arXiv:1701.06661v1 [math.OC] 23 Jan 2017

Mean Field Stochastic Games with Binary Action Spaces and Monotone Costs

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Abstract

This paper considers mean field games in a multi-agent Markov decision process (MDP) framework. Each player has a continuum state and binary action. By active control, a player can bring its state to a resetting point. All players are coupled through their cost functions. The structural property of the individual strategies is characterized in terms of threshold policies when the mean field game admits a solution. We further introduce a stationary equation system of the mean field game and analyze uniqueness of its solution under positive externalities.

Key words: dynamic programming, Markov decision process, mean field game, stationary distribution, threshold policy

1 Introduction

Mean field game theory studies stochastic decision problems with a large number of noncooperative players which are individually insignificant but collectively have a significant impact on a particular player. It provides a powerful methodology for reducing complexity in the analysis and design of strategies. With the aid of an infinite population model, one may apply consistent mean field approximations to construct a set of decentralized strategies for the original large but finite population model and show its ε-Nash equilibrium property [18, 19, 22]. A closely related approach is independently developed in [27]. Another related solution notion in Markov decision models is the oblivious equilibrium [41]. For nonlinear diffusion models [10, 22, 27], the analysis of mean field games depends on tools of Hamilton-Jacobi-Bellman (HJB) equations, Fokker-Planck equations, and McKean-Vlasov equations. For further literature in the stochastic analysis setting, see [11, 26]. To address mean field interactions with an agent possessing strong influences, mixed player models are studied in [8, 16, 34, 35]. The readers are referred to [8, 9, 14] for an overview on mean field game theory.

Mean field games have found applications in diverse areas such as power systems [25], large population electric vehicle recharging control [31, 36], economics and finance [1, 12, 30], stochastic growth theory [17], bio-inspired oscillator games [42].

This paper studies a class of mean field games in a multi-agent Markov decision process (MDP) framework. Dynamic games within an MDP setting are a classic area pioneered by Shapley under the name stochastic games [13, 37]. For MDP based mean field game modeling, see [1, 21, 41]. The players

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in our model have continuum state spaces and binary action spaces, and have coupling through their cost functions. The state of each player is used to model its risk (or distress) level which has random increase if no active control is taken. The one stage cost of a player depends on its own state, the population average state and its control effort. Naturally, the cost of a player is an increasing function of its own state. The motivation of this modeling framework comes from applications including network security investment games, and flue vaccination games [6, 24, 28, 32]; when the cost function is an increasing function of the population average state, it reflects positive externalities. Markov decision processes with binary action spaces also arise in control of queues and machine replacement problems [3, 7]. Our game model has connection with anonymous sequential games [23] which combine stochastic game modeling with a continuum of players. However, there is a subtle difference regarding treating individual behavior. In anonymous sequential games one determines the equilibrium as a joint state-action distribution of the population and leaves the individual strategies unspecified [23, Sec. 4], although there is an interpretation of randomized actions for players sharing a given state. Our approach works in the other direction by explicitly specifying the best response of an individual and using its closed-loop state distribution to determine the mean field. Our modeling starts with a finite population, and avoids certain measurability difficulties in directly treating a continuum of random processes [2].

A very interesting feature of our model is threshold policies for the solution of the mean field game. We consider a finite time horizon game, and identify conditions for the existence of a solution to the fixed point problem. The further analysis deals with the stationary equation of the game and addresses uniqueness under positive externalities, which is done by studying ergodicity of the closed-loop state process of an individual player. Proving uniqueness results in mean field games is a nontrivial task, particularly when attempting to seek less restrictive conditions. This work is perhaps the first to establish uniqueness by the route of exploiting externalities. For this paper, in order to maintain a balance in analyzing the finite horizon problem and the stationary equation system, the existence analysis of the latter is not included and will be reported in another work.

Although mean field games provide a powerful paradigm for substantially reducing complexity in designing strategies, except for the linear-quadratic (LQ) cases [18, 29, 39, 40] allowing simple computations, strategies in general nonlinear systems are often only implicitly determined, rarely taking simple forms. Their numerical solutions lead to high computational load. One of the objectives in this paper is to develop a modeling framework to obtain relatively simple solution structures.

This paper is an English version of [20]. All assumptions and results in Sections 2-6 and Appendices A and B of both papers are the same, but Appendices C and D have been rewritten.

The organization of the paper is as follows. The Markov decision process framework is introduced in Section 2. Section 3 solves the best response as a threshold policy. Section 4 shows an ϵ-Nash equilibrium property. The existence of a solution to the mean field equation system is analyzed in Section 5. Section 6 introduces the stationary equation system and analyzes uniqueness of its solution. Section 7 concludes the paper.

2 The Markov Decision Process Model

2.1 The system dynamics

The system consists of N players denoted by $\mathcal{A}_i$, $1 \le i \le N$. At time $t \in \mathbb{Z}_+ = \{0, 1, 2, \ldots\}$, the state of $\mathcal{A}_i$ is denoted by x_t^i, and its action by a_t^i.

For simplicity, we consider a population of homogeneous players. Each player has a state space $\mathbf{S} = [0, 1]$. A value of $\mathbf{S}$ may be interpreted as a risk (or "distress") level. All players have the same action space $\mathbf{A} = \{a_0, a_1\}$. A player can either do nothing (action a_0) or make an active effort (action a_1). For an interval I, let $\mathcal{B}(I)$ denote the Borel σ-algebra of I.

The state of each player evolves as a controlled Markov process which is affected only by its own action. For $t \ge 0$ and $x \in \mathbf{S}$, the state has transition kernel specified by

$$P(x_{t+1}^i \in B | x_t^i = x, a_t^i = a_0) = Q_0(B|x), \tag{1}$$
$$P(x_{t+1}^i = 0 | x_t^i = x, a_t^i = a_1) = 1, \tag{2}$$

where $Q_0(\cdot|x)$ is a stochastic kernel defined for $B \in \mathcal{B}(\mathbf{S})$ and $Q_0([0, x)|x) = 0$. The structure of Q_0 indicates that given $x_t^i = x$, $a_t^i = a_0$, the state has a transition into $[x, 1]$. In other words, the state of the player deteriorates if no active control is taken. We call $a_t^i = a_1$ and $0 \in \mathbf{S}$ a resetting action and a resetting point, respectively.

The vector process $(x_t^1, \ldots x_t^N)$ constitutes a controlled Markov process in higher dimension with its transition kernel determined as a product measure of the form

$$P(x_t^i \in B_i, i = 1, \ldots, N | x_t^i = x^{[i]}, a_t^i = a^{[i]}, i = 1, \ldots, N) = \prod_{i=1}^N P(x_t^i \in B_i | x_t^i = x^{[i]}, a_t^i = a^{[i]}),$$

where $B_i \in \mathcal{B}(\mathbf{S})$, $x^{[i]} \in \mathbf{S}$ and $a^{[i]} \in \mathbf{A}$. This product measure implies independent transitions of the N controlled Markov processes.

2.2 The individual costs

Define the population average state $x_t^{(N)} = \frac{1}{N} \sum_{i=1}^N x_t^i$. For $\mathcal{A}_i$, the one stage cost is given by

$$c(x_t^i, x_t^{(N)}, a_t^i) = R(x_t^i, x_t^{(N)}) + \gamma 1_{\{a_t^i = a_1\}},$$

where $\gamma > 0$ and $\gamma 1_{\{a_t^i = a_1\}}$ is the effort cost. The function $R \ge 0$ is defined on $\mathbf{S} \times \mathbf{S}$ and models the risk-related cost. For $0 < T < \infty$ and discount factor $\rho \in (0, 1)$, define the cost

$$J_i = E \sum_{t=0}^T \rho^t c(x_t^i, x_t^{(N)}, a_t^i), \quad 1 \le i \le N. \tag{3}$$

The following assumptions are introduced.

(A1) $\{x_0^i, i \ge 1\}$ are i.i.d. random variables taking values in $\mathbf{S}$ and $Ex_0^i = m_0$.

(A2) $R(x, z)$ is a continuous function on $\mathbf{S} \times \mathbf{S}$. For each fixed z, $R(\cdot, z)$ is strictly increasing.

(A3) There exists a random variable ξ taking values in $\mathbf{S}$ such that the measure $Q_0(\cdot|x)$ is the distribution of the random variable $x + (1 - x)\xi$. Furthermore, $P(\xi = 1) < 1$.

Denote the distribution function of ξ by F_ξ. To avoid triviality, we assume $P(\xi = 1) < 1$ in (A3).

We give some motivation about (A3). For $\mathbf{S} = [0,1]$, $1-x$ is the state margin from the maximal state 1. Denote $m_t^i = 1-x_t^i$. Then given m_t^i and $a_t^i = 0$, the state margin decays to $m_{t+1}^i = (1-\xi_{t+1}^i)m_t^i$ where ξ_{t+1}^i has the same distribution as ξ. In other words, the state margin decays exponentially if no active control is applied.

