Introduction to Mean Field Game Theory Part I

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Outline of talk

- Background and motivation for mean field game (MFG) theory
- Illustrative modeling examples
- Fundamental methodologies
- Applications and references
A mean field game is a situation of stochastic (dynamic) decision making where

- each agent interacts with the aggregate effect of all other agents;
- agents are non-cooperative.

Example: Hybrid electric vehicle recharging control (interacting through aggregate load/price)
Noncooperative games: historical background

1. Cournot duopoly  Cournot equilibrium, 1838
2. 2 person zero-sum  von Neumann’s minimax theorem, 1928
3. N-person nonzero-sum  Nash equilibrium, 1950

Antoine Cournot  (1801-1877)
John von Neumann  (1903-1957)
John Nash  (1928-2015)
Two directions for generalizations:

- Include **dynamics**
  
  **Examples:** L. Shapley (1953), MDP model; Rufus Isaacs (1965), *Differential games*, Wiley (work of 1950s)

- Consider **large populations** of players

Technical challenges arise ...

- the **Curse of Dimensionality** with a large number of players

- Example: 50 players, each having 2 states, 2 actions. Need a system configuration: $4^{50} = 1.27 \times 10^{30}$
Early efforts to deal with large populations

- J. W. Milnor and L.S. Shapley (1978); R. J. Aumann (1975). This is for cooperative games
- B. Jovanovic and R. W. Rosenthal (1988). Anonymous sequential games. It considers distributional strategies for an infinite population. However, individual behaviour is not addressed
- ...
The early motivation for mean field games

- A wireless network power control problem
Rate based power control (Huang, Caines, Malhamé’03)
\[ dp^i = u^i dt, \quad |u^i| \leq u_{\text{max}}, \quad 1 \leq i \leq N \]

Log-normal channel gain (in dB)
\[ dx^i = -a(x^i + b)dt + \sigma dW^i \quad \text{(Charalambous et al’99)} \]

\( W^i \): white noise (Brownian motion)

Cost function of user \( i \)
\[ J_i = E \int_0^T \left\{ \left[ p^i e^{x^i} - \gamma \left( \frac{\alpha}{N} \sum_{j \neq i}^N p^j e^{x^j} + \eta \right) \right]^2 + r(u^i)^2 \right\} dt \]

\[ \frac{\alpha}{N} \sum_{j \neq i}^N p^j e^{x^j} + \eta \]: total interference due to other users and thermal noise at receiver

Cocktail party model: each speaker deals with bkgrd noise; one against the mass

Control objective: keep signal to interference ratio around \( \gamma \)

**Figure**: A cell of \( N \) users
Computational load with $N = 2$ (Huang, Caines, Malhamé, IFAC’02, IEEE TAC’04):

- Grid size for the 4 spatial variables in $\mathbb{R}^4$: $90^4$
- Discretize time interval $[0, 1]$ with step size 0.1
- For the computation, memory allocation of 70 million sizeof(float) in C
- Very heavy computation via iterations!!

For large $N$, such iteration is no longer feasible

This motivated the development of a different approach to tackle dimensionality difficulty

See Aziz and Caines (2017) for more recent numerical analysis using MFG
Conceptual Modeling of mean field games

Model: Each player interacts with the empirical state distribution \( \delta_z^{(N)} \) of \( N \) players via dynamics and/or costs (see examples)

- Individual state and control \((z_i, u_i)\), \( \delta_z^{(N)}(t) = \frac{1}{N} \sum_{i=1}^{N} \delta_{z_i}(t) \)

Objective: Overcome dimensionality difficulty
Illustrative examples for MFG modeling

- Discrete time linear quadratic (LQ) model
- Continuous time LQ model
- MDP model
- Stochastic growth with relative performance
Example 1: Discrete time linear quadratic (LQ) games

\[ X_{t+1}^i = AX_t^i + Bu_t^i + GX_t^{(N)} + DW_{t+1}^i, \quad 1 \leq i \leq N \]

\[ J_i(u^i, u^{-i}) = E \sum_{t=0}^{\infty} \rho^t \left\{ |X_t^i - (\Gamma X_t^{(N)} + \eta)|_Q^2 + (u_t^i)^T R u_t^i \right\} \]

