Character identities in the twisted endoscopy of real reductive groups*

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Abstract

Suppose G is a real reductive algebraic group, θ is an automorphism of G, and ω is a quasicharacter of the group of real points $G(\mathbf{R})$. Under some additional assumptions, the theory of twisted endoscopy associates to this triple real reductive groups H. The Local Langlands Correspondence partitions the admissible representations of $H(\mathbf{R})$ and $G(\mathbf{R})$ into L-packets. We prove twisted character identities between L-packets of $H(\mathbf{R})$ and $G(\mathbf{R})$ comprised of essential discrete series or limits of discrete series.

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1 Introduction

The Local Langlands Correspondence is a conjectural relationship between two seemingly disparate realms. On the one hand, one has representations of a connected reductive algebraic group G over a $\mathfrak{p}$-adic or real field. On the other hand, one has data emanating from the absolute Galois group of the field. The conjectured relationship between these objects presents a mysterious and immense extension of local class field theory.

The theory of endoscopy emerges naturally from the Local Langlands Correspondence. In order to gain a sense of why this is so, let us approach the Local Langlands Correspondence over the field of real numbers, where it is proven ([Lan89]), in the knowledge that we shall have to put up with some imprecision. Under these circumstances, the "data emanating from the absolute Galois group" are homomorphisms $\varphi : W_{\mathbf{R}} \to {}^L G$ from the real Weil group to the L-group of G. It is the Weil group here which binds φ to the Galois group (§1 [Tat79]). By contrast, the L-group ${}^L G$ encodes information about how the complex group G is related to one of its real quasisplit forms (§3 [Bor79]).

To these homomorphisms correspond finite sets Π_φ of (equivalence classes of) irreducible representations of $G(\mathbf{R})$. These sets of representations, the *L-packets*, may be identified with their sets of characters. We may visualize the Local Langlands Correspondence as

$$\varphi \overset{LLC}{\longleftrightarrow} \Pi_\varphi.$$

Now, it may happen that the image of a homomorphism $\varphi : W_{\mathbf{R}} \to {}^L G$ is contained in a proper subgroup of ${}^L G$, and further that this proper subgroup is of the form ${}^L H$ for some (quasisplit) reductive algebraic group H. Let us assume that this holds. Then, to begin with, it is right to regard H as an *endoscopic group* of G. Moreover, one may define a "new" homomorphism $\varphi_H : W_{\mathbf{R}} \to {}^L H$ by simply renaming φ. The Local Langlands Correspondence imbues this simple procedure with deep consequences. Indeed, the homomorphism φ_H may itself correspond to an L-packet of representations of $H(\mathbf{R})$ as depicted by

$$\begin{tikzcd}
\varphi_H \arrow[r, leftrightarrow, "LLC"] \arrow[d, leftrightarrow] & \Pi_{\varphi_H} \arrow[d, leftrightarrow, dotted, "?"] \\
\varphi \arrow[r, leftrightarrow, "LLC"'] & \Pi_\varphi
\end{tikzcd}$$

The dotted arrow on the right indicates that one ought to expect a relationship between the characters of Π_{φ_H} and the characters of Π_φ. This relationship should be some sort of character identity which arises from the inclusion ${}^LH \hookrightarrow {}^LG$ relating φ_H to φ. What form could such a character identity take?

The answer to this question begins by thinking of the characters as (locally integrable) functions ([HC65b]). These functions are *class* functions, so an identity between characters is possible once there is a correspondence between the conjugacy classes of $H(\mathbf{R})$ and $G(\mathbf{R})$. The inclusion ${}^LH \hookrightarrow {}^LG$ underlying our assumptions furnishes such a correspondence in three steps. First, we assume that H is large enough for this inclusion to induce an isomorphism between a maximal torus of H and G. Second, this isomorphism of tori produces a bijection between semisimple conjugacy classes of H and G. This bijection of conjugacy classes takes place over the complex numbers. The third step is to interpret this bijection over the real numbers, where it reduces to a correspondence between finite sets of semisimple conjugacy classes of $H(\mathbf{R})$ and $G(\mathbf{R})$. The real conjugacy classes in these finite sets differ up to conjugacy over the complex groups H or G. They are the so-called *stable classes* and are the geometric analogues of the L-packets.

The correspondence between the stable classes provides a means of forming identities between the L-packets. This becomes apparent with the requisite notation. Suppose that $\gamma \in H(\mathbf{R})$ and $\delta \in G(\mathbf{R})$ are (regular) semisimple elements, and that their stable classes correspond to one another as above. Let Θ_π denote the character of a (tempered) representation π in Π_{φ_H} or Π_φ. Thinking of the characters in the L-packets as functions, one might hope that $\sum_{\pi_H \in \Pi_{\varphi_H}} \Theta_{\pi_H}$ and $\sum_{\pi \in \Pi_\varphi} \Theta_\pi$ are well-defined functions on stable classes. One might further hope that $\sum_{\pi_H \in \Pi_{\varphi_H}} \Theta_{\pi_H}(\gamma)$ is equal to $\sum_{\pi \in \Pi_\varphi} \Theta_\pi(\delta)$. In actuality, this second hope proves to be too much. One must introduce constants $\Delta(\boldsymbol{\varphi_H}, \pi)$ called *spectral transfer factors* so that

$$\sum_{\pi_H \in \Pi_{\varphi_H}} \Theta_{\pi_H}(\gamma) = \sum_{\pi \in \Pi_\varphi} \Delta(\boldsymbol{\varphi_H}, \pi)\ \Theta_\pi(\delta).$$

The homomorphism φ_H appears with boldface text in $\Delta(\boldsymbol{\varphi_H}, \pi)$ so as to indicate that the spectral transfer factors depend on φ_H only up to conjugacy.

We now see how one might come to expect endoscopic character identities from the Local Langlands Correspondence. The usual terminology for these identities is *spectral transfer*. For real groups, the usual form and proof of spectral transfer make a detour through what is called *geometric transfer*.

To give an overview, we begin with [Lan79], in which Langlands defines endoscopic groups H which are related to G through their geometry. This relationship furnishes a correspondence between stable classes of $\gamma \in H(\mathbf{R})$ and $\delta \in G(\mathbf{R})$ as above. Geometric transfer is tantamount to a map $f \mapsto f_H$, from suitable functions on $G(\mathbf{R})$ to functions on $H(\mathbf{R})$, which satisfies an identity between orbital integrals. This identity is of the form

$$\sum_{\gamma'} \mathcal{O}_{\gamma'}(f_H) = \sum_{\delta'} \Delta(\gamma, \delta')\ \mathcal{O}_{\delta'}(f),$$

where the first and second sums are taken over representatives in the stable classes of γ and δ respectively, $\mathcal{O}$ signifies the orbital integral, and the $\Delta(\gamma, \delta')$ are constants called *geometric transfer factors.* An initial proof of geometric transfer was given by Shelstad in [She82]. Afterwards, refinements were made to the geometric transfer factors ([LS87]), making them canonical in a manner we shall allude to later on. Shelstad has since provided a proof of geometric transfer using these canonical transfer factors ([She08]).

Taking geometric transfer for granted, one may express spectral transfer as an identity of the form

$$\sum_{\pi_H \in \Pi_{\varphi_H}} \Theta_{\pi_H}(f_H) = \sum_{\pi \in \Pi_{\varphi}} \Delta(\boldsymbol{\varphi_H}, \pi)\ \Theta_{\pi}(f). \tag{1}$$

Spectral transfer was first proved in [She82] for tempered L-packets. It was then reproved in [She10] with canonical spectral transfer factors.

What we have described so far is sometimes referred to as the theory of *ordinary* endoscopy for real groups. The word “ordinary” is present to distinguish it from the theory of *twisted* endoscopy. To see how the theory of twisted endoscopy comes into the picture, one must turn to the Arthur-Selberg trace formula. In truth, the driving force behind the theory of endoscopy was, and continues to be, the stabilization of the trace formula. This point is made in the foundational paper [Lan79] and is explained in the expository article [Art97]. More detail is given in §§27-30 [Art05], where it is shown how certain classical groups appear in the proposed stable trace formula for $\mathrm{GL}(N)$, when twisted by an outer automorphism. Without running too far afield, let us be content to presume that the theory of twisted endoscopy is necessitated by the stabilization of the twisted trace formula.

The theory of twisted endoscopy generalizes the theory of ordinary endoscopy by introducing two additional objects to G. The first is an algebraic automorphism θ of G, and the second is a quasicharacter ω of $G(\mathbf{R})$. The definition of endoscopic groups and the conjecture of geometric transfer in the twisted context was given in [KS99]. For real groups $G(\mathbf{R})$, Renard has given a proof of twisted geometric transfer when ω is trivial and θ is of finite order ([Ren03]). Shelstad has recently given a proof of twisted geometric transfer for real groups in general ([Shea]). Twisted geometric and spectral transfer were proved by Shelstad and Bouaziz respectively in the case that the real group has complex structure, the automorphism is that of complex conjugation, and the quasicharacter is trivial ([She84], [Bou89]). The goal of this work is to formulate and prove twisted spectral transfer for real groups. More precisely, the goal is to formulate and prove a twisted analogue of (1) for L-packets comprised essential discrete series (square-integrable) representations or essential limits of discrete series representations.

To appreciate the significance of twisted spectral transfer to Arthur's stabilization of the trace formula, we first refer the reader to §§1-2, 6 [Art08]. The spectral identities which appear in this reference are required for some twisted groups in the forthcoming work [Art], which classifies automorphic representations of special orthogonal and symplectic groups in terms of automorphic representations of the general linear group (see Theorem 2.2.3 [Art]). This classification exhibits the lofty principle of functoriality in the Langlands program. Generally speaking, endoscopic character identities are accessible examples of functoriality.

We continue by giving an outline the paper. The exact form of twisted spectral transfer will appear in course. The true work begins with a presentation of some of the foundational work of Kottwitz and Shelstad in twisted endoscopy. The definition of an endoscopic group H of the triple (G, θ, ω) is recollected together with a correspondence between conjugacy classes of H and twisted conjugacy classes of G. As in the ordinary case, this correspondence of twisted conjugacy classes may be adapted to elements of $G(\mathbf{R})$. The resulting concept is that of a *norm* of an element $\delta \in G(\mathbf{R})$, i.e. an element $\gamma \in H(\mathbf{R})$ whose stable conjugacy class corresponds the stable twisted conjugacy class of δ. With this setup, a terse statement of twisted geometric transfer is given and assumed to hold.

In section 4 we begin by describing the Local Langlands Correspondence for tempered representations of real groups. We then point out how the twisting data (θ, ω) may be combined with individual representations and with L-packets. Spectral transfer pertains to only those tempered representations and L-packets which are preserved under twisting.

In section 5 we restrict our attention to essentially square-integrable representations and L-packets, which are preserved under twisting. We describe how such a representation π may be recovered from a representation ϖ_1 of $G_{\mathrm{der}}(\mathbf{R})^0$, the identity component of the real points of the derived subgroup of G. As π is preserved under twisting, it has a twisted character (section 5.2). The twisted character of π may be recovered from a twisted character of ϖ_1 as well (Lemma 5). Restriction to the subgroup $G_{\mathrm{der}}(\mathbf{R})^0$ has two advantages. The first is that the restriction of the quasicharacter ω to this subgroup is trivial. The second is that the theory of twisted representations becomes a special case of the representation theory on the category of groups given by Duflo and Bouaziz ([Duf82], [Bou87]). This category allows for disconnected Lie groups and the particular groups we work with are semidirect products of $G_{\mathrm{der}}(\mathbf{R})^0$ by a cyclic group related to θ.

The leitmotif of our work is that the representation theory of connected real reductive groups relates to ordinary spectral transfer in the same manner that the representation theory of Duflo and Bouaziz relates to twisted spectral transfer. We show how Duflo's classification of discrete series representations ties in with ϖ_1 and how Bouaziz's character formula for discrete series ties in with the twisted character of ϖ_1 (section 5.4). Section 5 closes with a detailed description of the Weyl integration formula on a component of our disconnected Lie group.

Twisted spectral transfer for essentially square-integrable representations is the topic of section 6. The starting point is a fixed choice of endoscopic data. We must endure the technical necessity of a z-extension H_1 of the fixed endoscopic group H. We assume that we have associated L-parameters $\boldsymbol{\varphi_{H_1}}$ and $\boldsymbol{\varphi}$ for H_1 and G, which correspond to essentially square-integrable packets $\Pi_{\varphi_{H_1}}$ and Π_{φ} under the Local Langlands Correspondence. The L-packet Π_{φ} is assumed to be preserved under twisting, otherwise there is nothing to say. Finally, we assume that there is a θ-elliptic element $\delta \in G(\mathbf{R})$ which has a norm $\gamma \in H(\mathbf{R})$. In essence, this final assumption ensures that twisted elliptic tori in $G(\mathbf{R})$ have something to do with elliptic tori in $H(\mathbf{R})$. If this is not the case, we show how spectral transfer degenerates to an identity between zeros in section 6.5.

Working under the above assumptions, we parametrize the twisted conjugacy classes in $G(\mathbf{R})$ appearing in the stable class of δ (Lemma 15). The parametrizing set is a modification of the usual coset space of the Weyl group modulo the real Weyl group in ordinary endoscopy (see (66)). A slightly different modification of this coset space parametrizes the representations in Π_{φ}

which are preserved under twisting (Lemma 16). In particular, there exists a representation in Π_φ which is preserved under twisting (Corollary 2).

With these parametrizing sets in hand, we undertake in section 6.3 a comparison of characters for smooth functions $f \in C_c^\infty(G(\mathbf{R})\theta)$ with small elliptic support about $\delta\theta$. In keeping with our leitmotif, these functions are defined on the non-identity component $G(\mathbf{R})\theta$ in the semidirect product $G(\mathbf{R}) \rtimes \langle\theta\rangle$. This comparison leads to an identity of the following shape

$$(2) \quad \int_{H_1(\mathbf{R})/Z_1(\mathbf{R})} f_{H_1}(h) \sum_{\pi_{H_1} \in \Pi_{\varphi_{H_1}}} \Theta_{\pi_{H_1}}(h)\, dh = \sum_{\pi \in \Pi_\varphi} \Delta(\boldsymbol{\varphi_{H_1}}, \pi)\ \Theta_{\pi, \mathsf{U}_\pi}(f).$$

The terms $\Theta_{\pi_{H_1}}$ and $\Theta_{\pi,\mathsf{U}_\pi}$ are the (twisted) characters of the representations occurring in the packets. The terms $\Delta(\boldsymbol{\varphi_{H_1}}, \pi)$ are the spectral transfer factors. Initially, the spectral transfer factors are defined from the geometric transfer factors relative to the fixed element $\delta \in G(\mathbf{R})$ (see (115)-(117)). However, we show that the spectral transfer factors are independent of the choice of δ in section 6.3.1. *A priori*, the definition of our spectral transfer factors does depend on some data appearing in the geometric transfer factors. More will be said about this later.

The extension of character identity (2) to arbitrary functions in $C_c^\infty(G(\mathbf{R})\theta)$ is the desired twisted spectral transfer theorem for essentially square-integrable representations. The extension to $C_c^\infty(G(\mathbf{R})\theta)$ is effected in section 6.4. We prove that the distribution determined by (2) is an invariant eigendistribution when ω is trivial. One may then employ a twisted version of Harish-Chandra's uniqueness theorem (Theorem 15.1 [Ren97]) to extend from the θ-elliptic set to all of $G(\mathbf{R})\theta$ (Proposition 4). The assumption that ω is trivial is removed in Theorem 1, the first of two our main theorems.

In section 7 we move from essentially square-integrable representations to essential limits of discrete series representations of $G(\mathbf{R})$. The latter are defined in section 7.1 in terms of the notions of *coherent continuation* or *Zuckerman tensoring*. Spectral transfer for essential limits of discrete series is first proved when G is quasisplit (section 7.2). This is done by attaching the essential limits of discrete series to essentially square-integrable representations of a proper Levi subgroup of G, applying spectral transfer to the latter representations, and then using a twisted version of coherent continuation ([Duc02]) to recover spectral transfer on the level of limits of discrete series (Theorem 2). When G is not quasisplit the above procedure may be imitated for *relevant* Levi subgroups of G. Otherwise, spectral transfer is not defined. This is discussed in section 7.3, where the possibility of no norms is also considered.

Specialists in endoscopy will realize that the approach we have described is, in the main, the approach of Shelstad in [She82] and [She10]. Nonetheless, the technical obstacles created by twisting are considerable. Notable obstacles appear in the description of twisted transfer factors (section 6.2), the local elliptic comparison (section 6.3), and the theory of eigendistributions (in section 6.4). We have also given a detailed account of Shelstad's method for finding a Levi subgroup attached to limits of discrete series (§4.3 [She82]) in section 7.2.

Let us list three anticipated improvements of our spectral transfer theorems. We have worked under the assumptions of [KS99], with one exception. The bijection of stable (twisted) conjugacy classes mentioned earlier need not behave well under the action of the Galois group of $\mathbf{C}/\mathbf{R}$. This leads to a version of geometric transfer involving some twisting on the endoscopic groups (see §§5.4-5.5 [KS99]). We exclude this possibility by making an assumption on the map of conjugacy classes in section 3.3. Presumably, with some more effort, this assumption could be removed from the spectral transfer theorems.

Another spot for improvement is in the definition of the twisted spectral transfer factors. We have defined the spectral transfer factors by making a local comparison about an elliptic element $\delta \in G(\mathbf{R})$ and taking what was left over. Moreover, we have done this after fixing data (χ- and a-data) used in the definition of the geometric transfer factors. A much more satisfying approach is given in recent work by Shelstad ([Sheb]), where the spectral transfer factors are defined without reference to particular elements in $G(\mathbf{R})$ or particular choices of data for the geometric transfer factors. A cleaner version of twisted spectral transfer ought to be given in terms of these canonical spectral transfer factors.

Lastly, in the case of ordinary spectral transfer, the passage from essentially square-integrable or limits of discrete series representations to irreducible tempered representations involves a simple descent argument to a Levi subgroup of G (see §15 [She10])). In the case that the parabolic subgroup containing this Levi subgroup is preserved under θ one may imitate the descent argument (see §7.1 [Bou87]) to obtain the desired spectral transfer (see §11 [Sheb]). However, there are explicit examples in which the parabolic subgroups are not preserved by θ. The spectral transfer for such examples requires new methods. We expect they are near at hand.

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2 Notation

The set of integers, real numbers and complex numbers are denoted by $\mathbf{Z}$, $\mathbf{R}$ and $\mathbf{C}$ respectively.

In this section G is a real Lie group which acts upon a non-empty set J. For a subset $J_1 \subset J$ we set

$$N_G(J_1) = \{g \in G : g \cdot j \in J_1 \text{ for all } j \in J_1\},$$

$$Z_G(J_1) = \{g \in G : g \cdot j = j \text{ for all } j \in J_1\}.$$

and $\Omega(G, J_1)$ equal to the quotient group $N_G(J_1)/Z_G(J_1)$.

For an automorphism θ of G we set $\langle\theta\rangle$ equal to the group of automorphism generated by θ. There is a corresponding semidirect product $G \rtimes \langle\theta\rangle$. When elements of G are written side-by-side with elements in $\langle\theta\rangle$ we consider them to belong to this semidirect product.

The inner automorphism of an element $\delta \in G$ is defined by

$$\mathrm{Int}(\delta)(x) = \delta x \delta^{-1}, \ x \in G.$$

It shall be convenient to denote the fixed-point set of $\mathrm{Int}(\delta) \circ \theta$ by $G^{\delta\theta}$. In other words $G^{\delta\theta} = Z_{\langle \mathrm{Int}(\delta)\circ\theta\rangle}(G)$. In fact, we shall truncate the notation $\mathrm{Int}(\delta) \circ \theta$ to $\delta\theta$ habitually.

We almost always denote the real Lie algebra of a Lie group using Gothic script. For example the real Lie group of G is denoted by $\mathfrak{g}$. There is one exception in section 5.3.1 where $\mathfrak{u}$ is a complex Lie algebra. Suppose that J is Cartan subgroup of a reductive group G. Then the pair $(\mathfrak{g} \otimes \mathbf{C}, \mathfrak{j} \otimes \mathbf{C})$ determines a root system which we denote by $\mathrm{R}(\mathfrak{g} \otimes \mathbf{C}, \mathfrak{j} \otimes \mathbf{C})$. We denote the dual Lie algebra to $\mathfrak{g}$ by $\mathfrak{g}^*$. The differential of the inner automorphism $\mathrm{Int}(\delta)$ is the adjoint automorphism $\mathrm{Ad}(\delta)$ on $\mathfrak{g}$. The adjoint automorphism induces an automorphism on $\mathfrak{g}^*$ in the usual way. Often, it shall be convenient to write $\delta \cdot X$ in place of $\mathrm{Ad}(\delta)(X)$ for $X \in \mathfrak{g}$. Similarly, we write $\theta \cdot X$ to mean the differential of θ acting on $X \in \mathfrak{g}$. We extend this slightly abusive notation to the dual spaces, writing $\delta \cdot \lambda$ or even simply $\delta\lambda$ in place of the coadjoint action of δ on $\lambda \in \mathfrak{g}^*$.

Finally, if we take H to be an algebraic group defined over $\mathbf{R}$, we denote its identity component by H^0. The group of real points of H is denoted by $H(\mathbf{R})$. This is a real Lie group and we denote the identity component of $H(\mathbf{R})$ in the real manifold topology by $H(\mathbf{R})^0$.

3 The foundations of real twisted endoscopy

This section is a digest of some early material in [KS99], in the special case that the field of definition is equal to $\mathbf{R}$.