Remark 1 *In fact, (A3) implies the so-called stochastic monotonicity condition (see e.g. [3, 7]) for Q_0. We assume the specific form of Q_0, aiming to obtain more refined properties for the resulting threshold policies and sample path behavior of players.*

Example 1 A lab consists of N networked computers M_i, $1 \le i \le N$, each of which is assigned to a primary user U_i, $1 \le i \le N$, and is occasionally accessed by others for its specific resources. A computer has an unfitness state $x_t^i \in [0,1]$, which randomly degrades due to daily use and potential exposure to malwares, etc. A user U_i can take a maintenance action a_1 on M_i by installing or updating security software, scanning and cleaning up disk, freeing up memory space, etc., to bring it to an ideal condition $x_t^i = 0$. The one stage cost of U_i is $R(x_t^i, x_t^{(N)}) + \gamma 1_{\{a_t^i = a_1\}}$ where the dependence on $x_t^{(N)}$ is due to machine sharing and potential malware spreading from other machines. This model is called a labmate game.

3 The Mean Field Limit Model

3.1 The optimal control problem

Assume (A1)-(A3) for Section 3. A sequence $(b_s, \dots, b_t)$, $s \le t$, is denoted as $b_{s,t}$. Let x_t^i be given by (1)-(2). Let $x_t^{(N)}$ be approximated by a deterministic value z_t. Define

$$\bar{J}_i(z_{0,T}, a_{0,T}^i) = E\sum_{t=0}^{T} \rho^t c(x_t^i, z_t, a_t^i).$$

We call a_t^i a pure Markov policy (or strategy) at t if $a_t^i(x)$ is a mapping from $\mathbf{S}$ to $\mathbf{A}$. We say that a_t^i is a threshold policy with parameter $r \in [0,1]$ if $a_t^i(x) = a_1$ for $x \ge r$ and $a_t^i(x) = a_0$ for $x < r$; this gives a feedback policy. The analysis below will identify properties of the optimal policy.

3.2 The dynamic programming equation

Denote $a_{s,t}^i = (a_s^i, \dots, a_t^i)$ for $s \le t$. Fix the sequence $z_{0,T}$, where each $z_t \in [0,1]$. For $0 \le s \le T$ and $x \in \mathbf{S}$, define

$$\bar{J}_i(s, x, z_{0,T}, a_{s,T}^i) = E\left[\sum_{t=s}^{T} \rho^{t-s} c(x_t^i, z_t, a_t^i)\Big| x_s^i = x\right].$$

Define the value function $V(t,x) = \inf_{a_{t,T}^i} \bar{J}_i(t, x, z_{0,T}, a_{t,T}^i)$, where $a_{0,T}^i$ is from the set of all Markov policies. The dynamic programming equation takes the form

$$\begin{cases} V(t,x) = \min_{a_t^i}\left[c(x, z_t, a_t^i) + \rho E[V(t+1, x_{t+1}^i)|x_t^i = x]\right], \\ V(T,x) = R(x, z_T), \qquad 0 \le t < T. \end{cases} \tag{4}$$

Write (4) in the equivalent form

$$\begin{cases} V(t,x) = \min\left[\rho \int_0^1 V(t+1,y)Q_0(dy|x) + R(x,z_t), \quad \rho V(t+1,0) + R(x,z_t) + \gamma\right], \\ V(T,x) = R(x,z_T), \qquad 0 \le t < T. \end{cases} \tag{5}$$

Denote

$$G_t(x) = \int_0^1 V(t,y)Q_0(dy|x), \quad 0 \le t \le T. \tag{6}$$

Lemma 1 *For each $0 \le t \le T$, $V(t,x)$ is continuous on* $\mathbf{S}$.

Proof. We prove by induction. $V(T,x)$ is a continuous function of $x \in \mathbf{S}$. Suppose that $V(k,x)$ is a continuous function of x for $0 < k \le T$. By (A3),

$$G_k(x) = \int_0^1 V(k, x + (1-x)y)dF_\xi(y) = \int_0^1 V(k, (1-y)x + y)dF_\xi(y), \tag{7}$$

which combined with the induction hypothesis implies that $\rho G_k(x)$ is continuous in x.

Note that $R(x, z_{k-1})$ is continuous in x. On the other hand, if $g_1(x)$ and $g_2(x)$ are continuous on $[0,1]$, $\min\{g_1(x), g_2(x)\}$ is a continuous function of x. It follows from (5) that $V(k-1,x)$ is a continuous function of x.

By induction, we conclude that $V(t,x)$ is continuous in x for all $0 \le t \le T$. □

Lemma 2 *For each $0 \le t \le T$, $V(t,x)$ is strictly increasing on* $\mathbf{S}$.

Proof. For $t = T$, $V(T,x_1) < V(T,x_2)$ whenever $x_1 < x_2$. Suppose that for $0 < k \le T$,

$$V(k,x_1) < V(k,x_2), \qquad \text{for } x_1 < x_2. \tag{8}$$

For $0 \le x_1 < x_2 \le 1$,

$$R(x_1, z_{k-1}) < R(x_2, z_{k-1}).$$

By (7) and (8),

$$\rho G_k(x_1) + R(x_1, z_{k-1}) < \rho G_k(x_2) + R(x_2, z_{k-1}).$$

For $\alpha_1 < \alpha_2$ and $\beta_1 < \beta_2$, we have $\min\{\alpha_1, \beta_1\} < \min\{\alpha_2, \beta_2\}$. Taking

$$\alpha_i = \rho G_k(x_i) + R(x_i, z_{k-1}), \quad \beta_i = \rho V(k,0) + R(x_i, z_{k-1}) + \gamma,$$

we obtain $V(k-1,x_1) < V(k-1,x_2)$. By induction, $V(t,x)$ is strictly increasing for all $0 \le t \le T$. □

Lemma 3 *For $0 \le t \le T$, $G_t(x)$ is continuous and strictly increasing in x.*

Proof. The lemma follows from Lemmas 1 and 2, (7) and $P(\xi = 1) < 1$ in (A3). □

Lemma 4 *For $t \le T-1$, if*

$$\rho G_{t+1}(0) < \rho V(t+1,0) + \gamma < \rho G_{t+1}(1), \tag{9}$$

there exists a unique $x^* \in (0,1)$ such that $\rho G_{t+1}(x^*) = \rho V(t+1,0) + \gamma$.

Proof. The lemma follows from Lemma 3 and the intermediate value theorem. $\square$

Theorem 1 *If $t = T$, define $a^i_T = a_0$. For $t \le T-1$, define the policy $a^i_t(x)$ by the following rule.*
i) If $\rho G_{t+1}(1) \le \rho V(t+1,0) + \gamma$, take $a^i_t(x) = a_0$ for all $x \in \mathbf{S}$.
ii) If $\rho G_{t+1}(0) \ge \rho V(t+1,0) + \gamma$, take $a^i_t(x) = a_1$ for all $x \in \mathbf{S}$.
iii) If (9) holds, take a^i_t as a threshold policy with parameter x^* given in Lemma 4.
Then $a^i_{0,T}$ is an optimal policy.

Proof. It is easy to see that $a^i_T = a_0$ is optimal. Consider $t \le T-1$. By Lemma 3 and 4, we can verify that the minimum in (5) is attained when a^i_t is chosen according to i)-iii). $\square$

4 Solution of the Mean Field Game

Assume (A1)-(A3). To obtain a solution of the mean field game, we introduce the equation system

$$\begin{cases} V(t,x) = \min\Big[\rho \int_0^1 V(t+1,y)Q_0(dy|x) + R(x,z_t), \quad \rho V(t+1,0) + R(x,z_t) + \gamma\Big], \\ \qquad\qquad 0 \le t < T \\ V(T,x) = R(x,z_T), \\ z_t = Ex^i_t, \quad 0 \le t \le T. \end{cases} \tag{10}$$

By (A1), $z_0 = m_0$. We look for a solution $(\hat{z}_{0,T}, \hat{a}^i_{0T})$ for (10) such that $\{x^i_t, 0 \le t \le T\}$ is generated by $\{\hat{a}^i_t(x), 0 \le t \le T\}$ satisfying the rule in Theorem 1 after setting $z_{0,T} = \hat{z}_{0,T}$. The last equation is the standard consistency condition in mean field games.

Consider the game of N players specified by (1)-(3). Denote $a^{-i}_{0,T} = (a^1_{0,T}, \dots, a^{i-1}_{0,T}, a^{i+1}_{0,T}, \dots, a^N_{0,T})$. Write $J_i = J_i(a^i_{0,T}, a^{-i}_{0,T})$.

For the performance estimates, we consider the perturbation of a^i_t in a strategy space $\mathcal{U}_t$ consisting of all pure Markov strategies depending on $(x^1_t, \dots, x^N_t)$.

Definition 1 *A set of strategies $\{a^i_{0,T}, 1 \le i \le N\}$ for the N players is called an ϵ-Nash equilibrium with respect to the costs $\{J_i, 1 \le i \le N\}$, where $\epsilon \ge 0$, if for any $1 \le i \le N$,*

$$J_i(a^i_{0,T}, a^{-i}_{0,T}) \le J_i(b^i_{0,T}, a^{-i}_{0,T}) + \epsilon,$$

for any $b^i_{0,T} \in \prod_{t=0}^T \mathcal{U}_t$.