- \( X_t^i \): state; \( u_t^i \): control; \( u^{-i} \): control of all other agents; \( W_t^i \): noise;
- \( Q \geq 0, R > 0, \rho \in (0, 1); \quad |z|_Q = (z^T Q z)^{1/2} \)
- The **coupling term** \( X_t^{(N)} = \frac{1}{N} \sum_{k=1}^{N} X_t^k \)
- It is possible to consider different parameters such as \( A_i \) instead of \( A \), etc.
Example 1’: Continuous time linear quadratic (LQ)

\[
dX_t^i = (A_{\theta_i}X_t^i + B u_t^i + G X_t^{(N)}) dt + D dW_t^i, \quad 1 \leq i \leq N
\]

\[
J_i(u^i, u^{-i}) = E \int_0^\infty e^{-\rho t} \left\{ |X_t^i - (\Gamma X_t^{(N)} + \eta)|_Q^2 + (u_t^i)^T R u_t^i \right\} dt
\]

- \(X_t^i\): state; \(u_t^i\): control; \(W_t^i\): white noise as Brownian motion; \(\rho > 0\);
- \(|z|_Q = (z^T Q z)^{\frac{1}{2}}\)
- The coupling term \(X_t^{(N)} = \frac{1}{N} \sum_{k=1}^N X_t^k\)
- \(\theta_i\): dynamic parameter to indicate heterogeneity

Note: This is the continuous time counterpart of Example 1.
Example 2: Discrete time Markov decision processes (MDPs)

- $N$ MDPs

$(x^i_t, a^i_t), \ 1 \leq i \leq N,$

where state $x^i_t$ has transitions affected by action $a^i_t$ but not by $a^j_t, j \neq i$.

- Player $i$ has cost

$$J_i = E \sum_{t=0}^{T} \rho^t l(x^i_t, x^N_t, a^i_t), \quad \rho \in (0, 1].$$
Example 3: Growth with relative performance

Dynamics of $N$ agents:

$$dX^i_t = \left[ A(X^i_t)^\alpha - \delta X^i_t - C^i_t \right] dt - \sigma X^i_t dW^i_t, \quad 1 \leq i \leq N$$

- $X^i_t$: capital stock, $X^i_0 > 0$, $EX^i_0 < \infty$, $C^i_t$: consumption rate
- $A\alpha$, $\alpha \in (0, 1)$: Cobb-Douglas production function, $0 < \alpha < 1$, $A > 0$
- $\delta dt + \sigma dW^i_t$: stochastic depreciation (see e.g. Wälde'11, Feicht and Stummer'10 for stochastic depreciation modeling)
- $\{W^i_t, 1 \leq i \leq N\}$ are i.i.d. standard Brownian motions. The i.i.d. initial states $\{X^i_0, 1 \leq i \leq N\}$ are also independent of the $N$ Brownian motions
- Take the standard choice $\gamma = 1 - \alpha$, i.e., equalizing the coefficient of the relative risk aversion to capital share

See Huang and Nguyen (2016)
Example 3 (ctn) The utility functional of agent $i$:

$$J_i(C^1, \ldots, C^N) = E \left[ \int_0^T e^{-\rho t} U(C^i_t, C^{(N,\gamma)}_t) dt + e^{-\rho T} S(X_T) \right],$$

where $C^{(N,\gamma)}_t = \frac{1}{N} \sum_{i=1}^N (C^i_t)\gamma$, $\gamma \in (0,1)$. Take $S(x) = \frac{\eta x^{\gamma}}{\gamma}$, $\eta > 0$, and

$$U(C^i_t, C^{(N,\gamma)}_t) = \frac{1}{\gamma} (C^i_t)^{(1-\lambda)\gamma} \left[ \frac{(C^i_t)^{\gamma}}{C^{(N,\gamma)}_t} \right]^\lambda$$

$$= \left[ \frac{1}{\gamma} (C^i_t)\gamma \right]^{1-\lambda} \left[ \frac{1}{\gamma} \frac{(C^i_t)^{\gamma}}{C^{(N,\gamma)}_t} \right]^\lambda =: U_0^{1-\lambda} U_1^\lambda.$$  