3.1 Groups and automorphisms

Let G be a connected reductive algebraic group and let us fix a triple $(B, T, \{X\})$ in which B is a Borel subgroup of G, $T \subset B$ is a maximal torus of G, and $\{X\}$ is a collection of root vectors given by the simple roots determined by B and T . Such triples are called *splittings* of G. We take θ to be an algebraic automorphism of G.

Lemma 1 *Suppose the automorphism θ preserves the splitting $(B, T, \{X\})$ of G. Then the restriction of θ to the derived subgroup G_{der} is of finite order.*

Proof. The automorphism θ induces a graph automorphism of the Dynkin diagram attached to the system of simple roots determined by (B, T) (Corollary 2.14 [Spr79]). This graph automorphism has finite order, say k. As a result, θ^k is an automorphism of G whose differential fixes each vector in $\{X\}$. It follows in turn that θ^k fixes pointwise $T_{\mathrm{der}} = G_{\mathrm{der}} \cap T$, $\{X\}$, and a set of root vectors attached to the negative simple roots (Proposition 8.3 (f) [Hum94]). As G_{der} is generated by T_{der} and the exponentials of the aforementioned root vectors (Corollary 8.2.10 [Spr98]), we conclude that the restriction of θ^k to G_{der} is trivial.■

From now on we require that θ preserves the splitting $(B, T, \{X\})$. Lemma 1 ensures that $\theta_{|G_{\mathrm{der}}}$ is finite and therefore semisimple (i.e. θ is quasi-semisimple). We shall assume that G and θ are defined over the real numbers, and set $G(\mathbf{R})$ to be the group of real points of G. Let Γ be the Galois group of $\mathbf{C}/\mathbf{R}$ and σ be its non-trivial element. The Galois group Γ acts on G

and is trivial on $G(\mathbf{R})$. If $(B, T, \{X\})$ is preserved by Γ then it is called an $\mathbf{R}$-*splitting*.

There is a unique quasisplit group G^* of which G is an inner form (Lemma 16.4.8 [Spr98]). It follows that there is an isomorphism $\psi : G \to G^*$ and $\psi\sigma\psi^{-1}\sigma^{-1} = \mathrm{Int}(u')$ for some $u' \in G^*$. For the purpose of defining one of the geometric transfer factors, namely Δ_{III} (§§4.4, 5.4 [KS99]), we shall choose u_σ in the simply connected covering group G^*_{sc} of the derived group G^*_{der} of G^* so that its image under the covering map is u'. We shall then abuse notation slightly by identifying u_σ with u' in equations such as

$$\psi\sigma\psi^{-1}\sigma^{-1} = \mathrm{Int}(u_\sigma). \tag{3}$$

As G^* is quasisplit, there is a Borel subgroup B^* defined over $\mathbf{R}$. Applying Theorem 7.5 [Ste97] to B^* and σ, we obtain an $\mathbf{R}$-splitting $(B^*, T^*, \{X^*\})$. Following the convention made for $u_\sigma \in G^*_{\mathrm{sc}}$, we may choose $g_\theta \in G^*_{\mathrm{sc}}$ so that the automorphism

$$\theta^* = \mathrm{Int}(g_\theta)\psi\theta\psi^{-1} \tag{4}$$

preserves $(B^*, T^*, \{X^*\})$ (Theorems 6.2.7 and 6.4.1 [Spr98], §16.5 [Hum94]). Since

$$\sigma(\theta^*) = \sigma\theta^*\sigma^{-1} = \mathrm{Int}(\sigma(g_\theta u_\sigma)g_\theta^{-1}\theta^*(u_\sigma))\theta^*$$

preserves $(B^*, T^*, \{X^*\})$, and the only inner automorphisms which do so are trivial, it follows in turn that $\mathrm{Int}(\sigma(g_\theta u_\sigma)g_\theta^{-1}\theta^*(u_\sigma))$ is trivial and $\sigma(\theta^*) = \theta^*$. This means that the automorphism θ^* is defined over $\mathbf{R}$.

We wish to describe the action of θ induced on the L-group of G. Recall that $(B, T, \{X\})$ determines a based root datum (Proposition 7.4.6 [Spr98]) and an action of Γ on the Dynkin diagram of G (1.3 [Bor79]). To the dual based root datum there is attached a dual group $\hat{G}$ defined over $\mathbf{C}$, a Borel subgroup $\mathcal{B} \subset \hat{G}$ and a maximal torus $\mathcal{T} \subset \mathcal{B}$ (2.12 [Spr79]). Let us fix a splitting

$$(\mathcal{B}, \mathcal{T}, \{\mathcal{X}\}) \tag{5}$$

of $\hat{G}$. This allows us to transfer the action of Γ from the Dynkin diagram of $\hat{G}$ to an algebraic action of $\hat{G}$ (Proposition 2.13 [Spr79]). This action may be extended trivially to the Weil group $W_{\mathbf{R}}$, which as a set we write as $\mathbf{C}^\times \cup \sigma\mathbf{C}^\times$ (§9.4 [Bor79]). The L-group ${}^LG$ is defined by the resulting semidirect product ${}^LG = \hat{G} \rtimes W_{\mathbf{R}}$.

In a parallel fashion, θ induces an automorphism of the Dynkin diagram of G, which then transfers to an automorphism $\hat{\theta}$ on $\hat{G}$. We define ${}^L\theta$ to be the automorphism of ${}^LG$ equal to $\hat{\theta} \times 1_{W_{\mathbf{R}}}$. By definition, the automorphism $\hat{\theta}$ preserves $(\mathcal{B}, \mathcal{T}, \{\mathcal{X}\})$.

We close this section with some remarks concerning Weyl groups. Let T^1 be the identity component of $T^\theta \subset T$. It contains strongly regular elements (Lemma II.1.1 [Lab04]), so its centralizer in G is the maximal torus T. Setting the identity component of G^θ equal to G^1 and the Weyl group of G^1 relative to T^1 equal to $\Omega(G^1, T^1)$, we see that we have an embedding

$$\Omega(G^1, T^1) \to \Omega(G, T)^\theta$$

into the θ-fixed elements of the Weyl group $\Omega(G, T)$. In fact, this embedding is an isomorphism (Lemma II.1.2 [Lab04]). It is simple to verify that $\Omega(G, T)^\theta$ is equal to those elements of $\Omega(G, T)$ which stabilize T^θ.

3.2 Endoscopic data and z-pairs

Endoscopic data are defined in terms of the group G, the automorphism θ, and a cohomology class $\mathbf{a} \in H^1(W_{\mathbf{R}}, Z_{\hat{G}})$, where $Z_{\hat{G}}$ denotes the centre of $\hat{G}$. Let ω be the quasi-character of $G(\mathbf{R})$ determined by $\mathbf{a}$ (pp. 122-123 [Lan89]), and let us fix a one-cocycle a in the class $\mathbf{a}$. By definition (pp. 17-18 [KS99]), *endoscopic data* for $(G, \theta, \mathbf{a})$ consist of

1. a quasisplit group H defined over $\mathbf{R}$

2. a split topological group extension

$$1 \to \hat{H} \to \mathcal{H} \overset{c}{\leftrightarrows} W_{\mathbf{R}} \to 1,$$

whose corresponding action of $W_{\mathbf{R}}$ on $\hat{H}$ coincides with the action given by the L-group ${}^LH = \hat{H} \rtimes W_{\mathbf{R}}$

3. an element $\mathsf{s} \in \hat{G}$ such that $\mathrm{Int}(\mathsf{s})\hat{\theta}$ is a semisimple automorphism (§7 [Ste97])

4. an L-homomorphism (p. 18 [KS99]) $\xi : \mathcal{H} \to {}^LG$ satisfying

   (a) $\mathrm{Int}(\mathsf{s})\, {}^L\theta \circ \xi = a' \cdot \xi$ (8.5 [Bor79]) for some one-cocycle a' in the class $\mathbf{a}$

   (b) ξ maps $\hat{H}$ isomorphically onto the identity component of $\hat{G}^{\mathsf{s}\hat{\theta}}$, the group of fixed points of $\hat{G}$ under the automorphism $\mathrm{Int}(\mathsf{s})\hat{\theta}$.

Despite requirement 2 of this definition, it might not be possible to define an isomorphism between $\mathcal{H}$ and ${}^LH$ which extends the identity map on $\hat{H}$. One therefore introduces a z-extension (§2.2 [KS99], [Lan79])

$$1 \to Z_1 \to H_1 \stackrel{p_H}{\to} H \to 1 \tag{6}$$

in which H_1 is a connected reductive group containing a central torus Z_1. The surjection p_H restricts to a surjection $H_1(\mathbf{R}) \to H(\mathbf{R})$.

Dual to (6) is the extension

$$1 \to \hat{H} \to \hat{H}_1 \to \hat{Z}_1 \to 1. \tag{7}$$

Regarding $\hat{H}$ as a subgroup of $\hat{H}_1$, we may assume that ${}^LH$ embeds into ${}^LH_1$ and that $\hat{H}_1 \to \hat{Z}_1$ extends to an L-homomorphism

$$p : {}^LH_1 \to {}^LZ_1. \tag{8}$$

According to Lemma 2.2.A [KS99], there is an L-homomorphism $\xi_{H_1} : \mathcal{H} \to {}^LH_1$ which extends the inclusion of $\hat{H} \to \hat{H}_1$ and defines a topological isomorphism between $\mathcal{H}$ and $\xi_{H_1}(\mathcal{H})$. Kottwitz and Shelstad call (H_1, ξ_{H_1}) a *z-pair* for $\mathcal{H}$.

Observe that the composition

$$W_{\mathbf{R}} \stackrel{c}{\to} \mathcal{H} \stackrel{\xi_{H_1}}{\to} {}^LH_1 \stackrel{p}{\to} {}^LZ_1 \tag{9}$$

determines a quasicharacter λ_{Z_1} of $Z_1(\mathbf{R})$ via the Local Langlands Correspondence (§9 [Bor79]).

3.3 Norm mappings

Our goal here is to fix endoscopic data $(H, \mathcal{H}, \mathsf{s}, \xi)$ as defined in the previous section and to describe a map from the semisimple conjugacy classes of the endoscopic group H to the semisimple θ-conjugacy classes of G. The map uses the quasisplit form G^* as an intermediary. We basic reference for this section is chapter 3 [KS99].

Since we are interested in *semisimple* conjugacy classes, and semisimple elements lie in tori, we shall begin by defining maps between the tori of H and G^*. Suppose B_H is a Borel subgroup of H containing a maximal torus T_H and $(\mathcal{B}_H, \mathcal{T}_H, \{\mathcal{X}_H\})$ is the splitting of $\hat{H}$ used in the definition of

${}^LH$ (§3.1). Observe that we have already fixed a splitting (5) of $\hat{G} = \hat{G}^*$ (1.3 [Bor79]). Suppose further that B' is a Borel subgroup of G^* containing a maximal torus T'.[1] We may assume that $\mathsf{s} \in \mathcal{T}$, $\xi(\mathcal{T}_H) = (\mathcal{T}^{\hat{\theta}})^0$ and $\xi(\mathcal{B}_H) \subset \mathcal{B}$. The pairs $(\hat{B}_H, \hat{T}_H)$ and $(\mathcal{B}_H, \mathcal{T}_H)$ determine an isomorphism $\hat{T}_H \cong \mathcal{T}_H$. Similarly, through the pairs $(\hat{B}', \hat{T}')$ and $(\mathcal{B}, \mathcal{T})$, we conclude that $\hat{T}' \cong \mathcal{T}$. We may combine the former isomorphism with requirement 4b of §3.2 for the endoscopic map ξ to obtain isomorphisms

$$\hat{T}_H \cong \mathcal{T}_H \overset{\xi}{\cong} (\mathcal{T}^{\hat{\theta}})^0.$$

To connect $(\mathcal{T}^{\hat{\theta}})^0$ with T', we define $T'_{\theta^*} = T'/(1-\theta^*)T'$ and leave it as an exercise to prove that $((\hat{T}')^{\hat{\theta}})^0 \cong \widehat{T'_{\theta^*}}$. Combining this isomorphism with the earlier ones, we obtain in turn that

$$\hat{T}_H \cong \mathcal{T}_H \overset{\xi}{\cong} (\mathcal{T}^{\hat{\theta}})^0 \cong ((\hat{T}')^{\hat{\theta}})^0 \cong \widehat{T'_{\theta^*}}, \tag{10}$$

and $T_H \cong T'_{\theta^*}$.

The isomorphic groups T_H and T'_{θ^*} are related to the conjugacy classes, which we now define. The θ^**-conjugacy* class of an element $\delta \in G^*$ is defined as $\{g^{-1}\delta\theta^*(g) : g \in G^*\}$. The element δ is called a θ^**-semisimple* element if $\delta\theta^*$ is a semisimple element in $G^* \rtimes \langle\theta^*\rangle$, and a θ^**-semisimple* θ^**-conjugacy class* is a θ^*-conjugacy class of a θ^*-semisimple element. Let $Cl(G^*, \theta^*)$ be the set of all θ^*-conjugacy classes and $Cl_{\mathrm{ss}}(G^*, \theta^*)$ be the subset of θ^*-semisimple θ^*-conjugacy classes. With this notation in hand, we look to Lemma 3.2.A [KS99], which tells us that there is a bijection

$$Cl_{\mathrm{ss}}(G^*, \theta^*) \to T'_{\theta^*} / \, \Omega(G^*, T')^{\theta^*},$$

given by taking the coset of the intersection of a θ^*-conjugacy class with T'.

The aforementioned map specializes to give the bijections on either end of

$$Cl_{\mathrm{ss}}(H) \leftrightarrow T_H/\Omega(H, T_H) \to T'_{\theta^*}/\Omega(G^*, T')^{\theta^*} \leftrightarrow Cl_{\mathrm{ss}}(G^*, \theta^*). \tag{11}$$

To describe the remaining map in the middle of (11), recall from (10) that the isomorphism between T_H and T'_{θ^*} is obtained by way of ξ. Using these ingredients and the closing remarks of §3.1, we obtain maps

$$\Omega(H, T_H) \cong \Omega(\hat{H}, \hat{T}_H) \cong \Omega(\hat{H}, \mathcal{T}_H) \to \Omega(\hat{G}^*, \mathcal{T})^{\hat{\theta}} \cong \Omega(G^*, T')^{\theta^*}.$$

[1] Readers of [KS99] should note that we write T' for the torus T occurring there.

This completes the description of the map from $Cl_{\mathrm{ss}}(H)$ to $Cl_{\mathrm{ss}}(G^*, \theta^*)$.

We proceed by describing the a from $Cl_{\mathrm{ss}}(G^*, \theta^*)$ to $Cl_{\mathrm{ss}}(G, \theta)$. The function $m : G \to G^*$ defined by

$$m(\delta) = \psi(\delta) g_\theta^{-1}, \;\; \delta \in G$$

passes to a bijection from $Cl(G, \theta)$ to $Cl(G^*, \theta^*)$, since

$$m(g^{-1}\delta\theta(g)) = \psi(g)^{-1}\, m(\delta)\, \theta^*(\psi(g)).$$

We abusively denote this map on θ^*-conjugacy classes by m as well. It is pointed out in §3.1 [KS99] that this bijection need not be equivariant under the action of Γ. One of our key assumptions is that it is Γ-equivariant. Finally, we may combine this bijection with (11) to obtain a map

$$\mathcal{A}_{H\backslash G} : Cl_{\mathrm{ss}}(H) \to Cl_{\mathrm{ss}}(G, \theta).$$

In keeping with §3.3 [KS99], we define an element $\delta \in G$ to be *θ-regular* if the identity component of $G^{\delta\theta}$ is a torus. It is said to be *strongly θ-regular* if $G^{\delta\theta}$ itself is abelian. An element $\gamma \in H$ is said to be (*strongly*) *G-regular* if the elements in the image of its conjugacy class under $\mathcal{A}_{H\backslash G}$ are (strongly) regular. An element $\gamma \in H(\mathbf{R})$ is called a *norm* of an element $\delta \in G(\mathbf{R})$ if the θ-conjugacy class of δ equals the image of the conjugacy class of γ under $\mathcal{A}_{H\backslash G}$. It is possible for $\mathcal{A}_{H\backslash G}(\gamma)$ to be a θ-conjugacy class which contains no points in $G(\mathbf{R})$ even though $\gamma \in H(\mathbf{R})$. In this case one says that γ is *not a norm*. These definitions are carried to the z-extension H_1 in an obvious manner. For example, we say that $\gamma_1 \in H_1(\mathbf{R})$ is a norm of $\delta \in G(\mathbf{R})$ if the image of γ_1 in $H(\mathbf{R})$ under (6) is a norm of δ.

As in §3.3 [KS99], we conclude with a portrayal of the situation when a strongly regular element $\gamma \in H(\mathbf{R})$ is the norm of a strongly θ-regular element $\delta \in G(\mathbf{R})$. We may let $T_H = H^\gamma$ as γ is strongly regular. The maximal torus T_H is defined over $\mathbf{R}$ since γ lies in $H(\mathbf{R})$. Lemma 3.3.B [KS99] allows us to choose B_H, B' and T' as above so that both T' and the isomorphism $T_H \cong T'_{\theta^*}$ are defined over $\mathbf{R}$. The resulting isomorphism

$$T_H(\mathbf{R}) \cong T'_{\theta^*}(\mathbf{R}) \tag{12}$$

is called an *admissible embedding* in §3.3 [KS99] (or *toral data* in §7a [She10] for ordinary endoscopy). The image of γ under this admissible embedding defines a coset in $T'/\Omega(G^*, T')^{\theta^*}$. This coset corresponds to the θ^*-conjugacy class of $m(\delta)$. In fact, by Lemma 3.2.A [KS99] there exists some $g_{T'} \in G^*_{\mathrm{sc}}$ such that (after $g_{T'}$ has been identified with its image in G^*), this coset equals $g_{T'} m(\delta)\theta^*(g_{T'})^{-1}\Omega(G^*, T')^{\theta^*}$. The element

$$\delta^* = g_{T'}\, m(\delta)\, \theta^*(g_{T'})^{-1} \tag{13}$$

belongs to T' and it is an exercise to show that $\mathrm{Int}(g_{T'}) \circ \psi$ furnishes an isomorphism between $G^{\delta\theta}$ and $(G^*)^{\delta^*\theta^*}$. Since $\mathrm{Int}(\delta^*) \circ \theta^*$ preserves (B', T'), the torus $(G^*)^{\delta^*\theta^*}$ contains strongly G-regular elements of T' (see pp. 227-228 [Art88]) so we see in turn that the centralizer of $(G^*)^{\delta^*\theta^*}$ in G^* is T', and $(G^*)^{\delta^*\theta^*} = (T')^{\theta^*}$. By (3.3.6) [KS99], the resulting isomorphism

$$G^{\delta\theta} \stackrel{\mathrm{Int}(g_{T'})\psi}{\longrightarrow} (T')^{\theta^*} \tag{14}$$

is defined over $\mathbf{R}$.

3.4 Twisted geometric transfer

The underlying assumption of this work is twisted geometric transfer, which is laid out generally in §5.5 [KS99]. For real groups, it has been partially proven in [Ren03] and completely proven by different means in [Shea]. It shall be convenient for us to state this assumption in the framework of orbital integrals on the component $G(\mathbf{R})\theta$ of the group $G(\mathbf{R}) \rtimes \langle\theta\rangle$. Let $\delta \in G(\mathbf{R})$ be a θ-semisimple and strongly θ-regular and assume that the quasi-character ω is trivial on $G^{\delta\theta}(\mathbf{R})$. Let $C_c^\infty(G(\mathbf{R})\theta)$ be the space of smooth compactly supported functions on the component $G(\mathbf{R})\theta$. Define the *twisted orbital integral* of $f \in C_c^\infty(G(\mathbf{R})\theta)$ at $\delta\theta \in G(\mathbf{R})\theta$ to be

$$O_{\delta\theta}(f) = \int_{G^{\delta\theta}(\mathbf{R})\backslash G(\mathbf{R})} \omega(g)\, f(g^{-1}\delta\theta g)\, dg.$$

This integral depends on a choice of quotient measure dg.

We wish to match functions in $C_c^\infty(G(\mathbf{R})\theta)$ with functions on the z-extension H_1. Specifically, let $C_c^\infty(H_1(\mathbf{R}), \Lambda_{Z_1})$ be the space of smooth functions f_{H_1} on $H_1(\mathbf{R})$ whose support is compact modulo $Z_1(\mathbf{R})$ and which satisfy

$$f_{H_1}(zh) = \lambda_{Z_1}(z)^{-1} f_{H_1}(h),\ z \in Z_1(\mathbf{R}),\ h \in H_1(\mathbf{R}) \tag{15}$$

(see the end of §3.2). The definition of orbital integrals easily carries over to functions of this type at semisimple regular elements.

Suppose $\gamma_1 \in H_1(\mathbf{R})$ is a norm of a θ-semisimple strongly θ-regular element $\delta \in G(\mathbf{R})$. Our geometric transfer assumption is that for every $f \in C_c^\infty(G(\mathbf{R})\theta)$ there exists a function $f_{H_1} \in C^\infty(H_1(\mathbf{R}), \lambda Z_1)$ as above such that

$$\sum_{\gamma_1'} O_{\gamma_1'}(f_{H_1}) = \sum_{\delta'} \Delta(\gamma_1, \delta')\, O_{\delta'\theta}(f). \tag{16}$$

The sum on the left is taken over representatives in $H_1(\mathbf{R})$ of $H_1(\mathbf{R})$-conjugacy classes contained in the H_1-conjugacy class of γ_1. The sum on the right is taken over representatives in $G(\mathbf{R})$ of θ-conjugacy classes under $G(\mathbf{R})$ contained in the θ-conjugacy class of δ. The terms $\Delta(\gamma_1, \delta')$ are geometric transfer factor and are defined in chapter 4 [KS99]. We will examine the geometric transfer factors in more detail in 6.2. For the time being, let us simply remark that they are complex numbers which require some sort of normalization. Normalization is also required for the measures in the orbital integrals to be compatible (page 71 [KS99]). We also assume that the map $f \mapsto f_{H_1}$ is a continuous linear map in the topology given by the seminorms defined in §3.5 [Bou87].