Theorem 2 *Suppose that (10) has a solution $(\hat{z}_{0,T}, \hat{a}^i_{0,T})$. Then $(\hat{a}^1_{0,T}, \dots, \hat{a}^N_{0,T})$ is an ϵ-Nash equilibrium, i.e.,*

$$J_i(\hat{a}^i_{0,T}, \hat{a}^{-i}_{0,T}) - \epsilon \le \inf_{a^i_{0,T}} J_i(a^i_{0,T}, \hat{a}^{-i}_{0,T}) \le J_i(\hat{a}^i_{0,T}, \hat{a}^{-i}_{0,T}), \quad 1 \le i \le N,$$

where $a^i_{0,T} \in \prod_{t=0}^T \mathcal{U}_t$ and $\epsilon \to 0$ as $N \to \infty$.

Proof. For $(a^i_{0,T}, \hat{a}^{-i}_{0,T})$, denote the corresponding states by x^i_t, and $\hat{x}^j_t$, $j \ne i$. We have

$$\lim_{N\to\infty} \max_{0\le t\le T} |x^{(N)}_t - \hat{z}_t| = 0, \quad a.s. \tag{11}$$

where $x_t^{(N)} = \frac{1}{N}(\sum_{j\neq i} \hat{x}_t^j + x_t^i)$. Denote

$$\epsilon_{1,N} = \sup_{a_{0,T}^i} |J_i(a_{0,T}^i, \hat{a}_{0,T}^{-i}) - \bar{J}_i(\hat{z}_{0,T}, a_{0,T}^i)|.$$

Then by (11), $\lim_{N\to\infty} \epsilon_{1,N} = 0$. Furthermore,

$$\begin{aligned} J_i(a_{0,T}^i, \hat{a}_{0,T}^{-i}) =& \bar{J}_i(\hat{z}_{0,T}, a_{0,T}^i) + J_i(a_{0,T}^i, \hat{a}_{0,T}^{-i}) - \bar{J}_i(\hat{z}_{0,T}, a_{0,T}^i) \\ \geq& \bar{J}_i(\hat{z}_{0,T}, a_{0,T}^i) - \epsilon_{1,N} \geq \bar{J}_i(\hat{z}_{0,T}, \hat{a}_{0,T}^i) - \epsilon_{1,N}. \end{aligned}$$

On the other hand, denoting $\epsilon_{2,N} = |J_i(\hat{a}_{0,T}^i, \hat{a}_{0,T}^{-i}) - \bar{J}_i(\hat{z}_{0,T}, \hat{a}_{0,T}^i)|$, we have $\lim_{N\to\infty} \epsilon_{2,N} = 0$. Therefore,

$$J_i(a_{0,T}^i, \hat{a}_{0,T}^{-i}) \geq J_i(\hat{a}_{0,T}^i, \hat{a}_{0,T}^{-i}) - (\epsilon_{1,N} + \epsilon_{2,N}).$$

The theorem follows by taking $\epsilon = \epsilon_{1,N} + \epsilon_{2,N}$. □

5 Existence Result

Denote $\mathcal{Z}_T^{m_0} = \{z_{0,T} | z_0 = m_0, z_t \in [0,1] \text{ for } 1 \leq t \leq T\}$. We introduce the following assumptions.

(H1) ξ has a probability density function denoted by f_ξ.

(H2) Consider the optimal control problem with cost function $\bar{J}_i(z_{0,T}, a_{0,T}^i) = E\sum_{t=0}^T \rho^t c(x_t^i, z_t, a_t^i)$. For any $z_{0,T} \in \mathcal{Z}_T^{m_0}$, there exists $c > 0$ such that the optimal policy satisfies $a_t^i(x) = a_0$ for all $x \in [0, c]$ and $0 \leq t \leq T$.

We call (H2) the uniformly positive threshold condition for the family of optimal control problems. When the state of the player is small, the effort cost outweighs the extra benefit in further reducing the risk by active control. This holds uniformly with respect to $z_{0,T}$.

Define the class $\mathcal{P}_0$ of probability measures on $\mathbf{S}$ as follows. $\nu \in \mathcal{P}_0$ if there exist a constant $c_\nu \geq 0$ and a measurable function $g(x) \geq 0$ defined on $[0,1]$ such that

$$\nu(B) = \int_B g(x)dx + c_\nu 1_B(0),$$

where $B \in \mathcal{B}(\mathbf{S})$ and 1_B is the indicator function of B. When restricted to $(0,1]$, ν is absolutely continuous with respect to the Lebesgue measure μ^{Leb}.

Assume (A1)-(A3) and (H1)-H2) hold, and the distribution of x_0^i is $\mu_0 \in \mathcal{P}_0$ for this section.

For given $z_{0,T} \in \mathcal{Z}_T^{m_0}$, let the optimally controlled state process be x_t^i with distribution μ_t. Define $w_t = \int_0^1 x\mu_t(dx)$ and the mapping Φ from $[0,1]^T$ to $[0,1]^T$:

$$(w_1, \ldots, w_T) = \Phi(z_1, \ldots, z_T).$$

Lemma 5 *Φ is continuous.*

Proof. Let $z_{0,T} \in \mathcal{Z}_T^{m_0}$ be fixed, and denote the optimal policy by $a_{0,T}^i$ and the state process by x_t^i. Select $z'_{0,T} \in \mathcal{Z}_T^{m_0}$, and denote the corresponding optimal policy by $b_{0,T}^i$ and the state process by y_t^i. Let the distribution of x_t^i and y_t^i be μ_t and μ'_t, respectively. Here $\mu_0 = \mu'_0 \in \mathcal{P}_0$. By Lemmas A.1 and A.2, both μ_t and μ'_t are in $\mathcal{P}_0$ for $t \leq T$. This ensures that μ_t has a small perturbation when the associated positive threshold parameters have a small perturbation. By Lemmas A.3 and A.4, we can first show that

$$\lim_{z'_{0,T}\to z_{0,T}} \sup_{B\in\mathcal{B}(\mathbf{S})} |\mu_1(B) - \mu'_1(B)| = 0.$$

Repeating the estimate, we further obtain

$$\lim_{z'_{0,T}\to z_{0,T}} \sup_{B\in\mathcal{B}(\mathbf{S})} |\mu_t(B) - \mu'_t(B)| = 0, \quad 0 \le t \le T.$$

Subsequently,

$$\lim_{z'_{0,T}\to z_{0,T}} \int_0^1 x\mu'_t(dx) = \int_0^1 x\mu_t(dx), \quad 0 \le t \le T.$$

This proves continuity. □

Theorem 3 *There exists a solution* $(\hat{a}^i_{0,T}, \hat{z}_{0,T})$ *to* (10).

Proof. The theorem follows from Lemma 5 and Brouwer's fixed point theorem. □

6 The Stationary Equations

6.1 The stationary form

Assume (A1)-(A3). This section introduces a stationary version of (10). Take $z \in \mathbf{S}$. The value function is independent of time t and so denoted as $V(x)$. The dynamic programming equation becomes

$$V(x) = \min_{a^i}[c(x, z, a^i) + \rho E[V(x^i_{t+1})|x^i_t = x]],$$

which gives

$$V(x) = \min \Big[\rho \int_0^1 V(y)Q_0(dy|x) + R(x, z), \quad \rho V(0) + R(x, z) + \gamma\Big]. \tag{12}$$

We introduce another equation

$$z = \int_0^1 x\pi(dx) \tag{13}$$

for the probability measure π. We say $(\hat{z}, \hat{a}^i, \hat{\pi})$ is a stationary solution to (12)-(13) if i) the feedback policy $\hat{a}^i$ is the best response with respect to $\hat{z}$ in (12), ii) $\{x^i_t, t \ge 0\}$ under the policy $\hat{a}^i$ has the stationary distribution $\hat{\pi}$, and iii) $(\hat{z}, \hat{\pi})$ satisfies (13).

The equation system (12)-(13) can be interpreted as follows. For the finite horizon problem, suppose that T is increasing toward ∞. If the family of solutions (indexed by different values of T) could settle down to a steady-state, we expect for very large t, $V(t, x)$ and z_t will be nearly independent of time. This motivates us to introduce (12)-(13) as the stationary version of (10).

6.2 Value function with general z

Consider a general $z \in \mathbf{S}$ not necessarily satisfying (12)-(13) simultaneously, and further determine $V(x)$ by (12). Denote $G(x) = \int_0^1 V(y)Q_0(dy|x)$.

Lemma 6 *i) Equation* (12) *has a unique solution* $V \in C([0,1], \mathbb{R})$.
ii) V *is strictly increasing.*
iii) The optimal policy can be determined as follows:
a) If $\rho G(1) \le \rho V(0) + \gamma$, $a^i(x) \equiv a_0$.
b) If $\rho G(0) \ge \rho V(0) + \gamma$, $a^i(x) \equiv a_1$.
c) If $\rho G(0) < \rho V(0) + \gamma < \rho G(1)$, *there exists a unique* $x^* \in (0,1)$ *and* a^i *is a threshold policy with parameter* x^*.