Interpretation of $U$: as a **weighted geometric mean** of $U_0$ (own utility) and $U_1$ (relative utility).
Try conventional methods to solve, for instance, Example 1’:

\[ dX^i_t = (A_{\theta_i}X^i_t + Bu^i_t + G\mathbf{X}^{(N)}_t)dt + Dw^i_t, \quad 1 \leq i \leq N \]

\[ J_i(u^i, u^{-i}) = E \int_0^\infty e^{-\rho t} \{ |X^i_t - (\Gamma \mathbf{X}^{(N)}_t + \eta)|^2 + (u^i_t)^T R u^i_t \} dt \]

- Form large vector \( X_t := [X^1_t, \ldots, X^N_t]^T \). Similarly, \( u_t = [u^1_t, \ldots, u^N_t]^T \)
- This gives a standard dynamic game

**Immediate difficulties** (for large \( N \)):

- Solve a very large scale dynamic programming problem
- In the resulting solution, each \( u^i_t \) will be a function of \( X_t \), which is too much information and may be unnecessary

**Such difficulties** motivate the development of MFG theory!
The fundamental diagram of MFG theory

Problem $P_0$: $N$ player Nash game
- states: $(x_1, x_2, \ldots, x_N)$
- strategies: $(u_1, u_2, \ldots, u_N)$
- costs: $(J_1, J_2, \ldots, J_N)$

A large-scale coupled equation system
- Example: $N$ coupled dynamic programming equations

Problem $P_\infty$: Optimal control of a single player
- state: $x_i$
- control: $u_i$
- mean field $\mu_i$ is fixed and not controlled by $u_i$

MFG equation system:
- 1 equation of optimal control;
- 1 equation of mean field dynamics (for $\mu_i$)
- Example: HJB PDE; FPK PDE (or McKean-Vlasov SDE)

- blue: direct approach (Lasry and Lions, 06, 07);
- red: fixed point approach (Huang, Caines, Malhamé, 03, 07), (Huang, Malhamé, Caines, 06);
see overview of the two approaches in (Caines, Huang, Malhamé, 17)
Before analyzing MFG, we mention the math toolkit for optimal control

- Dynamic programming (DP) (in Markov control models)
- Stochastic maximum principle (SMP)/calculus of variations

SMP is useful when the optimal control problem does not have Markov properties (example: random model coefficients)

Here we give brief preliminaries on continuous time DP
Consider the optimal control problem –

Dynamics and cost:

\[ dX_t = (AX_t + Bu_t + G\bar{X}_t)ds + DdW_t, \]

\[ J(u(\cdot)) = E \int_0^T e^{-\rho t}L(X_t, u_t)dt, \]

where \( L = |X_t - (\Gamma \bar{X}_t + \eta)|^2 Q + u_t^T Ru_t, \rho > 0, Q \geq 0, R > 0. \)

\( \bar{X} \in C[0, T]. \) Moreover \( |\bar{X}_t| = O(e^{t(\rho-\epsilon)/2}) \) for some small \( \epsilon > 0 \)
(depending on \( \bar{X} \)) if \( T = \infty; \) denote \( \bar{X} \in C_{\rho/2}[0, \infty). \)

\( u_t \) is \( \mathcal{F}_t^W \)-adapted (i.e., adapted to the filtration of the BM). Alternatively, one may take \( u_t \) to be \( X_t \) dependent (i.e. state feedback).