4 The Local Langlands Correspondence

We wish to outline the Local Langlands Correspondence for tempered representations of $G(\mathbf{R})$ as given in §3 [Lan89]. Roughly speaking, this correspondence is a bijection between *L-parameters* and *L-packets*. More precisely, it is a bijection between $\hat{G}$-conjugacy classes $\boldsymbol{\varphi}$ of homomorphisms $\varphi : W_{\mathbf{R}} \to {}^LG$ and sets of (equivalence classes of) irreducible tempered representations Π_φ.

We begin our outline with a closer look at the homomorphisms. The homomorphisms $\varphi : W_{\mathbf{R}} \to {}^LG$ of the correspondence must be *admissible* (8.2 [Bor79]). In our context, this is equivalent to the following four requirements.

1. φ is continuous
2. the composition of φ with the projection of ${}^LG$ onto $W_{\mathbf{R}}$ is the identity map
3. the elements in the projection of $\varphi(\mathbf{C}^\times)$ to $\hat{G}$ are semisimple
4. the smallest Levi subgroup of ${}^LG$ containing $\varphi(W_{\mathbf{R}})$ is the Levi subgroup of a *relevant* parabolic subgroup (§3 [Bor79]).

An application of Theorem 5.16 [SS70], allows us to assume that $\varphi(W_{\mathbf{R}})$ normalizes a maximal torus of $\hat{G}$. Since we are only interested in the $\hat{G}$-conjugacy class of φ, we may assume that $\varphi(W_{\mathbf{R}})$ normalizes $\mathcal{T}$. By requirement 1 above, $\varphi(\mathbf{C}^\times)$ is a connected subgroup. Since the group of algebraic automorphisms of $\mathcal{T}$ is discrete, the subgroup $\varphi(\mathbf{C}^\times)$ centralizes $\mathcal{T}$. Requirement 2 then ensures that $\varphi(\mathbf{C}^\times)$ is a subgroup of $\mathcal{T}$.

Let $X^*(T)$ be the group of rational characters of T and $X_*(\hat{T})$ be the group of cocharacters of $\hat{T}$ so that $X^*(T) \cong X_*(\hat{T})$ (3.2.1 [Spr98], 2.1 [Bor79]). It is left as an exercise to the reader to show that every continuous homomorphism of $\mathbf{C}^\times$ into $\mathcal{T}$ is of the form

$$z \mapsto z^\mu \bar{z}^\nu, \ z \in \mathbf{C}^\times$$

for some $\mu, \nu \in X^*(T) \otimes \mathbf{C}$ satisfying $\mu - \nu \in X^*(T)$. Suppose φ is given by such a pair. Then it follows from

$$\varphi(-\bar{z}) = \varphi(\sigma)\varphi(z)\varphi(\sigma), \ z \in \mathbf{C}^\times$$

that $\nu = \varphi(\sigma) \cdot \mu$. This shows that the behaviour of φ on $\mathbf{C}^\times$ is determined entirely by the parameter μ. The behaviour of φ on $\sigma \in W_{\mathbf{R}}$ is parametrized as follows. We may write $\varphi(\sigma) = (a, \sigma)$ for some $a \in \hat{G}$. If there is $\lambda' \in X_*(T) = X^*(\mathcal{T})$ such that $\langle \lambda', \alpha \rangle = 0$ for every root α of (G, T) then λ' a rational character of $\hat{G}$, i.e. $\lambda' \in X^*(\hat{G})$ (Corollary 8.1.6, Proposition 8.1.8 [Spr79]). It therefore makes sense to write $\lambda'(a)$ and to choose $\lambda \in X^*(T) \otimes \mathbf{C}$ such that

$$\lambda(a) = e^{2\pi i \langle \lambda', \lambda \rangle}.$$

It is clear from this equation that the choice of λ is not unique. Nevertheless, the ensuing constructions do not depend on this lack of uniqueness.

4.1 Essentially square-integrable L-packets

In this section we assume that φ is an admissible homomorphism as above, and additionally, that $\varphi(W_{\mathbf{R}})$ is not contained in any proper Levi subgroup of ${}^L G$ (3.3 [Bor79]). Then it is not contained in any proper parabolic subgroup for ${}^L G$, for its non-trivial elements are not unipotent by requirement 3. Now, $\beta + \varphi(\sigma) \cdot \beta$ is fixed by $\varphi(\sigma)$ for any root β of $(\hat{G}, \mathcal{T})$. As a result, the subgroup generated by $\varphi(W_{\mathbf{R}})$ and the root spaces of those roots β_1 satisfying $\langle \beta + \varphi(\sigma) \cdot \beta, \beta_1^\vee \rangle \geq 0$ is a parabolic subgroup of ${}^L G$ (3.3 [Bor79], proof of Proposition 8.4.5 [Spr98]). Since $\varphi(W_{\mathbf{R}})$ is not contained in a proper parabolic subgroup, this implies that $\langle \beta + \varphi(\sigma) \cdot \beta, \beta_1^\vee \rangle \geq 0$ for *all* roots β_1. Consequently,

$$\varphi(\sigma) \cdot \beta = -\beta \tag{17}$$

for any root β (Proposition 8.1.8 [Spr98]).

Define ι to be the half-sum of the positive roots of $(\mathcal{B}, \mathcal{T})$. Set $\mu_0 = \mu - \iota$ so that $\varphi(\sigma) \cdot \mu_0 = \nu + \iota$. It is then a consequence of Lemma 3.2 [Lan89] (see page 132 [Lan89]) that

$$\frac{(\mu_0 - \varphi(\sigma) \cdot \mu_0)}{2} + (\lambda + \varphi(\sigma) \cdot \lambda) \in X_*(\mathcal{T}). \tag{18}$$

This is precisely the condition necessary for (μ_0, λ) to define a quasicharacter on a real torus via the Local Langlands Correspondence on tori (Lemma 2.8 [Lan89]). The quasicharacter is regular in the sense that $\langle \mu, \alpha \rangle \neq 0$ for all roots $\alpha \in R(\hat{G}, \mathcal{T})$ (Lemma 3.3 [Lan89]). The real torus in question is determined by taking its L-group to be the group generated by $\mathcal{T}$ and $\varphi(W_{\mathbf{R}})$.

As it happens, G itself possesses a maximal torus S, defined over $\mathbf{R}$, whose L-group is isomorphic to the group generated by $\mathcal{T}$ and $\varphi(W_{\mathbf{R}})$ (page 132 and Lemma 3.1 [Lan89]). Those familiar with the L-groups of real tori (9.4 [Bor79]) will see that (17) is equivalent to $S(\mathbf{R})$ being an *elliptic* torus, i.e. $S(\mathbf{R})$ is compact modulo the centre $Z_G(\mathbf{R})$ of $G(\mathbf{R})$. One may transfer the pair $\mu_0, \lambda \in X^*(T) \otimes \mathbf{C}$ to a pair in $X^*(S) \otimes \mathbf{C}$. We will abuse notation by denoting this pair on $X^*(S) \otimes \mathbf{C}$ as μ_0 and λ as well. This transfer is not unique. It depends on conjugation in $\hat{G}$, and so it is unique only up to the choice of an element in $\Omega(G, S) \cong \Omega(\hat{G}, \hat{S})$. In this way we obtain an $\Omega(G, S)$-orbit of pairs

$$\{(w^{-1}\mu_0, \lambda) : w \in \Omega(G, S)\} \tag{19}$$

to which we attach a set

$$\{\Lambda(w^{-1}\mu_0, \lambda) = w^{-1}\Lambda(\mu_0, \lambda) : w \in \Omega(G, S)\} \tag{20}$$

of quasicharacters of $S(\mathbf{R})$ through the Local Langlands Correspondence for tori.

To any of these quasicharacters on $S(\mathbf{R})$, Harish-Chandra associated an irreducible *essentially square-integrable* representation of $G(\mathbf{R})$ ([HC66]), i.e. an irreducible representation of $G(\mathbf{R})$ whose matrix coefficients are square-integrable modulo $Z_G(\mathbf{R})$. Moreover, any two of these representations on

$G(\mathbf{R})$ are equivalent if and only if their corresponding quasicharacters are conjugate under an element of $\Omega_{\mathbf{R}}(G,S)$, the subgroup of elements in $\Omega(G,S)$ which have a representative in $G(\mathbf{R})$ (*cf.* page 134 [Lan89]).

Let us summarize. The admissible homomorphism $\varphi : W_{\mathbf{R}} \to {}^{L}G$ may be identified with a pair $\mu, \lambda \in X^*(S) \otimes \mathbf{C}$. This pair is attached to the set of quasicharacters (20) of $S(\mathbf{R})$. Each quasicharacter is of the form $w^{-1}\Lambda(\mu_0, \lambda)$, where $\mu_0 = \mu - \iota$, and corresponds to an irreducible essentially square-integrable representation $\pi_{w^{-1}\Lambda(\mu_0,\lambda)}$ of $G(\mathbf{R})$. Up to equivalence, the set of these representations is of the form

$$\{\pi_{w^{-1}\Lambda(\mu_0,\lambda)} : w \in \Omega(G,S)/\Omega_{\mathbf{R}}(G,S)\}. \tag{21}$$

This set of (equivalence classes of) representations is called the *L-packet* corresponding to the $\hat{G}$-conjugacy class of φ and is denoted by Π_{φ}. Langlands proved that different L-packets are disjoint and that the union of such L-packets exhausts the collection of irreducible essentially square-integrable representations up to equivalence (pages 132-135 [Lan89]).

4.2 Tempered L-packets

Let us now change our assumption on φ by demanding that $\varphi(W_{\mathbf{R}})$ be bounded in ${}^{L}G$. If G is semisimple and $\varphi(W_{\mathbf{R}})$ is not contained in a proper Levi subgroup of ${}^{L}G$ then $\varphi(W_{\mathbf{R}})$ is bounded. This is amusing to prove for $G(\mathbf{R}) = \mathrm{SL}(2,\mathbf{R})$, and the general case follows from it. Therefore, up to considerations involving the centre of G, our present assumption on φ may be viewed as a weakening of the one in §4.1.

Suppose $\tilde{P}$ is a minimal parabolic subgroup of ${}^{L}G$ containing $\varphi(W_{\mathbf{R}})$. By requirement 4 in §4, the parabolic subgroup $\tilde{P}$ is relevant, that is to say $\tilde{P}$ corresponds to a parabolic $\mathbf{R}$-subgroup P of G (§3 [Bor79]). As was mentioned in §4.1, the subgroup $\varphi(W_{\mathbf{R}})$ necessarily lies in a Levi subgroup of $\tilde{P}$. This Levi subgroup may be identified with the L-group ${}^{L}M$ of a Levi subgroup M of P (3.4 [Bor79]). We may now apply the correspondence of §4.1 to obtain an L-packet $\Pi_{\varphi,M}$ of essentially square-integrable representations of $M(\mathbf{R})$.

Let us scrutinize the central characters of the representations in $\Pi_{\varphi,M}$. First off, they are all equal. Indeed, the representations in $\Pi_{\varphi,M}$ are distinguished by the action $\Omega(M,S)$, where $S(\mathbf{R})$ is a maximal elliptic torus in $M(\mathbf{R})$, and the parameters affecting $Z_M(\mathbf{R})$ are are insensitive to this action, given as it is by conjugation. Our next wish is to show that the central character under scrutiny is unitary. Since $Z_M \subset S$, we may restrict our attention to the maximal $\mathbf{R}$-split subtorus S_0 of S. If the central character is obtained from parameters μ_0, λ (§4) which produce a bounded set $\varphi(W_{\mathbf{R}})$ then the projection of μ_0 to $X^*(S_0) \otimes \mathbf{C}$ must be imaginary and so corresponds to a unitary character of $S_0(\mathbf{R})$ (9.4 (b) [Bor79]).

The central character of any representation $\pi \in \Pi_{\varphi,M}$ is unitary, one knows that any irreducible subrepresentation of the (normalized, parabolically)induced representation $\mathrm{ind}_{P(\mathbf{R})}^{G(\mathbf{R})}\pi$ is tempered (Proposition 3.7 Chapter IV [BW80]). The L-packet Π_φ is the set of (equivalence classes of) irreducible subrepresentations of $\mathrm{ind}_{P(\mathbf{R})}^{G(\mathbf{R})}\pi$ as π runs over $\Pi_{\varphi,M}$ (p. 153 [Lan89]). Again, Langlands proved that different L-packets are disjoint and that the union of such L-packets exhausts the collection of irreducible tempered representations up to equivalence (§4 [Lan89]).

4.3 L-packets stable under twisting

Now that the tempered L-packets have been defined, we shall single out those that are preserved by twisting with ω and θ (§2.2 [KS99]). Suppose Π_φ is a tempered L-packet of $G(\mathbf{R})$ as in §4.2. We write π^θ for $\pi \circ \theta$, and write $\omega \otimes (\Pi_\varphi \circ \theta)$ for the set

$$\{\omega \otimes \pi^\theta : \pi \in \Pi_\varphi\}.$$

Lemma 2 *Suppose that the quasicharacter ω is unitary. Then the set $\omega \otimes (\Pi_\varphi \circ \theta)$ is a tempered L-packet equal to $\Pi_{a\cdot({}^L\theta\circ\varphi)}$. It is essentially square-integrable if and only if Π_φ is essentially square-integrable.*

Proof. Suppose first that Π_φ is an essentially square-integrable L-packet and ω is trivial. Then the representations in $\Pi_\varphi \circ \theta$ correspond to a set of characters of the form $\Lambda(w\mu_0, \lambda) \circ \theta$ of an elliptic torus $\theta^{-1}(S(\mathbf{R}))$ of $G(\mathbf{R})$ (see (20)). The elliptic torus $\theta^{-1}(S(\mathbf{R}))$ is $G(\mathbf{R})$-conjugate to $S(\mathbf{R})$ (Corollary 4.35 [Kna96], Corollary 5.31 [Spr79]). This implies that there exists an element $x \in G(\mathbf{R})$ so that $\mathrm{Int}(x)\theta$ preserves the subgroup $S(\mathbf{R})$ (and any fixed splitting). The isomorphism between ${}^LS$ and the group generated by $\varphi(W_{\mathbf{R}})$ and $\mathcal{T}$ then transports the set of characters $\Lambda(w\mu_0, \lambda) \circ \mathrm{Int}(x)\theta$ to the data $\mu_0 \circ \theta$, $\lambda \circ \theta \in X^*(T)$. Finally, these data correspond to ${}^L\theta \circ \varphi$. This proves that $\Pi_\varphi \circ \theta = \Pi_{{}^L\theta\circ\varphi}$.

Now let us assume merely that Π_φ is a tempered L-packet. Then each representation in $\Pi_\varphi \circ \theta$ is an irreducible subrepresentation of an induced representation $(\mathrm{ind}_{P(\mathbf{R})}^{G(\mathbf{R})} \pi) \circ \theta$ for some essentially square-integrable representation $\pi \in \Pi_{\varphi, M}$ (see §4.2). As can be seen in the proof of Proposition 3.1 [Mez07], the induced representation $(\mathrm{ind}_{P(\mathbf{R})}^{G(\mathbf{R})} \pi) \circ \theta$ is equivalent to $\mathrm{ind}_{\theta^{-1}(P(\mathbf{R}))}^{G(\mathbf{R})} (\pi \circ \theta)$. By the previous paragraph we know that $\pi \circ \theta \in \Pi_{{}^L\theta \circ \varphi, \theta^{-1}(M)}$. This proves that

$$\Pi_\varphi \circ \theta = \Pi_{{}^L\theta \circ \varphi}. \tag{22}$$

Evidently, the set $\varphi(W_{\mathbf{R}})$ is not contained in a proper Levi subgroup of ${}^LG$ if and only if the same is true for ${}^L\theta \circ \varphi$. Equivalently (§4.1), $\Pi_\varphi \circ \theta$ is essentially square-integrable if and only if Π_φ is.

It remains to prove the lemma for non-trivial ω. Recall from §3.2, that ω corresponds to a cocycle $a \in Z^1(W_{\mathbf{R}}, Z_{\hat{G}})$. According to condition (iii) on page 125 [Lan89], we have

$$\Pi_{a \cdot \varphi} = \omega \otimes \Pi_\varphi.$$

The first assertion of the lemma follows from this identity combined with (22),

$$\omega \otimes \Pi_\varphi \circ \theta = \Pi_{a \cdot \varphi} \circ \theta = \Pi_{{}^L\theta \circ (a \cdot \varphi)}.$$

Given our observation on the square-integrability under a twist by θ above, the second assertion follows from the fact that Π_φ is essentially square-integrable if and only if $\Pi_{a \cdot \varphi}$ is. Indeed, an equivalent statement is that φ is not contained in a proper parabolic subgroup of ${}^LG$ if and only if $a \cdot \varphi$ is not contained in a proper parabolic subgroup of ${}^LG$, and we know that $Z_{\hat{G}}$ is a subgroup of every parabolic subgroup of ${}^LG$.■

5 Tempered essentially square-integrable representations

The only tempered L-packets Π_φ which are meaningful to the spectral transfer theorem for twisted endoscopy are those which are preserved by the twisting data ω and θ. We record this property here for future reference.

$$\Pi_\varphi = \Pi_{a \cdot ({}^L\theta \circ \varphi)} = \omega \otimes (\Pi_\varphi \circ \theta). \tag{23}$$

Suppose Π_φ is a tempered essentially square-integrable L-packet. In other words, the representations of Π_φ are unitary and essentially square-integrable. Suppose further that Π_φ satisfies (23) and let $\pi \in \Pi_\varphi$. It is possible for $\omega \otimes \pi^\theta$ to be inequivalent to π. These representations of Π_φ do not contribute to the character identities of the spectral transfer theorem. We are therefore interested in representations $\pi \in \Pi_\varphi$ which are equivalent to $\omega \otimes \pi^\theta$.

Let us assume for the remainder of §5 that $\pi \in \Pi_\varphi$ is equivalent to $\omega \otimes \pi^\theta$. This means that there is an intertwining operator T on the space V_π of π which satisfies

$$\mathsf{U} \circ \omega^{-1}(x)\pi(x) = \pi^\theta(x) \circ \mathsf{U}, \; x \in G(\mathbf{R}). \tag{24}$$

By Schur's lemma we know that the operator U is unique only up to multiplication by a scalar in $\mathbf{C}^\times$. Schur's lemma also tells us that

$$\omega^{-1}_{|Z_G(\mathbf{R})}\, \chi_\pi = \chi_\pi^\theta, \tag{25}$$

where χ_π denotes the central character of π. Observe that since π is tempered, it is unitary, and so χ_π is also unitary.

5.1 Reduction to the subgroup $G_{\mathrm{der}}(\mathbf{R})^0$

To further dissect the representation π, we turn to the derived subgroup G_{der} of G. Let $G_{\mathrm{der}}(\mathbf{R})^0$ denote the identity component of $G_{\mathrm{der}}(\mathbf{R})$ regarded as a *Lie group*, i.e. as a real manifold. Then according to Lemma 3.5 [Lan89], there exists an irreducible essentially square-integrable subrepresentation ϖ of the restriction of π to the subgroup $Z_G(\mathbf{R})G_{\mathrm{der}}(\mathbf{R})^0$. The group $Z_G(\mathbf{R}) \cap G_{\mathrm{der}}(\mathbf{R})^0$ may be non-trivial, we may identify ϖ with a representation on $Z_G(\mathbf{R}) \times G_{\mathrm{der}}(\mathbf{R})^0$ and shall abuse notation by writing

$$\varpi = \chi_\pi \otimes \varpi_1, \tag{26}$$

where ϖ_1 is the restriction of ϖ to $G_{\mathrm{der}}(\mathbf{R})^0$. Since $G_{\mathrm{der}}(\mathbf{R})^0$ is semisimple (Corollary 8.1.6 (ii) [Spr98]), its centre is finite, and the representation ϖ_1 is square-integrable.

To see how we may recover π from ϖ we appeal to Theorem 16 [HC66]. This theorem assures us that for any two representatives $\delta_1, \delta_2 \in G(\mathbf{R})$ of distinct cosets in $G(\mathbf{R})/Z_G(\mathbf{R})G_{\mathrm{der}}(\mathbf{R})^0$ the representations ϖ^{δ_1} and ϖ^{δ_2} are inequivalent. By Mackey's criterion, the finitely induced representation

$\mathrm{ind}_{Z_G(\mathbf{R})G_{\mathrm{der}}(\mathbf{R})^0}^{G(\mathbf{R})} \varpi$ is irreducible. An application of Frobenius reciprocity then implies that

$$\pi \cong \mathrm{ind}_{Z_G(\mathbf{R})G_{\mathrm{der}}(\mathbf{R})^0}^{G(\mathbf{R})} \varpi.$$

Combining this argument with decomposition (26), we also see that

$$\pi \cong \mathrm{ind}_{Z_G(\mathbf{R})G_{\mathrm{der}}(\mathbf{R})^0}^{G(\mathbf{R})} (\chi_\pi \otimes \varpi_1). \tag{27}$$

The close relationship between π and ϖ recorded above suggests that we may reduce our study of π to that of ϖ_1. To follow this strategy, it will be necessary to reconsider the group $G(\mathbf{R})/Z_G(\mathbf{R})G_{\mathrm{der}}(\mathbf{R})^0$.