Proof. Part i) will be shown by a fixed point argument. Define the dynamic programming operator

$$(\mathcal{L}g)(x) = \min\Big[\rho \int_0^1 g(y)Q_0(dy|x) + R(x,z), \quad \rho g(0) + R(x,z) + \gamma\Big],$$

where $g \in C([0,1], \mathbb{R})$. By the method in proving Lemma 1, it can be shown that $\mathcal{L}g \in C([0,1], \mathbb{R})$. Now take $g_1, g_2 \in C([0,1], \mathbb{R})$. Denote $\hat{x} = \arg\max |(\mathcal{L}g_2)(x) - (\mathcal{L}g_1)(x)|$. Without loss of generality, assume $(\mathcal{L}g_2)(\hat{x}) - (\mathcal{L}g_1)(\hat{x}) \ge 0$.

Case 1) $(\mathcal{L}g_1)(\hat{x}) = \rho \int_0^1 g_1(y)Q_0(dy|\hat{x}) + R(\hat{x}, z)$. We obtain

$$\begin{aligned} 0 \le (\mathcal{L}g_2)(\hat{x}) - (\mathcal{L}g_1)(\hat{x}) &\le \rho \int_0^1 (g_2(y) - g_1(y))Q_0(dy|\hat{x}) \\ &\le \rho \|g_2 - g_1\|. \end{aligned}$$

Case 2) $(\mathcal{L}g_1)(\hat{x}) = \rho g_1(0) + R(\hat{x}, z) + \gamma$. It follows that

$$\begin{aligned} 0 \le (\mathcal{L}g_2)(\hat{x}) - (\mathcal{L}g_1)(\hat{x}) &\le \rho |g_2(0) - g_1(0)| \\ &\le \rho \|g_2 - g_1\|. \end{aligned}$$

Combining the two cases, we conclude that $\mathcal{L}$ is a contraction and has a unique fixed point V.

To show ii), Define $g_{k+1} = \mathcal{L}g_k$ for $k \ge 0$, and $g_0 = 0$. By the method in Lemma 2 and induction, it can be shown that each g_k is increasing on $[0,1]$. Since $\lim_{k\to\infty} \|g_k - V\| = 0$, V is increasing. Recalling (12), we claim that V is strictly increasing. This proves ii). By showing that $G(x)$ is strictly increasing, we further obtain iii). □

For given z, Lemma 6 shows the structure of the optimal policy. Now we specify an optimal policy $a^i(x)$ in terms of a threshold parameter $\theta(z)$ by the following rule. i) If $\rho V(0) + \gamma \le \rho G(0)$, then $a^i(x) \equiv a_1$ with $\theta(z) = 0$; ii) if $\rho G(0) < \rho V(0) + \gamma < \rho G(1)$, $a^i(x)$ is a threshold policy with $\theta(z) \in (0,1)$; iii) if $\rho V(0) + \gamma = \rho G(1)$, then $a^i(x)$ has $\theta(z) = 1$; iv) if $\rho V(0) + \gamma > \rho G(1)$, $a^i(x) \equiv a_0$ for which we formally denote $\theta(z) = 1^+$.

Remark 2 *For the case* $\rho V(0) + \gamma = \rho G(1)$, *the above rule gives* $a^i(1) = a_1$ *and* $a^i(x) = a_0$ *for* $x < 1$, *which is slightly different from Lemma 6 but still attains optimality.*

6.3 Stationary distribution for a given threshold policy

Suppose that a^i is a threshold policy with parameter $\theta \in (0,1)$. Denote the corresponding state process by $\{x_t^{i,\theta}, t \geq 0\}$, which is a Markov process. Let the probability measure $P^t(x,\cdot)$ on $\mathcal{B}(\mathbf{S})$ be the distribution of $x_t^{i,\theta}$ given $x_0^{i,\theta} = x \in \mathbf{S}$.

We introduce a further condition on ξ.

(A4) ξ has a probability density function $f_\xi(x) > 0$ a.e. on $\mathbf{S}$.

Theorem 4 *For $\theta \in (0,1)$, $\{x_t^{i,\theta}, t \geq 0\}$ is uniformly ergodic with stationary probability distribution π_θ, i.e.,*

$$\sup_{x\in\mathbf{S}} \|P^t(x,\cdot) - \pi_\theta\|_{\mathrm{TV}} \leq K r^t \tag{14}$$

for some constants $K > 0$ and $r \in (0,1)$, where $\|\cdot\|_{\mathrm{TV}}$ is the total variation norm of signed measures.

Proof. See appendix B. □

6.4 Comparison theorems

Denote $z(\theta) = \int_0^1 x\pi_\theta(dx)$. We have the first comparison theorem on monotonicity.

Theorem 5 *$z(\theta_1) \leq z(\theta_2)$ for $0 < \theta_1 < \theta_2 < 1$.*

Proof. See appendix D. □

In the further analysis, we consider the case where R takes the product form $R(x,z) = R_1(x)R_2(z)$, and where R still satisfies (A2) and $R_1 \geq 0$, $R_2 > 0$. We further assume

(A5) $R_2 > 0$ is strictly increasing on $\mathbf{S}$.

This assumption indicates positive externalities since an individual benefits from the decrease of the population average state. This condition has a crucial role in the uniqueness analysis.

Given the product form of R, now (12) takes the form

$$V(x) = \min\Big[\rho\int_0^1 V(y)Q_0(dy|x) + R_1(x)R_2(z), \quad \rho V(0) + R_1(x)R_2(z) + \gamma\Big].$$

Consider $0 \leq z_2 < z_1 \leq 1$ and

$$V_l(x) = \min\Big[\rho\int_0^1 V_l(y)Q_0(dy|x) + R_1(x)R_2(z_l), \quad \rho V_l(0) + R_1(x)R_2(z_l) + \gamma\Big]. \tag{15}$$

Denote the optimal policy as a threshold policy with parameter θ_l in $[0,1]$ or equal to 1^+, where we follow the rule in Section 6.2 to interpret $\theta_l = 1^+$. We state the second comparison theorem about the threshold parameters under different mean field parameters z_l.

Theorem 6 *θ_1 and θ_2 in (15) are specified according to the following scenarios:*

i) If $\theta_1 = 0$, then we have either $\theta_2 \in [0,1]$ or $\theta_2 = 1^+$.

ii) If $\theta_1 \in (0,1)$, we have either a) $\theta_2 \in (\theta_1, 1)$, or b) $\theta_2 = 1$, or c) $\theta_2 = 1^+$.

iii) If $\theta_1 = 1$, $\theta_2 = 1^+$.

iv) If $\theta_1 = 1^+$, $\theta_2 = 1^+$.

Proof. Since $R_2(z_1) > R_2(z_2) > 0$, we divide both sides of (15) by $R_2(z_l)$ and define $\gamma_l = \frac{\gamma}{R_2(z_l)}$. Then $0 < \gamma_1 < \gamma_2$. The dynamic programming equation reduces to (C.2). Subsequently, the optimal policy is determined according to Lemma C.4. □

6.5 Uniqueness

We look for a solution (z, a^i, π) from the class $\mathcal{C}$ of solutions where $z \in \mathbf{S}$ and a^i is a threshold policy with parameter $\theta \in [0,1]$ or $\theta = 1^+$.

Theorem 7 *Under (A1)-(A5) with $R(x,z) = R_1(x)R_2(z)$, the equation system* (12)-(13) *has at most one solution in $\mathcal{C}$.*

Proof. Assume two different solutions

$$(z_1, a^i, \pi) \neq (z_2, b^i, \nu). \tag{16}$$

If $z_1 = z_2$, (12) ensures $a^i = b^i$, and subsequently $\pi = \nu$. This is a contradiction to two different solutions. Now we can assume

$$0 \leq z_2 < z_1 \leq 1. \tag{17}$$

We check different scenarios listed in Theorem 6. If $\theta_1 \in (0,1)$ so that $\theta_2 \in (\theta_1, 1)$, Theorem 5 implies $z_1 \leq z_2$, which contradicts (17). For all remaining scenarios, it is easy to show $z_1 \leq z_2$, which again contradicts (17). Therefore, assumption (16) does not hold. Uniqueness follows. □

7 Conclusion

This paper considers mean field games in a framework of multi-agent Markov decision processes (MDP). Each player has a monotone cost function and can apply resetting control to the state process. Decentralized strategies are obtained as threshold policies. We further examine a system of stationary equations of the mean field game and study uniqueness of the solution under positive externalities.