We consider two cases:  

i) \( T \) is finite,  

ii) \( T = \infty \)
By the method of DP, we try to solve a family of optimization problems with initial pair \((t, x)\). Consider

\[
dX_r = (AX_r + Bu_r + \bar{G}X_r)dr + DdW_r, \quad r \geq t, \quad X_t = x,
\]

\[
J(t, x, u(\cdot)) = E_{t,x} \int_t^T e^{-\rho(r-t)}L(X_r, u_r)dr,
\]

where \(E_{t,x}\) indicates \(X_t = x\), and \(u_r\) is \(\mathcal{F}_r^W\)-adapted.

Define the value function

\[
V(t, x) = \inf_u E_{t,x} \int_t^T e^{-\rho(r-t)}L(X_r, u_r)dr.
\]
By DP, for $\delta > 0$,

$$V(t, x) = \inf E \left[ \int_t^{t+\delta} e^{-\rho(r-t)} L(X_r, u_r) dr + e^{-\rho\delta} V(t + \delta, X_{t+\delta}) \right],$$

and we further obtain the HJB equation

$$\rho V = V_t + V_x^T (Ax + \bar{G}\bar{X}_t) + |x - (\bar{\Gamma}\bar{X}_t + \eta)|^2_Q$$

$$+ \frac{1}{2} Tr(D^T V_{xx} D) + \min_u [u^T Ru + V_x^T Bu]$$

$$= V_t - \frac{1}{4} V_x^T BR^{-1} B^T V_x + V_x^T (Ax + \bar{G}\bar{X}_t)$$

$$+ |x - (\bar{\Gamma}\bar{X}_t + \eta)|^2_Q + \frac{1}{2} Tr(D^T V_{xx} D).$$

And $V(T, x) = 0$ when $T$ is finite. $V_t, V_x, V_{xx}$: partial derivatives.

See e.g. Fleming and Rishel (1975), Yong and Zhou (1999)
i) For finite $T$, denote

$$V(t, x) = x^T P(t)x + 2s^T(t)x + q(t).$$

We derive differential Riccati equation (DRE)

$$\begin{cases}
\rho P(t) = \dot{P}(t) - P(t)BR^{-1}B^T P(t) + P(t)A + A^T P(t) + Q, \\
P(T) = 0.
\end{cases}$$

$$\begin{cases}
\rho s(t) = \dot{s}(t) + (A^T - PBR^{-1}B^T)s(t) + PG\bar{X}_t - Q(\Gamma\bar{X}_t + \eta), \\
s(T) = 0.
\end{cases}$$

**Fact:** The DRE has a unique solution (due to $Q \geq 0$ and $R > 0$). Use the minimizer in the HJB to give the optimal control

$$u^\text{opt}_t = R^{-1}B^T(P(t)X_t + s(t)).$$
ii) For $T = \infty$,

- the DRE is replaced by an algebraic Riccati equation (ARE) which has a unique solution $II \geq 0$.
- In this case, the ODE of $s$ has no terminal condition;
- a growth condition can be imposed to determine a unique solution for $s$. 
Solve Example 1’ by the fixed point approach

- We use the LQ model to illustrate this methodology
- The procedure here is very robust and can be applied to different models
Fixed approach to the LQ MFG (as in Ex. 1'): Problem $P_0$ –

$$dX_t^i = (A_{\theta_i}X_t^i + Bu_t^i + GX_t^{(N)})dt + DdW_t^i, \quad 1 \leq i \leq N$$

$$J_i(u^i, u^{-i}) = E \int_0^\infty e^{-\rho t} \left\{ |X_t^i - (\Gamma X_t^{(N)} + \eta)|_Q^2 + (u_t^i)^T R u_t^i \right\} dt$$

Assume $x_0^{(N)} = \frac{1}{N} \sum_{i=1}^N EX_0^i \to \bar{X}_0$.