Let S be a maximal torus of G as in §4.1. Then $S_{\mathrm{der}} = S \cap G_{\mathrm{der}}$ is a maximal torus of G_{der} such that $S_{\mathrm{der}}(\mathbf{R})$ is compact in $G_{\mathrm{der}}(\mathbf{R})$ (Lemma 3.1 [Lan89]). Since compact real tori are products of circle groups, which are connected, the compact maximal torus $S_{\mathrm{der}}(\mathbf{R})$ lies in $G_{\mathrm{der}}(\mathbf{R})^0$. It now follows from

$$G = G_{\mathrm{der}}\, Z_G^0 \tag{28}$$

(Proposition 7.3.1 and Corollary 8.1.6 of [Spr98]) that

$$S = (Z_G^0 \cap S)(G_{\mathrm{der}} \cap S) = Z_G^0 S_{\mathrm{der}},$$

$$S(\mathbf{R}) = (Z_G^0 S_{\mathrm{der}})(\mathbf{R}) = Z_G^0(\mathbf{R}) S_{\mathrm{der}}(\mathbf{R}) \tag{29}$$

and

$$Z_G(\mathbf{R})G_{\mathrm{der}}(\mathbf{R})^0 = Z_G(\mathbf{R})S_{\mathrm{der}}(\mathbf{R})G_{\mathrm{der}}(\mathbf{R})^0 = S(\mathbf{R})G_{\mathrm{der}}(\mathbf{R})^0. \tag{30}$$

As an aside, we remark that equations (28) and (25) imply that the quasicharacter ω is unitary.

Fix a maximal compact subgroup K of $G_{\mathrm{der}}(\mathbf{R})^0$ containing $S_{\mathrm{der}}(\mathbf{R})$, and fix a positive system for the root system $\mathrm{R}(\mathfrak{k} \otimes \mathbf{C}, \mathfrak{s}_{\mathrm{der}} \otimes \mathbf{C})$.

Lemma 3 *There exists a complete set of representatives $\{\delta_1, \ldots, \delta_k\}$ for the cosets of $G(\mathbf{R})/Z_G(\mathbf{R})G_{\mathrm{der}}(\mathbf{R})^0$ such that for every $1 \leq j \leq k$,*

1. $\mathrm{Int}(\delta_j)\theta(K) = K$
2. $\mathrm{Int}(\delta_j)\theta(S_{\mathrm{der}}(\mathbf{R})) = S_{\mathrm{der}}(\mathbf{R})$
3. *The differential of* $\mathrm{Int}(\delta_j)\theta$ *preserves the fixed positive system of* $\mathrm{R}(\mathfrak{k} \otimes \mathbf{C}, \mathfrak{s}_{\mathrm{der}} \otimes \mathbf{C})$.

Proof. By the argument given at the beginning of the proof of Theorem 6.9.5 [Wal88], each coset in $G(\mathbf{R})/Z_G(\mathbf{R})G_{\mathrm{der}}(\mathbf{R})^0$ contains a representative which normalizes $S(\mathbf{R})$, and so $G(\mathbf{R})/Z_G(\mathbf{R})G_{\mathrm{der}}(\mathbf{R})^0$ may be identified with a subset of $\Omega(G,S)$. Consequently, we may choose a finite and complete set of representatives $\delta_1',\ldots,\delta_k' \in G(\mathbf{R})$ for $G(\mathbf{R})/Z_G(\mathbf{R})G_{\mathrm{der}}(\mathbf{R})^0$. Suppose $1 \leq j \leq k$.

Both K and $\mathrm{Int}(\delta_j')\theta(K)$ are maximal compact subgroups of $G_{\mathrm{der}}(\mathbf{R})^0$. By Corollary 5.31 [Spr79], we may choose $y_1,\ldots,y_k \in G_{\mathrm{der}}(\mathbf{R})^0$ such that

$$\mathrm{Int}(y_j\delta_j')\theta(K) = K,\ 1 \leq j \leq k.$$

Similarly, both $S_{\mathrm{der}}(\mathbf{R})$ and $\mathrm{Int}(y_j\delta_j')\theta(S_{\mathrm{der}}(\mathbf{R}))$ are maximal tori in the connected (Theorem 6.46 [Kna96]) compact group K. By Corollary 4.35 [Kna96], we may choose $u_1,\ldots u_k \in K$ such that

$$\mathrm{Int}(u_jy_j\delta_j')\theta(S_{\mathrm{der}}(\mathbf{R})) = S_{\mathrm{der}}(\mathbf{R}).$$

Set $\delta_j = u_jy_j\delta_j'$. In this way, we obtain a complete set of representatives for $G(\mathbf{R})/Z_G(\mathbf{R})G_{\mathrm{der}}(\mathbf{R})^0$ such that the first two assertions hold.

We make an additional adjustment to the representatives $\delta_1,\ldots,\delta_k$, but this time keeping the same notation. The Weyl group $\Omega(\mathfrak{k}\otimes\mathbf{C},\mathfrak{s}_{\mathrm{der}}\otimes\mathbf{C})$ is isomorphic to $\Omega(K,S_{\mathrm{der}}(\mathbf{R}))$ (Theorem 4.41 [Kna86]). Therefore, after possibly multiplying each of $\delta_1,\ldots,\delta_k$ by an element in $N_K(S_{\mathrm{der}}(\mathbf{R}))$, we may assume that the third assertion holds (Theorem 10.3 (b) [Hum94]).■

Fix $\delta_1,\ldots\delta_k \in G(\mathbf{R})$ satisfying Lemma 3.

Lemma 4 *There exists a unique $1 \leq m \leq k$ such that the representation ϖ_1 is equivalent to $\varpi_1^{\delta_m\theta}$, where*

$$\varpi_1^{\delta_m\theta}(x) = \varpi_1(\delta_m\theta(x)\delta_m^{-1}),\ x \in G_{\mathrm{der}}(\mathbf{R})^0.$$

The restriction $\pi(\delta_m)T_{|V_\varpi}$ of $\pi(\delta_m)T$ to the subspace $V_\varpi \subset V_\pi$ of ϖ is an isomorphism of V_ϖ which intertwines ϖ_1 with $\varpi_1^{\delta_m\theta}$, that is

$$\pi(\delta_m)T\varpi_1(x) = \varpi_1^{\delta_m\theta}(x)\,\pi(\delta_m)T_{|V_\varpi},\ x \in G_{\mathrm{der}}(\mathbf{R})^0. \tag{31}$$

Proof. According to Clifford's theorem, equation (25) and our earlier decompositions we have

$$\begin{aligned}(\omega^{-1}\otimes\pi)_{|Z_G(\mathbf{R})G_{\mathrm{der}}(\mathbf{R})^0} &\cong \omega^{-1}_{|Z_G(\mathbf{R})}\otimes\left(\oplus_{j=1}^k\varpi^{\delta_j}\right)\\ &\cong \omega^{-1}_{|Z_G(\mathbf{R})}\,\chi_\pi\otimes(\oplus_{j=1}^k\varpi_1^{\delta_j})\\ &\cong \chi_\pi^\theta\otimes(\oplus_{j=1}^k\varpi_1^{\delta_j}).\end{aligned} \tag{32}$$

In particular, we see that $\chi_\pi^\theta \otimes \varpi_1$ is an irreducible subrepresentation of the restriction of $\omega \otimes \pi$ to $Z_G(\mathbf{R})G_{\mathrm{der}}(\mathbf{R})^0$. We may therefore follow the justification for (27) to arrive at

$$(33)\quad \mathrm{ind}_{Z_G(\mathbf{R})G_{\mathrm{der}}(\mathbf{R})^0}^{G(\mathbf{R})}(\chi_\pi^\theta \otimes \varpi_1) \cong \omega \otimes \pi \cong \pi^\theta \cong \mathrm{ind}_{Z_G(\mathbf{R})G_{\mathrm{der}}(\mathbf{R})^0}^{G(\mathbf{R})}(\chi_\pi^\theta \otimes \varpi_1^\theta).$$

By Frobenius reciprocity, we have

$$1 = \dim \mathrm{Hom}(\omega \otimes \pi, \pi^\theta) = \dim \mathrm{Hom}(\chi_\pi^\theta \otimes (\oplus_{j=1}^k \varpi_1^{\delta_j}), \chi_\pi^\theta \otimes \varpi_1^\theta).$$

In consequence, there is a unique $1 \leq b \leq k$ such that $\varpi_1^{\delta_b} \cong \varpi_1^\theta$. The first assertion of the lemma follows by taking m to be the unique integer such that δ_m is the representative of the coset of δ_b^{-1}.

To prove the second assertion, take $1 \leq r \leq k$ and let $P_rT(V_\varpi)$ be the orthogonal projection of $T(V_\varpi)$ onto the subspace $\pi(\delta_r)(V_\varpi)$, which is irreducible under $\pi_{|Z_G(\mathbf{R})G_{\mathrm{der}}(\mathbf{R})^0}$. It is easily verified that $P_rT(V_\varpi)$ is invariant under $\pi_{|Z_G(\mathbf{R})G_{\mathrm{der}}(\mathbf{R})^0}$. The irreducibility of $\pi(\delta_r)(V_\varpi)$ implies that the projection $P_rT(V_\varpi)$ is either zero or all of $\pi(\delta_r)(V_\varpi)$. By Clifford's Theorem, the representation space of $\pi_{|Z_G(\mathbf{R})G_{\mathrm{der}}(\mathbf{R})^0}$ is equal to $\oplus_{j=1}^k \pi(\delta_j)(V_\varpi)$ so we may assume that $\pi(\delta_r)(V_\varpi) = T(V_\varpi)$ for some unique r. In this way the operator $\pi(\delta_r^{-1})T_{|V_\varpi}$ is an isomorphism of V_ϖ. In addition, for all $x \in G_{\mathrm{der}}(\mathbf{R})^0$ and $v \in V_\varpi$ we have

$$\begin{aligned} \pi(\delta_r^{-1})T\varpi(x)v &= \pi(\delta_r^{-1})T\pi(x)v \\ &= \pi^{\delta_r^{-1}\theta}(x)\pi(\delta_r^{-1})Tv \\ &= \varpi^{\delta_r^{-1}\theta}(x)\pi(\delta_r^{-1})Tv. \end{aligned}$$

In other words, $\pi(\delta_r^{-1})T_{|V_\varpi}$ intertwines ϖ_1 with $\varpi_1^{\delta_r^{-1}\theta}$ By the proof of the first assertion of the lemma, we have $r = b$ so that we may replace δ_r^{-1} with δ_m.■

5.2 Twisted characters

We continue our study of the representation $\pi \cong \omega \otimes \pi^\theta$ of the previous section by defining the notion of a twisted character. According to Harish-Chandra (§4 [HC54]), the operator

$$\int_{G(\mathbf{R})} f(x)\, \pi(x)\, dx$$

is trace-class for every $f \in C_c^\infty(G(\mathbf{R}))$. Since the space of trace-class operators forms a two-sided ideal in the space of bounded operators ([RS75]), we may define the *twisted character* $\Theta_{\pi,\mathsf{U}}$ by

$$f \mapsto \operatorname{tr} \int_{G(\mathbf{R})} f(x\theta)\, \pi(x)\, \mathsf{U}\, dx, \;\; f \in C_c^\infty(G(\mathbf{R})\theta). \tag{34}$$

We would like to express this twisted character in terms of an analogous twisted character for ϖ_1. To this end, set

$$\mathsf{U}_1 = \pi(\delta_m)\, \mathsf{U}_{|V_\varpi}$$

as in (31). One may regard $G_{\mathrm{der}}(\mathbf{R})^0$ as a characteristic subgroup of $G(\mathbf{R}) \rtimes \langle\theta\rangle$. In this way, the subgroup generated by $G_{\mathrm{der}}(\mathbf{R})^0$ and $\delta_m\theta$ is a Lie group. We may define $C_c^\infty(G_{\mathrm{der}}(\mathbf{R})^0\delta_m\theta)$ and $\Theta_{\varpi_1,\mathsf{U}_1}$ by tailoring the above ideas to $G(\mathbf{R})$, θ, π and U above.

Lemma 5 *Suppose $f \in C_c^\infty(G(\mathbf{R})\theta)$ and define $f_1 \in C_c^\infty(G_{\mathrm{der}}(\mathbf{R})^0\delta_m\theta)$ by*

$$f_1(x\delta_m\theta) = \frac{1}{|G_{\mathrm{der}}(\mathbf{R})^0 \cap Z_G(\mathbf{R})|} \sum_{r=1}^{k} \omega(\delta_r) \int_{Z_G(\mathbf{R})} f(z\delta_r^{-1}x\delta_m\theta\delta_r)\, \chi_\pi(z)\, dz,$$

for all $x \in G_{\mathrm{der}}(\mathbf{R})^0$. Then

$$\Theta_{\pi,\mathsf{U}}(f) = \Theta_{\varpi_1,\mathsf{U}_1}(f_1).$$

Proof. By Proposition 7.3.1 [Spr98], the intersection $G_{\mathrm{der}}(\mathbf{R})^0 \cap Z_G(\mathbf{R})$ is finite, and so we may write

$$\begin{aligned}
&\operatorname{tr} \int_{G(\mathbf{R})} f(x\theta)\, \pi(x)\, \mathsf{U}\, dx \\
&= \operatorname{tr} \int_{Z_G(\mathbf{R})G_{\mathrm{der}}(\mathbf{R})^0} \sum_{j=1}^{k} f(x\delta_j\theta)\pi(x\delta_j)\, \mathsf{U}\, dx \\
&= \operatorname{tr} \int_{G_{\mathrm{der}}(\mathbf{R})^0/G_{\mathrm{der}}(\mathbf{R})^0\cap Z_G(\mathbf{R})} \sum_{j=1}^{k} \int_{Z_G(\mathbf{R})} f(zx\delta_j\theta)\chi_\pi(z)\, dz\; \pi(x\delta_j)T\, \mathsf{U} dx \\
&= \operatorname{tr} \int_{G_{\mathrm{der}}(\mathbf{R})^0} \frac{1}{|G_{\mathrm{der}}(\mathbf{R})^0 \cap Z_G(\mathbf{R})|} \sum_{j=1}^{k} \int_{Z_G(\mathbf{R})} f(zx\delta_j\theta)\chi_\pi(z)\, dz\; \pi(x\delta_j)\, \mathsf{U}\, dx.
\end{aligned}$$

In the proof of Lemma 4 it was pointed out that the restriction of π to $G_{\mathrm{der}}(\mathbf{R})^0$ is a unitary representation whose space is equal to the orthogonal sum $\oplus_{j=1}^k \pi(\delta_j)V_\varpi$. Let $\langle\cdot,\cdot\rangle$ denote the inner product on this space and let B_ϖ be an orthonormal basis of V_ϖ. Then

$$\begin{aligned}
&\mathrm{tr}\int_{G_{\mathrm{der}}(\mathbf{R})^0} \frac{1}{|G_{\mathrm{der}}(\mathbf{R})^0 \cap Z_G(\mathbf{R})|} \sum_{j=1}^k \int_{Z_G(\mathbf{R})} f(zx\delta_j\theta)\chi_\pi(z)\,dz\,\pi(x\delta_j)\,\mathsf{U}\,dx \\
&= \sum_{v\in B_\varpi}\sum_{r=1}^k \int_{G_{\mathrm{der}}(\mathbf{R})^0} \frac{1}{|G_{\mathrm{der}}(\mathbf{R})^0 \cap Z_G(\mathbf{R})|} \sum_{j=1}^k \int_{Z_G(\mathbf{R})} f(zx\delta_j\theta)\chi_\pi(z)\,dz \\
&\quad \cdot\langle \pi(x\delta_j)\mathsf{U}\pi(\delta_r)v, \pi(\delta_r)v\rangle dx \\
&= \sum_{v\in B_\varpi}\sum_{r=1}^k \int_{G_{\mathrm{der}}(\mathbf{R})^0} \frac{1}{|G_{\mathrm{der}}(\mathbf{R})^0 \cap Z_G(\mathbf{R})|} \sum_{j=1}^k \int_{Z_G(\mathbf{R})} f(zx\delta_j\theta)\chi_\pi(z)\,dz \\
&\quad \cdot\, \omega(\delta_r^{-1})\langle \pi(\delta_r^{-1}x\delta_j\theta(\delta_r))\mathsf{U}v, v\rangle dx \\
&= \sum_{v\in B_\varpi}\sum_{r=1}^k \int_{G_{\mathrm{der}}(\mathbf{R})^0} \frac{1}{|G_{\mathrm{der}}(\mathbf{R})^0 \cap Z_G(\mathbf{R})|} \sum_{j=1}^k \int_{Z_G(\mathbf{R})} f(z\delta_r x\delta_j\theta\delta_r^{-1})\chi_\pi(z)\,dz \\
&\quad \cdot\, \omega(\delta_r^{-1})\langle \pi(x)\,\pi(\delta_j)\mathsf{U}v, v\rangle dx.
\end{aligned}$$

Here we have used (24) and a change of variable. From Lemma 4 we know that the only operator $\pi(\delta_j)\,\mathsf{U}$ which preserves the subspace V_ϖ is $\pi(\delta_m)\,\mathsf{U}$. Therefore, the previous expression reduces to the integral over $G_{\mathrm{der}}(\mathbf{R})^0$ of

$$\frac{1}{|G_{\mathrm{der}}(\mathbf{R})^0 \cap Z_G(\mathbf{R})|} \sum_{v\in B_\varpi}\sum_{r=1}^k \omega(\delta_r^{-1}) \int_{Z_G(\mathbf{R})} f(z\delta_r x\delta_m\theta\delta_r^{-1})\chi_\pi(z)\,dz\langle \varpi_1(x)\mathsf{U}_1 v, v\rangle.$$

This is easily seen to be equal to $\Theta_{\varpi_1,\mathsf{U}_1}(f_1)$.■

5.3 Square-integrable representations on a class of disconnected Lie groups

At this point, we find it advantageous to set

$$\delta = \delta_m. \tag{35}$$

Our intention in this section is to present enough of the work of Duflo ([Duf82]) and Bouaziz ([Bou87]) to be able to represent the twisted character $\Theta_{\pi,\mathsf{U}}$ of Lemma 5 as a locally integrable function on the regular subset of $G_{\mathrm{der}}(\mathbf{R})^0$. Rather than working with the notationally burdensome

$\langle G_{\mathrm{der}}(\mathbf{R})^0, \delta\theta\rangle$, we shall work with a semisimple Lie group L of the kind prescribed in [Bou87]. To be precise, the Lie group L is prescribed to have a semisimple Lie algebra and to posses an abelian subgroup A which centralizes the identity component L^0. Moreover, the subgroup AL^0 is of finite index in L.

Working in this level of generality is suitable for our purposes. Indeed, the group $G_{\mathrm{der}}(\mathbf{R})^0$ is a connected semisimple Lie group (Corollary 8.1.6 (ii) [Spr98]). To discover the abelian subgroup corresponding to A, we recall from Lemma 1 that θ has finite order on $G_{\mathrm{der}}(\mathbf{R})^0$. Therefore, the element $(\delta\theta)^{|\theta|}$ acts as inner automorphism on $G_{\mathrm{der}}(\mathbf{R})^0$ which normalizes $S(\mathbf{R})$ (Lemma 3 and (29)). As such, the element $(\delta\theta)^{|\theta|}$ corresponds to a unique element of the finite group $\Omega(G(\mathbf{R}), S(\mathbf{R}))$, and for some minimal positive exponent ℓ there exists $s_0 \in S_{\mathrm{der}}(\mathbf{R})$ such that

$$(\mathrm{Int}(\delta)\theta_{|G_{\mathrm{der}}(\mathbf{R})^0})^{\ell} = \mathrm{Int}(s_0). \tag{36}$$

We may now take the group corresponding to A above to be $\langle s_0^{-1}(\delta\theta)^{\ell}\rangle$. Indeed, the group

$$\langle G_{\mathrm{der}}(\mathbf{R})^0, s_0^{-1}(\delta\theta)^{\ell}\rangle = G_{\mathrm{der}}(\mathbf{R})^0\langle s_0^{-1}(\delta\theta)^{\ell}\rangle$$

is of finite index in the group $\langle G_{\mathrm{der}}(\mathbf{R})^0, \delta\theta\rangle$.

5.3.1 Some work of Harish-Chandra and Duflo

Suppose that ϖ_1 is an irreducible square-integrable representation of L^0. In [Duf82], Duflo parametrizes the set of representations of L (up to equivalence), whose restriction to L^0 is equal to ϖ_1. In section 5.4, we specialize to $L = \langle G_{\mathrm{der}}(\mathbf{R})^0, \delta\theta\rangle$, and Duflo's parametrization will be conjoined with the intertwining operators occurring in Lemma 4. A discussion similar to this one may be found in §18 [Ren97].

Let us recall some of the work of Harish-Chandra ([HC66], IX and XII [Kna86]). There exists a compact maximal torus H in L^0 (Theorem 12.20 [Kna86]), contained in a maximal compact subgroup K of L^0. Let $\mathfrak{l}$, $\mathfrak{k}$ and $\mathfrak{h}$ be the real Lie algebras of L, K and H respectively. We may fix a maximal nilpotent subalgebra $\mathfrak{u}$ of $\mathfrak{l}\otimes\mathbf{C}$ such that $\mathfrak{b}_1 = (\mathfrak{h}\otimes\mathbf{C})\oplus\mathfrak{u}$ is a Borel subalgebra of $\mathfrak{l}\otimes\mathbf{C}$. This Borel subalgebra determines a positive system of roots for $\mathrm{R}(\mathfrak{l}\otimes\mathbf{C}, \mathfrak{h}\otimes\mathbf{C})$, and thereby also determines a positive system of roots for $\mathrm{R}(\mathfrak{k}\otimes\mathbf{C}, \mathfrak{h}\otimes\mathbf{C})$. Let ρ_1 be the half-sum of the positive roots of $(\mathfrak{l}\otimes\mathbf{C}, \mathfrak{h}\otimes\mathbf{C})$.

