$. By Remark 3.5, we may further write an equivalent set of n decoupled equations of the form

$$\bar{\Gamma}_1 f_{2,k} + f_{2,k}' = f_{2,k}, \quad 1 \le k \le n,$$

each of which has a unique solution since $\bar{\Gamma}_1$ is a contraction. Hence $\Gamma_2 f_2 = f_2$ has a unique solution. So (3.4) has a unique solution. ∎

4. Decentralized strategies. Throughout this section we assume that there exists a solution $(f_1, f_2, g) \in C_{\mathrm{NCE}}$ to the NCE equation system (3.4). Consider (1.1)–(1.2). Let the control laws of $\mathcal{A}_0$ and $\mathcal{A}_i$, $1 \le i \le N$, be given by

$$\hat{u}_0(t) = R_0^{-1} B_0^T \big[- P_0(t) x_0(t) + \nu_0(t) \big], \tag{4.1}$$

$$\hat{u}_i(t) = R^{-1} B^T(\theta_i) \big[- P_{\theta_i}(t) x_i(t) + \nu_{\theta_i}(t) \big], \quad 1 \le i \le N, \tag{4.2}$$

where $P_0(t)$ and $P_{\theta_i}(t)$ are determined by (2.8) and (2.23), and $\nu_0(t)$ and $\nu_{\theta_i}(t)$ are determined by (2.10) and (2.25) with (f_1, f_2, g) being a solution to (3.4). Their explicit solutions are given by (2.14) and (2.28). The control laws (4.1)–(4.2) use the states of the system of $N+1$ players and are different from (2.11) and (2.26). Since each strategy depends on the current state information and the major player's Brownian motion via the solution of the backward SDE, we call it a partial state feedback strategy.

After the control laws (4.1)–(4.2) are applied, the dynamics of $\mathcal{A}_0$ and $\mathcal{A}_i$ may be written as

$$dx_0(t) = \Big[\mathbb{A}_0(t) x_0(t) + B_0 R_0^{-1} B_0^T P_0(t) \nu_0(t) + F_0 x^{(N)}(t) \Big] dt + D_0 dW_0(t), \tag{4.3}$$

$$\begin{aligned} dx_i(t) = {} & \Big[\mathbb{A}_{\theta_i}(t) x_i(t) + B(\theta_i) R^{-1} B^T(\theta_i) P_{\theta_i}(t) \nu_{\theta_i}(t) + F(\theta_i) x^{(N)}(t) \Big] dt \\ & + D(\theta_i) dW_i(t), \end{aligned} \tag{4.4}$$

where $1 \le i \le N$ and $x^{(N)} = (1/N) \sum_{i=1}^N x_i$. For further performance estimates, we construct the auxiliary equation system for the $N+1$ players

$$d\bar{x}_0(t) = \Big[\mathbb{A}_0(t) \bar{x}_0(t) + B_0 R_0^{-1} B_0^T P_0(t) \nu_0(t) + F_0 z(t) \Big] dt + D_0 dW_0(t), \tag{4.5}$$

$$\begin{aligned} d\bar{x}_i(t) = {} & \Big[\mathbb{A}_{\theta_i}(t) \bar{x}_i(t) + B(\theta_i) R^{-1} B^T(\theta_i) P_{\theta_i}(t) \nu_{\theta_i}(t) + F(\theta_i) z(t) \Big] dt \\ & + D(\theta_i) dW_i(t), \quad 1 \le i \le N, \end{aligned} \tag{4.6}$$

where $\bar{x}_0(0) = x_0(0)$ and $\bar{x}_i(0) = x_i(0)$. From (2.12)–(2.21) and (2.27)–(2.38), we have

$$\begin{aligned} \bar{x}_0(t) &= [\Gamma_{0,1} f_1](t) + [\Gamma_{0,2} f_2](t) x_0(0) + \int_0^t [\Lambda_0 g](t,s) dW_0(s), \\ \bar{x}_i(t) &= [\Gamma_{\theta_i,1} f_1](t) + [\Gamma_{\theta_i,2} f_2](t) x_0(0) + \Phi_{\theta_i}(t,0) x_i(0) \\ &\quad + \int_0^t [\Lambda_{\theta_i} g](t,s) dW_0(s) + \int_0^t \Phi_{\theta_i}(t,s) D(\theta_i) dW_i(s), \quad 1 \le i \le N. \end{aligned} \tag{4.7}$$

Denote $\bar{x}^{(N)}(t) = (1/N) \sum_{i=1}^N \bar{x}_i(t)$.

Define $\epsilon_N \geq 0$ by

$$\epsilon_N^2 = \epsilon_{f_1,N}^2 + \epsilon_{f_2,N}^2 + \epsilon_{g,N}^2,$$

where

$$\begin{aligned}
\epsilon_{f_1,N}^2 = \int_0^T \bigg| \int_\Theta [\Gamma_{\theta,1} f_1](t) d\mathbf{F}_N(\theta) + \int_{\Theta\times\mathbb{R}^n} \Phi_\theta(t,0) x d\mathbf{F}_N(\theta,x) \\
- \int_\Theta [\Gamma_{\theta,1} f_1](t) d\mathbf{F}(\theta) - \int_{\Theta\times\mathbb{R}^n} \Phi_\theta(t,0) x d\mathbf{F}(\theta,x) \bigg|^2 dt, \\
\epsilon_{f_2,N}^2 = \int_0^T \bigg| \int_\Theta [\Gamma_{\theta,2} f_2](t) d\mathbf{F}_N(\theta) - \int_\Theta [\Gamma_{\theta,2} f_2](t) d\mathbf{F}(\theta) \bigg|^2 dt, \\
\epsilon_{g,N}^2 = \int_0^T \int_0^t \bigg| \int_\Theta [\Lambda_\theta g](t,s) d\mathbf{F}_N(\theta) - \int_\Theta [\Lambda_\theta g](t,s) d\mathbf{F}(\theta) \bigg|^2 ds dt.
\end{aligned}$$

LEMMA 4.1. *Assume* (A1)–(A3). *We have* $\lim_{N\to\infty} \epsilon_N = 0$.

Proof. Note that

$$\begin{aligned}
\epsilon_{f_j,N}^2 &= \int_0^T \left|[\Gamma_j^{(N)} f_j](t) - [\Gamma_j f_j](t)\right|^2 dt, \quad j = 1,2, \\
\epsilon_{g,N}^2 &= \int_0^T \int_0^t \left|[\Lambda^{(N)} g](t,s) - [\Lambda g](t,s)\right|^2 ds dt,
\end{aligned}$$

where $\Gamma_j^{(N)} f_j$ and $\Lambda^{(N)} g$ are defined in (3.8)–(3.10), and $\Gamma_j f_j$ and Λg are defined in (3.1)–(3.3).

Since uniform equicontinuity and pointwise convergence imply uniform convergence, by Remarks 3.3 and 3.4 we conclude

$$\begin{aligned}
\sup_{t\in[0,T]} |[\Gamma_j^{(N)} f_j](t) - [\Gamma_j f_j](t)| \to 0, \quad j = 1,2, \\
\sup_{(t,s)\in\Delta} |[\Lambda^{(N)} g](t,s) - [\Lambda g](t,s)| \to 0, \quad \text{as } N \to \infty.
\end{aligned}$$

The lemma follows. ∎

Let $(f_1, f_2, g) \in C_{\text{NCE}}$ be a solution to (3.4) and

$$z(t) = f_1(t) + f_2(t) x_0(0) + \int_0^t g(t,s) dW_0(s), \quad 0 \leq t \leq T. \tag{4.8}$$

We have the error estimate on the mean field approximation.

LEMMA 4.2. *Assume* (A1)–(A3) *and let* z *be given by* (4.8). *Then*

$$E \int_0^T \left|z(t) - \bar{x}^{(N)}(t)\right|^2 dt = O\Big(\epsilon_N^2 + \frac{1}{N}\Big).$$

Proof. We have $\frac{1}{N}\sum_{i=1}^N \Phi_{\theta_i}(t,0)Ex_i(0) = \int_{\Theta\times\mathbb{R}^n}\Phi_\theta(t,0)xd\mathbf{F}_N(\theta,x)$. Then (3.4) and (4.7) yield

$$\begin{aligned}
z(t) &= \left[\int_\Theta[\Gamma_{\theta,1}f_1](t)d\mathbf{F}(\theta) + \int_{\Theta\times\mathbb{R}^n}\Phi_\theta(t,0)xd\mathbf{F}(\theta,x)\right] + \int_\Theta[\Gamma_{\theta,2}f_2](t)d\mathbf{F}(\theta)x_0(0)\\
&\quad + \int_0^t\int_\Theta[\Lambda_\theta g](t,s)d\mathbf{F}(\theta)dW_0(s),\\
\bar{x}^{(N)}(t) &= \left[\int_\Theta[\Gamma_{\theta,1}f_1](t)d\mathbf{F}_N(\theta) + \int_{\Theta\times\mathbb{R}^n}\Phi_\theta(t,0)xd\mathbf{F}_N(\theta,x)\right]\\
&\quad + \int_\Theta[\Gamma_{\theta,2}f_2](t)d\mathbf{F}_N(\theta)x_0(0) + \int_0^t\int_\Theta[\Lambda_\theta g](t,s)d\mathbf{F}_N(\theta)dW_0(s)\\
&\quad + \frac{1}{N}\sum_{i=1}^N\Phi_{\theta_i}(t,0)[x_i(0)-Ex_i(0)] + \frac{1}{N}\sum_{i=1}^N\int_0^t\Phi_{\theta_i}(t,s)D(\theta_i)dW_i(s).
\end{aligned}$$