Taking **approximation**: Optimal Control Problem $P_\infty$ –

$$dX_t^i = (A_{\theta_i}X_t^i + Bu_t^i + G\bar{X}_t)dt + DdW_t^i, \quad 1 \leq i \leq N$$

$$J_i(u^i, u^{-i}) = E \int_0^\infty e^{-\rho t} \left\{ |X_t^i - (\Gamma \bar{X}_t + \eta)|_Q^2 + (u_t^i)^T R u_t^i \right\} dt$$

- **Step 1.** Solve $P_\infty$ assuming the deterministic $\bar{X}_t$ were known
- **Step 2.** Use **consistency condition** of MFG to find an equation for $\bar{X}_t$
Step 1. Denote (we consider $A_{\theta_i} \equiv A$ just for simplicity)

Riccati equation: $\rho \Pi = \Pi A + A^T \Pi - \Pi BR^{-1} B^T \Pi + Q$

and (for $s \in \mathbb{C}_{\rho/2}$; no given initial condition)

ODE: $\rho s_t = \frac{ds_t}{dt} + (A^T - \Pi BR^{-1} B^T) s_t + \Pi G \bar{X}_t - Q(\Gamma \bar{X}_t + \eta)$

Best response (BR): $\hat{u}_t^i = -R^{-1}B^T(\Pi X_t^i + s_t)$.

Step 2. Closed-loop for $P_\infty$:

$$dX_t^i = (A - BR^{-1}B^T \Pi) X_t^i dt - BR^{-1}B^T s_t dt + G \bar{X}_t dt + DdW_t^i$$

$$\Rightarrow \frac{d\bar{X}_t}{dt} = (A - BR^{-1}B^T \Pi) \bar{X}_t + G \bar{X}_t - BR^{-1}B^T s_t,$$

which is obtained from averaging $X_t^i$ with $\bar{X}_0$ determined.
Summarizing, the MFG equation system is

\[
\begin{cases}
\frac{d\tilde{X}_t}{dt} = (A - BR^{-1}B^T\Pi + G)\tilde{X}_t - BR^{-1}B^Ts_t, \\
\rho s_t = \frac{ds_t}{dt} + (A^T - \Pi BR^{-1}B^T)s_t + \Pi G\tilde{X}_t - Q(\Gamma \tilde{X}_t + \eta),
\end{cases}
\]

where $\tilde{X}_0$ is given. Look for $s \in C_{\rho/2}[0, \infty)$.

**Analysis and results:**

This is carried out in the form of a list of Q&A’s.
Question 1: Does there exist a unique solution $(\tilde{X}, s)$?

Note: $s$ only has a growth condition

Answer: under some conditions, there exists a unique solution.

Method: use fixed point theorem (write $\tilde{X}$ as the fixed point of a linear operator), or subspace decomposition of the ODE system.
Question 2. How is the solution used in the $N$ player model?

Answer: provide **decentralized strategies**

The set of **decentralized strategies**, denoted as $\hat{u}$, for the $N$ players:

$$\hat{u}_t^i = -R^{-1}B^T(\Pi X_t^i + s_t), \quad 1 \leq i \leq N.$$ 

**Key Feature**: Player $i$ does not need the state or sample path information of other players.
Question 3: What is the **performance** of the set of decentralized strategies?

To analyze performance, we need to introduce strategy spaces.

\[ \mathcal{U}^i_{\text{centr}} \text{ consists of strategies of the form } u^i(t, x^1, \ldots, x^N); \]
\[ |u^i(t, x^1, \ldots, x^N)| \leq C(1 + |x|); \text{ and } u^i \text{ is Lipschitz in } x = (x^1, \ldots, x^N). \] This ensures well defined SDEs.

- \( u^i \) is a **centralized strategy**, here depending on states of all players
- \( u^i \) is a Markov feedback strategy

Although decentralized strategies are desired, the consideration of \( \mathcal{U}^i_{\text{centr}} \) is useful for characterizing performance.
To answer Q3 on performance –

Main assumptions:

i) $X_0^i, i \geq 1$, are independent and also independent of the BMs $W^i, i \geq 1$.

ii) $\sup_i E|X_0^i|^2 \leq C$.

iii) The average of initial means $x_0^{(N)} \rightarrow \bar{X}_0$ as $N \rightarrow \infty$. 
Question 3 (restated): What is the performance of the set of decentralized strategies?