Appendix A: Technical Lemmas for Section 5

Let X be a random variable with distribution $\nu \in \mathcal{P}_0$. Set $x_t^i = X$. Define $Y_0 = x_{t+1}^i$ by applying $a_t^i \equiv a_0$. Further define $Y_1 = x_{t+1}^i$ by applying the threshold policy a_t^i with parameter $r \in (0,1)$. Then

$$P(Y_0 \in B) = \int_0^1 Q_0(B|x)\nu(dx), \quad B \in \mathcal{B}(\mathbf{S}). \tag{A.1}$$

Lemma A.1 *The distribution of Y_0 is in $\mathcal{P}_0$.*

Proof. We can directly show that the probability density function of Y_0 is

$$g(y) = \int_{0 \leq x < y} \frac{1}{1-x} f_\xi \left(\frac{y-x}{1-x} \right) \nu(dx), \quad y \in (0,1).$$

In this case $P(Y_0 = 0) = 0$. □

Lemma A.2 *The distribution of Y_1 is in $\mathcal{P}_0$.*

Proof. It is clear that $P(Y_1 = 0) = P(X \geq r)$. The distribution of Y_1 restricted on $(0, 1]$ is absolutely continuous with respect to μ^{Leb}. Denote

$$g(y) = \int_{0 \leq x < y \wedge r} \frac{1}{1-x} f_\xi \left(\frac{y-x}{1-x} \right) \nu(dx).$$

Then for $B \in \mathcal{B}(\mathbf{S})$,

$$P(Y_1 \in B) = \int_B g(y)dy + P(X \geq r) 1_B(0).$$

The lemma follows. $\square$

Let $z_{0,T} \in \mathcal{Z}_T^{m_0}$ be fixed, and denote the associated optimal policy by $a_{0,T}^i$. Select another sequence $z'_{0,T} \in \mathcal{Z}_T^{m_0}$ and let the optimal policy be denoted by $b_{0,T}^i$. Denote $d(z'_{0,T}, z_{0,T}) = \sum_{k=1}^T |z'_k - z_k|$. Write $z'_{0,T} \to z_{0,T}$ when $d(z'_{0,T}, z_{0,T}) \to 0$. Fix any $t \leq T-1$. Based on (H2), we consider two cases.

Case A) $a_t^i(x) = a_0$ for all $x \in \mathbf{S}$.

Lemma A.3 *For case A) and any $\epsilon > 0$, there exists $\delta > 0$ such that for all $z'_{0,T}$ satisfying $d(z'_{0,T}, z_{0,T}) \leq \delta$, we have either*

i) $b_t^i(x) = a_0$ for all $x \in \mathbf{S}$, or

ii) there exists $r' \in (0, 1)$ such that b_t^i is a threshold policy with parameter r' and $0 < 1 - r' < \epsilon$.

Proof. When $z'_{0,T} \in \mathcal{Z}_T^{m_0}$ is used in the optimal control problem in (H2), denote the value function by $\acute{V}(t, x)$. Define $\acute{G}_t(x)$ in place of $G_t(x)$. Then $\acute{G}_t(x)$ is continuous and strictly increasing for each t. Since V depends on $z_{0,T}$ continuously,

$$\lim_{z'_{0,T} \to z_{0,T}} \sup_{x \in \mathbf{S}} |\acute{V}(t, x) - V(t, x)| = 0. \tag{A.2}$$

Consider any $0 < \epsilon < 1$. We only need to treat the following two scenarios.

1) $\rho G_{t+1}(1) < \rho V(t + 1, 0) + \gamma$.

By (A.2), there exists $\delta > 0$ such that for all $z'_{0,T}$ satisfying $d(z'_{0,T}, z_{0,T}) \leq \delta$, we have

$$\rho \acute{G}_{t+1}(1) < \rho \acute{V}(t + 1, 0) + \gamma. \tag{A.3}$$

Then $b_t^i(x) = a_0$ for all $x \in \mathbf{S}$. So i) holds.

2) $\rho G_{t+1}(1) = \rho V(t + 1, 0) + \gamma$.

Then

$$\rho G_{t+1}(1 - \epsilon) < \rho V(t + 1, 0) + \gamma. \tag{A.4}$$

By (A.4), there exists $\delta > 0$ such that for all $z'_{0,T}$ satisfying $d(z'_{0,T}, z_{0,T}) \leq \delta$, we have

$$\rho \acute{G}_{t+1}(1 - \epsilon) < \rho \acute{V}(t + 1, 0) + \gamma. \tag{A.5}$$

For such $z'_{0,T}$, if $\rho \acute{G}_{t+1}(1) \leq \rho \acute{V}(t + 1, 0) + \gamma$, we select $b_t^i(x) = a_0$ for all $x \in \mathbf{S}$ and then i) holds. If $z'_{0,T}$ results in

$$\rho \acute{G}_{t+1}(1) > \rho \acute{V}(t + 1, 0) + \gamma,$$

by (A.5), we can find $r' \in (1 - \epsilon, 1)$ such that

$$\rho \acute{G}_{t+1}(r') = \rho \acute{V}(t + 1, 0) + \gamma,$$

which further determines b_t^i as a threshold policy with parameter r'. $\square$

Case B). There exists $r \in (0, 1)$ such that a_t^i is a threshold policy with parameter r.

Lemma A.4 *For case B), when $d(z'_{0,T}, z_{0,T})$ is sufficiently small, b^i_t is a threshold policy with parameter $r' \in (0,1)$ and in addition, $r' \to r$ as $z'_{0,T} \to z_{0,T}$.*

Proof. We have $\rho G_{t+1}(r) = \rho V(t+1,0) + \gamma$. Fix a small $\epsilon > 0$ such that $(r-\epsilon, r+\epsilon) \subset (0,1)$. Then

$$\rho G_{t+1}(r-\epsilon) < \rho V(t+1,0) + \gamma, \quad \rho G_{t+1}(r+\epsilon) > \rho V(t+1,0) + \gamma.$$

By (A.2), we may select $\delta > 0$ such that for all $z'_{0,T}$ satisfying $d(z'_{0,T}, z_{0,T}) \leq \delta$, we have

$$\rho \acute{G}_{t+1}(r-\epsilon) < \rho \acute{V}(t+1,0) + \gamma, \quad \rho \acute{G}_{t+1}(r+\epsilon) > \rho \acute{V}(t+1,0) + \gamma.$$

Since $\acute{G}_{t+1}(x)$ is strictly increasing, there exists a unique $r' \in (r-\epsilon, r+\epsilon)$ such that

$$\rho \acute{G}_{t+1}(r') = \rho \acute{V}(t+1,0) + \gamma,$$

where r' depends on $z'_{0,T}$. This in turn determines the threshold policy b^i_t with parameter r'. Since ϵ can be arbitrarily small, the last part of the lemma follows. □

Appendix B. Proof of Theorem 4

Consider $0 < \theta < 1$. The definitions of irreducibility, aperiodicity and a small set follow those in [33]. Let δ_x be the dirac measure at $x \in \mathbb{R}$. Let $\varphi := \delta_0$. So $\delta_0(B) = 1_B(0)$ for $B \in \mathcal{B}(\mathbf{S})$.

Throughout this appendix, we write $x_t := x^{i,\theta}_t$ in order to keep the notation light.

Lemma B.1 *$\{x_t, t \geq 0\}$ is φ-irreducible.*

Proof. We can directly verify that

$$\begin{aligned} P(x_2 = 0 | x_0 = x) &> 0, \quad x \in [0, \theta), \\ P(x_1 = 0 | x_0 = x) &= 1, \quad x \in [\theta, 1]. \end{aligned}$$

The above probabilities of the process are calculated by setting the distribution for x_0 as the dirac measure δ_x. This implies that $\{x_t, t \geq 0\}$ is φ-irreducible. □

Lemma B.2 *$\{x_t, t \geq 0\}$ is aperiodic.*

Proof. Define $C_s = \{0\}$. Denote $\epsilon_0 = \int_\theta^1 f_\xi(y)dy > 0$ and the measure $\nu = \epsilon_0 \delta_0$. Then

$$\begin{aligned} P(x_2 = 0 | x_0 = 0) &\geq P(x_2 = 0, x_1 \geq \theta | x_0 = 0) \\ &= P(x_1 \geq \theta | x_0 = 0) \\ &= \epsilon_0. \end{aligned}$$

For any $B \in \mathcal{B}(\mathbf{S})$, then

$$P(x_2 \in B | x_0 = 0) \geq \nu(B). \tag{B.1}$$

Therefore, we can take C_s as a small set with $\nu(C_s) = \epsilon_0$. Given $x_0 = 0 \in C_s$, we further check

$$\begin{aligned} P(x_3 = 0|x_0 = 0) &\geq P(x_3 = 0, x_2 \geq \theta, x_1 < \theta|x_0 = 0) \\ &= P(x_2 \geq \theta, x_1 < \theta|x_0 = 0). \end{aligned}$$

Let ξ, ξ_1, ξ_2 be i.i.d. random variables. Then

$$\begin{aligned} P(x_2 \geq \theta, x_1 < \theta|x_0 = 0) &= P(\xi_1 + (1 - \xi_1)\xi_2 \geq \theta, \xi_1 < \theta) \\ &\geq P(\xi_2 \geq \theta, \xi_1 < \theta). \end{aligned}$$

Hence

$$P(x_3 = 0|x_0 = 0) \geq \int_0^\theta f_\xi(y)dy \int_\theta^1 f_\xi(y)dy.$$

Denote $\epsilon_1 = \int_0^\theta f_\xi(y)dy$. Then for any $B \in \mathcal{B}(\mathbf{S})$,

$$P(x_3 \in B|x_0 = 0) \geq \epsilon_1 \nu(B). \tag{B.2}$$

Since time indices 2 and 3 in (B.1) and (B.2) have the greatest common divisor equal to 1, $\{x_t, t \geq 0\}$ is aperiodic [33, pp. 112-114]. □