The representation ϖ_1 corresponds to a unique regular element Λ_1 in the dual Lie algebra $\mathfrak{h}^* \subset \mathfrak{l}^*$, such that $i\Lambda_1$ (the *Harish-Chandra parameter*) lies in the positive chamber relative to $\mathfrak{b}_1$ (Theorem 9.20 and V §3 [Kna86]). The sum $i\Lambda_1 + \rho_1$ is the differential of a character of H (Proposition 4.13 [Kna86]).

We will sketch how the objects of the previous paragraph find their place in II [Duf82]. The regularity of $\Lambda_1 \in \mathfrak{h}^*$ translates to the technical condition that Λ_1 is *bien polarisible* (Theorem 2.9′ [Kna96], Lemma 7 II [Duf82]). One of the consequences of this condition is that the bilinear form

$$X, Y \mapsto \Lambda_1([X, Y]), \ X, Y \in \mathfrak{l}$$

induces a symplectic form on $(\mathfrak{l}/\mathfrak{h}) \otimes \mathbf{C}$. Another consequence is that the image of the Borel subalgebra $\mathfrak{b}_1 = (\mathfrak{h} \otimes \mathbf{C}) \oplus \mathfrak{u}$ in $(\mathfrak{l}/\mathfrak{h}) \otimes \mathbf{C}$ is equal to its orthogonal complement with respect to this form, i.e. the image of $\mathfrak{b}_1$ is a *Lagrangian subspace* (see the proof of Lemma 7 II [Duf82]). To any such Lagrangian subspace, Duflo attaches a a complex linear form on $\mathfrak{h}$ ((5) II.1 [Duf82]). The linear form for $\mathfrak{b}_1$ is the half-sum of the roots ρ_1 of $(\mathfrak{b}_1, \mathfrak{h} \otimes \mathbf{C})$ (see (8) II.1 [Duf82]), and the fact that the sum $i\Lambda_1 + \rho_1$ lifts to a character of H is equivalent to the condition that Λ_1 be *admissible* (Remark 2 II.2 [Duf82]) for the group L^0.

For L^0, the notion of admissibility pertains to a metaplectic covering group

$$1 \to \{1, \varepsilon\} \to \tilde{H} \to H \to 1$$

(chapter I [Duf82]). By definition, the element Λ_1 is *admissible* for L^0 if there exists a unitary representation τ of $\tilde{H}$ such that $\tau(\varepsilon) = -1$ and the differential of τ equals $i\Lambda_1$ (II.2 [Duf82]). We denote the set of (equivalence classes of) all such representations which are irreducible by $X^{\text{irr}}_{L^0}(\Lambda_1)$. For each $\tau \in X^{\text{irr}}_{L^0}(\Lambda_1)$, Duflo constructs a unitary representation of L^0 (II.4 [Duf82]). In III.3 [Duf82] it is shown that there is only one such representation and it is equal to ϖ_1.

The notion of admissibility extends to the group L in a similar fashion. Let $L(\Lambda_1)$ be the subgroup $\{x \in L : x \cdot (\Lambda_1) = \Lambda_1\}$ of L. By the regularity of Λ_1 we know that $L^0(\Lambda_1) = H$. However, for the group L the notion of admissibility pertains to a metaplectic covering group

$$1 \to \{1, \varepsilon\} \to \widetilde{L(\Lambda_1)} \to L(\Lambda_1) \to 1$$

and $X^{\text{irr}}_L(\Lambda_1)$ is defined as before, but with $L(\Lambda_1)$ in place of H. For each $\tau \in X^{\text{irr}}_L(\Lambda_1)$ Duflo constructs a unitary representation of $L(\Lambda_1)L^0$ ((4) III.5

[Duf82]) whose restriction to L^0 is equal to ϖ_1. These representations are of the form

$$\tau \otimes S_{\Lambda_1}\varpi_1,$$

where S_{Λ_1} is a representation of $\widetilde{L(\Lambda_1)}$ whose space is identical to that of ϖ_1 (Lemma 6 III.5 [Duf82]). To be more precise, for any $y \in L^0$, $x \in L(\Lambda_1)$ and preimage $\tilde{x}$ of x in $\widetilde{L(\Lambda_1)}$

$$\tau \otimes S_{\Lambda_1}\varpi_1(xy) = \tau(\tilde{x}) \otimes S_{\Lambda_1}(\tilde{x})\varpi_1(y), \tag{37}$$

where the expression on the right does not depend on the choice of pre-image $\tilde{x}$.

5.3.2 Some work of Bouaziz

Let us now assume that $L(\Lambda_1)L^0 = L$, and turn to the work of Bouaziz which expresses the character of the representation (37) as a locally integrable function. Fix $\tau \in X_L^{\mathrm{irr}}(\Lambda_1)$ and let ϖ_τ be the representation of L defined by (37). The integrals

$$\Theta_{\varpi_\tau}(f) = \mathrm{tr} \int_L f(x)\varpi_\tau(x)\,dx,\ \ f \in C_c^\infty(L)$$

converge (*cf.* §5.1). It follows from Theorem 2.1.1 [Bou87] that Θ_{ϖ_τ} is given by a locally integrable function.

We shall only be interested in the values of this function at particular elements of L. Specifically, we shall evaluate Θ_{ϖ_τ} only at elements $x \in L(\Lambda_1)$ which are regular in the sense of 1.3 [Bou87] and satisfy $\mathrm{Ad}(x)\mathfrak{k} = \mathfrak{k}$. According to Proposition 6.1.2 and page 60 [Bou87], the value $\Theta_{\varpi_\tau}(x)$ is of the form

$$\sum_{w \in \Omega(L^0,H)^x} \frac{(-1)^{q^{w\cdot\Lambda_1}}}{\det(1-\mathrm{Ad}(x))_{|\mathfrak{u}}} \mathrm{tr}(\tau^w\tilde{\rho}_1)(x). \tag{38}$$

Let us briefly explicate the terms of this formula. The expression $\Omega(L^0,H)^x$ denotes the subgroup of elements in the Weyl group $\Omega(L^0,H)$ whose action on H commutes with that of $Int(x)$ (see page 53 [Bou87]). The term $q^{w\cdot\Lambda_1}$ is the number of negative eigenvalues of the matrix given by the Hermitian form

$$X \mapsto i(w\cdot\Lambda_1)([X,\bar{X}]),\ \ X \in \mathfrak{u}$$

(see §5.1 [Bou87]). One may compute that q^{Λ_1} is the number of positive compact imaginary roots of $\mathrm{R}(\mathfrak{l}\otimes\mathbf{C},\mathfrak{h}\otimes\mathbf{C})$, and that $q^{w\cdot\Lambda_1}$ is the sum of the number of positive compact imaginary roots which remain positive under the action of w and the number of positive noncompact imaginary roots which become negative under the action of w. It follows from Lemma A §10.3 [Hum94] that

$$(-1)^{q^{w\cdot\Lambda_1}} = \det(w)(-1)^{q^{\Lambda_1}}. \tag{39}$$

Finally, $\tilde{\rho}_1$ is a character of $\widetilde{L(\Lambda_1)}$ which satisfies

$$\tilde{\rho}_1(\tilde{x})^2 = \det(\mathrm{Ad}x)_{|\mathfrak{u}} \tag{40}$$

for every $\tilde{x} \in \widetilde{L(\Lambda_1)}$ whose projection to $L(\Lambda_1)$ is x ((5.2.1) [Bou87]). The product $\tau^w\tilde{\rho}_1$ descends to a representation of $L(\Lambda_1)$ (§5.2 [Bou87]).

5.4 Twisted characters again

We return once more to the context and notation of §5.1. In particular π is an essentially square-integrable representation of $G(\mathbf{R})$ whose restriction to $Z_G(\mathbf{R})G_{\mathrm{der}}(\mathbf{R})^0$ contains an irreducible subrepresentation ϖ. Equation (26) furnishes us with a related irreducible square-integrable representation ϖ_1 of the semisimple connected group $G_{\mathrm{der}}(\mathbf{R})^0$. Lemma 4 tells us that $\varpi_1 \cong \varpi_1^{\delta\theta}$ and also provides us with an explicit intertwining operator $\mathsf{U}_1 = \pi(\delta)\mathsf{U}_{|V_\varpi}$. We now wish to apply the work of Duflo and Bouaziz summarized above to the representation ϖ_1.

We begin by hearkening back to the discussion at the beginning of §5.3 and setting $L = \langle G_{\mathrm{der}}(\mathbf{R})^0, \delta\theta\rangle$. As we know from that discussion, there is a least positive integer ℓ such that (36) holds. This implies that U_1^ℓ intertwines ϖ_1 with $\varpi_1^{s_0}$. Schur's Lemma then tells us that U^ℓ is a scalar multiple of $\varpi_1(s_0^{\ell_1})$. Since the intertwining operator U of (24) is only defined up to a scalar multiple, we may normalize U so that $\mathsf{U}_1^\ell = \varpi_1(s_0^\ell)$. This normalization is unique only up to multiplication by an ℓ^{th} root of unity. Once U is so fixed, the representation ϖ_1 may be extended to the group L by defining $\varpi_1(\delta\theta) = \mathsf{U}_1$. In this manner one obtains ℓ inequivalent unitary representations $\bar{\varpi}_1$ of L–one for each root of unity. There is another manner in which our normalization is not canonical. The element $s_0 \in S_{\mathrm{der}}(\mathbf{R})$ is only unique up to multiplication by an element in the centre of the semisimple analytic group $G_{\mathrm{der}}(\mathbf{R})^0$, which is finite (Proposition 7.9 [Kna96]). This non-uniqueness may mean that additional roots of unity are possible in the normalization of U. However, the possibility of several extensions to L will be accounted for in our character identities and we have chosen not to include the roots of unity in the notations for $\bar{\varpi}_1$, U_1 and U.

We wish to find $\tau \in X_L^{\mathrm{irr}}(\Lambda_1)$ as in §5.3.1 such that ϖ_τ is equivalent to $\bar{\varpi}_1$. We may specialize the framework of §5.3.1 by setting $L^0 = G_{\mathrm{der}}(\mathbf{R})^0$, $\mathfrak{l} = \mathfrak{g}_{\mathrm{der}}$, $H = S_{\mathrm{der}}(\mathbf{R})$, and $\mathfrak{h} = \mathfrak{s}_{\mathrm{der}}$. In addition, in §5.1 we have fixed a maximal compact subgroup $K \supset S_{\mathrm{der}}(\mathbf{R})$ of $G_{\mathrm{der}}(\mathbf{R})^0$ and a positive system for $\mathrm{R}(\mathfrak{k} \otimes \mathbf{C}, \mathfrak{s}_{\mathrm{der}} \otimes \mathbf{C})$. We may associate to ϖ_1 a unique regular $\Lambda_1 \in \mathfrak{s}_{\mathrm{der}}^*$ in the positive chamber determined by the positive system for $\mathrm{R}(\mathfrak{k}\otimes\mathbf{C}, \mathfrak{s}_{\mathrm{der}}\otimes\mathbf{C})$. We then fix $\mathfrak{b}_1$ to be the Borel subalgebra $(\mathfrak{g}_{\mathrm{der}} \otimes \mathbf{C}, \mathfrak{s}_{\mathrm{der}} \otimes \mathbf{C})$ determined by Λ_1 (Theorem′ 10.1 [Hum94]).

Lemma 6 *Under the assumptions of this section, the subgroup $L(\Lambda_1)$ of $L = \langle G_{\mathrm{der}}(\mathbf{R})^0, \delta\theta\rangle$ is the subgroup generated by $S_{\mathrm{der}}(\mathbf{R})$ and $\delta\theta$.*

Proof. Lemma 4 tells us that ϖ_1 is equivalent to $\varpi_1^{\delta\theta}$. By inspecting the character of $\varpi_1^{\delta\theta}$ (Theorem 12.7 (a) [Kna86]) and recalling equation 2 of Lemma 3, one sees that the parameter associated to $\varpi_1^{\delta\theta}$ is given by $\delta\theta \cdot \Lambda_1 \in \mathfrak{s}_{\mathrm{der}}^*$, the differential of $\mathrm{Int}(\delta)\theta$ applied to Λ_1. According to Harish-Chandra's classification of square-integrable representations ([HC66], Theorem 9.20 [Kna86]), the equivalence $\varpi_1 \cong \varpi_1^{\delta\theta}$ implies that there is some $w_1 \in \Omega(K, S_{\mathrm{der}}(\mathbf{R})) \cong \Omega(\mathfrak{k} \otimes \mathbf{C}, \mathfrak{s}_{\mathrm{der}} \otimes \mathbf{C})$ such that $\Lambda_1 = w_1\delta\theta \cdot \Lambda_1$. As the representative δ was chosen so that $\delta\theta$ preserves the positive roots of $\mathrm{R}(\mathfrak{k}\otimes\mathbf{C}, \mathfrak{s}_{\mathrm{der}}\otimes\mathbf{C})$, and Λ_1 lies in the positive Weyl chamber relative to these roots, Lemma B §10.3 [Hum94] applies and we may conclude that $\Lambda_1 = \delta\theta\cdot\Lambda_1$. It follows in turn that $\delta\theta \in L(\Lambda_1)$ and $L(\Lambda_1) = \langle S_{\mathrm{der}}(\mathbf{R}), \delta\theta\rangle$.■

It is immediate from Lemma 6 that $L(\Lambda_1)L^0 = L$. Thus, for each representation $\tau \in X_L^{\mathrm{irr}}(\Lambda_1)$ we obtain from (37) an operator $\tau \otimes S_{\Lambda_1}(\delta\theta)$ which intertwines ϖ_1 with $\varpi_1^{\delta\theta}$. We describe the representation τ by first recalling that $\Lambda_1 = \delta\theta \cdot \Lambda_1$ (Lemma 6) and setting

$$\tau_0(\exp(X)) = e^{(i\Lambda_1+\rho_1)(X)},\ X \in \mathfrak{s}_{\mathrm{der}} \tag{41}$$

This defines a character in $L^0(\Lambda_1)$. We may extend it to a character of $L(\Lambda_1)$ by recalling (36), choosing an ℓth root of unity ζ, and setting

$$\tau_0(\delta\theta) = \zeta\ e^{(i\Lambda_1+\rho_1)(\log(s_0)/\ell)}. \tag{42}$$

This produces a character τ_0 of $L(\Lambda_1)$ which depends on a choice of an ℓth root of unity. If δ happens to belong to $S_{\mathrm{der}}(\mathbf{R})$ then the $\delta\theta$-invariance of Λ_1 and ρ_1 imply in turn that $(i\Lambda_1+\rho_1)(\log(s_0)) = (i\Lambda_1+\rho_1)(\ell\log(\delta))$ and

$$\tau_0(\delta\theta) = \zeta\ e^{(i\Lambda_1+\rho_1)(\log(\delta))}. \tag{43}$$

The product $\tau_0\tilde{\rho}_1{}^{-1}$ is a character of $\widetilde{L(\Lambda_1)}$ which lies in $X_L^{\mathrm{irr}}(\Lambda_1)$ (§5.3.2, §5.2 [Bou87]). By Schur's Lemma and (37), the intertwining operator $\tau_0\tilde{\rho}_1^{-1}\otimes S_{\Lambda_1}(\delta\theta)$ is a scalar multiple of U_1. After possibly dividing $\tau_0(\delta\theta)$ by this scalar, which is an ℓth root of unity, we obtain a character $\tau = \tau_0\tilde{\rho}_1{}^{-1}$ in $X_L^{\mathrm{irr}}(\Lambda_1)$ such that $\tau\otimes S_{\Lambda_1}(\delta\theta) = \mathsf{U}_1$. As a result, the representation $\varpi_\tau = \tau\otimes S_{\Lambda_1}\varpi_1$ is equal to the extension $\bar{\varpi}_1$ defined above.

The upshot of this identification of $\bar{\varpi}_1$ with ϖ_τ is that we may express some values of the twisted character $\Theta_{\varpi_1,\mathsf{U}_1}$ of §5.2 by using the work of Bouaziz. Indeed, by (38) and (39), we deduce that for any $x\in S_{\mathrm{der}}(\mathbf{R})$ such that $x\delta\theta$ is regular in L, the value $\Theta_{\varpi_1,\mathsf{U}_1}(x\delta\theta)$ is equal to

$$\begin{aligned}
&\sum_{w\in\Omega(G_{\mathrm{der}}(\mathbf{R})^0,S_{\mathrm{der}}(\mathbf{R}))^{x\delta\theta}} \frac{\det(w)(-1)^{q^{\Lambda_1}}}{\det(1-\mathrm{Ad}(x\delta\theta))_{|\mathfrak{u}}}\mathrm{tr}(\tau^w\tilde{\rho}_1)(x\delta\theta)\\
=&\sum_{w\in\Omega(G_{\mathrm{der}}(\mathbf{R})^0,S_{\mathrm{der}}(\mathbf{R}))^{\delta\theta}} \frac{\det(w)(-1)^{q^{\Lambda_1}}}{\det(1-\mathrm{Ad}(x\delta\theta))_{|\mathfrak{u}}}(\tau_0^w(\tilde{\rho}_1{}^{-1})^w\tilde{\rho}_1)(x\delta\theta).
\end{aligned}$$

The expression $(\tau_0^w(\tilde{\rho}_1{}^{-1})^w\tilde{\rho}_1)(x\delta\theta)$ may be made plainer. It follows from (40) that for $w\in\Omega(G_{\mathrm{der}}(\mathbf{R})^0,S_{\mathrm{der}}(\mathbf{R}))^{\delta\theta}$, $\tilde{x}\in\widetilde{L(\Lambda_1)}$ which projects to $x\in S_{\mathrm{der}}(\mathbf{R})$ and $X\in\mathfrak{s}_{\mathrm{der}}$ such that $x=\exp(X)$ we have

$$\begin{aligned}
\tau^w\tilde{\rho}_1(x^2) &= \tau_0^w(\tilde{\rho}_1{}^{-1})^w\tilde{\rho}_1(x^2)\\
&= \tau_0^w(\tilde{\rho}_1{}^{-1})^w\tilde{\rho}_1(\tilde{x}^2)\\
&= e^{wi\Lambda_1(2X)}e^{w2\rho_1(X)}\det(\mathrm{Ad}wx^{-1}w^{-1})_{|\mathfrak{u}}\det(\mathrm{Ad}x)_{|\mathfrak{u}}\\
&= e^{wi\Lambda_1(2X)}\det(\mathrm{Ad}wxw^{-1})_{|\mathfrak{u}}\det(\mathrm{Ad}wx^{-1}w^{-1})_{|\mathfrak{u}}e^{\rho_1(2X)}\\
&= e^{(wi\Lambda_1+\rho_1)(2X)}.
\end{aligned}$$

As every element of the compact connected torus $S_{\mathrm{der}}(\mathbf{R})$ is a square, this proves that

$$\tau^w\tilde{\rho}_1(x) = \tau_0^w(\tilde{\rho}_1{}^{-1})^w\tilde{\rho}_1(x) = e^{(wi\Lambda_1+\rho_1)(X)}. \tag{44}$$

Likewise, we compute $\tau^w \tilde{\rho}_1(\delta\theta) = \tau_0^w (\tilde{\rho}_1{}^{-1})^w \tilde{\rho}_1(\delta\theta)$. We choose a representative $\dot{w} \in G_{\mathrm{der}}(\mathbf{R})^0$ of w in the normalizer of $S_{\mathrm{der}}^{\delta\theta}(\mathbf{R})$ and highlight the element

$$\dot{w}(\delta\theta)\dot{w}^{-1}(\delta\theta)^{-1} \in G_{\mathrm{der}}(\mathbf{R})^0 \tag{45}$$

which we denote by $e(\dot{w})$. By our choice of w, the element $e(\dot{w})$ acts trivially on $S_{\mathrm{der}}(\mathbf{R})$ under conjugation, and is therefore actually an element in $S_{\mathrm{der}}(\mathbf{R})$. According to (40),

$$\begin{aligned} (\tilde{\rho}_1^{-1})^w \tilde{\rho}_1(\delta\theta) &= \pm \det(\mathrm{Ad}\dot{w}\delta\theta\dot{w}^{-1})_{|\mathfrak{u}}^{-1/2} \pm \det(\mathrm{Ad}\delta\theta)_{|\mathfrak{u}}^{1/2} \\ &= \det(\mathrm{Ad}e(\dot{w}))_{|\mathfrak{u}}^{-1/2}. \end{aligned}$$

Choose $E(\dot{w}) \in \mathfrak{s}_{\mathrm{der}}(\mathbf{R})$ such that $\exp(E(\dot{w})) = e(\dot{w})$ and apply (41) to derive

$$\tau_0^w(\delta\theta) = \tau_0(e(\dot{w})\,\delta\theta) = e^{(i\Lambda_1+\rho_1)E(\dot{w})}\tau_0(\delta\theta).$$

Combining this equation with our computation for $(\tilde{\rho}_1^{-1})^w \tilde{\rho}_1(\delta\theta)$ we find that

$$\tau^w \tilde{\rho}_1(\delta\theta) = \tau_0(\delta\theta)\, e^{(i\Lambda_1+\rho_1)E(\dot{w})} \det(\mathrm{Ad}e(\dot{w}))_{|\mathfrak{u}}^{-1/2} = \tau_0(\delta\theta)\, e^{i\Lambda_1 E(\dot{w})} \tag{46}$$

Observe that $\Lambda_1 E(\dot{w})$ is independent of our choice of representative as

$$e(s\dot{w}) = e(\dot{w})\, e(s),\ \ s \in S_{\mathrm{der}}(\mathbf{R})$$

and

$$\Lambda_1 E(s) = \Lambda_1(\log(s)) - \delta\theta \cdot \Lambda_1(\log(s)) = 0,\ \ s \in S_{\mathrm{der}}(\mathbf{R})$$

(Lemma 6). We collect these simplifications in the proof of the following lemma.