Hence

$$\begin{aligned}
&E\int_0^T\left|z(t)-\bar{x}^{(N)}(t)\right|^2dt\\
&\le 5\int_0^T\left|\int_\Theta[\Gamma_{\theta,1}f_1](t)d\mathbf{F}(\theta) + \int_{\Theta\times\mathbb{R}^n}\Phi_\theta(t,0)xd\mathbf{F}(\theta,x)\right.\\
&\qquad\qquad \left. - \int_\Theta[\Gamma_{\theta,1}f_1](t)d\mathbf{F}_N(\theta) - \int_{\Theta\times\mathbb{R}^n}\Phi_\theta(t,0)xd\mathbf{F}_N(\theta,x)\right|^2dt\\
&\quad + 5E|x_0(0)|^2\int_0^T\left|\int_\Theta[\Gamma_{\theta,2}f_2](t)d\mathbf{F}(\theta) - \int_\Theta[\Gamma_{\theta,2}f_2](t)d\mathbf{F}_N(\theta)\right|^2dt\\
&\quad + 5E\int_0^T\left|\int_0^t\int_\Theta\big\{[\Lambda_\theta g](t,s)d\mathbf{F}(\theta) - [\Lambda_\theta g](t,s)d\mathbf{F}_N(\theta)\big\}dW_0(s)\right|^2dt\\
&\quad + \frac{5}{N^2}E\int_0^T\left|\sum_{i=1}^N\Phi_{\theta_i}(t,0)\big[x_i(0)-Ex_i(0)\big]\right|^2dt\\
&\quad + \frac{5}{N^2}E\int_0^T\left|\sum_{i=1}^N\int_0^t\Phi_{\theta_i}(t,s)D(\theta_i)dW_i(s)\right|^2dt\\
&\le 5(1+E|x_0(0)|^2)\big(\epsilon_{f_1,N}^2+\epsilon_{f_2,N}^2+\epsilon_{f_3,N}^2\big) + \frac{5}{N^2}(NC)\\
&\le C\Big(\epsilon_N^2+\frac{1}{N}\Big).
\end{aligned}\tag{4.9}$$

The second inequality follows from the independence of $\{W_i, i\ge 1\}$, the independence of $\{x_i(0), i\ge 1\}$ and the boundedness of $\Phi_\theta(t,s)$ as implied by Remark A.4. The last inequality follows from (A1). ∎

By using Lemma 4.2 we may further establish the next theorem.

THEOREM 4.3. *Assume* (A1)–(A3). *We have*

$$E\int_0^T\Big[\big|z(t)-x^{(N)}(t)\big|^2 + \sup_{0\le j\le N}\big|x_j(t)-\bar{x}_j(t)\big|^2\Big]dt = O\Big(\epsilon_N^2+\frac{1}{N}\Big). \tag{4.10}$$

Proof. See Appendix B. ∎

Consider the system of $N+1$ players described by (1.1)–(1.2). Denote the class $\mathcal{U}_W$ consisting of all $\mathbb{R}^{(N+1)n}$-valued processes y_W of the form

$$y_W(t) = \int_0^t [h_0(t,s)dW_0(s); h_1(t,s)dW_1(s); \dots ; h_N(t,s)dW_N(s)],$$

where each h_j is an $\mathbb{R}^{n\times n_2}$-valued bounded measurable function on Δ.

Denote $\xi = (x_0, x_1, \dots, x_N, y_W)$. For any $j = 0, \dots, N$, the admissible control set $\mathcal{U}_j$ of player $\mathcal{A}_j$ consists of controls u_j of the form $f(t, \xi_t)$ for some $y_W \in \mathcal{U}_W$ and some continuous function f, Lipschitz continuous in ξ_t. So the control of a player may not be purely in a feedback form since the noise process may be used via y_W; this general form of controls is necessary in order to include the decentralized strategies (4.1)–(4.2). Since the control still uses the players' states, $(u_0, u_1, \dots, u_n)$ will be called a set of partial state feedback strategies. Note that $\mathcal{U}_j$ is not restricted to be decentralized. Given each set of strategies in $\mathcal{U}_0 \times \cdots \times \mathcal{U}_N$, the closed-loop system has a unique strong solution. Recall $u_{-j} = (u_0, u_1, \dots, u_{j-1}, u_{j+1}, \dots, u_N)$ for $j = 0, \dots, N$.

Definition 4.4. *A set of controls $u_j \in \mathcal{U}_j$, $0 \le j \le N$, for the $N+1$ players is called an ε-Nash equilibrium with respect to the costs J_j, $0 \le j \le N$, where $\varepsilon \ge 0$, if for any j, $0 \le j \le N$, $J_j(u_j, u_{-j}) \le J_j(u'_j, u_{-j}) + \varepsilon$, when any alternative u'_j is applied by player $\mathcal{A}_j$.*

Theorem 4.5. *Assume* (A1)–(A3). *Let $\bar{u}_0$ and $\bar{u}_j$ be the optimal controls in Problems* (I) *and* (II). *For $0 \le j \le N$,*

$$\Big| J_j(\hat{u}_j, \hat{u}_{-j}) - \bar{J}_j(\bar{u}_j) \Big| = O\Big(\epsilon_N + \frac{1}{\sqrt{N}}\Big). \tag{4.11}$$

Proof. See Appendix C. ∎

Theorem 4.6. *Assume* (A1)–(A3). *Then the set of controls $\hat{u}_j$, $0 \le j \le N$, for the $N+1$ players is an ε-Nash equilibrium, i.e., for $0 \le j \le N$,*

$$J_j(\hat{u}_j, \hat{u}_{-j}) - \varepsilon \le \inf_{u_j \in \mathcal{U}_j} J_j(u_j, \hat{u}_{-j}) \le J_j(\hat{u}_j, \hat{u}_{-j}), \tag{4.12}$$

where $0 \le \varepsilon = O(\epsilon_N + 1/\sqrt{N})$.

Proof. See Appendix C. ∎

5. The scalar model and numerical solutions. The dynamics of the $N+1$ players are given by

$$dx_0 = [a_0 x_0(t) + b_0 u_0(t)]dt + D_0 dw_0(t), \quad t \ge 0, \tag{5.1}$$

$$dx_i = [a_i x_i(t) + b u_i(t)]dt + D dw_i(t), \quad 1 \le i \le N, \tag{5.2}$$

where x_0, u_0, w_0, x_i, u_i, and w_i are all scalar processes. For simplicity, we only let a_i be dependent on the players, and furthermore, $Ex_i(0) = 0$ for $i \ge 1$. Suppose $a_i \in \Theta := [\underline{a}, \bar{a}]$, which is a compact interval.

The cost functions for $\mathcal{A}_0$ and $\mathcal{A}_i$, $1 \le i \le N$, are given by

$$J_0(u_0, u_{-0}) = E\int_0^T \big\{ q_0(x_0(t) - h_0 x^{(N)}(t) - \eta_0)^2 + u_0^2(t) \big\} dt,$$

$$J_i(u_i, u_{-i}) = E\int_0^T \big\{ q(x_i(t) - h x_0(t) - \hat{h} x^{(N)}(t) - \eta)^2 + u_i^2(t) \big\} dt.$$

We introduce the following assumption to replace (A2).

(A2′) There exists a distribution function $\mathbf{F}(a)$ on $\mathbb{R}$ such that the sequence of empirical distribution functions $\mathbf{F}_N(a) = \frac{1}{N}\sum_{i=1}^N 1_{\{a_i \le a\}}$, $N \ge 1$, converges to $\mathbf{F}(a)$ weakly.

Let $P_0(t) \ge 0$ be the solution to the Riccati equation

$$\dot{P}_0(t) + 2a_0 P_0(t) - b_0^2 P_0^2(t) + q_0 = 0, \quad P_0(T) = 0.$$

Then

$$P_0(t) = \frac{q_0\Big(e^{\alpha(T-t)} - e^{\alpha(t-T)}\Big)}{\big(\alpha + a_0\big)e^{\alpha(t-T)} + \big(\alpha - a_0\big)e^{\alpha(T-t)}} \ge 0, \quad 0 \le t \le T,$$

where $\alpha = \sqrt{a_0^2 + b_0^2 q_0}$. Denote

$$\mathbf{a}_0(t) = a_0 - P_0(t) b_0^2, \quad 0 \le t \le T. \tag{5.3}$$

Since $P_0(t) \ge 0$ for $0 \le t \le T$, we have $\mathbf{a}_0(t) \le a_0$ for $0 \le t \le T$. Note that in the present setup, (A.2) with $M_0(t) = \mathbf{a}_0(t)$ gives $\Phi_0(t,s) = \exp\{\int_s^t \mathbf{a}_0(s_1)ds_1\}$. Following the steps as in (2.5)–(2.21), we have $\bar{x}_0(t) = f_{\bar{x}_0,1}(t) + f_{\bar{x}_0,2}(t)x_0(0) + \int_0^t g_{\bar{x}_0}(t,s)dw_0(s)$.