The Markov process $\{x_t, t \geq 0\}$ is said to satisfy Doeblin's condition if there exist a probability measure ϕ on $\mathcal{B}(\mathbf{S})$ and $\epsilon < 1$, $\eta > 0$, $m \geq 0$, such that $\phi(B) > \epsilon$ implies

$$\inf_{x \in \mathbf{S}} P(x_m \in B|x_0 = x) \geq \eta.$$

Lemma B.3 *Doeblin's condition holds for $\{x_t, t \geq 0\}$.*

Proof. We take $\phi = \delta_0$, and in this case $\phi(B) > 0$ implies $0 \in B$. It suffices to show

$$\inf_{x \in \mathbf{S}} P(x_4 = 0|x_0 = x) \geq \eta.$$

For $x \in [\theta, 1]$,

$$\begin{aligned} P(x_4 = 0|x_0 = x) &\geq P(x_4 = 0, x_3 \geq \theta, x_2 < \theta, x_1 = 0|x_0 = x) \\ &= P(x_3 = 0, x_2 \geq \theta, x_1 < \theta|x_0 = 0) \\ &= P(x_2 \geq \theta, x_1 < \theta|x_0 = 0) \\ &\geq \epsilon_0 \epsilon_1. \end{aligned} \tag{B.3}$$

Let ξ, ξ_k, $k = 1, 2, 3$, be i.i.d. For $x \in [0, \theta)$,

$$\begin{aligned} P(x_4 = 0|x_0 = x) &\geq P(x_4 = 0, x_3 \geq \theta, x_2 = 0, x_1 \geq \theta|x_0 = x) \\ &= P(x_3 \geq \theta, x_2 = 0, x_1 \geq \theta|x_0 = x) \\ &= P(\xi_3 \geq \theta)P(x_2 = 0, x_1 \geq \theta|x_0 = x) \\ &= P(\xi_3 \geq \theta)P(x + (1 - x)\xi_1 \geq \theta) \\ &\geq P(\xi_3 \geq \theta)P(\xi_1 \geq \theta) = \epsilon_0^2. \end{aligned} \tag{B.4}$$

By (B.3) and (B.4), Doeblin's condition holds with $\epsilon = \frac{1}{2}$, $m = 4$, $\eta = \epsilon_0^2 \epsilon_1 > 0$. □

Proof of Theorem 4. Since $\{x_t, t \geq 0\}$ is aperiodic and satisfies Doeblin's condition, by [33, pp. 394, Theorem 16.0.2], (14) holds. □

Appendix C: An Auxiliary MDP

Assume (A1)-(A4). This appendix introduces an auxiliary optimal control problem to show the effect of the effort cost on the threshold parameter of the optimal policy. The state and control processes $\{(x_t^i, a_t^i), t \geq 0\}$ are specified by (1)-(2). The cost has the form

$$J_i^r = E \sum_{t=0}^{\infty} \rho^t (R_1(x_t^i) + r 1_{\{a_t^i = a_1\}}), \tag{C.1}$$

where R_1 is continuous and strictly increasing on $[0, 1]$ and $r \in (0, \infty)$. Let r take two different values $0 < \gamma_1 < \gamma_2$ and write the corresponding dynamic programming equation

$$v_l(x) = \min \left\{ \rho \int_0^1 v_l(y) Q_0(dy|x) + R_1(x), \quad \rho v_l(0) + R_1(x) + \gamma_l \right\}, \quad l = 1, 2, \; x \in \mathbf{S}. \tag{C.2}$$

By the method in proving Lemma 6, it can be shown that there exists a unique solution $v_l \in C([0, 1], \mathbb{R})$ and that the optimal policy $a^{i,l}(x)$ is a threshold policy. If $\rho \int_0^1 v_l(y) Q_0(dy|1) < \rho v_l(0) + \gamma_l$, $a^{i,l}(x) \equiv a_0$, and we follow the notation in Section 6.2 to formally denote it as a threshold policy with parameter $\theta_l = 1^+$. Otherwise, $a^{i,l}(x)$ is a threshold policy with parameter $\theta_l \in [0, 1]$, i.e., $a^{i,l}(x) = a_1$ if $x \geq \theta_l$, and $a^{i,l}(x) = a_0$ if $x < \theta_l$.

Lemma C.1 *If $\theta_1 \in (0, 1)$, $\theta_2 \neq \theta_1$.*

Proof. We prove by contradiction. Suppose for some $\theta \in (0, 1)$,

$$\theta_1 = \theta_2 = \theta. \tag{C.3}$$

Under assumption (C.3), the resulting optimal policy leads to the representation (see e.g. [15, pp. 22])

$$v_l(x) = E \sum_{t=0}^{\infty} \rho^t \left[R_1(x_t^i) + \gamma_l 1_{\{a_t^i = a_1\}} \right], \quad l = 1, 2,$$

where $\{x_t^i, t \geq 0\}$ is generated by the threshold policy $a_t^i(x_t^i)$ with parameter θ and $x_0^i = x$. Denote $\delta_{21} = \gamma_2 - \gamma_1$.

For any fixed $x \geq \theta$ and $x_0^i = x$, denote the resulting optimal state and control processes by $\{(\hat{x}_t^i, \hat{a}_t^i), t \geq 0\}$. Then $\hat{a}_0^i = a_1$ w.p.1., and

$$v_2(x) - v_1(x) = \delta_{21} + \delta_{21} E \sum_{t=1}^{\infty} \rho^t 1_{\{\hat{a}_t^i = a_1\}}, \qquad x \geq \theta.$$

Next consider $x_0^i = 0$ and denote the optimal state and control processes by $\{(\check{x}_t^i, \check{a}_t^i), t \geq 0\}$. Then

$$v_2(0) - v_1(0) = \delta_{21} E \sum_{t=0}^{\infty} \rho^t 1_{\{\check{a}_t^i = a_1\}} =: \Delta.$$

It is clear that $\hat{x}_1^i = 0$ w.p.1. By the optimality principle, $\{(\hat{x}_t^i, \hat{a}_t^i), t \geq 1\}$ may be interpreted as the optimal state and control processes of the MDP with initial state 0 at $t = 1$. Hence the two processes

$\{(\hat{x}_t^i, \hat{a}_t^i), t \geq 1\}$ and $\{(\check{x}_t^i, \check{a}_t^i), t \geq 0\}$, where $\check{x}_0^i = 0$, have the same finite dimensional distributions. In particular, $\hat{a}_{t+1}^i$ and $\check{a}_t^i$ have the same distribution for $t \geq 0$. Therefore,

$$E \sum_{t=1}^{\infty} \rho^{t-1} 1_{\{\hat{a}_t^i = a_1\}} = E \sum_{t=0}^{\infty} \rho^t 1_{\{\check{a}_t^i = a_1\}}.$$

It follows that

$$v_2(x) - v_1(x) = \delta_{21} + \rho\Delta, \qquad \forall x \geq \theta. \tag{C.4}$$

Combining (C.2) and (C.3) gives

$$\rho \int_0^1 v_l(y) Q_0(dy|\theta) = \rho v_l(0) + \gamma_l, \qquad l = 1, 2,$$

which implies

$$\rho \int_0^1 [v_2(x) - v_1(x)] Q_0(dx|\theta) = \delta_{21} + \rho\Delta. \tag{C.5}$$

By $Q_0([0, \theta)|\theta) = 0$ and (C.4), (C.5) further yields

$$\rho(\delta_{21} + \rho\Delta) = \delta_{21} + \rho\Delta,$$

which is impossible since $0 < \rho < 1$ and $\delta_{21} + \rho\Delta > 0$. Therefore, (C.3) does not hold. This completes the proof. □

For the MDP with cost (C.1), we continue to analyze the dynamic programming equation

$$v_r(x) = \min \Big[\rho \int_0^1 v_r(y) Q_0(dy|x) + R_1(x), \quad \rho v_r(0) + R_1(x) + r \Big]. \tag{C.6}$$

For each fixed $r \in (0, \infty)$, we obtain the optimal policy as a threshold policy with parameter $\theta(r)$. By evaluating the cost (C.1) associated with the two policies $a_t^i(x_t^i) \equiv a_0$ and $a_t^i(x_t^i) \equiv a_1$, respectively, we have the prior estimate

$$v_r(x) \leq \min \left\{ \frac{R_1(1)}{1-\rho}, \ R_1(x) + \frac{r + \rho R_1(0)}{1-\rho} \right\}. \tag{C.7}$$

On the other hand, let $\{x_t^i, t \geq 0\}$ with $x_0^i = x$ be generated by any fixed Markov policy. Then

$$E \sum_{t=0}^{\infty} \rho^t (R_1(x_t^i) + r 1_{\{a_t^i = a_1\}}) \geq R_1(x) + \sum_{t=1}^{\infty} \rho^t R_1(0),$$

which implies

$$v_r(x) \geq R_1(x) + \frac{\rho R_1(0)}{1-\rho}. \tag{C.8}$$

If $r > \frac{\rho R_1(1)}{1-\rho}$, it follows from (C.7) that

$$\rho \int_0^1 v_r(y) Q_0(dy|x) < \rho v_r(0) + r, \quad \forall x, \tag{C.9}$$

i.e., $\theta(r) = 1^+$.