Lemma 7 *Suppose $x \in S_{\mathrm{der}}(\mathbf{R})$ such that $x\delta\theta$ is regular in L, $w \in \Omega(G_{\mathrm{der}}(\mathbf{R})^0, S_{\mathrm{der}}(\mathbf{R}))^{\delta\theta}$ and $\dot{w} \in G_{\mathrm{der}}(\mathbf{R})^0$ is any representative for w. Suppose further that $X \in \mathfrak{s}_{\mathrm{der}}$ satisfies* $\exp(X) = x$. *Then $\Theta_{\varpi_1,\mathsf{U}_1}(x\delta\theta)$ is equal to*

$$(-1)^{q^{-\Lambda_1}}\,\bar{\tau}_0(\delta\theta) \sum_{w\in\Omega(G_{\mathrm{der}}(\mathbf{R})^0, S_{\mathrm{der}}(\mathbf{R}))^{\delta\theta}} \frac{\det(w)\ \ e^{(wi\Lambda_1-\rho_1)X}\, e^{i\Lambda_1 E(\dot{w})}}{\det(1-\mathrm{Ad}(x\delta\theta))_{|\bar{\mathfrak{u}}}}, \tag{47}$$

where $\bar{\tau}_0$ is the character of $L(\Lambda_1)$ which satisfies

$$\bar{\tau}_0(\exp(X)) = e^{(i\Lambda_1-\rho_1)(X)},\ \ X \in \mathfrak{s}_{\mathrm{der}}$$

and

$$\bar{\tau}_0(\delta\theta) = \zeta e^{(i\Lambda_1-\rho_1)(\log(s_0)/\ell)} \tag{48}$$

*(*cf. *(41) and (42)).*

Proof: By the discussion above we know that $\Theta_{\varpi_1,\mathsf{U}_1}(x\delta\theta)$ is equal to

$$(-1)^{q^{\Lambda_1}} \sum_{w\in\Omega(G_{\mathrm{der}}(\mathbf{R})^0, S_{\mathrm{der}}(\mathbf{R}))^{x\delta\theta}} \frac{\det(w)\,\tau^w \tilde{\rho}_1(x\delta\theta)}{\det(1-\mathrm{Ad}(x\delta\theta))_{|\mathfrak{u}}}$$

and it is easily computed that $\det(1-\mathrm{Ad}(x\delta\theta))_{|\mathfrak{u}}$ is equal to

$$(-1)^{\dim\mathfrak{u}}\, e^{2\rho_1(X)} \det\mathrm{Ad}(\delta\theta)_{|\mathfrak{u}}\ \det(1-\mathrm{Ad}(x\delta\theta))_{|\bar{\mathfrak{u}}}. \tag{49}$$

If w_0 is the longest Weyl group element of $\mathrm{R}(\mathfrak{l}\otimes\mathbf{C}, \mathfrak{s}_{\mathrm{der}}\otimes\mathbf{C})$ then equation (39) implies

$$(-1)^{\dim\mathfrak{u}}(-1)^{q^{\Lambda_1}} = \det(w_0)(-1)^{q^{\Lambda_1}} = (-1)^{q^{w_0\Lambda_1}} = (-1)^{q^{-\Lambda_1}}.$$

The remaining portions of the lemma follow by drawing together (49) with (44) and (46) and setting

$$\bar{\tau}_0 = \frac{\tau_0}{\det\mathrm{Ad}_{|\mathfrak{u}}}. \blacksquare$$

5.4.1 The twisted Weyl integration formula

One typically computes the character of a representation by applying the Weyl integration formula to an explicit locally integrable function, such as the one given in (47). Our goal here is to compute the character distribution value $\Theta_{\varpi_1,\mathsf{U}_1}(f_1)$ of Lemma 5 using this strategy. We shall compute this value under the assumption that δ is strongly θ-regular in G and $G^{\delta\theta} = S^{\delta\theta}$, continuing with the notations of §5.4. In particular, $L = \langle G_{\mathrm{der}}(\mathbf{R})^0, \delta\theta\rangle$. Some of computations here are implicit in 1.4.2 and (7.1.8) [Bou87].

The basic map for the Weyl integration formula in our situation is $\Psi : G_{\mathrm{der}}(\mathbf{R})^0 \times {}'S^{\delta\theta}_{\mathrm{der}}(\mathbf{R}) \to L$, where

$$\Psi(g,s) = gs\delta\theta g^{-1},\ g\in G_{\mathrm{der}}(\mathbf{R})^0,\ s\in {}'S^{\delta\theta}_{\mathrm{der}}(\mathbf{R})$$

and ${}'S^{\delta\theta}_{\mathrm{der}}(\mathbf{R})$ denotes the subset of θ-regular elements in $S^{\delta\theta}_{\mathrm{der}}(\mathbf{R})$.

Lemma 8 *The map Ψ is a submersion.*

Proof. We must prove that the differential $d\Psi_{g,s} : \mathfrak{g}_{\mathrm{der}} \times \mathfrak{s}^{\delta\theta}_{\mathrm{der}} \to \mathfrak{g}_{\mathrm{der}}$ at each point $(g,s)\in G_{\mathrm{der}}(\mathbf{R})^0 \times {}'S^{\delta\theta}_{\mathrm{der}}(\mathbf{R})$ is surjective. By following the computation of (4.45) [Kna96] and §2 [Ren97], we see that

(50) $d\Psi_{g,s}(X,Y) = \mathrm{Ad}(g)((\mathrm{Ad}(s\delta\theta)^{-1} - I)X + Y),\ X\in\mathfrak{g}_{\mathrm{der}},\ Y\in\mathfrak{s}^{\delta\theta}_{\mathrm{der}}.$

The image of this map contains $\mathfrak{q} = (I - \mathrm{Ad}(\delta\theta))\mathfrak{g}_{\mathrm{der}}$ and $\mathfrak{s}^{\delta\theta}_{\mathrm{der}}$. According to (36), the element $(\delta\theta)^{\ell}$ gives rise to a semisimple inner automorphism of the semisimple group $G_{\mathrm{der}}(\mathbf{R})^0$. This is equivalent to $\delta\theta$ itself giving rise to a semisimple automorphism of $G_{\mathrm{der}}(\mathbf{R})^0$ and so $\mathrm{Ad}(\delta\theta)$ is a semisimple automorphism of $\mathfrak{g}_{\mathrm{der}}$. This implies that we may decompose $\mathfrak{g}_{\mathrm{der}}$ as

$$\mathfrak{g}_{\mathrm{der}} = \mathfrak{g}^{\delta\theta}_{\mathrm{der}} \oplus \mathfrak{q} = \mathfrak{s}^{\delta\theta}_{\mathrm{der}} \oplus \mathfrak{q}. \blacksquare \tag{51}$$

We determine the extent to which the submersion Ψ fails to be injective. As before, $\Omega(G_{\mathrm{der}}(\mathbf{R})^0, S^{\delta\theta}_{\mathrm{der}}(\mathbf{R}))$ is the subgroup of the Weyl group $\Omega(G_{\mathrm{der}}(\mathbf{R})^0, S_{\mathrm{der}}(\mathbf{R}))$ comprised of elements which commute with the action of $\delta\theta$. Set

$$`S^{\delta\theta}_{\mathrm{der}}(\mathbf{R}) = \{s \in S^{\delta\theta}_{\mathrm{der}}(\mathbf{R}) : \det(I - \mathrm{Ad}(s\delta_m\theta))_{|\mathfrak{q}} \neq 0\}, \tag{52}$$

where $\mathfrak{q} = (I - \mathrm{Ad}(\delta\theta))\mathfrak{g}_{\mathrm{der}}$. The set $`S^{\delta\theta}_{\mathrm{der}}(\mathbf{R})$ contains a dense open subset of $S^{\delta\theta}_{\mathrm{der}}(\mathbf{R})^0$ which also contains regular elements in $G_{\mathrm{der}}(\mathbf{R})^0$.

Lemma 9 *Suppose $g, g' \in G_{\mathrm{der}}(\mathbf{R})^0$, $s, s' \in {}`S^{\delta\theta}_{\mathrm{der}}(\mathbf{R}) \cap S^{\delta\theta}_{\mathrm{der}}(\mathbf{R})^0$ and*

$$\Psi(g, s) = \Psi(g', s').$$

Then

$$(g^{-1}g')\, S^{\delta\theta}_{\mathrm{der}}(\mathbf{R})^0 \delta\theta\, (g^{-1}g')^{-1} = (S^{\delta\theta}_{\mathrm{der}})(\mathbf{R})^0\delta\theta.$$

Proof. By hypothesis, we have

$$s\delta\theta = (g^{-1}g')s'\delta\theta(g^{-1}g')^{-1}. \tag{53}$$

The centralizer in $\mathfrak{g}_{\mathrm{der}}$ of the left-hand side of this equation is equal to $\mathfrak{s}^{\delta\theta}_{\mathrm{der}}$. The centralizer of the right-hand side is equal to $\mathrm{Ad}(g^{-1}g')\mathfrak{s}^{\delta\theta}_{\mathrm{der}}$. After exponentiating, this identity of centralizers translates to

$$S^{\delta\theta}_{\mathrm{der}}(\mathbf{R})^0 = (g^{-1}g')\, S^{\delta\theta}_{\mathrm{der}}(\mathbf{R})^0\, (g^{-1}g')^{-1}.$$

This identity shows that $g^{-1}g'$ normalizes $S^{\delta\theta}_{\mathrm{der}}(\mathbf{R})^0$. We may also rearrange equation (53) to obtain

$$\delta\theta(g^{-1}g')^{-1}\delta\theta^{-1}(g^{-1}g') = (s')^{-1}(g^{-1}g')^{-1}s(g^{-1}g').$$

Since $g^{-1}g'$ normalizes $S^{\delta\theta}_{\mathrm{der}}(\mathbf{R})^0$, the right-hand side belongs to $S^{\delta\theta}_{\mathrm{der}}(\mathbf{R})^0$. It now follows that

$$(g^{-1}g')S^{\delta\theta}_{\mathrm{der}}(\mathbf{R})^0\delta\theta(g^{-1}g')^{-1} = (g^{-1}g')S^{\delta\theta}_{\mathrm{der}}(\mathbf{R})^0(g^{-1}g')^{-1}\delta\theta = S^{\delta\theta}_{\mathrm{der}}(\mathbf{R})^0\delta\theta. \blacksquare$$

In view of Lemma 9, we define $N_{G_{\mathrm{der}}(\mathbf{R})^0}(S_{\mathrm{der}}^{\delta\theta}(\mathbf{R})^0\delta\theta)$ to be the group

$$\{g \in G_{\mathrm{der}}(\mathbf{R})^0 : gS_{\mathrm{der}}(\mathbf{R})\delta\theta g^{-1} = S_{\mathrm{der}}(\mathbf{R})\delta\theta\},$$

and we define $\Omega(G_{\mathrm{der}}(\mathbf{R})^0, S_{\mathrm{der}}^{\delta\theta}(\mathbf{R})^0\delta\theta)$ to be $N_{G_{\mathrm{der}}(\mathbf{R})^0}(S_{\mathrm{der}}^{\delta\theta}(\mathbf{R})^0\delta\theta)/S_{\mathrm{der}}^{\delta\theta}(\mathbf{R})$. According to Lemma 1.3.2 [Bou87], the group $\Omega(G_{\mathrm{der}}(\mathbf{R})^0, S_{\mathrm{der}}^{\delta\theta}(\mathbf{R})^0\delta\theta)$ is finite. Evidently, Ψ passes to a map on $G_{\mathrm{der}}(\mathbf{R})^0/S_{\mathrm{der}}^{\delta\theta}(\mathbf{R}) \times {}^{\backprime}S_{\mathrm{der}}^{\delta\theta}(\mathbf{R})$ and we shall likewise denote this map by Ψ.

Lemma 10 *The restriction of Ψ to $G_{\mathrm{der}}(\mathbf{R})^0/S_{\mathrm{der}}^{\delta\theta}(\mathbf{R}) \times {}^{\backprime}S_{\mathrm{der}}^{\delta\theta}(\mathbf{R}) \cap S_{\mathrm{der}}^{\delta\theta}(\mathbf{R})^0$ is $|\Omega(G_{\mathrm{der}}(\mathbf{R})^0, S_{\mathrm{der}}^{\delta\theta}(\mathbf{R})^0\delta\theta)|$-to-one.*

Proof. We follow Lemma 8.57 [Kna96]. Define an element of $G_{\mathrm{der}}(\mathbf{R})^0/S_{\mathrm{der}}^{\delta\theta}(\mathbf{R})\times {}^{\backprime}S_{\mathrm{der}}^{\delta\theta}(\mathbf{R}) \cap S_{\mathrm{der}}^{\delta\theta}(\mathbf{R})^0$ to be equivalent to $(g'S_{\mathrm{der}}^{\delta\theta}(\mathbf{R}), s') \in G_{\mathrm{der}}(\mathbf{R})^0/S_{\mathrm{der}}^{\delta\theta}(\mathbf{R}) \times {}^{\backprime}S_{\mathrm{der}}^{\delta\theta}(\mathbf{R})$ if it is of the form $(g'wS_{\mathrm{der}}^{\delta\theta}(\mathbf{R}), w^{-1}s'\delta\theta(w))$ for some $w \in N_{G_{\mathrm{der}}(\mathbf{R})^0}(S_{\mathrm{der}}^{\delta\theta}(\mathbf{R})^0\delta\theta)$. Apparently, equivalent elements have the same image under Ψ, and there are $|\Omega(G_{\mathrm{der}}(\mathbf{R})^0, S_{\mathrm{der}}^{\delta\theta}(\mathbf{R})^0\delta\theta)|$ elements in any equivalence class.

Conversely, suppose that $\Psi(gS_{\mathrm{der}}^{\delta\theta}(\mathbf{R}), s) = \Psi(g'S_{\mathrm{der}}(\mathbf{R}), s')$. We are to show that $(gS_{\mathrm{der}}^{\delta\theta}(\mathbf{R}), s)$ is equivalent to $(g'S_{\mathrm{der}}(\mathbf{R}), s')$. By Lemma 9, the element $g^{-1}g'$ belongs to $N_{G_{\mathrm{der}}(\mathbf{R})^0}(S_{\mathrm{der}}^{\delta\theta}(\mathbf{R})^0\delta\theta)$, and equation (53) implies that $(g^{-1}g')s'\delta\theta((g^{-1}g')^{-1}) = s$. Setting $w = g^{-1}g'$, we see that $(g'S_{\mathrm{der}}(\mathbf{R}), s')$ is equal to $(gwS_{\mathrm{der}}(\mathbf{R}), s')$. The latter element is equivalent to $(gS_{\mathrm{der}}(\mathbf{R}), ws'\delta\theta(w^{-1}))$, and this final element is equal to $(gS_{\mathrm{der}}(\mathbf{R}), s)$.■

Proposition 1 (The twisted Weyl integration formula for $S_{\mathrm{der}}(\mathbf{R})$) *Suppose f is a continuous function on $G_{\mathrm{der}}(\mathbf{R})^0\delta\theta$ with support in the closure of $\Psi(G_{\mathrm{der}}(\mathbf{R})^0 \times {}^{\backprime}S_{\mathrm{der}}^{\delta\theta}(\mathbf{R}) \cap S_{\mathrm{der}}^{\delta\theta}(\mathbf{R})^0)$. Then the integral $\int_{G_{\mathrm{der}}(\mathbf{R})^0} f(x\delta\theta)\,dx$ is equal to*

$$\frac{1}{|\Omega(G_{\mathrm{der}}(\mathbf{R})^0, S_{\mathrm{der}}^{\delta\theta}(\mathbf{R})^0\delta\theta)|} \int_{S_{\mathrm{der}}^{\delta\theta}(\mathbf{R})^0} \int_{G_{\mathrm{der}}(\mathbf{R})^0/S_{\mathrm{der}}^{\delta\theta}(\mathbf{R})} f(gs\delta\theta g^{-1})|\det(1-\mathrm{Ad}(s\delta\theta))_{|\mathfrak{q}}|\,dg\,ds.$$

Moreover, for $s \in {}^{\backprime}S_{\mathrm{der}}^{\delta\theta}(\mathbf{R}) \cap S_{\mathrm{der}}^{\delta\theta}(\mathbf{R})^0$ we have

$$\begin{aligned} &|\det(1-\mathrm{Ad}(s\delta\theta))_{|\mathfrak{q}}| \\ &= |\det(1-\mathrm{Ad}(\delta\theta))_{|(\mathfrak{s}_{\mathrm{der}}/\mathfrak{s}_{\mathrm{der}}^{\delta\theta})\otimes\mathbf{C}}|\;|\det(1-\mathrm{Ad}(s\delta\theta))_{|\mathfrak{u}}|^2. \end{aligned} \tag{54}$$

Proof. The first assertion follows from Proposition 8.19 [Kna96], (8.59) [Kna96] and Lemma 10 if the absolute value of the determinant of the differential $d\Psi_{g,s}$ is equal to $|\det(1-\mathrm{Ad}(s\delta\theta))_{|\mathfrak{q}}|$ for $s \in {}^{\backprime}S_{\mathrm{der}}^{\delta\theta}(\mathbf{R})$. This final point is established from the fact that

$$|\det(d\Psi_{g,s})| = |\det \mathrm{Ad}(g)||\det(1-\mathrm{Ad}(s\delta\theta))_{|\mathfrak{q}}|$$

(see (50)), and that $|\det \mathrm{Ad}(g)| = 1$ (Lemma 4.28 [Kna96]).

As for the second assertion, it is shown in the proof of Lemma 2.3.3 [Bou87] that for s in the component of ${}^{\backprime}S^{\delta\theta}_{\mathrm{der}}(\mathbf{R})$ containing the identity, the term $|\det(1 - \mathrm{Ad}(s\delta\theta))_{|\mathfrak{q}}|$ is equal to

$$|\det(1 - \mathrm{Ad}(\delta\theta))_{|\mathfrak{q}\otimes\mathbf{C}\cap\mathfrak{s}_{\mathrm{der}}\otimes\mathbf{C}} \ \det \mathrm{Ad}(\delta\theta)_{|\bar{\mathfrak{u}}} \ e^{-2\rho_1(\log(s))} \ \det(1 - \mathrm{Ad}(s\delta\theta))^2_{|\mathfrak{u}}|.$$

Here, $\bar{\mathfrak{u}}$ is the nilpotent radical of the Borel subalgebra opposite to $\mathfrak{b}_1$. The map

$$s \mapsto |e^{-2\rho_1(\log(s))}| = |\det \mathrm{Ad}(s)_{|\bar{\mathfrak{u}}}|, \ \ s \in S_{\mathrm{der}}(\mathbf{R})$$

is a continuous homomorphism of a compact connected group into $\mathbf{R}^\times$, so its image is one. In addition, from §5.3 we know that $(\delta\theta)^\ell$ lies in $Z_G(\mathbf{R})S_{\mathrm{der}}(\mathbf{R})$. In particular, $|(\mathrm{Ad}(\delta\theta)_{|\bar{\mathfrak{u}}})^\ell| = 1$ which implies that $|(\mathrm{Ad}(\delta\theta)_{|\bar{\mathfrak{u}}})| = 1$. For $s \in {}^{\backprime}S^{\delta\theta}_{\mathrm{der}}(\mathbf{R}) \cap S^{\delta\theta}_{\mathrm{der}}(\mathbf{R})^0$ in the component containing the identity, the second assertion now follows from $\mathfrak{q}\cap\mathfrak{s}_{\mathrm{der}} \cong \mathfrak{s}_{\mathrm{der}}/\mathfrak{s}^{\delta\theta}_{\mathrm{der}}$ (see (51)). The second assertion follows for any $s \in {}^{\backprime}S^{\delta\theta}_{\mathrm{der}}(\mathbf{R})\cap S^{\delta\theta}_{\mathrm{der}}(\mathbf{R})^0$ by observing that the above arguments remain true when $\delta\theta$ is replaced by $s'\delta\theta$ for any $s' \in S^{\delta\theta}_{\mathrm{der}}(\mathbf{R})^0$.■

The Weyl integration formula provides us with an expression of the character $\Theta_{\pi,\mathsf{U}}(f) = \Theta_{\varpi_1,\mathsf{U}_1}(f_1)$ of Lemma 5 in terms of twisted orbital integrals and the character formula (47). As Proposition 1 applies only to functions with support in the image of Ψ, a few words are in order concerning the support of $f \in C_c^\infty(G(\mathbf{R})\theta)$. According to Lemma 8, the image of Ψ is open in $G_{\mathrm{der}}(\mathbf{R})^0\delta\theta$. Since the quotient $G(\mathbf{R})/Z_G(\mathbf{R})G_{\mathrm{der}}(\mathbf{R})^0$ is finite (§5), the set $Z_G(\mathbf{R})^0\Psi(G_{\mathrm{der}}(\mathbf{R})^0 \times {}^{\backprime}S^{\delta\theta}_{\mathrm{der}}(\mathbf{R}) \cap S^{\delta\theta}_{\mathrm{der}}(\mathbf{R})^0)$ is open in $G(\mathbf{R})\theta$. This implies that the union of the $G(\mathbf{R})$-conjugates of $Z_G(\mathbf{R})^0({}^{\backprime}S^{\delta\theta}_{\mathrm{der}}(\mathbf{R}) \cap S^{\delta\theta}_{\mathrm{der}}(\mathbf{R})^0)\delta\theta$ is open in $G(\mathbf{R})\theta$. Since the intersection of ${}^{\backprime}S^{\delta\theta}_{\mathrm{der}}(\mathbf{R})$ with $S^{\delta\theta}_{\mathrm{der}}(\mathbf{R})^0$ is dense in $S^{\delta\theta}_{\mathrm{der}}(\mathbf{R})^0$ (see (52)), this open set is dense in the union of the $G(\mathbf{R})$-conjugates of $Z_G(\mathbf{R})^0S^{\delta\theta}_{\mathrm{der}}(\mathbf{R})^0\delta\theta$.