**Theorem:** If the MFG equation system has a unique solution, the set of decentralized strategies is an $\epsilon$-Nash equilibrium, i.e., for each $i$,

$$\inf_{u_i \in U_{i_{\text{centr}}}} J_i(u^i, \hat{u}^{-i}) \geq J_i(\hat{u}^i, \hat{u}^{-i}) - \epsilon,$$

where $\epsilon = O(|x_0^{(N)} - \bar{X}_0| + 1/\sqrt{N})$.

Interpretation: using **full information** by one deviating player offers little benefit to itself.
Question 4: How much **efficiency loss** occurs in the MFG?

In order to answer this question, we need to compare with a social optimization problem.

This is doable; we postpone this until introducing the social optimization problem to minimize $J_{soc}^{(N)} = \sum_i J_i$. 

Question 5: **Heterogeneous** agents?

Answer: the MFG equation system will be modified.

Assume empirical distribution for the parameter sequence \( \{\theta_i, i \geq 1\} \)

The intuition behind is to average non-uniform “particles”
Nonlinear case – Interacting particle modeling for MFG:

\[ N \text{ agents (players), } 1 \leq i \leq N. \text{ Dynamics and costs:} \]

\[
dx_i = \frac{1}{N} \sum_{j=1}^{N} f(x_i, u_i, x_j)dt + \frac{1}{N} \sum_{j=1}^{N} \sigma(x_i, u_i, x_j)dw_i, \quad \text{each } x_j \in \mathbb{R}^n
\]

\[
J_i(u_i, u_{-i}) = E \int_0^T \frac{1}{N} \sum_{j=1}^{N} L(x_i, u_i, x_j)dt, \quad 1 \leq i \leq N.
\]

Assume i.i.d. initial states. \((u_{-i}: \text{controls of all players other than player } i)\)

As in the LQ case, a key step is to solve a special single agent optimal control problem.

How to construct such a problem?
Recall

\[ dx_i = \frac{1}{N} \sum_{j=1}^{N} f(x_i, u_i, x_j) dt + \frac{1}{N} \sum_{j=1}^{N} \sigma(x_i, u_i, x_j) dw_i, \quad \text{each } x_j \in \mathbb{R}^n \]

\[ J_i(u_i, u_{-i}) = E \int_0^T \frac{1}{N} \sum_{j=1}^{N} L(x_i, u_i, x_j) dt, \quad 1 \leq i \leq N. \]

Write \( \delta_{x}^{(N)} = \frac{1}{N} \sum_{j=1}^{N} \delta_{x_j} \), \( \delta_{x} : \text{dirac measure.} \) For \( \varphi = f, \sigma, L, \) denote

\[ \frac{1}{N} \sum_{j=1}^{N} \varphi(x_i, u_i, x_j) = \int_{\mathbb{R}^n} \varphi(x_i, u_i, y) \delta_{x}^{(N)}(dy) := \varphi[x_i, u_i, \delta_{x}^{(N)}] \]

Idea: approximate \( \delta_{x}^{(N)} \) by a deterministic measure \( \mu_t \)
For simplicity, consider constant $\sigma$ and scalar state. The optimal control problem with dynamics and cost:

\[
dx_i = f[x_i, u_i, \mu_t]dt + \sigma dw_i, \quad \text{each } x_j \in \mathbb{R}
\]

\[
\bar{J}_i(u_i) = E \int_0^T L[x_i, u_i, \mu_t]dt,
\]

where $\mu_t$ is called the mean field.