Lemma C.2 *There exists $\delta > 0$ such that for all $0 < r < \delta$,*

$$\rho \int_0^1 v_r(y) Q_0(dy|x) > \rho v_r(0) + r, \quad \forall x, \tag{C.10}$$

and so $\theta(r) = 0$.

Proof. By (C.8),

$$\begin{aligned}\rho \int_0^1 v_r(y) Q_0(dy|x) &\geq \rho \int_0^1 R_1(y) Q_0(dy|x) + \frac{\rho^2 R_1(0)}{1-\rho} \\ &\geq \rho \int_0^1 R_1(y) Q_0(dy|0) + \frac{\rho^2 R_1(0)}{1-\rho},\end{aligned}$$

and (C.7) gives

$$\rho v_r(0) + r \leq \frac{\rho R_1(0)}{1-\rho} + \frac{r}{1-\rho}.$$

Since $R_1(x)$ is strictly increasing,

$$C_{R_1} := \int_0^1 R_1(y) Q_0(dy|0) - R_1(0) > 0.$$

and

$$\rho \int_0^1 v_r(y) Q_0(dy|x) - (\rho v_r(0) + r) \geq \rho C_{R_1} - \frac{r}{1-\rho}.$$

It suffices to take $\delta = \rho(1-\rho) C_{R_1}$. ☐

Define the nonempty sets

$$\mathcal{R}_{a_0} = \{r > 0 | \text{(C.9) hods}\}, \quad \mathcal{R}_{a_1} = \{r > 0 | \text{(C.10) holds}\}.$$

Remark C.1 *We have $(\frac{\rho R_1(1)}{1-\rho}, \infty) \subset \mathcal{R}_{a_0}$ and $(0, \delta) \subset \mathcal{R}_{a_1}$.*

Lemma C.3 *Let (r, v_r) be the parameter and the associated solution in* (C.6).

i) If $r > 0$ satisfies

$$\rho \int_0^1 v_r(y) Q_0(dy|x) \leq \rho v_r(0) + r, \quad \forall x, \tag{C.11}$$

then any $r' > r$ is in $\mathcal{R}_{a_0}$.

ii) If $r > 0$ satisfies

$$\rho \int_0^1 v_r(y) Q_0(dy|x) \geq \rho v_r(0) + r, \quad \forall x, \tag{C.12}$$

then any $r' \in (0, r)$ is in $\mathcal{R}_{a_1}$.

Proof. i) For $r' > r$, $v_{r'}$ is uniquely solved from (C.6) with r' in place of r. We can use (C.11) to verify

$$v_r(x) = \min \left[\rho \int_0^1 v_r(y) Q_0(dy|x) + R_1(x), \quad \rho v_r(0) + R_1(x) + r' \right].$$

Hence $v_{r'} = v_r$ for all $x \in [0,1]$. It follows that $\rho \int_0^1 v_{r'}(y) Q_0(dy|x) < \rho v_{r'}(0) + r'$ for all x. Hence $r' \in \mathcal{R}_{a_0}$.

ii) By (C.6) and (C.12),

$$v_r(0) = \frac{R_1(0) + r}{1-\rho},$$

and subsequently,

$$\begin{aligned} v_r(x) &= \rho v_r(0) + R_1(x) + r \\ &= \frac{\rho R_1(0) + r}{1-\rho} + R_1(x). \end{aligned}$$

By substituting $v_r(0)$ and $v_r(x)$ into (C.12), we obtain

$$\rho R_1(0) + r \leq \rho \int_0^1 R_1(y) Q_0(dy|x), \quad \forall x. \tag{C.13}$$

Now for $0 < r' < r$, we construct $v_{r'}(x)$, as a candidate solution to (C.6) with r replaced by r', to satisfy

$$v_{r'}(0) = \rho v_{r'}(0) + R_1(0) + r', \quad v_{r'}(x) = \rho v_{r'}(0) + R_1(x) + r', \tag{C.14}$$

which gives

$$v_{r'}(x) = \frac{\rho R_1(0) + r'}{1-\rho} + R_1(x). \tag{C.15}$$

We show that $v_{r'}(x)$ in (C.15) satisfies

$$\rho v_{r'}(0) + r' < \rho \int_0^1 v_{r'}(y) Q_0(dy|x), \quad \forall x, \tag{C.16}$$

which is equivalent to

$$\rho R_1(0) + r' < \rho \int_0^1 R_1(y) Q_0(dy|x), \quad \forall x,$$

which in turn follows from (C.13). By (C.14) and (C.16), $v_{r'}$ indeed satisfies (C.6) with r replaced by r'. So $r' \in \mathcal{R}_{a_1}$. $\square$

Further define

$$\underline{r} = \sup \mathcal{R}_{a_1}, \quad \overline{r} = \inf \mathcal{R}_{a_0}.$$

Lemma C.4 *i)* $\underline{r}$ *satisfies*

$$\rho \int_0^1 v_{\underline{r}}(y) Q_0(dy|0) = \rho v_{\underline{r}}(0) + \underline{r},$$

and $\theta(\underline{r}) = 0$.

ii) $\overline{r}$ *satisfies*

$$\rho \int_0^1 v_{\overline{r}}(y) Q_0(dy|1) = \rho v_{\overline{r}}(1) = \rho v_{\overline{r}}(0) + \overline{r}.$$

and $\theta(\overline{r}) = 1$.

iii) We have $0 < \underline{r} < \overline{r} < \infty$.

iv) $\theta(r)$ *is continuous and strictly increasing on* $[\underline{r}, \overline{r}]$.

Proof. i)-ii) By Lemmas C.2 and C.3, we have $0 < \underline{r} \le \infty$ and $0 \le \overline{r} < \infty$. Assume $\underline{r} = \infty$; then $\mathcal{R}_{a_1} = (0,\infty)$ giving $\mathcal{R}_{a_0} = \emptyset$, a contradiction. So $0 < \underline{r} < \infty$. For $\delta > 0$ in Lemma C.2, we have $(0,\delta) \subset \mathcal{R}_{a_1}$. Therefore, $0 < \bar{r} < \infty$. Note that v_r depends on the parameter r continuously, i.e., $\lim_{|r'-r|\to 0} \sup_x |v_{r'}(x) - v_r(x)| = 0$. Hence

$$\rho \int_0^1 v_{\underline{r}}(y) Q_0(dy|0) \ge \rho v_{\underline{r}}(0) + \underline{r}.$$

Now assume

$$\rho \int_0^1 v_{\underline{r}}(y) Q_0(dy|0) > \rho v_{\underline{r}}(0) + \underline{r}. \tag{C.17}$$

Then there exists a sufficiently small $\epsilon > 0$ such that (C.17) still holds when $(\underline{r}+\epsilon, v_{\underline{r}+\epsilon})$ replaces $(\underline{r}, v_{\underline{r}})$; since $g(x) = \int_0^1 v_{\underline{r}+\epsilon}(y) Q_0(dy|x)$ is increasing in x, then $\underline{r}+\epsilon \in \mathcal{R}_{a_1}$, which is impossible. Hence (C.17) does not hold, and this proves i). ii) can be shown in a similar manner.

To show iii), assume

$$0 < \overline{r} < \underline{r} < \infty. \tag{C.18}$$

Then, recalling Remark C.1, there exist $r' \in \mathcal{R}_{a_0}$ and $r'' \in \mathcal{R}_{a_1}$ such that

$$0 < \overline{r} < r' < r'' < \underline{r} < \infty.$$

By Lemma C.3-i), $r' < r'' \in \mathcal{R}_{a_0}$, and then $r'' \in \mathcal{R}_{a_0} \cap \mathcal{R}_{a_1} = \emptyset$, which is impossible. Therefore, (C.18) doe not hold and we conclude $0 < \underline{r} \le \overline{r} < \infty$. We further assume $\underline{r} = \overline{r}$. Then i)-ii) would imply $\int_0^1 v_{\underline{r}}(y) Q_0(dy|0) = v_{\underline{r}}(1)$, which is impossible since $v_{\underline{r}}$ is strictly increasing on $[0,1]$ and (A3) holds. This proves iii).

iv) By the definition of $\underline{r}$ and $\overline{r}$, it can be shown using (C.6) that $\theta(r) \in (0,1)$ for $r \in (\underline{r}, \overline{r})$. By the continuous dependence of the function $v_r(\cdot)$ on r and the method of proving Lemma A.4, we can show the continuity of $\theta(r)$ on $(0,1)$, and further show $\lim_{r\to\underline{r}^+} \theta(r) = 0$ and $\lim_{r\to\overline{r}^-} \theta(r) = 1$. So $\theta(r)$ is continuous on $[\underline{r}, \overline{r}]$. If $\theta(r)$ were not strictly increasing on $[\underline{r}, \overline{r}]$, there would exist $\underline{r} < r_1 < r_2 < \overline{r}$ such that

$$\theta(r_1) \ge \theta(r_2). \tag{C.19}$$

If $\theta(r_1) > \theta(r_2)$ in (C.19), by the continuity of $\theta(r)$, $\theta(\underline{r}) = 0$, $\theta(\overline{r}) = 1$, and the intermediate value theorem we may find $r' \in (\underline{r}, r_1)$ such that $\theta(r_1') = \theta(r_2)$. Next, we replace r_1 by r_1'. Thus if $\theta(r)$ is not strictly increasing, we may find $r_1 < r_2$ from $(\underline{r}, \overline{r})$ such that $\theta(r_1) = \theta(r_2) \in (0,1)$, which is a contradiction to Lemma C.1. This proves iv). $\square$

Remark C.2 *By Lemmas C.3 and C.4,* $\mathcal{R}_{a_1} = (0, \underline{r})$ *and* $\mathcal{R}_{a_0} = (\overline{r}, \infty)$.