Corollary 1 *Suppose $f \in C_c^\infty(G(\mathbf{R})\theta)$ has support in the union of the $G(\mathbf{R})$-conjugates of $Z_G(\mathbf{R})^0S^{\delta\theta}_{\mathrm{der}}(\mathbf{R})^0\delta\theta$. Then $f_1 \in C_c^\infty(G_{\mathrm{der}}(\mathbf{R})^0\delta\theta)$, as defined in Lemma 5, has support in the closure of the image of Ψ and $\Theta_{\pi,\mathsf{U}}(f)$ is equal to the product of*

$$\frac{|\det(1 - \mathrm{Ad}(\delta\theta))_{|\mathfrak{s}/\mathfrak{s}^{\delta\theta}\otimes\mathbf{C}}|}{|\Omega(G_{\mathrm{der}}(\mathbf{R})^0, S^{\delta\theta}_{\mathrm{der}}(\mathbf{R})^0\delta\theta)|}$$

with the integral over $s \in S^{\delta\theta}_{\mathrm{der}}(\mathbf{R})^0$ of three terms. The first term is the integral over $z \in Z^{\delta\theta}_G(\mathbf{R})$ of $\chi_\pi(z)$ times the twisted orbital integral (§5.5

[KS99])

$$\mathcal{O}_{zs\delta\theta}(f) = \int_{G(\mathbf{R})/S^{\delta\theta}(\mathbf{R})} \omega(g) f(g^{-1} zs\delta\theta g)\, dg.$$

The second term is the character $\Theta_{\varpi_1,\mathsf{U}_1}(x\delta\theta)$ of Lemma 7. The third term is $|\det(1-\mathrm{Ad}(s\delta\theta))_{|\mathfrak{u}}|^2$.

Proof. Lemma 5 and Theorem 2.1.1 [Bou87] imply that

$$\Theta_{\pi,\mathsf{U}}(f) = \Theta_{\varpi_1,\mathsf{U}_1}(f_1) = \int_{G_{\mathrm{der}}(\mathbf{R})^0} f_1(x\delta\theta)\Theta_{\varpi_1,\mathsf{U}_1}(x\delta\theta)\, dx.$$

for some locally integrable $G_{\mathrm{der}}(\mathbf{R})^0$-invariant function $\Theta_{\varpi_1,\mathsf{U}_1}$. We may therefore apply Proposition 1 to deduce that $\Theta_{\pi,\mathsf{U}}(f)$ is equal to

$$1/|\Omega(G_{\mathrm{der}}(\mathbf{R})^0, S^{\delta\theta}_{\mathrm{der}}(\mathbf{R})^0\delta\theta)|$$

times

$$(55)\quad \int_{S^{\delta\theta}_{\mathrm{der}}(\mathbf{R})^0} \int_{G_{\mathrm{der}}(\mathbf{R})^0/S^{\delta\theta}_{\mathrm{der}}(\mathbf{R})} f_1(gs\delta\theta g^{-1})\, dg \Theta_{\varpi_1,\mathsf{U}_1}(s\delta\theta) |\det(1-\mathrm{Ad}(s\delta\theta))_{|\mathfrak{q}}|\, ds.$$

It is now clear that (47) delivers the second term in the assertion. To obtain the remaining terms, observe that

$$\begin{aligned}&\int_{G_{\mathrm{der}}(\mathbf{R})^0/S^{\delta\theta}_{\mathrm{der}}(\mathbf{R})} f_1(gs\delta\theta g^{-1})\, dg \\ &= \int_{G_{\mathrm{der}}(\mathbf{R})^0/S^{\delta\theta}_{\mathrm{der}}(\mathbf{R})} \frac{1}{|G_{\mathrm{der}}(\mathbf{R})^0 \cap Z_G(\mathbf{R})|} \sum_{r=1}^{k} \omega(\delta_r) \int_{Z_G(\mathbf{R})^0} f(\delta_r^{-1} gzs\delta\theta g^{-1}\delta_r)\, \chi_\pi(z)\, dz\, dg\end{aligned}$$

We may decompose the Lie algebra $\mathfrak{z}$ of $Z_G(\mathbf{R})$ as $\mathfrak{z} = \mathfrak{z}^{\delta\theta} \oplus (1-\mathrm{Ad}(\delta\theta))\mathfrak{z}$. Writing $z = z_1 z_2$ accordingly, and making a change of variable from z_2 to $z_2\delta\theta z_2^{-1}\delta\theta^{-1}$ we find that the integral over $Z_G(\mathbf{R})^0$ equals

$$|\det(1-\mathrm{Ad}(\delta\theta))_{|\mathfrak{z}/\mathfrak{z}^{\delta\theta}\otimes\mathbf{C}}| \int_{Z_G^{\delta\theta}(\mathbf{R})^0} \int_{Z_G(\mathbf{R})^0/Z_G^{\delta\theta}(\mathbf{R})^0} \omega(z_2) f(\delta_r^{-1} g z_2 z_1 s\delta\theta z_2^{-1} g^{-1}\delta_r)\, dz_2 \chi_\pi(z_1)\, dz_1.$$

The first and second integrals here may be taken over $Z_G^{\delta\theta}(\mathbf{R})$ and $Z_G(\mathbf{R})/Z_G^{\delta\theta}(\mathbf{R})$ respectively, because of the support hypothesis on f. Combining the integral over $Z_G(\mathbf{R})/Z_G^{\delta\theta}(\mathbf{R})$ with the integral over $G_{\mathrm{der}}(\mathbf{R})^0/S_{\mathrm{der}}^{\delta\theta}(\mathbf{R})$ recalling from §5.1 that $\cup_{r=1}^k \delta_r^{-1} Z_G(\mathbf{R}) G_{\mathrm{der}}(\mathbf{R})^0 = G(\mathbf{R})$ we see that

$$\begin{aligned}
&\int_{G_{\mathrm{der}}(\mathbf{R})^0/S_{\mathrm{der}}^{\delta\theta}(\mathbf{R})} f_1(gs\delta\theta g^{-1})\, dg \\
&= |\det(1-\mathrm{Ad}(\delta\theta))_{|\mathfrak{z}/\mathfrak{z}^{\delta\theta}\otimes\mathbf{C}}| \int_{Z_G^{\delta\theta}(\mathbf{R})} \int_{G(\mathbf{R})/Z_G^{\delta\theta}(\mathbf{R})S_{\mathrm{der}}^{\delta\theta}(\mathbf{R})} \omega(g) f(gzs\delta\theta g^{-1})\, dg\ \chi_\pi(z)\, dz \\
&= |\det(1-\mathrm{Ad}(\delta\theta))_{|\mathfrak{z}/\mathfrak{z}^{\delta\theta}\otimes\mathbf{C}}| \int_{Z_G^{\delta\theta}(\mathbf{R})} \mathcal{O}_{zs\delta\theta}(f)\, dz.
\end{aligned}$$

The remaining terms of the corollary result from (54) and the decomposition $\mathfrak{s} = \mathfrak{z} \oplus \mathfrak{s}_{\mathrm{der}}$.■

6 Spectral transfer for essentially square-integrable representations

We provide more detail to the description of spectral transfer. Fix endoscopic data $(H, \mathcal{H}, \mathsf{s}, \xi)$, a z-pair (H_1, ξ_{H_1}), and a quasi-character λ_{Z_1} of $Z_1(\mathbf{R})$ as in §3.2. Suppose that $\boldsymbol{\varphi_{H_1}}$ is an $\hat{H}_1$-conjugacy class of an admissible homomorphism $\varphi_{H_1} : W_{\mathbf{R}} \to {}^L H_1$ whose composition with $p : {}^L H \to {}^L Z_1$ (see (8)) equals homomorphism (9). In other words, $p \circ \varphi_{H_1}$ corresponds to λ_{Z_1} under the Local Langlands Correspondence. This assumption is equivalent to the assumption that

(56) $$\chi_{\pi_{H_1}}(z) = \lambda_{Z_1}(z),\ z \in Z_1(\mathbf{R})$$

for the central character of any representation $\pi_{H_1} \in \Pi_{\varphi_{H_1}}$ (cf. 10.1 [Bor79] and §4.1).

We wish to associate to the L-parameter $\boldsymbol{\varphi_{H_1}}$ an L-parameter $\boldsymbol{\varphi}$ of $G(\mathbf{R})$. In the special case that $H = H_1$ and $\xi : {}^L H \to {}^L G$ is inclusion, one may define an L-parameter for $G^*(\mathbf{R})$ by defining $\varphi^* = \xi \circ \varphi_{H_1}$. One may then set $\varphi = \varphi^*$ in the case that φ^* is relevant with respect to the inner form $G(\mathbf{R})$ (requirement 4 in §4).

The general case involving non-trivial z-extensions necessitates the use of the intermediate map $\xi_{H_1} : \mathcal{H} \to {}^L H_1$. We shall verify that $\varphi_{H_1}(W_{\mathbf{R}})$ is contained in $\xi_{H_1}(\mathcal{H})$. Once this is done, we may use the isomorphism $\mathcal{H} \cong \xi_{H_1}(\mathcal{H})$ of Lemma 2.2.A [KS99] to define $\varphi^* = \xi \circ \xi_{H_1}^{-1} \circ \varphi_{H_1}$. If φ^* is relevant, we define $\boldsymbol{\varphi}$ to be the $\hat{G}$-conjugacy class of φ^*. There is also the matter of showing that this process does not depend on the choice of representative in $\boldsymbol{\varphi_{H_1}}$, but let us first take care of proving that $\varphi_{H_1}(W_{\mathbf{R}}) \subset \xi_{H_1}(\mathcal{H})$.

Looking back to (8) and (9) we see that our hypothesis on φ_{H_1} is

$$p \circ \varphi_{H_1} = p \circ \xi_{H_1} \circ c.$$

This implies that $p(\varphi_{H_1}(w)\xi_{H_1}(c(w^{-1})))$ is trivial for every $w \in W_{\mathbf{R}}$. Since ξ_{H_1} is an L-homomorphism (§15 [Bor79]), the element $\varphi_{H_1}(w)\xi_{H_1}(c(w^{-1}))$ belongs to $\hat{H}_1$. Combining this observation with the fact that p is an extension of $\hat{H}_1 \to \hat{Z}_1$ (see (7)), yields $\varphi_{H_1}(w)\xi_{H_1}(c(w^{-1})) \in \hat{H}$. It now follows from the fact that ξ_{H_1} extends $\hat{H} \to \hat{H}_1$ (Lemma 2.2.A [KS99]) that

$$\varphi_{H_1}(w) \in \xi_{H_1}(c(w))\hat{H} \subset \xi_{H_1}(\mathcal{H})\hat{H} = \xi_{H_1}(\mathcal{H}). \tag{57}$$

This proves $\varphi_{H_1}(W_{\mathbf{R}}) \subset \xi_{H_1}(\mathcal{H})$.

Lemma 11 *The map from $\boldsymbol{\varphi_{H_1}}$ to $\boldsymbol{\varphi}^*$ described above is well-defined.*

Proof. We are to prove that the map does not depend on the choice of representative φ_{H_1} in its definition. This is shown in §2 [She10]. We shall provide a more detailed argument here. Suppose then that $\varphi'_{H_1} \in \boldsymbol{\varphi_{H_1}}$ also satisfies

$$p \circ \varphi'_{H_1} = p \circ \xi_{H_1} \circ c.$$

This means that there is some $h \in \hat{H}_1$ such that $\varphi'_{H_1} = \mathrm{Int}(h) \circ \varphi_{H_1}$ and

$$p(h)p(\varphi_{H_1}(w))p(h)^{-1} = p(\varphi'_{H_1}(w)) = p(\xi_{H_1}(c(w))) = p(\varphi_{H_1}(w)),\ w \in W_{\mathbf{R}}.$$

This equation implies that $p(h)$ belongs to the Γ-fixed elements $\hat{Z}_1^{\Gamma}$ in $\hat{Z}_1$. It follows from (1.8.1) [Kot84] that there exists $z \in Z_{\hat{H}_1}$ such that $p(zh) = 1$. Thus, zh belongs to $\hat{H}$ and

$$z\varphi'_{H_1}(w)z^{-1} = \begin{cases} \varphi'_{H_1}(w), & w \in \mathbf{C}^{\times} \\ z\sigma(z)^{-1}\,\varphi'_{H_1}(w), & \text{otherwise} \end{cases}. \tag{58}$$

Notice that $z\sigma(z)^{-1}$ belongs to $Z_{\hat{H}}$, as $p(z) = p(h^{-1}) \in \hat{Z}_1^{\Gamma}$ ((1.8.1) [Kot84]). The map $\sigma \mapsto z\sigma(z)^{-1}$ therefore determines a class in $H^1(\Gamma, Z_{\hat{H}})$. Using the long exact sequence of Tate cohomology (§6.2 [Wei94]), we obtain an exact sequence

$$\hat{Z}_1^{\Gamma}/(1+\sigma)\hat{Z}_1 \to H^1(\Gamma, Z_{\hat{H}}) \to H^1(\Gamma, Z_{\hat{H}_1}). \tag{59}$$

The class in $H^1(\Gamma, Z_{\hat{H}})$ given above lies in the kernel of the second map of (59). Moreover, Γ permutes a $\mathbf{Z}$-basis of the rational character group $X^*(Z_1)$ (p. 20 [KS99]) and $\hat{Z}_1 \cong X^*(Z_1) \otimes_{\mathbf{Z}} \mathbf{C}$. Therefore,

$$\hat{Z}_1^{\Gamma} \cong X^*(Z_1)^{\Gamma} \otimes_{\mathbf{Z}} \mathbf{C} = (1+\sigma) X^*(Z_1) \otimes_{\mathbf{Z}} \mathbf{C} \cong (1+\sigma)\hat{Z}_1,$$

which means that the second map of (59) is injective. As a result, the aforementioned class in $H^1(\Gamma, Z_{\hat{H}})$ is trivial, i.e. there exists $z' \in Z_{\hat{H}}$ such that $z'\sigma(z')^{-1} = z\sigma(z)^{-1}$. Consequently, $(z')^{-1}zh$ is an element in $\hat{H}$ and using (58)

$$((z')^{-1}zh)\varphi_{H_1}(w)((z')^{-1}zh)^{-1} = ((z')^{-1}z)\varphi'_{H_1}(w)((z')^{-1}z)^{-1} = \varphi'_{H_1}(w).$$

Finally, since ξ_{H_1} extends $\hat{H} \to \hat{H}_1$, we see that

$$\varphi'_{H_1}(W_{\mathbf{R}}) = (z'zh)\varphi_{H_1}(W_{\mathbf{R}})(z'zh)^{-1} \subset \xi_{H_1}(\mathcal{H})$$

and

$$\xi \circ \xi_{H_1}^{-1} \circ \varphi'_{H_1} = \mathrm{Int}(\xi(\xi_{H_1}^{-1}(z'zh))) \circ \varphi^* \in \boldsymbol{\varphi}^*. \blacksquare$$

We now make two assumptions that will remain in force for the remainder of this section.

Assumption 1 We first assume that for any $\varphi^* \in \boldsymbol{\varphi}^*$ the subgroup $\varphi^*(W_{\mathbf{R}}) \subset {}^LG$ is bounded and not contained in any proper Levi subgroup of ${}^LG$. This assumption has several consequences. To start, any $\varphi^* \in \boldsymbol{\varphi}^*$ is relevant (§3 [Bor79], 3.6 [Spr79]) so the L-parameter $\boldsymbol{\varphi}$ of $G(\mathbf{R})$ is defined. In addition, the L-packet Π_{φ} consists of unitary essentially square-integrable representations (§§4.1-4.2). This implies that $G(\mathbf{R})$ is *cuspidal* (see p. 135 [Kna86] and §5.3.1). That is, G contains a maximal torus S which is defined over $\mathbf{R}$ and elliptic (§4.1).

Assumption 2 We assume that $\Pi_{\varphi} = \omega \otimes (\Pi_{\varphi} \circ \theta)$.

While we have the connection between φ_{H_1} and φ^* freshly in mind, it is worth recording a further consequence of the first assumption.

Lemma 12 *Every representation in the packet $\Pi_{\varphi_{H_1}}$ is essentially square-integrable.*

Proof. As discussed in section 4.1, it suffices to show that $\varphi_{H_1}(W_{\mathbf{R}})$ is not contained in a proper Levi subgroup of $\hat{H}_1$. By way of contradiction, suppose that $\varphi_{H_1}(W_{\mathbf{R}})$ is contained in a proper Levi subgroup. Then, by §3 [Bor79] and §3.6 [Spr79], there exists a Γ-stable set of simple roots $\{\beta_1, \ldots, \beta_r\} \subset R(\hat{H}_1, \hat{T}_{H_1})$ such that $\varphi_{H_1}(W_{\mathbf{R}})$ lies in the centralizer in $\hat{H}_1$ of $\cap_{j=1}^{r} \ker \beta_j$. It follows that $\varphi_{H_1}(W_{\mathbf{R}})$ permutes the positive roots not generated by $\{\beta_1, \ldots, \beta_r\}$. Let $\beta \in X^*(\hat{T}_{H_1})$ be the sum of the positive roots not generated by $\{\beta_1, \ldots, \beta_r\}$. We may identify the set of positive roots in $R(\hat{H}_1, \hat{T}_{H_1})$ with the set of positive roots in $R(\hat{H}, \mathcal{T}_H)$ with respect to the Borel subgroup $\mathcal{B}_H$. The fact that ξ_{H_1} extends the embedding $\hat{H} \to \hat{H}_1$ implies that $\xi_{H_1}^{-1} \circ \varphi_{H_1}(W_{\mathbf{R}})$ still lies in the corresponding centralizer in $\hat{H}$. The map ξ allows us to identify $R(\hat{H}, \mathcal{T}_H)$ with a subsystem of $R_{\mathrm{res}}(\hat{G}^*, \mathcal{T}) = R(((\hat{G}^*)^{\hat{\theta}})^0, (\mathcal{T}^{\hat{\theta}})^0)$. Accordingly, the element $\beta \in X^*(\mathcal{T}_H)$ corresponds to an element $\beta_{\mathrm{res}} \in X^*((\mathcal{T}^{\hat{\theta}})^0)$ which is generated by positive roots in $R_{\mathrm{res}}(\hat{G}^*, \mathcal{T})$. Clearly, the element β_{res} is fixed by $\xi \circ \xi_{H_1}^{-1} \circ \varphi_{H_1}(W_{\mathbf{R}}) = \varphi^*(W_{\mathbf{R}})$. This implies that $\varphi^*(W_{\mathbf{R}})$ is contained in the proper parabolic subgroup of ${}^L G$ determined by the Γ-fixed element β_{res} (§8.4 [Spr98]), contradicting our assumption on φ^*.■

The main goal of this section is to make precise and prove an identity of the form

$$\int_{H_1(\mathbf{R})/Z_1(\mathbf{R})} f_{H_1}(h) \sum_{\pi_{H_1} \in \Pi_{\varphi_{H_1}}} \Theta_{\pi_{H_1}}(h)\, dh = \sum_{\pi \in \Pi_{\varphi}} \Delta(\boldsymbol{\varphi_{H_1}}, \pi)\ \Theta_{\pi, \mathsf{U}_\pi}(f),\ \ f \in C_c^\infty(G(\mathbf{R})\theta) \tag{60}$$

(*cf.* (16)). Here, the expressions $\Delta(\boldsymbol{\varphi_{H_1}}, \pi)$ are complex numbers. They are the *spectral transfer factors*. The terms $\Theta_{\pi, \mathsf{U}_\pi}$ are distributions arising from twisted characters. These distributions and spectral transfer factors shall be defined in the course of this section. As we shall see, the the distributions in (60) are determined by their behaviour on elliptic elements. We shall require the concept of θ-elliptic elements in $G(\mathbf{R})$. A strongly θ-regular element $\delta \in G(\mathbf{R})$ is said to be *θ-elliptic* if the identity component of $G^{\delta\theta}/Z_G^\theta$ is anisotropic over $\mathbf{R}$ (page 5 [KS99]). The torus $(G^{\delta\theta})^0$ lies in a maximal torus S which is defined and maximally compact over $\mathbf{R}$. Hence, $S(\mathbf{R})$ is elliptic (Proposition 6.61 [Kna96] and section 4.1). As S is contained in the centralizer of $G^{\delta\theta}$ in G and $G^{\delta\theta}$ contains G-regular elements (see pp. 227-228 [Art88]), the maximal torus S is uniquely determined by δ.