Similarly, for $a \in [\underline{a}, \bar{a}]$, let $P_a(t) \ge 0$ be the solution to the Riccati equation

$$\dot{P}_a(t) + 2aP_a(t) - b^2 P_a^2(t) + q = 0, \quad P_a(T) = 0,$$

and denote

$$\mathbf{a}(t) = a - P_a(t) b^2, \quad 0 \le t \le T. \tag{5.4}$$

Then, $\mathbf{a}(t) \le a$ for $a \in [\underline{a}, \bar{a}]$. Now (A.8) with $M_a(t) = \mathbf{a}(t)$ gives $\Phi_a(t,s) = \exp\{\int_s^t \mathbf{a}(s_1)ds_1\}$. Following the steps as in (2.22)–(2.38), we have

$$\bar{x}_i(t) = f_{\bar{x}_i,1}(t) + f_{\bar{x}_i,2}(t)x_0(0) + f_{\bar{x}_i,3}(t)x_i(0) + \int_0^t g_{\bar{x}_i}(t,s)dw_0(s) + \int_0^t h_{\bar{x}_i}(t,s)dw_i(s).$$

5.1. Nash certainty equivalence equation system. Following the notation in section 3, for $f \in C([0,T],\mathbb{R})$ and $g \in C(\Delta,\mathbb{R})$, denote the linear operators

$$\begin{aligned}
[\bar{\Gamma}_0 f](t) &= q_0 b_0^2 h_0 \int_0^t \int_{s_1}^T e^{\int_{s_1}^t \mathbf{a}_0(s_3)ds_3} e^{\int_{s_1}^{s_2} \mathbf{a}_0(s_4)ds_4} f(s_2)ds_2 ds_1,\\
[\bar{\Lambda}_0 g](t,s) &= q_0 b_0^2 h_0 \int_s^t \int_{s_1}^T e^{\int_{s_1}^t \mathbf{a}_0(s_3)ds_3} e^{\int_{s_1}^{s_2} \mathbf{a}_0(s_4)ds_4} g(s_2,s)ds_2 ds_1,\\
[\bar{\Gamma} f](t) &= qb^2 \int_0^t \int_{s_1}^T \int_{\underline{a}}^{\bar{a}} e^{\int_{s_1}^t \mathbf{a}(s_3)ds_3} e^{\int_{s_1}^{s_2} \mathbf{a}(s_4)ds_4} (h[\bar{\Gamma}_0 f](s_2) + \hat{h}f(s_2))d\mathbf{F}(a)ds_2 ds_1,\\
[\bar{\Lambda} g](t,s) &= qb^2 \int_s^t \int_{s_1}^T \int_{\underline{a}}^{\bar{a}} e^{\int_{s_1}^t \mathbf{a}(s_3)ds_3} e^{\int_{s_1}^{s_2} \mathbf{a}(s_4)ds_4} \big(h[\bar{\Lambda}_0 g](s_2,s)\\
&\qquad + \hat{h}g(s_2,s)\big)d\mathbf{F}(a)ds_2 ds_1,
\end{aligned}$$

where $\mathbf{a}_0(t)$ and $\mathbf{a}(t)$ are defined in (5.3) and (5.4), respectively. For the scalar case, $\bar{\Gamma}_{0,1}$ and $\bar{\Gamma}_{0,2}$ defined in (3.11) coincide. Similarly, $\bar{\Gamma}_1$ and $\bar{\Gamma}_2$ defined in (3.12) coincide.

Denote

$$\varphi_0(t) = q_0 b_0^2 \eta_0 \int_0^t \int_{s_1}^T e^{\int_{s_1}^t \mathbf{a}_0(s_3)ds_3} e^{\int_{s_1}^{s_2} \mathbf{a}_0(s_4)ds_4} ds_2 ds_1,$$

$$\varphi_1(t) = qb^2 \int_0^t \int_{s_1}^T \int_{\underline{a}}^{\bar{a}} e^{\int_{s_1}^t \mathbf{a}(s_3)ds_3} e^{\int_{s_1}^{s_2} \mathbf{a}(s_4)ds_4} (h\varphi_0(s_2) + \eta) d\mathbf{F}(a) ds_2 ds_1,$$

$$\varphi_2(t) = qb^2 h \int_0^t \int_{s_1}^T \int_{\underline{a}}^{\bar{a}} e^{\int_{s_1}^t \mathbf{a}(s_3)ds_3} e^{\int_{s_1}^{s_2} \mathbf{a}(s_4)ds_4} e^{\int_0^{s_2} \mathbf{a}_0(s_5)ds_5} d\mathbf{F}(a) ds_2 ds_1,$$

$$\psi(t,s) = qb^2 h D_0 \int_s^t \int_{s_1}^T \int_{\underline{a}}^{\bar{a}} e^{\int_{s_1}^t \mathbf{a}(s_3)ds_3} e^{\int_{s_1}^{s_2} \mathbf{a}(s_4)ds_4} e^{\int_s^{s_2} \mathbf{a}_0(s_5)ds_5} d\mathbf{F}(a) ds_2 ds_1.$$

Taking into account $Ex_i(0) = 0$, $1 \le i \le N$, we obtain the NCE equation system

$$\begin{cases} f_j(t) = [\bar{\Gamma} f_j](t) + \varphi_j(t), \quad j = 1, 2, \\ g(t,s) = [\bar{\Lambda} g](t,s) + \psi(t,s). \end{cases} \tag{5.5}$$

COROLLARY 5.1. *The NCE equation system* (5.5) *has a unique solution if*

$$\frac{qb^2T^2}{2}\Big(|\hat{h}| + \frac{|h_0 h| q_0 b_0^2 T^2}{2} e^{2Ta_0 \vee 0}\Big) \int_{\underline{a}}^{\bar{a}} e^{2Ta\vee 0} d\mathbf{F}(a) < 1.$$

Proof. By Theorem 3.6, it suffices to show $\|\bar{\Gamma}\| < 1$ and $\|\bar{\Lambda}\| < 1$. Since $\mathbf{a}_0(t) \le a_0$ for $0 \le t \le T$,

$$\|\bar{\Gamma}_0\| \le q_0 b_0^2 |h_0| \int_0^t \int_{s_1}^T e^{(t-s_1)a_0} e^{(s_2-s_1)a_0} ds_2 ds_1 \le q_0 b_0^2 |h_0| \frac{T^2}{2} e^{2Ta_0\vee 0}.$$

Since $\mathbf{a}(t) \le a$ for $0 \le t \le T$, we have

$$\begin{aligned} \|\bar{\Gamma}\| &\le qb^2(|h|\|\bar{\Gamma}_0\| + |\hat{h}|) \int_{\underline{a}}^{\bar{a}} \int_0^t \int_{s_1}^T e^{(t-s_1)a} e^{(s_2-s_1)a} ds_2 ds_1 d\mathbf{F}(a) \\ &\le \frac{qb^2T^2}{2}\Big(|\hat{h}| + \frac{|h_0 h| q_0 b_0^2 T^2}{2} e^{2Ta_0\vee 0}\Big) \int_{\underline{a}}^{\bar{a}} e^{2Ta\vee 0} d\mathbf{F}(a) < 1. \end{aligned}$$

Similarly, $\|\bar{\Lambda}\| < 1$. This completes the proof. ∎

In the numerical solution for (5.5), we take the parameters $T = 1$,

$$[a_0, b_0, D_0, q_0, h_0, \eta_0] = [0.5, 1, 1, 1, 0.6, 1.5],$$
$$[\underline{a}, \bar{a}, b, D, q, h, \hat{h}, \eta] = [0.1, 0.4, 1, 1, 1.2, 0.5, 0.4, 0.5],$$

and $\mathbf{F}(a)$ is a uniform distribution on $[\underline{a}, \bar{a}]$. It can be verified that the condition of Corollary 5.1 holds for this example. For the iteration of the fixed point equations in (5.5), we take a step size of 0.05 to discretize t and a, and the solutions of f_1, f_2, and g are shown in Figure 1.

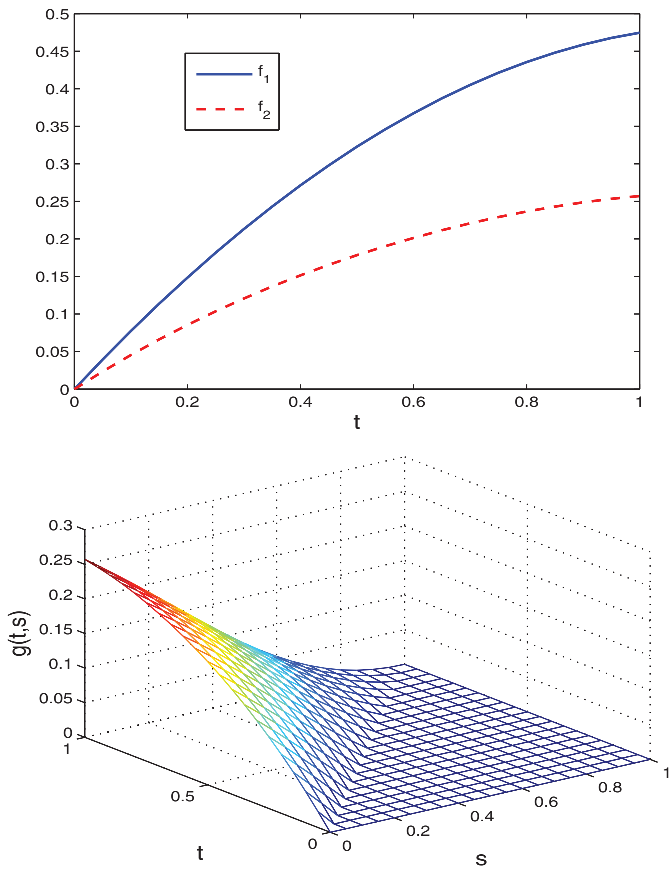

FIG. 1. *The numerical solution of $f_1(t)$, $f_2(t)$, and $g(t,s)$.*

6. Concluding remarks. This paper considers mean field LQG games with a major player and continuum-parametrized minor players. We introduce random mean field approximations, solve the resulting limiting problems as stochastic optimal control with random coefficients, and further obtain decentralized strategies for the players.

For the model analyzed in the paper, the minor players are affected by the major player only via their cost functions. The mean field in the closed loop is nonresponsive to the strategy change of the major player when the minor players implement their ε-Nash strategies.

