Linear quadratic mean field games: Asymptotic solvability and relation to the fixed point approach

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***Abstract*—Mean field game theory has been developed largely following two routes. One of them, called the direct approach, starts by solving a large-scale game and next derives a set of limiting equations as the population size tends to infinity. The second route is to apply mean field approximations and formalize a fixed point problem by analyzing the best response of a representative player. This paper addresses the connection and difference of the two approaches in a linear quadratic (LQ) setting. We first introduce an asymptotic solvability notion for the direct approach, which means for all sufficiently large population sizes, the corresponding game has a set of feedback Nash strategies in addition to a mild regularity requirement. We provide a necessary and sufficient condition for asymptotic solvability and show that in this case the solution converges to a mean field limit. This is accomplished by developing a re-scaling method to derive a low dimensional ordinary differential equation (ODE) system, where a non-symmetric Riccati ODE has a central role. We next compare with the fixed point approach which determines a two point boundary value (TPBV) problem, and show that asymptotic solvability implies feasibility of the fixed point approach, but the converse is not true. We further address non-uniqueness in the fixed point approach and examine the long time behavior of the non-symmetric Riccati ODE in the asymptotic solvability problem.**

***Index Terms*—Asymptotic solvability, direct approach, fixed point approach, linear quadratic, mean field game, re-scaling, Riccati differential equation.**

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