Remark C.3 *If (A4) is replaced by* $P(\xi \in (0,1)) > 0$ *without assuming a probability density function* f_ξ*, we still have* $C_{R_1} > 0$ *in the proof of Lemma C.2 and all results in this appendix hold.*

Appendix D: Proof of Theorem 5

Let $\{x_t^{i,\theta}, t \geq 0\}$ be the Markov chain generated by the threshold policy with parameter $0 < \theta < 1$, where $x_0^{i,\theta}$ is given. Define $m_t^{i,\theta} = 1 - x_t^{i,\theta}$. By Theorem 4, $\{x_t^{i,\theta}, t \geq 0\}$ and $\{m_t^{i,\theta}, t \geq 0\}$ are ergodic.

To facilitate further computation, we define an auxiliary Markov chain $\{Y_t, t \geq 0\}$. Let ξ be specified in (A3) and define $\tilde{\xi} = 1 - \xi$. Let $\{\tilde{\xi}, \tilde{\xi}_t, t \geq 1\}$ be i.i.d. random variables. For $\lambda \in (0, 1)$, define $\{Y_t, t \geq 0\}$ as follows:

$$Y_0 = 1, \quad Y_t = \tilde{\xi}_t Y_{t-1} \quad \text{for} \quad 1 \leq t \leq \tau, \tag{D.1}$$

where

$$\tau = \inf\{t | Y_t \leq \lambda\}.$$

By (A4), $P(\tau < \infty) = 1$, and moreover, $E\tau < \infty$. Set $Y_{\tau+1} = 1$ and the process $\{Y_t, t \geq 0\}$ further evolves from state 1 at time $\tau + 1$ as a Markov chain with a stationary transition probability kernel. Denote $S_t = \sum_{i=0}^t Y_i$ for $t \geq 0$.

Lemma D.1 *We have*

$$\lim_{k\to\infty} \frac{1}{k} \sum_{t=0}^{k-1} Y_t = \frac{ES_\tau}{1 + E\tau} \qquad \text{w.p.1.} \tag{D.2}$$

Proof. Take $\lambda = 1 - \theta$. Since $\{Y_t, t \geq 0\}$ has the same transition probability kernel as $\{m_t^{i,\theta}, t \geq 0\}$, it is ergodic, and therefore the left hand side of (D.2) has a constant limit w.p.1. Define $T_0 = 0$ and T_n as the time for $\{Y_t, t \geq 0\}$ to return to state 1 for the nth time. Define $B_n = \sum_{t=T_{n-1}}^{T_n - 1} Y_t$ for $n \geq 1$. We observe that $\{Y_t, t \geq 0\}$ is a regenerative process (see e.g. [4, 38] and [5, Theorem 4]) with regeneration times $\{T_n, n \geq 1\}$ and that $\{B_n, n \geq 1\}$ is a sequence of i.i.d. random variables. Note that $B_1 = S_\tau$ is the sum of $\tau + 1$ terms. By the strong law of large numbers for regenerative processes [4, pp. 177], the lemma follows. □

Define another Markov chain $\{Y_t', t \geq 0\}$ after replacing λ in (D.1) by $\lambda' \in (0, \lambda)$, and τ by $\tau' = \inf\{t | Y_t' \leq \lambda'\}$. The initial state is $Y_0' = 1$. Let $S_{\tau'}' = \sum_{t=0}^{\tau'} Y_t'$.

Lemma D.2 *We have*

$$\frac{ES_\tau}{1 + E\tau} \geq \frac{ES_{\tau'}'}{1 + E\tau'}. \tag{D.3}$$

Proof. Denote $\zeta_k = \prod_{t=1}^k \tilde{\xi}_t$ for $k \geq 1$. If $k \geq 2$, $\{\tau \geq k\} = \{\zeta_{k-1} > \lambda\}$. For the given λ, we have

$$E\tau = \sum_{k=1}^\infty kP(\tau = k) = \sum_{k=1}^\infty P(\tau \geq k) = 1 + \sum_{k=1}^\infty P(\zeta_k > \lambda). \tag{D.4}$$

Denote $\alpha = E\tilde{\xi}$. Then $\alpha \in (0, 1)$ by (A4). We obtain

$$\begin{aligned} ES_\tau &= E\sum_{k=1}^\infty S_k 1_{\{\tau=k\}} \\ &= EY_0 + EY_1 + E\sum_{k=2}^\infty Y_k 1_{\{\tau \geq k\}}. \end{aligned}$$

By the independence of $\tilde{\xi}_k$ and $\zeta_{k-1}1_{\{\zeta_{k-1}>\lambda\}}$ for $k \geq 2$, it follows that

$$E(Y_k 1_{\{\tau\geq k\}}) = E(\tilde{\xi}_k \zeta_{k-1} 1_{\{\zeta_{k-1}>\lambda\}}) = \alpha E(\zeta_{k-1} 1_{\{\zeta_{k-1}>\lambda\}}).$$

This gives

$$ES_\tau = 1 + \alpha + \alpha \sum_{k=1}^{\infty} E(\zeta_k 1_{\{\zeta_k>\lambda\}}).$$

For $k \geq 1$, denote

$$p_k = P(\zeta_k > \lambda), \quad r_k = E(\zeta_k 1_{\{\zeta_k>\lambda\}}), \quad \delta_k = P(\lambda' < \zeta_k \leq \lambda), \quad \Delta_k = E(\zeta_k 1_{\{\lambda'<\zeta_k\leq\lambda\}}).$$

We have

$$\frac{ES_\tau}{1+E\tau} = \frac{1+\alpha+\alpha\sum_{k=1}^{\infty} r_k}{2+\sum_{k=1}^{\infty} p_k}$$

and

$$\frac{ES'_{\tau'}}{1+E\tau'} = \frac{1+\alpha+\alpha\sum_{k=1}^{\infty}(r_k+\Delta_k)}{2+\sum_{k=1}^{\infty}(p_k+\delta_k)}.$$

The inequality (D.3) is equivalent to

$$\alpha \left(\sum_{k=1}^{\infty} \Delta_k\right)\left(2+\sum_{k=1}^{\infty} p_k\right) \leq \left(1+\alpha+\alpha\sum_{k=1}^{\infty} r_k\right)\left(\sum_{k=1}^{\infty}\delta_k\right). \tag{D.5}$$

Clearly, $\Delta_k \leq \lambda\delta_k$ for $k \geq 1$. To prove (D.5), it suffices to show

$$\alpha\lambda\left(2+\sum_{k=1}^{\infty} p_k\right) \leq 1+\alpha+\alpha\sum_{k=1}^{\infty} r_k.$$

Since $r_k \geq \lambda p_k$ for $k \geq 1$, we only need to show

$$2\alpha\lambda \leq 1+\alpha,$$

which follows from $0<\alpha<1,\ 0<\lambda<1$. □

Suppose $0<\theta<\theta'<1$. Then there exist two constants $C_\theta, C_{\theta'}$ such that

$$\lim_{k\to\infty}\frac{1}{k}\sum_{t=0}^{k-1} x_t^{i,\theta} = C_\theta, \qquad \lim_{k\to\infty}\frac{1}{k}\sum_{t=0}^{k-1} x_t^{i,\theta'} = C_{\theta'}, \quad \text{w.p.1.}$$

Lemma D.3 *We have $C_\theta \leq C_{\theta'}$.*

Proof. The lemma follows from the relation $x_t^{i,\theta} = 1 - m_t^{i,\theta}$ for $t \geq 0$, Lemmas D.1 and D.2. □

Remark D.1 *Due to the ergodicity of the Markov chains in Lemma D.3, the initial states $x_0^{i,\theta}$ and $x_0^{i,\theta'}$ can be arbitrary.*

Proof of Theorem 5. By the ergodicity of $\{x_t^{i,\theta}, t\geq 0\}$, we have $z(\theta_l) = \lim_{k\to\infty}\frac{1}{k}\sum_{t=0}^{k-1} x_t^{i,\theta_l}$ w.p.1. Lemma D.3 implies $z(\theta_1) \leq z(\theta_2)$. □

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