If the state of the major player appears in the dynamics of each minor player, the mean field will become responsive to the major player's control. More specifically, the strategy change of the major player causes a change in the major player's state and consequently a change in each minor player's state, leading to a change of the mean field. Thus, there is a strong coupling between the major player's strategy change and the mean field evolution. In this case, the design of the major player's strategy should address its ability in simultaneously perturbing its own state process and the mean field; some initial progress has been made in the recent work [27] by introducing a procedure called anticipative variational calculations for the major player's limiting control problem. The analysis in [27] treated homogeneous minor players and it is of interest to generalize it to continuum-parametrized minor players.

Appendix A. Auxiliary lemmas. To solve the forward-backward SDE systems in section 2, we introduce two general lemmas.

LEMMA A.1. (i) *Let $M_0 \in C([0,T], \mathbb{R}^{n\times n})$, $D_0 \in \mathbb{R}^{n\times n_2}$, and*

$$\xi_0(t) = f_{\xi_0,1}(t) + f_{\xi_0,2}(t)x_0(0) + \int_0^t g_{\xi_0}(t,s)dW_0(s),$$

where $f_{\xi_0,1} \in C([0,T],\mathbb{R}^n)$, $f_{\xi_0,2} \in C([0,T],\mathbb{R}^{n\times n})$, $g_{\xi_0} \in C(\Delta, \mathbb{R}^{n\times n_2})$ are deterministic functions and W_0 is an n_2-dimensional standard Brownian motion. Let x_0 be the solution of the SDE

$$dx_0(t) = \big(\xi_0(t) + M_0(t)x_0(t)\big)dt + D_0 dW_0(t) \tag{A.1}$$

with the initial condition $x_0(0)$ independent of W_0. Then

$$x_0(t) = f_{x_0,1}(t) + f_{x_0,2}(t)x_0(0) + \int_0^t g_{x_0}(t,s)dW_0(s),$$

where

$$\begin{aligned} f_{x_0,1}(t) &= \int_0^t \Phi_0(t,s_1) f_{\xi_0,1}(s_1)ds_1, \\ f_{x_0,2}(t) &= \Phi_0(t,0) + \int_0^t \Phi_0(t,s_1) f_{\xi_0,2}(s_1)ds_1, \\ g_{x_0}(t,s) &= \Phi_0(t,s)D_0 + \int_s^t \Phi_0(t,s_1) g_{\xi_0}(s_1,s)ds_1, \end{aligned}$$

and $\Phi_0(\cdot,\cdot)$ is the unique solution of the system

$$\begin{cases} d\Phi_0(t,s) = M_0(t)\Phi_0(t,s)dt, \\ \Phi_0(s,s) = I, \qquad 0 \le s,t \le T. \end{cases} \tag{A.2}$$

(ii) *Let M_0, W_0, $x_0(0)$, and Φ_0 be the same as in* (i), *and $\zeta_0(t) = f_{\zeta_0,1}(t) + f_{\zeta_0,2}(t)x_0(0) + \int_0^t g_{\zeta_0}(t,s)dW_0(s)$, where $f_{\zeta_0,1} \in C([0,T],\mathbb{R}^n)$, $f_{\zeta_0,2} \in C([0,T],\mathbb{R}^{n\times n})$, $g_{\zeta_0} \in C(\Delta,\mathbb{R}^{n\times n_2})$ and $x_0(0)$ is independent of W_0. The backward SDE*

$$\begin{cases} d\nu_0(t) = \big(\zeta_0(t) - M_0^T(t)\nu_0(t)\big)dt + \mu_0(t)dW_0(t), \\ \nu_0(T) = 0 \end{cases} \tag{A.3}$$

has a unique solution (ν_0,μ_0) and $\nu_0(t) = f_{\nu_0,1}(t) + f_{\nu_0,2}(t)x_0(0) + \int_0^t g_{\nu_0}(t,s)dW_0(s)$, where

$$\begin{aligned} f_{\nu_0,j}(t) &= -\int_t^T \Phi_0^T(s_1,t) f_{\zeta_0,j}(s_1)ds_1, \quad j=1,2, \\ g_{\nu_0}(t,s) &= -\int_t^T \Phi_0^T(s_1,t) g_{\zeta_0}(s_1,s)ds_1. \end{aligned}$$

Proof. (i) By Fubini's theorem for stochastic integrals,

$$\begin{aligned}
x_0(t) &= \int_0^t \Phi_0(t,s_1)\Big(f_{\xi_0,1}(s_1) + f_{\xi_0,2}(s_1)x_0(0) + \int_0^{s_1} g_{\xi_0}(s_1,s)dW_0(s)\Big)ds_1 \\
&\quad + \Phi_0(t,0)x_0(0) + \int_0^t \Phi_0(t,s)D_0 dW_0(s) \\
&= \int_0^t \Phi_0(t,s_1) f_{\xi_0,1}(s_1)ds_1 + \Big(\Phi_0(t,0) + \int_0^t \Phi_0(t,s_1) f_{\xi_0,2}(s_1)ds_1\Big)x_0(0) \\
&\quad + \int_0^t \Big(\Phi_0(t,s)D_0 + \int_s^t \Phi_0(t,s_1) g_{\xi_0}(s_1,s)ds_1\Big)dW_0(s).
\end{aligned}$$

This gives the desired identities.

(ii) The linear backward SDE has a unique solution [5, 36]. Let $\Psi_0(t,s) = \Phi_0^T(s,t)$ for $t \geq 0$ and $s \geq 0$. Then by [7, Chapter 2],

$$\begin{cases} d\Psi_0(t,s) = -M_0^T(t)\Psi_0(t,s)dt, \\ \Psi_0(s,s) = I, \qquad t \geq 0,\ s \geq 0. \end{cases} \tag{A.4}$$

Denote $\hat{\zeta}_0(t) = \Psi_0(0,t)\zeta_0(t)$, $\hat{\nu}_0(t) = \Psi_0(0,t)\nu_0(t)$, and $\hat{\mu}_0(t) = \Psi_0(0,t)\mu_0(t)$. Then

$$\hat{\zeta}_0(t) = f_{\hat{\zeta}_0,1}(t) + f_{\hat{\zeta}_0,2}(t)x_0(0) + \int_0^t g_{\hat{\zeta}_0}(t,s)dW_0(s),$$

where

$$f_{\hat{\zeta}_0,1}(t) = \Psi_0(0,t)f_{\zeta_0,1}(t), \quad f_{\hat{\zeta}_0,2}(t) = \Psi_0(0,t)f_{\zeta_0,2}(t), \quad g_{\hat{\zeta}_0}(t,s) = \Psi_0(0,t)g_{\zeta_0}(t,s). \tag{A.5}$$

By virtue of (A.4), $d\Psi_0(0,t) = \Psi_0(0,t)M_0^T(t)dt$. Thus, by Ito's formula,

$$\begin{cases} d\hat{\nu}_0(t) = \hat{\zeta}_0(t)dt + \hat{\mu}_0(t)dW_0(t), \\ \hat{\nu}_0(T) = 0. \end{cases} \tag{A.6}$$

By (A.6) and Fubini's theorem,

$$\begin{aligned}
\hat{\nu}_0(T) - \hat{\nu}_0(t) &= \int_t^T f_{\hat{\zeta}_0,1}(s)ds + \int_t^T f_{\hat{\zeta}_0,2}(s)dsx_0(0) + \int_0^t \int_t^T g_{\hat{\zeta}_0}(s,s_1)dsdW_0(s_1) \\
&\quad + \int_t^T \int_{s_1}^T g_{\hat{\zeta}_0}(s,s_1)dsdW_0(s_1) + \int_t^T \hat{\mu}_0(s)dW_0(s).
\end{aligned}$$

Taking conditional expectation with respect to $\sigma\{x_0(0), W_0(s), s \leq t\}$ yields

$$\hat{\nu}_0(t) = -\int_t^T f_{\hat{\zeta}_0,1}(s)ds - \int_t^T f_{\hat{\zeta}_0,2}(s)dsx_0(0) - \int_0^t \int_t^T g_{\hat{\zeta}_0}(s,s_1)dsdW_0(s_1). \tag{A.7}$$

This implies that $\hat{\nu}_0(0) = -\int_0^T f_{\hat{\zeta}_0,1}(s)ds - \int_0^T f_{\hat{\zeta}_0,2}(s)dsx_0(0)$ and

$$\begin{aligned}
\hat{\nu}_0(t) - \hat{\nu}_0(0) &= \int_0^t f_{\hat{\zeta}_0,1}(s)ds + \int_0^t f_{\hat{\zeta}_0,2}(s)dsx_0(0) - \int_0^t \int_t^T g_{\hat{\zeta}_0}(s,s_1)dsdW_0(s_1) \\
&= \int_0^t \hat{\zeta}_0(s)ds + \int_0^t \hat{\mu}_0(s)dW_0(s),
\end{aligned}$$

where the second equality follows from (A.6). By using Fubini's theorem again we obtain $\hat{\mu}_0(s) = -\int_s^T g_{\hat{\zeta}_0}(t,s)dt$ as a deterministic function. Since $\Psi_0(t,s) = \Psi_0(t,0)\Psi_0(0,s)$ and $\Psi_0(t,0) = \Psi_0^{-1}(0,t)$, we have $\nu_0(t) = \Psi_0(t,0)\hat{\nu}_0(t)$, $\zeta_0(t) = \Psi_0(t,0)\hat{\zeta}_0(t)$, and $\mu_0(t) = \Psi_0(t,0)\hat{\mu}_0(t)$. Therefore, from (A.5) and (A.7) we have

$$\begin{aligned}\nu_0(t) = &-\int_t^T \Psi_0(t,s)f_{\zeta_0,1}(s)ds - \int_t^T \Psi_0(t,s)f_{\zeta_0,2}(s)dsx_0(0)\\ &-\int_0^t\int_t^T \Psi_0(t,s)g_{\zeta_0}(s,s_1)dsdW_0(s_1).\end{aligned}$$