Again, following the method in the LQ MFG case –

- Assume $\{\mu_t, 0 \leq t \leq T\}$ were known; solve the optimal control $\hat{u}_i$.
- **Consistency condition** – the closed-loop state process under $\hat{u}_i$ has its marginal equal to $\mu_t$.
- The analysis can be done for a finite classes/types of agents.
HJB–McKV/FPK for the nonlinear diffusion model:

- HJB equation (scaler state for simplicity):

\[ \frac{\partial V}{\partial t} = \inf_{u \in U} \left\{ f[x, u, \mu_t] \frac{\partial V}{\partial x} + L[x, u, \mu_t] \right\} + \frac{\sigma^2}{2} \frac{\partial^2 V}{\partial x^2} \]

\[ V(T, x) = 0, \quad (t, x) \in [0, T) \times \mathbb{R}. \]

\[ \Downarrow \]

**Best Response:** \( u_t = \varphi(t, x|\mu.), \quad (t, x) \in [0, T) \times \mathbb{R}. \)

- Closed-loop McK-V equation (which can be written as a Fokker-Planck equation):

\[ dx_t = f[x_t, \varphi(t, x|\mu.), \mu_t]dt + \sigma d\omega_t, \quad 0 \leq t \leq T. \]

Find a solution \((x_t, \mu_t)\) in McK-V sense, i.e., Law\((x_t) = \mu_t.\)

Refs. Huang, Caines and Malhame’06, Lasry and Lions (06, 07), Cardaliaguet’12
Lasry and Lions (2006, 2007) considered dynamics and costs

$$dx_t^i = -u^i_t dt + \sigma dw_t^i, \quad x^i \in \mathbb{R}^d, \sigma > 0,$$

$$J^i(u^1, \ldots, u^N) = \lim inf_{T \to \infty} \frac{1}{T} E \int_0^T (L(x^i, u^i) + V[x^i, \mu^{N,-i}_t])dt,$$

where $\mu^{N,-i}_t$ is the empirical distribution of states of other agents. The second argument of $V$ means it is a functional of the distribution for given $x^i$. Assume periodicity of $F$ and $V$ in the cost. Denote $Q = [0, 1]^d$.

Solve the $N$-player game, and show the solution converges to a limiting HJB-FPK equation system

$$-\frac{\sigma^2}{2} \Delta \nu(x) = H(x, \nabla \nu) + \lambda = V[x, m],$$

$$-\frac{\sigma^2}{2} \Delta m - \text{div} \left( \frac{\partial H}{\partial p} (x, \nabla \nu) m \right) = 0,$$

where $\int_Q \nu(x) dx = 0, \int_Q m(x) = 1, m \geq 0$.

$H$: Hamiltonian, $\lambda$: long run average cost, $\nu(x)$: differential cost
Finite horizon can also be considered for this model.

Dynamics of agent $i$:

$$dx_t^i = u_t^i dt + \sqrt{2} dw_t^i, \quad x_t^i \in \mathbb{R}^d;$$

cost functional:

$$J^N_i(u_i, u_{-i}) = E \int_0^T [(u_t^i)^2 + F(x_t^i, \mu_t^{N,-i})] dt + EG(x_t^i, \mu_t^{N,-i}),$$

where $\mu_t^{N,-i}$ is the empirical distribution of the states of all other agents.
See Lasry and Lions’07, also notes Cardaliaguet’12
The MFG equations:

\[
\begin{align*}
\partial_t V - \Delta V + \frac{1}{2}|DV|^2 &= F(x, m), \quad (x, t) \in \mathbb{R}^d \times (0, T) \\
\partial_t m - \Delta m - \text{div}(mDV) &= 0, \\
m(0) &= m_0, \quad V(x, T) = G(x, m(T)), \quad x \in \mathbb{R}^d,
\end{align*}
\]

where $V(t, x)$ and $m(t, x)$ are the value function and the density of the state distribution, respectively.