Since $\Psi_0(t,s) = \Phi_0^T(s,t)$, this completes the proof. ∎

LEMMA A.2. (i) *Let W_0, W_i be two independent standard Brownian motions, which are also independent of $\mathbb{R}^n$-valued random vectors $x_0(0)$ and $x_i(0)$. Let $M_i \in C([0,T],\mathbb{R}^{n\times n})$, $D_i \in \mathbb{R}^{n\times n_2}$, and*

$$\xi_i(t) = f_{\xi_i,1}(t) + f_{\xi_i,2}(t)x_0(0) + \int_0^t g_{\xi_i}(t,s)dW_0(s),$$

where $f_{\xi_i,1} \in C([0,T],\mathbb{R}^n)$, $f_{\xi_i,2} \in C([0,T],\mathbb{R}^{n\times n})$, $g_{\xi_i} \in C(\Delta,\mathbb{R}^{n\times n_2})$. Let x_i be the solution of

$$dx_i(t) = (\xi_i(t) + M_i(t)x_i(t))dt + D_i dW_i(t).$$

Then

$$x_i(t) = f_{x_i,1}(t)+f_{x_i,2}(t)x_0(0)+f_{x_i,3}(t)x_i(0)+\int_0^t g_{x_i}(t,s)dW_0(s)+\int_0^t h_{x_i}(t,s)dW_i(s),$$

where

$$\begin{aligned}&f_{x_i,1}(t) = \int_0^t \Phi_i(t,s)f_{\xi_i,1}(s)ds, \quad f_{x_i,2}(t) = \int_0^t \Phi_i(t,s)f_{\xi_i,2}(s)ds,\\ &f_{x_i,3}(t) = \Phi_i(t,0), \quad g_{x_i}(t,s) = \int_s^t \Phi_i(t,s_1)g_{\xi_i}(s_1,s)ds_1,\\ &h_{x_i}(t,s) = \Phi_i(t,s)D_i,\end{aligned}$$

and $\Phi_i(\cdot,\cdot)$ is the unique solution of

$$\begin{cases} d\Phi_i(t,s) = M_i(t)\Phi_i(t,s)dt,\\ \Phi_i(s,s) = I, \qquad 0 \le s,t \le T.\end{cases} \tag{A.8}$$

(ii) *Let $(M_i, W_0, W_i, \Phi_i, x_0(0))$ be the same as in* (i), *and $\zeta_i(t) = f_{\zeta_i,1}(t) + f_{\zeta_i,2}(t)x_0(0) + \int_0^t g_{\zeta_i}(t,s)dW_0(s)$, where $f_{\zeta_i,1} \in C([0,T],\mathbb{R}^n)$, $f_{\zeta_i,2} \in C([0,T],\mathbb{R}^{n\times n})$, $g_{\zeta_i} \in C(\Delta,\mathbb{R}^{n\times n_2})$. Then*

$$\begin{cases} d\nu_i(t) = (\zeta_i(t) - M_i^T(t)\nu_i(t))dt + \mu_i(t)dW_0(t) + \lambda_i(t)dW_i(t),\\ \nu_i(T) = 0,\end{cases}$$

has a unique solution (ν_i,μ_i,λ_i) which satisfies

$$\nu_i(t) = f_{\nu_i,1}(t) + f_{\nu_i,2}(t)x_0(0) + \int_0^t g_{\nu_i}(t,s)dW_0(s),$$

where

$$f_{\nu_i,j}(t) = -\int_t^T \Phi_i^T(s_1,t) f_{\zeta_i,j}(s_1)ds_1, \quad j=1,2,$$
$$g_{\nu_i}(t,s) = -\int_t^T \Phi_i^T(s_1,t) g_{\zeta_i}(s_1,s)ds_1.$$

Proof. The proof of this lemma is similar to that of Lemma A.1 and we omit the details. ∎

LEMMA A.3. *Let $A(\theta)$ and $B(\theta)$ be continuous matrix-valued functions on a compact set Θ. For each $\theta \in \Theta$, let $P_\theta(t)$ be the solution to the Riccati equation*

$$\begin{cases} \dot{P}_\theta(t) + P_\theta(t)A(\theta) + A^T(\theta)P_\theta(t) - P_\theta(t)B(\theta)R^{-1}B^T(\theta)P_\theta(t) + Q = 0, \\ P_\theta(T) = 0. \end{cases} \tag{A.9}$$

Then $P_\theta(t)$ is continuous in (t,θ), and there exists a constant C such that

$$\sup_{\theta\in\Theta, 0\le t\le T} |P_\theta(t)| \le C.$$

Proof. We use the method of Bernoulli substitution to transform (A.9) into linear equations [24, 29]. Let $U_\theta(t)$ and $V_\theta(t)$ be determined by

$$\begin{cases} \dot{U}_\theta(t) = \big[A(\theta) - B(\theta)R^{-1}B^T(\theta)P_\theta(t)\big]U_\theta(t), \\ U_\theta(T) = I, \end{cases} \qquad \begin{cases} V_\theta(t) = P_\theta(t)U_\theta(t), \\ V_\theta(T) = 0. \end{cases}$$

It is clear that U_θ^{-1} exists for $(t,\theta) \in [0,T]\times\Theta$, and (U_θ, V_θ) satisfies the ODE

$$\begin{bmatrix} \dot{U}_\theta(t) \\ \dot{V}_\theta(t) \end{bmatrix} = \begin{bmatrix} A(\theta) & -B(\theta)R^{-1}B^T(\theta) \\ -Q & -A^T(\theta) \end{bmatrix} \begin{bmatrix} U_\theta(t) \\ V_\theta(t) \end{bmatrix}, \qquad \begin{bmatrix} U_\theta(T) \\ V_\theta(T) \end{bmatrix} = \begin{bmatrix} I \\ 0 \end{bmatrix}.$$

Since $A(\cdot)$ and $B(\cdot)$ are continuous functions of θ, by [9, Theorem 7.4, p. 29], $U_\theta(t)$ and $V_\theta(t)$ are continuous in $(t,\theta)\in[0,T]\times\Theta$. Then $U_\theta^{-1}(t)$ is continuous in (t,θ), so is $P_\theta(t) = V_\theta(t)U_\theta^{-1}(t)$. By the compactness of Θ, we obtain the desired bound for $|P_\theta(t)|$. ∎

Remark A.4. Let Θ be a compact set, and $M_\theta(t)$ a continuous $\mathbb{R}^{n\times n}$-valued function of $(t,\theta)\in[0,T]\times\Theta$. Let $\Phi_\theta(t,s)$ be the solution of $d\Phi_\theta(t,s) = M_\theta(t)\Phi_\theta(t,s)dt$, $\Phi_\theta(s,s) = I$ for $0\le s,t\le T,\ \theta\in\Theta$. Then $\Phi_\theta(t,s)$ is continuous in (t,s,θ) and hence $\sup_{\theta\in\Theta}\sup_{0\le t,s\le T}|\Phi_\theta(t,s)| < \infty$.

Appendix B. Proof of Theorem 4.3. Let $\mathbb{A}_0(t)$ and $\mathbb{A}_{\theta_i}(t)$ be defined as in (2.9) and (2.24), respectively. By (4.3) and (4.5),

$$x_0(t) = \bar{x}_0(t) + \int_0^t \Phi_0(t,s_1)F_0\big(x^{(N)}(s_1) - z(s_1)\big)ds_1. \tag{B.1}$$

Next, by (4.4) and (4.6),

$$x_i(t) = z(t) + \big(\bar{x}_i(t) - z(t)\big) + \int_0^t \Phi_{\theta_i}(t,s_1)F(\theta_i)(x^{(N)}(s_1) - z(s_1))ds_1. \tag{B.2}$$

By adding both sides of (B.2) for $i = 1, \ldots, N$ and normalizing by $1/N$, we obtain

$$x^{(N)}(t) - z(t) = \big(\bar{x}^{(N)}(t) - z(t)\big) + \frac{1}{N}\sum_{i=1}^{N}\int_0^t \Phi_{\theta_i}(t, s_1)F(\theta_i)(x^{(N)}(s_1) - z(s_1))ds_1.$$

By (A3) and Remark A.4,

$$|x^{(N)}(t) - z(t)| \le \big|\bar{x}^{(N)}(t) - z(t)\big| + C\int_0^t \big|z(s_1) - x^{(N)}(s_1)\big|ds_1.$$

Hence, for $0 \le t \le T$,

$$\begin{aligned} E\int_0^t |x^{(N)}(s) - z(s)|^2 ds &\le 2E\int_0^t |\bar{x}^{(N)}(s) - z(s)|^2 ds \\ &\quad + 2C^2 E\int_0^t \bigg(\int_0^s \big|x^{(N)}(s_1) - z(s_1)\big|ds_1\bigg)^2 ds. \end{aligned}$$

Denote $\alpha(t) = E\int_0^t |x^{(N)}(s) - z(s)|^2 ds$. By Lemma 4.2, we have

$$\alpha(t) \le C\Big(\epsilon_N^2 + \frac{1}{N}\Big) + 2C^2 T\int_0^t \alpha(s)ds, \quad 0 \le t \le T.$$

By Gronwall's inequality,

$$E\int_0^t |x^{(N)}(s) - z(s)|^2 ds \le C\Big(\epsilon_N^2 + \frac{1}{N}\Big)e^{2C^2T^2}. \tag{B.3}$$