See an $\epsilon$-Nash equilibrium result in Cardaliaguet’12

This relates the MFG solution to finite population games
Problem $P_0$: $N$ player Nash game
states: $\left(x_1, x_2, \ldots, x_N\right)$
strategies: $\left(u_1, u_2, \ldots, u_N\right)$
costs: $\left(J_1, J_2, \ldots, J_N\right)$

A large-scale coupled equation system
Example: $N$ coupled dynamic programming equations

Problem $P_{\infty}$: Optimal control of a single player
state: $x_i$ control: $u_i$
mechanical $\mu_i$ is fixed and not controlled by $u_i$

MFG equation system:
1 equation of optimal control;
1 equation of mean field
dynamics (for $\mu_i$)
Example: HJB PDE; FPK PDE (or McKean-Vlasov SDE)

So for nonlinear diffusion models, the diagram reduces to the structure on next page
The basic framework of MFGs: Math. Aspects

\[ P_0 — \text{Game with } N \text{ players; Example} \]
\[
dx_i = f(x_i, u_i, \delta_x^{(N)}) dt + \sigma(\cdots) dw_i
\]
\[
J_i(u_i, u_{-i}) = E \int_0^T l(x_i, u_i, \delta_x^{(N)}) dt
\]
\[
\delta_x^{(N)}: \text{empirical distribution of } (x_j)_{j=1}^N
\]

\[ \downarrow \text{construct} \quad \downarrow \text{performance?} \]

\[ P_\infty — \text{Limiting problem, 1 player} \]
\[
dx_i = f(x_i, u_i, \mu_t) dt + \sigma(\cdots) dw_i
\]
\[
J_i(u_i) = E \int_0^T l(x_i, u_i, \mu_t) dt
\]
\[
\text{Freeze } \mu_t, \text{ as approx. of } \delta_x^{(N)}
\]

\[ \downarrow \text{construct} \quad \downarrow \text{performance?} \]

\[ \hat{u}_i(t, x_i): \text{optimal response} \]
\[ HJB \text{ (with } v(T, \cdot) \text{ given):} \]
\[
-v_t = \inf_{u_i} (f^T v_{x_i} + l + \frac{1}{2} \text{Tr} [\sigma \sigma^T v_{x_i} x_i])
\]
\[ \text{Fokker-Planck-Kolmogorov:} \]
\[
p_t = -\text{div}(fp) + \sum ((\frac{\sigma \sigma^T}{2})_{jk} p)_{x_i^j x_i^k}
\]
\[ \text{Coupled via } \mu_t \text{ (w. density } p_t; p_0 \text{ given)} \]

The consistency based approach (red) is more popular; related to ideas in statistical physics (McKean-Vlasov eqn); FPK can be replaced by an MV-SDE.
Applications:

- Economic theory
- Finance
- Communication networks
- Social opinion formation
- Power systems
- Electric vehicle recharging control
- Public health (vaccination games)
- ......
Some applications of MFGs to economic growth and finance.

- Guéant, Lasry and Lions (2011): human capital optimization
- Lucas and Moll (2011): Knowledge growth and allocation of time
- Carmona and Lacker (2013): Investment of $n$ brokers
- Huang (2013): capital accumulation with congestion effect
- Lachapelle et al. (2013): price formation
- Espinosa and Touzi (2013): Optimal investment with relative performance concern (depending on $\frac{1}{N-1} \sum_{j \neq X} X_j$)
- Jaimungal and Nourian (2014): Optimal execution
- Chan and Sircar (2015): Bertrand and Cournot models
- more ......
Only some papers are sampled and inserted into the lecture.

For further literature, see these papers and books, and references therein.

Bensoussan, Frehse, and Yam. Springer Brief 2013.
Gomes and Saud. Mean field games models – a brief survey. 2014.
Huang, Caines, and Malhame. 2003 CDC, IEEE TAC'07.
Huang, Malhame and Caines, CIS’06.
Weintraub, Benkard, and Van Roy (2008). Econometrica; for oblivious equilibria
Thank you!
Note added July 20, 2018

The primary goal of this tutorial (Workshop on Game Theory, Institute for Math. Sci., NUS, June 2018, organizing co-chairs: Profs. Y.-C. Chen and Y. Sun) is to present the basic ideas of MFGs to researchers who have interest but are non-specialists in this area. I try to make a modest usage of technical tools, and mostly use dynamic programming. The important approach via maximum principle is not covered here and can be checked from the references supplied at the end.

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