It follows from (B.1) and Remark A.4 that

$$\begin{aligned} E\int_0^T \big|x_0(t) - \bar{x}_0(t)\big|^2 dt &= E\int_0^T \bigg|\int_0^t \Phi_0(t, s_1)F_0\big(x^{(N)}(s_1) - z(s_1)\big)ds_1\bigg|^2 dt \\ &\le CE\int_0^T t\int_0^t \big|x^{(N)}(s_1) - z(s_1)\big|^2 ds_1 dt \\ &= O\Big(\epsilon_N^2 + \frac{1}{N}\Big). \end{aligned} \tag{B.4}$$

Similarly, by virtue of (B.2) and Remark A.4 we have

$$\sup_{1\le i\le N} E\int_0^T \big|x_i(t) - \bar{x}_i(t)\big|^2 dt = O\Big(\epsilon_N^2 + \frac{1}{N}\Big). \tag{B.5}$$

Combining (B.3)–(B.5), we obtain (4.10). ∎

Appendix C. Proof of Theorems 4.5–4.6. Before proving the two theorems, we give two auxiliary lemmas. Suppose that $\xi_a(\cdot)$ and $\xi_b(\cdot)$ are random processes which are adapted to the filtration $\{\mathcal{F}_t\}_{t\ge 0}$ and satisfy

$$\epsilon_a = \bigg(\int_0^T E|\xi_a(t)|^2 dt\bigg)^{1/2} < \infty, \quad \epsilon_b = \bigg(\int_0^T E|\xi_b(t)|^2 dt\bigg)^{1/2} < \infty. \tag{C.1}$$

C.1. A perturbed version of Problem (I) for the major player. Let $z(t)$ be given as in (4.8). Recall that for the optimal control problem with dynamics and cost

$$dx_0(t) = \big(A_0x_0(t) + B_0u_0(t) + F_0z(t)\big)dt + D_0dW_0(t), \tag{C.2}$$

$$\bar{J}_0(u_0) = E\int_0^T \big(\big|x_0(t) - H_0z(t) - \eta_0\big|^2_{Q_0} + u_0^T(t)R_0u_0(t)\big)dt, \tag{C.3}$$

the optimal control law is $\bar{u}_0(t) = R_0^{-1}B_0^T(-P_0(t)x_0(t) + \nu_0(t))$, where $P_0(t)$ and $\nu_0(t)$ are determined by (2.8) and (2.14)–(2.17).

Subsequently, we consider a perturbed version of (C.2)–(C.3) having the dynamics and cost

$$dx_0(t) = (A_0x_0(t) + B_0u_0(t) + F_0z(t) + \xi_a(t))dt + D_0dW_0(t), \quad t \geq 0, \tag{C.4}$$

$$J_0^\xi(u_0) = E\int_0^T \Big\{\big|x_0(t) - H_0z(t) + \xi_b(t) - \eta_0\big|^2_{Q_0} + u_0^T(t)R_0u_0(t)\Big\}dt. \tag{C.5}$$

LEMMA C.1. *Let $\bar{u}_0$ be the optimal control of the control problem* (C.2)–(C.3). *For any u_0 adapted to $\{\mathcal{F}_t\}_{t\geq 0}$, we have*

$$J_0^\xi(u_0) \geq \bar{J}_0(\bar{u}_0) - O(\epsilon_a + \epsilon_b),$$

provided that for a fixed constant C_0, the pair (x_0, u_0) in (C.4) *has the prior upper bound*

$$E\int_0^T (|x_0(t)|^2 + |u_0(t)|^2)dt \leq C_0. \tag{C.6}$$

Proof. We write (C.4) in the form

$$dx_0(t) = \Big(\mathbb{A}_0(t)x_0(t) + B_0u_0'(t) + B_0R_0^{-1}B_0^T\nu_0(t) + F_0z(t) + \xi_a(t)\Big)dt + D_0dW_0(t), \tag{C.7}$$

where $u_0'(t) = u_0(t) + R_0^{-1}B_0^T(P_0(t)x_0(t) - \nu_0(t))$. By (C.6), $E\int_0^T |u_0'(t)|^2dt \leq C_1$, where $C_1 < \infty$ depends on C_0. For given $x_0(0)$ and u_0, u_0' is a well-defined process adapted to $\{\mathcal{F}_t\}_{t\geq 0}$. We construct the control problem with dynamics and cost

$$\begin{aligned} d\tilde{x}_0(t) &= \Big(\mathbb{A}_0(t)\tilde{x}_0(t) + B_0u_0''(t) + B_0R_0^{-1}B_0^T\nu_0(t) + F_0z(t)\Big)dt + D_0dW_0(t), \\ \tilde{J}_0(u_0'') &= E\int_0^T \Big\{\big|\tilde{x}_0 - H_0z - \eta_0\big|^2_{Q_0} + \big|u_0'' - R_0^{-1}B_0^T(P_0(t)\tilde{x}_0(t) - \nu_0(t))\big|^2_{R_0}\Big\}dt, \end{aligned} \tag{C.8}$$

where $\tilde{x}_0(0) = x_0(0)$ and u_0'' is adapted to $\{\mathcal{F}_t\}_{t\geq 0}$. It is clear that $\tilde{J}_0(u_0'')$ attains its minimum when $u_0'' \equiv 0$ and the corresponding cost is equal to $\bar{J}_0(\bar{u}_0)$. Hence

$$\tilde{J}_0(u_0') \geq \tilde{J}_0(0) = \bar{J}_0(\bar{u}_0). \tag{C.9}$$

Take $u_0'' = u_0'$ in (C.8) and let $\tilde{x}_0'$ be the associated solution. Then $\tilde{x}_0'(t) = \bar{x}_0(t) + \int_0^t \Phi_0(t,s)u_0'(s)ds$, where $\bar{x}_0(t)$ is the optimal state process of (C.2) as determined in (4.5). This implies $E\int_0^T |\tilde{x}_0'(t)|^2dt \leq C_2$. It follows from (C.7)–(C.8) that

$$E\int_0^T |x_0(t) - \tilde{x}_0'(t)|^2dt \leq C\epsilon_a^2, \tag{C.10}$$

where C is independent of ξ_a and (x_0, u_0). Following the proving argument in [15, Lemma A.3] we may use an expansion of (x_0, u_0) around $(\tilde{x}_0', u_0')$ and (C.10) to further show $|J_0^\xi(u_0) - \tilde{J}_0(u_0')| \le C(\epsilon_a + \epsilon_b)$, where C does not change with (ξ_a, ξ_b). This combined with (C.9) proves the lemma. □

C.2. A perturbed version of Problem (II) for the minor player. Let $z(t)$ be given as in (4.8) and $\bar{x}_0(t)$ given as in (2.12) and (2.19)–(2.21). Now we consider a perturbed version of Problem (II). Let the dynamics and cost be given by

$$dx_i(t) = \Big(A(\theta_i)x_i(t)+B(\theta_i)u_i(t)+F(\theta_i)z(t)+\xi_a(t)\Big)dt+D(\theta_i)dW_i(t), \quad t \ge 0, \tag{C.11}$$

and

$$J_i^\xi(u_i) = E\int_0^T \Big(\big|x_i(t) - H\bar{x}_0(t) - \hat{H}z(t) + \xi_b(t) - \eta\big|_Q^2 + u_i^T(t)Ru_i(t)\Big)dt. \tag{C.12}$$

The proof of the next lemma is similar to that of Lemma C.1 and hence is omitted.

LEMMA C.2. *Let $\bar{u}_i$ be the optimal control of Problem* (II) *with z given by* (4.8) *and $\bar{x}_0$ given by* (2.12) *and* (2.19)–(2.21). *For any u_i adapted to $\{\mathcal{F}_t\}_{t\ge 0}$, we have $J_i^\xi(u_i) \ge \bar{J}_i(\bar{u}_i) - O(\epsilon_a+\epsilon_b)$, provided that for a fixed constant C_0, the pair (x_i, u_i) in* (C.11) *satisfies $E\int_0^T (|x_i(t)|^2 + |u_i(t)|^2)dt \le C_0$.*

C.3. Proof of Theorem 4.5. Let x_j, $0 \le j \le N$, be determined by (4.3)–(4.4), which is associated with the controls $\hat{u}_j$, $0 \le j \le N$, given by (4.1)–(4.2). We first show $|J_0(\hat{u}_0, \hat{u}_{-0}) - \bar{J}_0(\bar{u}_0)| = O\big(\epsilon_N + 1/\sqrt{N}\big)$. We have

$$\begin{aligned}
&|J_0(\hat{u}_0, \hat{u}_{-0}) - \bar{J}_0(\bar{u}_0)| \\
&\quad \le E\int_0^T \Big| |x_0 - H_0x^{(N)} - \eta_0|_{Q_0}^2 - |\bar{x}_0 - H_0z - \eta_0|_{Q_0}^2 \Big| dt \\
&\qquad + E\int_0^T \Big| |\hat{u}_0|_{R_0}^2 - |\bar{u}_0|_{R_0}^2 \Big| dt \\
&\quad \le E\int_0^T |Q_0| \Big\{ \Big|(x_0 - \bar{x}_0) - H_0(x^{(N)} - z)\Big|^2 \\
&\qquad\qquad + 2\big|\bar{x}_0 - H_0z - \eta_0\big| \Big|(x_0 - \bar{x}_0) - H_0(x^{(N)} - z)\Big| \Big\} dt \\
&\qquad + E\int_0^T |R_0| \Big\{ \big|\hat{u}_0 - \bar{u}_0\big|^2 + 2|\bar{u}_0|\big|\hat{u}_0 - \bar{u}_0\big| \Big\} dt.
\end{aligned} \tag{C.13}$$

It follows from Theorem 4.3 that

$$E\int_0^T \big|(x_0 - \bar{x}_0) - H_0(x^{(N)} - z)\big|^2 dt = O\Big(\epsilon_N^2 + \frac{1}{N}\Big). \tag{C.14}$$

Next, by Schwarz's inequality,

$$\begin{aligned}
&E\int_0^T \big|\bar{x}_0 - H_0z - \eta_0\big| \Big|(x_0 - \bar{x}_0) + H_0(x^{(N)} - z)\Big| dt \\
&\quad \le \Big(\int_0^T E\big|\bar{x}_0 - H_0z - \eta_0\big|^2 dt\Big)^{1/2} \Big(\int_0^T E\Big|(x_0 - \bar{x}_0) - H_0(x^{(N)} - z)\Big|^2 dt\Big)^{1/2} \\
&\quad = O\Big(\epsilon_N + \frac{1}{\sqrt{N}}\Big).
\end{aligned} \tag{C.15}$$

We use (2.11), (4.1), and (4.10) to obtain

$$E\int_0^T |\hat u_0-\bar u_0|^2dt = E\int_0^T \big|R_0^{-1}B_0^TP_0(t)(x_0-\bar x_0)\big|^2dt = O\Big(\epsilon_N^2+\frac{1}{N}\Big),$$

and subsequently,

$$E\int_0^T |R_0|\Big\{\big|\hat u_0-\bar u_0\big|^2+2|\bar u_0|\big|\hat u_0-\bar u_0\big|\Big\}dt = O\Big(\epsilon_N+\frac{1}{\sqrt{N}}\Big). \tag{C.16}$$

By (C.13)–(C.16), it follows that $|J_0(\hat u_0,\hat u_{-0})-\bar J_0(\bar u_0)| = O\big(\epsilon_N+1/\sqrt{N}\big)$.

Similarly, we can prove that $|J_i(\hat u_i,\hat u_{-i})-\bar J_i(\bar u_i)| = O\big(\epsilon_N+1/\sqrt{N}\big)$ for $i=1,\dots,N$. ∎

C.4. Proof of Theorem 4.6. It suffices to show the first inequality, and the second is evident.

Step 1. *The case for the major player $\mathcal{A}_0$ to use an alternative strategy u_0.* Each minor player takes the control law $\hat u_i$ given in (4.2). Denote $\xi_a(t)=F_0(x^{(N)}(t)-z(t))$ and $\xi_b(t)=H_0(z(t)-x^{(N)}(t))$. The dynamics and cost of $\mathcal{A}_0$ may be written as

$$dx_0(t) = (A_0x_0(t)+B_0u_0(t)+F_0z(t)+\xi_a(t))dt + D_0dW_0(t)$$

and

$$J_0(u_0,\hat u_{-0}) = E\int_0^T \Big\{\big|x_0(t)-H_0z(t)+\xi_b(t)-\eta_0\big|_{Q_0}^2 + u_0^T(t)R_0u_0(t)\Big\}dt.$$

We follow the notation in (C.1) to define ϵ_a and ϵ_b. Note that when the set of strategies $(u_0,\hat u_1,\dots,\hat u_N)$ is applied, $z(t)-x^{(N)}(t)$ is the same as in Theorem 4.3 since (x_0,u_0) does not appear in the dynamics of x_i, $i\ge 1$. Hence,

$$\epsilon_a+\epsilon_b = \Big(\int_0^T E|\xi_a(t)|^2dt\Big)^{1/2} + \Big(\int_0^T E|\xi_b(t)|^2dt\Big)^{1/2} = O\Big(\epsilon_N+\frac{1}{\sqrt{N}}\Big). \tag{C.17}$$

If u_0 is simply taken as $\hat u_0$, we may apply Theorem 4.5 to show that $J_0(\hat u_0,\hat u_{-0})$ is upper bounded by a constant C_0 independent of N. So, it suffices to restrict our attention to all u_0 such that $J_0(u_0,\hat u_{-0})\le C_0$, which further ensures that $E\int_0^T(|x_0(t)|^2+|u_0(t)|^2)dt \le C_1$ for some constant C_1 independent of N. In view of Lemma C.1, $J_0(u_0,\hat u_{-0}) \ge \bar J_0(\bar u_0) - O(\epsilon_a+\epsilon_b)$. Therefore, by (C.17) and Theorem 4.5,

$$J_0(u_0,\hat u_{-0}) \ge \bar J_0(\bar u_0) - O\Big(\epsilon_N+\frac{1}{\sqrt{N}}\Big) \ge J_0(\hat u_0,\hat u_{-0}) - O\Big(\epsilon_N+\frac{1}{\sqrt{N}}\Big).$$

This proves the lower bound in (4.12).

Step 2. *The case for any given minor player $\mathcal{A}_i$ to use an alternative strategy u_i.* After all players, except $\mathcal{A}_i$, apply the control laws given in (4.1)–(4.2), the dynamics of $\mathcal{A}_0$ and $\mathcal{A}_j$, $j\ne i$, may be written in the form

$$dx_0(t) = \Big(\mathbb{A}_0(t)x_0(t)+B_0R_0^{-1}B_0^T\nu_0(t)+F_0z(t)\Big)dt + D_0dW_0(t) \\ + F_0\big(x^{(N)}(t)-z(t)\big)dt, \tag{C.18}$$

$$dx_j(t) = \Big(\mathbb{A}_{\theta_j}(t)x_j(t)+B(\theta_j)R^{-1}B^T(\theta_j)\nu_{\theta_j}(t)+F(\theta_j)z(t)\Big)dt + D(\theta_j)dW_j(t) \\ + F(\theta_j)\big(x^{(N)}(t)-z(t)\big)dt,\quad 1\le j\le N,\ j\ne i. \tag{C.19}$$

We write the dynamics of $\mathcal{A}_i$ as follows:

$$\begin{aligned} dx_i(t) = &\Big(\mathbb{A}_{\theta_i}(t)x_i(t) + B(\theta_i)R^{-1}B^T(\theta_i)\nu_{\theta_i}(t) + F(\theta_i)z(t)\Big)dt + D(\theta_i)dW_i(t) \\ &+ F(\theta_i)\big(x^{(N)}(t) - z(t)\big)dt + B(\theta_i)\big(u_i(t) - \hat{u}_i(t)\big)dt, \end{aligned} \tag{C.20}$$

where $\hat{u}_i(t) = R^{-1}B^T(\theta_i)\big[-P_{\theta_i}(t)x_i(t) + \nu_{\theta_i}(t)\big]$ depends on x_i and hence is affected by u_i. Since $J_i(\hat{u}_i, \hat{u}_{-i}) \le C_0$ for some fixed C_0 independent of N, it suffices to consider u_i such that $J_i(u_i, \hat{u}_{-i}) \le C_0$. This implies that $E\int_0^T |u_i(t)|^2 dt \le C_1$ for some fixed C_1. So by (4.2) there exists C_2 such that for all $t \in [0, T]$,

$$E\int_0^t |u_i(s) - \hat{u}_i(s)|^2 ds \le C_2 + C_2 E\int_0^t |x_i(s)|^2 ds. \tag{C.21}$$

In parallel with (B.1),

$$x_0(t) = \bar{x}_0(t) + \int_0^t \Phi_0(t, s_1)F_0\big(x^{(N)}(s_1) - z(s_1)\big)ds_1, \tag{C.22}$$

$$x_j(t) = \bar{x}_j(t) + \int_0^t \Phi_{\theta_j}(t, s_1)F(\theta_j)\big(x^{(N)}(s_1) - z(s_1)\big)ds_1, \quad 1 \le j \le N,\ j \ne i, \tag{C.23}$$

$$\begin{aligned} x_i(t) = \bar{x}_i(t) &+ \int_0^t \Phi_{\theta_i}(t, s_1)F(\theta_i)\big(x^{(N)}(s_1) - z(s_1)\big)ds \\ &+ \int_0^t \Phi_{\theta_i}(t, s_1)B(\theta_i)\big(u_i(s_1) - \hat{u}_i(s_1)\big)ds_1. \end{aligned} \tag{C.24}$$

Adding up the N equations in (C.23)–(C.24), we have

$$\begin{aligned} x^{(N)}(t) - z(t) = \bar{x}^{(N)} - z(t) &+ \frac{1}{N}\sum_{j=1}^N \int_0^t \Phi_{\theta_j}(t, s_1)F(\theta_j)\big(x^{(N)}(s_1) - z(s_1)\big)ds_1 \\ &+ \frac{1}{N}\int_0^t \Phi_{\theta_i}(t, s_1)B(\theta_i)\big(u_i(s_1) - \hat{u}_i(s_1)\big)ds_1. \end{aligned}$$

Therefore, for $0 \le t \le T$,

$$|x^{(N)}(t) - z(t)| \le |\bar{x}^{(N)} - z(t)| + C\int_0^t \big|x^{(N)}(s) - z(s)\big|ds + \frac{C}{N}\int_0^t \big|u_i(s) - \hat{u}_i(s)\big|ds.$$

By Schwarz's inequality,

$$\begin{aligned} &E\int_0^t |x^{(N)}(s) - z(s)|^2 ds \\ &\le 3E\int_0^t |\bar{x}^{(N)}(s) - z(s)|^2 ds + 3CE\int_0^t \left(\int_0^s \big|x^{(N)}(s_1) - z(s_1)\big|ds_1\right)^2 ds \\ &\quad + \frac{3C}{N}E\int_0^t \left(\int_0^s \big|u_i(s_1) - \hat{u}_i(s_1)\big|ds_1\right)^2 ds \\ &\le C\left(\epsilon_N^2 + \frac{1}{N} + \frac{1}{N}\int_0^t E|x_i(s)|^2 ds\right) + CE\int_0^t\int_0^s \big|x^{(N)}(s_1) - z(s_1)\big|^2 ds_1 ds. \end{aligned} \tag{C.25}$$

By Gronwall's inequality,

$$E\int_0^t |x^{(N)}(s) - z(s)|^2 ds \le C\left(\epsilon_N^2 + \frac{1}{N} + \frac{1}{N}\int_0^t E|x_i(s)|^2 ds\right). \tag{C.26}$$

Combining (C.20), (C.21), and (C.26), we may use Gronwall's inequality to show $\sup_{0\le t\le T} E|x_i(t)|^2 \le C_3$ for some constant C_3. Therefore,

$$E\int_0^T |x^{(N)}(s)-z(s)|^2 ds \le C\Big(\epsilon_N^2+\frac{1}{N}\Big). \tag{C.27}$$

It follows from (C.22) and (C.27) that

$$E\int_0^T |x_0(s)-\bar{x}_0(s)|^2 ds \le C\Big(\epsilon_N^2+\frac{1}{N}\Big). \tag{C.28}$$

Next, denote $\xi_a(t)=F(\theta_i)(x^{(N)}(t)-z(t))$ and $\xi_b(t)=H(\bar{x}_0(t)-x_0(t))+\hat{H}(z(t)-x^{(N)}(t))$. Then the dynamics and cost of player $\mathcal{A}_i$ may be written as

$$dx_i(t)=\Big(A(\theta_i)x_i(t)+B(\theta_i)u_i(t)+F(\theta_i)z(t)+\xi_a(t)\Big)dt+D(\theta_i)dW_i(t), \quad t\ge 0,$$
$$J_i(u_i)=E\int_0^T \Big\{\big|x_i(t)-H\bar{x}_0(t)-\hat{H}z(t)+\xi_b(t)-\eta\big|_Q^2+u_i^T(t)Ru_i(t)\Big\}dt.$$

It is clear that $J_i(u_i)=J_i^{\xi}(u_i)$ which is defined in (C.12). In view of (C.27)–(C.28),

$$\epsilon_a+\epsilon_b=\left\{\int_0^T E|\xi_a(t)|^2dt\right\}^{1/2}+\left\{\int_0^T E|\xi_b(t)|^2dt\right\}^{1/2}=O\Big(\epsilon_N+\frac{1}{\sqrt{N}}\Big). \tag{C.29}$$

Since $E\int_0^T(|x_i(t)|^2+|u_i(t)|^2)dt \le C$ for some constant C independent of N, by Lemma C.2, $J_i(u_i,\hat{u}_{-i})\ge \bar{J}_i(\bar{u}_i)-O(\epsilon_a+\epsilon_b)$. By (C.29) and Theorem 4.5,

$$J_i(u_i,\hat{u}_{-i})\ge \bar{J}_i(\bar{u}_i)-O\Big(\epsilon_N+\frac{1}{\sqrt{N}}\Big)\ge J_i(\hat{u}_i,\hat{u}_{-i})-O\Big(\epsilon_N+\frac{1}{\sqrt{N}}\Big).$$

This proves the lower bound in (4.12) for player $\mathcal{A}_i$. ∎

REFERENCES

[1] Y. Achdou and I. Capuzzo-Dolcetta, *Mean field games: Numerical methods*, SIAM J. Numer. Anal., 48 (2010), pp. 1136–1162.

[2] S. Adlakha, R. Johari, G. Weintraub, and A. Goldsmith, *Oblivious equilibrium for large-scale stochastic games with unbounded costs*, in Proceedings of the 47th IEEE Conference on Decision and Control, Cancun, Mexico, 2008, pp. 5531–5538.

[3] N. I. Al-Najjar, *Aggregation and the law of large numbers in large economies*, Games Econom. Behav., 47 (2004), pp. 1–35.

[4] D. Andersson and B. Djehiche, *A maximum principle for SDEs of mean-field type*, Appl. Math. Optim., 63 (2011), pp. 341–356.

[5] J.-M. Bismut, *Linear quadratic optimal stochastic control with random coefficients*, SIAM J. Control Optim., 14 (1976), pp. 419–444.

[6] R. Buckdahn, P. Cardaliaguet, and M. Quincampoix, *Some recent aspects of differential game theory*, Dynam. Games Appl., 1 (2010), pp. 74–114.

[7] F. M. Callier and C. A. Desoer, *Linear System Theory*, Springer-Verlag, New York, 1991.

[8] S. Chen and X. Y. Zhou, *Stochastic linear quadratic regulation with indefinite control weight costs*. II, SIAM J. Control Optim., 39 (2000), pp. 1065–1081.

[9] E. A. Coddington and N. Levinson, *Theory of Ordinary Differential Equations*, McGraw-Hill, New York, 1955.

[10] D. Dawson and J. Vaillancourt, *Stochastic McKean-Vlasov equations*, Nonlinear Differential Equations Appl., 2 (1995), pp. 199–229.

[11] C. Dogbé, *Modeling crowd dynamics by the mean field limit approach*, Math. Comput. Modelling, 52 (2010), pp. 1506–1520.

[12] Z. Galil, *The nucleolus in games with major and minor players*, Internat. J. Game Theory, 3 (1974), pp. 129–140.

[13] O. Haimanko, *Nonsymmetric values of nonatomic and mixed games*, Math. Oper. Res., 25 (2000), pp. 591–605.

[14] S. Hart, *Values of mixed games*, Internat. J. Game Theory, 2 (1973), pp. 69–86.

[15] M. Huang, *Large-population LQG games involving a major player: The Nash certainty equivalence principle*, SIAM J. Control Optim., 48 (2010), pp. 3318–3353.

[16] M. Huang, P. E. Caines, and R. P. Malhamé, *Large-population cost-coupled LQG problems with nonuniform agents: Individual-mass behavior and decentralized ε-Nash equilibria*, IEEE Trans. Automat. Control, 52 (2007), pp. 1560–1571.

[17] M. Huang, P. E. Caines, and R. P. Malhamé, *Social optima in mean-field LQG control: Centralized and decentralized strategies*, in Proceedings of the 47th Allerton Conference on Communication, Control and Computing, Monticello, IL, 2009, pp. 322–329.

[18] M. Huang, R. P. Malhamé, and P. E. Caines, *Large population stochastic dynamic games: Closed-loop McKean-Vlasov systems and the Nash certainty equivalence principle*, Commun. Inf. Syst., 6 (2006), pp. 221–251.

[19] B. Jovanovic and R. W. Rosenthal, *Anonymous sequential games*, J. Math. Econom., 17 (1988), pp. 77–87.

[20] J.-M. Lasry and P.-L. Lions, *Mean field games*, Japan. J. Math., 2 (2007), pp. 229–260.

[21] T. Li and J.-F. Zhang, *Asymptotically optimal decentralized control for large population stochastic multiagent systems*, IEEE Trans. Automat. Control, 53 (2008), pp. 1643–1660.

[22] J. Ma and J. Yong, *Forward-Backward Stochastic Differential Equations and Their Applications*, Springer, Berlin, 1999.

[23] Z. Ma, D. Callaway, and I. Hiskens, *Decentralized charging control for large populations of plug-in electric vehicles*, in Proceedings of the 49th IEEE Conference on Decision and Control, Atlanta, GA, 2010, pp. 206–212.

[24] W. M. McEneaney, *A new fundamental solution for differential Riccati equations arising in control*, Automatica, 44 (2008), pp. 920–936.

[25] S. L. Nguyen and M. Huang, *Mean field LQG games with mixed players: The backward SDE approach*, in Abstracts, Optimization Days, Montreal, Quebéc, Canada, 2011, p. 25.

[26] S. L. Nguyen and M. Huang, *Mean field LQG games with a major player: Continuum-parameters for minor players*, in Proceedings of the 50th IEEE Conference on Decision and Control, Orlando, FL, 2011, pp. 1012–1017.

[27] S. L. Nguyen and M. Huang, *Mean field LQG games with mass behavior responsive to a major player*, to be presented at the 51st IEEE Conference on Decision and Control, Maui, HI, 2012.

[28] M. Nourian, R. P. Malhamé, M. Huang, and P. E. Caines, *Mean field (NCE) formulation of estimation based leader-follower collective dynamics*, Int. J. Robotics Automat., 26 (2011), pp. 120–129.

[29] L. Ntogramatzidis and A. Ferrante, *On the solution of the Riccati differential equation arising from the LQ optimal control problem*, Systems Control Lett., 59 (2010), pp. 114–121.

[30] E. D. Sontag, *Mathematical Control Theory: Deterministic Finite-Dimensional Systems*, 2nd ed., Springer-Verlag, New York, 1998.

[31] S. Tang, *General linear quadratic optimal stochastic control problems with random coefficients: Linear stochastic Hamilton systems and backward stochastic Riccati equations*, SIAM J. Control Optim., 42 (2003), pp. 53–75.

[32] H. Tembine, Q. Zhu, and T. Başar, *Risk-sensitive mean-field stochastic differential games*, in Proceedings of the 18th IFAC World Congress, Milan, Italy, 2011.

[33] G. Y. Weintraub, C. L. Benkard, and B. Van Roy, *Markov perfect industry dynamics with many firms*, Econometrica, 76 (2008), pp. 1375–1411.

[34] H. Yin, P. G. Mehta, S. P. Meyn, and U. V. Shanbhag, *Synchronization of coupled oscillators is a game*, IEEE Trans. Automat. Control, 57 (2012), pp. 920–935.

[35] J. Yong, *A leader-follower stochastic linear quadratic differential game*, SIAM J. Control Optim., 41 (2002), pp. 1015–1041.

[36] J. Yong and X. Y. Zhou, *Stochastic Controls: Hamiltonian Systems and HJB Equations*, Springer-Verlag, New York, 1999.