Lemma 13 *Suppose $\gamma \in H(\mathbf{R})$ is a strongly G-regular element which is a norm of $\delta \in G(\mathbf{R})$. Then γ is elliptic if and only if δ is θ-elliptic.*

Proof. By Lemma 3.3.C (2) [KS99], the element γ is strongly H-regular. Therefore the $T_H = H^\gamma$ is a maximal torus of H which is real and elliptic. Let us describe the following sequence of homomorphisms

$$(G^{\delta\theta})^0 \cong (G^{\delta^*\theta^*})^0 = ((T')^{\theta^*})^0 \to T'_{\theta^*} \cong T_H.$$

The isomorphism on the left is given by (14) and is defined over $\mathbf{R}$. The homomorphism in the middle is induced by the quotient map $T' \to T'/(1-\theta^*)T'$. Since θ^* preserves the pair (B^*, T^*) (see (4)) and the characteristic of $\mathbf{C}$ is zero, the automorphism θ^* is semisimple (§9 [Ste97]) and

$$\mathfrak{t}' \otimes \mathbf{C} = ((\mathfrak{t}')^{\theta^*} \otimes \mathbf{C}) \oplus ((1-\theta^*)\mathfrak{t}' \otimes \mathbf{C}).$$

This decomposition implies that the homomorphism $((T')^{\theta^*})^0 \to T'_{\theta^*}$ is dominant (Theorem 4.3.6 (i) [Spr98]) and has zero-dimensional kernel (Theorem 5.1.6 [Spr98]). Therefore the kernel is finite (Proposition 2.2.1 (i) [Spr98]), the homomorphism is surjective (Propositions 2.2.1 and 2.2.5 (ii) [Spr98]) and is defined over $\mathbf{R}$ (Theorem 12.2.1 [Spr98]).

The final isomorphism $T'_{\theta^*} \cong T_H$ is that of (12). It is also defined over $\mathbf{R}$. In summary, we have a surjection $(G^{\delta\theta})^0(\mathbf{R}) \to T_H(\mathbf{R})$ with finite kernel. The same sequence of homomorphisms induces

$$Z_G^\theta \cong Z_{G^*}^{\theta^*} \to Z_{G^*,\theta^*} \to Z_H$$

(see (5.4.1) [KS99]), so that we obtain a surjection from the connected component of $(G^{\delta\theta}/Z_G)(\mathbf{R})$ to $(T_H/Z_H)(\mathbf{R})$ with finite kernel. In conclusion, the former quotient is compact if and only if the latter quotient is compact. The lemma follows.■

We proceed by first considering (60) in the case that there exists a strongly θ-regular and θ-elliptic element $G(\mathbf{R})$ which has a norm in $H(\mathbf{R})$ (section 3.3). We shall treat the case that no such elements exist in section 6.5.

Lemma 14 *Suppose there is a Borel subgroup B' of G^* containing a maximal torus T', which is defined over $\mathbf{R}$ and elliptic. Suppose further that θ^* preserves the pair (B', T'). Then there exists $\pi \in \Pi_{\varphi^*} = \omega \otimes (\Pi_{\varphi^*} \circ \theta^*)$ such that $\pi \cong \omega \otimes \pi^\theta$.*

Proof. Recall that the set Π_{φ^*} is of the form (21) (with $S = T'$). We may therefore choose $\pi \in \Pi_{\varphi^*}$ such that the differential of its corresponding quasicharacter Λ' on $T'(\mathbf{R})$ lies in the Weyl chamber determined by B'. As $\Pi_{\varphi^*} = \omega \otimes (\Pi_{\varphi^*} \circ \theta^*)$, there exists $w \in \Omega(G^*, T')$ such that $w^{-1}\Lambda' = \omega_{|T'(\mathbf{R})}(\Lambda' \circ \theta^*_{|T'(\mathbf{R})})$. Since $\omega_{|T'(\mathbf{R})}$ is trivial on $T'_{\mathrm{der}}(\mathbf{R})$ (see (29)) and θ^* preserves B', the differential of $w^{-1}\Lambda'$ also belongs to the chamber determined by B'. As Λ' is regular (section 4.1) and the Weyl group acts simply transitively on the chambers, the element w is trivial. Consequently, the representation π, corresponding to $\Lambda' = w^{-1}\Lambda'$, is equivalent to the representation $\omega \otimes \pi^\theta$, which corresponds to $\omega_{|T'(\mathbf{R})}(\Lambda' \circ \theta^*_{|T'(\mathbf{R})})$.■

Corollary 2 *Suppose that there exists a strongly θ-regular and θ-elliptic element $\delta \in G(\mathbf{R})$ which has a norm $H(\mathbf{R})$. Then there exists $\pi \in \Pi_\varphi = \omega \otimes (\Pi_\varphi \circ \theta)$ such that π is equivalent to $\omega \otimes \pi^\theta$.*

Proof. By the hypothesis of the corollary and Lemma 3.3.B [KS99], there exists a Borel subgroup B' of G^* containing a maximal torus T', defined over $\mathbf{R}$, and both are preserved by θ^*. Isomorphism (14) extends to an $\mathbf{R}$-isomorphism between T' and the elliptic torus S (see (149)) so that T' is elliptic. Therefore the hypotheses of Lemma 14 hold. The composition of the quasicharacter Λ' of $T'(\mathbf{R})$ of Lemma 14 with $\mathrm{Int}(g_{T'})\psi$ defines a quasi-character Λ on $S(\mathbf{R})$ such that

$$
\begin{aligned}
\Lambda \circ \mathrm{Int}(\delta)\theta &= \Lambda' \circ \mathrm{Int}(g_{T'})\psi \circ \mathrm{Int}(\delta)\theta \\
&= \Lambda' \circ \mathrm{Int}\delta^*\theta^* \circ \mathrm{Int}(g_{T'})\psi \\
&= \Lambda' \circ \theta^* \circ \mathrm{Int}(g_{T'})\psi \\
&= (\omega_{|T'(\mathbf{R})}\Lambda') \circ \mathrm{Int}(g_{T'})\psi \\
&= \omega_{|S(\mathbf{R})}\,\Lambda
\end{aligned}
$$

(see (4) and (13)) Therefore the essentially square-integrable representation $\pi_\Lambda \in \Pi_\varphi$ (see (21)) satisfies

$$\pi_\Lambda^\theta \cong \pi_\Lambda^{\delta\theta} \cong \pi_{\delta\theta\cdot\Lambda} = \pi_{\omega_{|S(\mathbf{R})}\Lambda} = \omega \otimes \pi_\Lambda. \blacksquare$$

From this point until section 6.5 we will assume that

Assumption 3 $\delta \in G(\mathbf{R})$ is a strongly θ-regular θ-elliptic element with norm $\gamma \in H(\mathbf{R})$.

The results of sections 3 and 4 apply with S taken to be the unique elliptic maximal torus containing $G^{\delta\theta}$. In order to employ the results of section 5, we must specify the objects fixed in 5.1 so as to be compatible with δ.

We may choose a maximal compact subgroup of $G(\mathbf{R}) \rtimes \langle\theta\rangle$ containing the compact subgroup $S_{\mathrm{der}}(\mathbf{R})\langle\delta\theta\rangle$. This maximal compact subgroup is the fixed point set of some Cartan involution on the real algebraic group $G \rtimes \langle\theta\rangle$ (For the theory of Cartan involutions we refer to §§1.1-1.6 [BHC62]). The Cartan involution is algebraic, and so preserves the characteristic subgroup $G_{\mathrm{der}}(\mathbf{R})$ and its identity component $G_{\mathrm{der}}(\mathbf{R})^0$. Let K be the maximal compact subgroup of $G_{\mathrm{der}}(\mathbf{R})^0$ equal to the fixed points in $G_{\mathrm{der}}(\mathbf{R})^0$ of the Cartan involution. Then the subgroup

$$(\delta\theta)K(\delta\theta)^{-1} = \mathrm{Int}(\delta)\theta(K) \subset G_{\mathrm{der}}(\mathbf{R})^0$$

also lies in the fixed point set of the chosen Cartan involution and this means that $\mathrm{Int}(\delta)\theta(K) = K$.

We fix the positive system in $\mathrm{R}(\mathfrak{g} \otimes \mathbf{C}, \mathfrak{s} \otimes \mathbf{C})$ determined by the Borel subgroup of G equal to the image of the Borel subgroup $B' \subset G^*$ of section 3.3 under $(\mathrm{Int}(g_{T'})\psi)^{-1}$ (*cf.* 14). This also fixes positive system on the set of roots $\mathrm{R}(\mathfrak{k} \otimes \mathbf{C}, \mathfrak{s}_{\mathrm{der}} \otimes \mathbf{C})$.

With these objects in place, we may assume that δ is one of the representatives listed in Lemma 3. In addition, by choosing $\pi \in \Pi_\varphi$ in section 5.1 to be the representation given in the proof of Corollary 2, we may assume that $\delta = \delta_m$ as in 4. Indeed, the arguments in the proof of Corollary 2 imply that the Harish-Chandra parameter of ϖ_1 is fixed by $\delta\theta$.

We may use these assumptions to construct a bridge between the maximal torus $T_H = H^\gamma$ of H and the maximal torus S arising from Π_φ (§4.1). This bridge comes in three pieces. The first piece is isomorphism (14) which passes to an isomorphism

$$S^{\delta\theta}(\mathbf{R})^0 \stackrel{\mathrm{Int}(g_{T'})\psi}{\longrightarrow} (T')^{\theta^*}(\mathbf{R})^0.$$

The second piece is the homomorphism from $(T')^{\theta^*}(\mathbf{R})^0$ to $T'_{\theta^*}(\mathbf{R})^0$ defined by

$$t \mapsto t\,(1-\theta^*)T'(\mathbf{R}),\ t \in (T')^{\theta^*}(\mathbf{R})^0. \tag{61}$$

This homomorphism is surjective and has finite kernel (see the proof of Lemma 13 or Lemma 4.11 [Ren03]). The third piece is the restriction of the admissible embedding (12) to $T'_{\theta^*}(\mathbf{R})^0$ which yields an isomorphism

$$T'_{\theta^*}(\mathbf{R})^0 \cong T_H(\mathbf{R})^0.$$

We denote the composition of these three maps by

$$\eta : S^{\delta\theta}(\mathbf{R})^0 \to T_H(\mathbf{R})^0. \tag{62}$$

The map η is not canonical, depending as it does on the choices for B_H, B', $g_{T'}$, etc. However, our results are independent of these choices. Although η need not be an isomorphism, it is a *local* isomorphism. That is to say, there is an open subset of the identity $\mathcal{V} \subset S^{\delta\theta}(\mathbf{R})^0$ such that η maps $\mathcal{V}$ homeomorphically onto $\eta(\mathcal{V})$.

Let T_{H_1} denote the pre-image of T_H under the projection p_H in (6). On the set $\mathcal{V}$ we may extend η to a map $\eta_1 : \mathcal{V} \to T_{H_1}(\mathbf{R})^0$ by noting that (6) induces a split exact sequence of Lie algebras

$$0 \to \mathfrak{z}_1 \to \mathfrak{t}_{H_1} \to \mathfrak{t}_H \to 0 \tag{63}$$

and composing η with resulting local isomorphism between $T_H(\mathbf{R})^0$ and $T_{H_1}(\mathbf{R})^0$.

According to Proposition 4.12 [Ren03], $\eta_1(x)\gamma_1$ is a norm of $x\delta$ for every $x \in S^{\delta\theta}(\mathbf{R})^0$ such that $x\delta$ is strongly θ-regular. As the latter elements form a dense subset of $S^{\delta\theta}(\mathbf{R})^0$, it follows that the set of norms of elements in $S^{\delta\theta}(\mathbf{R})^0\delta$ forms a dense subset of $T_H(\mathbf{R})^0$.

Corollary 3 *The torus $T_H(\mathbf{R})$ is elliptic.*

Proof. This is immediate from the assumption that δ is θ-elliptic and Lemma 13.■

6.1 A parameterization of stable data

In the matching of orbital integrals ((5.5.1) [KS99]) geometric transfer factors of the form $\Delta(\delta_H, \delta)$ are present. In our setup, this becomes $\Delta(\gamma_1, \delta')$, where $\delta' \in G(\mathbf{R})$ runs over a set of representatives for the θ-conjugacy classes under $G(\mathbf{R})$ of elements whose norm is γ_1. Every δ' is of the form $x^{-1}\delta\theta(x)$ for some $x \in G$. Our first goal is to show that the set of representatives δ' is to some extent parameterized by the set

$$(\Omega(G,S)/\Omega_{\mathbf{R}}(G,S))^{\delta\theta}$$

which we define to be the set of cosets in $\Omega(G,S)/\Omega_{\mathbf{R}}(G,S)$ whose representatives $w \in \Omega(G,S)$ satisfy

$$w^{-1}\,\delta\theta\,w(\delta\theta)^{-1} \in \Omega_{\mathbf{R}}(G,S). \tag{64}$$

This property does not depend on the representative w. Indeed, if $w_1 \in \Omega_{\mathbf{R}}(G,S)$ and (64) holds then $w_1^{-1}, w^{-1}\delta\theta w(\delta\theta)^{-1}, (\delta\theta)w_1(\delta\theta)^{-1} \in \Omega_{\mathbf{R}}(G,S)$ and so

$$w_1^{-1}w^{-1}\delta\theta w w_1(\delta\theta)^{-1} = w_1^{-1}\left(w^{-1}\delta\theta w(\delta\theta)^{-1}\right)(\delta\theta)w_1(\delta\theta)^{-1} \in \Omega_{\mathbf{R}}(G,S).$$

A more refined version of the set $(\Omega(G,S)/\Omega_{\mathbf{R}}(G,S))^{\delta\theta}$ is obtained by taking the elements x in the normalizer $N_G(S)$ which satisfy

$$x^{-1}\delta\theta x(\delta\theta)^{-1} \in G(\mathbf{R}). \tag{65}$$

As before, this property passes to cosets in $N_G(S)/N_{G(\mathbf{R})}(S)$. We define $(N_G(S)/N_{G(\mathbf{R})}(S))^{\delta\theta}$ to be the collection of cosets in $N_G(S)/N_{G(\mathbf{R})}(S)$ whose representatives satisfy (65). One may then consider the double cosets

$$S^{\delta\theta}\backslash(N_G(S)/N_{G(\mathbf{R})}(S))^{\delta\theta}. \tag{66}$$

Lemma 15 *Suppose $x \in N_G(S)$ satisfies (65). Then the map defined by*

$$x \mapsto x^{-1}\delta\theta(x)$$

passes to a bijection from (66) to the collection of θ-conjugacy classes under $G(\mathbf{R})$ of elements in $G(\mathbf{R})$ whose norm is γ_1.

Proof. Suppose $x \in N_G(S)$ satisfies (65). Since δ belongs to $G(\mathbf{R})$, property (65) is equivalent to $x^{-1}\delta\theta(x) \in G(\mathbf{R})$. As γ_1 is a norm of δ it is by definition also a norm of $x^{-1}\delta\theta(x)$ (see section 3.3). It is simple to verify that any element in the double coset $S^{\delta\theta}\backslash x/N_{G(\mathbf{R})}(S)$ maps to an element which is θ-conjugate to $x^{-1}\delta\theta(x)$ under $G(\mathbf{R})$. Thus, we have a map from (66) to the desired collection of θ-conjugacy classes.

To show that this map is surjective, suppose now that $x \in G$ is any element satisfying $x^{-1}\delta\theta(x) \in G(\mathbf{R})$, that is, an element in $G(\mathbf{R})$ whose norm is γ_1 (section 3.3). The automorphism $\mathrm{Int}(x^{-1}\delta\theta(x))\theta$ is defined over $\mathbf{R}$. Therefore, the group $G^{x^{-1}\delta\theta(x)\theta}$ is defined over $\mathbf{R}$. Since γ_1 is a norm of $x^{-1}\delta\theta(x)$, there is a surjection $G^{x^{-1}\delta\theta(x)\theta}(\mathbf{R}) \to T_H(\mathbf{R})$ analogous to (62). The quotient $G^{x^{-1}\delta\theta(x)\theta}(\mathbf{R})/Z_G^{\theta}(\mathbf{R}) = x^{-1}(G^{\delta\theta}(\mathbf{R})/Z_G^{\theta}(\mathbf{R}))x$ is compact, for δ is θ-elliptic. Using Corollary 4.35 [Kna96] and Corollary 5.31 [Spr79], one may then show that there exists $g \in G(\mathbf{R})$ such that $g^{-1}G^{x^{-1}\delta\theta(x)\theta}(\mathbf{R})g$ lies in the elliptic torus $S(\mathbf{R})$. Hence,

$$S \supset g^{-1}G^{x^{-1}\delta\theta(x)\theta}g = (xg)^{-1}G^{\delta\theta}\,xg = (xg)^{-1}S^{\delta\theta}\,xg.$$

The group $S^{\delta\theta}$ contains strongly G-regular elements (pp. 227-228 [Art88]). The previous containment therefore implies that xg normalizes S. It is now clear that $xg \in N_G(S)$ maps to the same θ-conjugacy class as $x^{-1}\delta\theta(x)$ under $G(\mathbf{R})$, and surjectivity is proven.

To prove injectivity, suppose that $x_1, x_2 \in G$ are representatives for double cosets in (66) such that $x_1^{-1}\delta\theta(x_1)$ and $x_2^{-1}\delta\theta(x_2)$ belong to the same θ-conjugacy class under $G(\mathbf{R})$. Then there exists $g \in G(\mathbf{R})$ such that

$$x_1^{-1}\delta\theta(x_1) = (x_2 g)^{-1}\delta\theta(x_2 g)$$

and it follows that

$$x_2 g x_1^{-1} \in G^{\delta\theta} = S^{\delta\theta}.$$

In other words, x_1 and x_2 represent the same double coset in (66).■

Lemma 15 parametrizes the elements over which one sums in (16) with the double coset space $S^{\delta\theta}\backslash(N_G(S)/N_{G(\mathbf{R})}(S))^{\delta\theta}$. It is therefore pertinent to geometric transfer. As we shall soon see, the pertinent parameter space for spectral transfer is $(\Omega(G,S)/\Omega_{\mathbf{R}}(G,S))^{\delta\theta}$. Let us clarify the discrepancy between these two parameter spaces.

Proposition 2 *There is a canonical surjection*

$$S^{\delta\theta}\backslash(N_G(S)/N_{G(\mathbf{R})}(S))^{\delta\theta} \to (\Omega(G,S)/\Omega_{\mathbf{R}}(G,S))^{\delta\theta}$$

whose fibres are orbits of the kernel of the homomorphism

$$S/S(\mathbf{R})S^{\delta\theta} \overset{\delta\theta-1}{\longrightarrow} S/S(\mathbf{R})$$

induced by $\mathrm{Int}(\delta)\theta - 1$. *Moreover, one may choose representatives z_1 for elements of this kernel such that $(\delta\theta - 1)(z_1)$ are involutions and $z_1 \in A_G$, where A_G is the split component of the centre of G.*

Proof. It is a simple exercise to show that there is a canonical surjection of $S^{\delta\theta}\backslash(N_G(S)/N_{G(\mathbf{R})}(S))^{\delta\theta}$ onto $S\backslash S(N_G(S)/N_{G(\mathbf{R})}(S))^{\delta\theta}$ and that there is a canonical bijection from $S\backslash S(N_G(S)/N_{G(\mathbf{R})}(S))^{\delta\theta}$ onto $(\Omega(G,S)/\Omega_{\mathbf{R}}(G,S))^{\delta\theta}$. The composition of these two maps is the canonical surjection of the proposition.

To determine the fibres of this map, observe that two representatives of cosets in $S^{\delta\theta}\backslash(N_G(S)/N_{G(\mathbf{R})}(S))^{\delta\theta}$ map to the same element in $(\Omega(G,S)/\Omega_{\mathbf{R}}(G,S))^{\delta\theta}$

if and only if they differ by a left-multiple of an element in S. We must therefore determine the elements $s \in S$ which satisfy

$$(sx)^{-1}\delta\theta sx(\delta\theta)^{-1} \in G(\mathbf{R}) \tag{67}$$

for a given representative $x \in N_G(S)$ satisfying (65). An element $s \in S$ satisfies (67) if and only if

$$\begin{aligned} & x^{-1}s^{-1}\delta\theta s(\delta\theta)^{-1}x(x^{-1}\delta\theta x(\delta\theta)^{-1}) \in G(\mathbf{R}) \\ \Leftrightarrow\ & x^{-1}s^{-1}\delta\theta s(\delta\theta)^{-1}x \in G(\mathbf{R}) \\ \Leftrightarrow\ & s^{-1}\delta\theta s(\delta\theta)^{-1} \in S(\mathbf{R}) \\ \Leftrightarrow\ & (\delta\theta - 1)(s) \in S(\mathbf{R}) \end{aligned}$$

as conjugation on the elliptic torus S by elements in $N_G(S)$ is defined over $\mathbf{R}$ (Lemma 6.4.1 [Lab08]). The final membership is equivalent to the coset $sS(\mathbf{R})S^{\delta\theta}$ belonging to the kernel of the map

$$S/S(\mathbf{R})S^{\delta\theta} \xrightarrow{\delta\theta-1} S/S(\mathbf{R}).$$

This proves the first assertion concerning the fibres of our canonical surjection.

To prove the second assertion about the fibres, we introduce some Galois cohomology. Given $s \in S$ with $(\delta\theta - 1)(s) \in S(\mathbf{R})$, one obtains a cocycle $a_s \in Z^1(\Gamma, S^{\delta\theta})$ by setting $a_s(\sigma) = s^{-1}\sigma(s)$. Indeed, $\delta\theta$ is defined over $\mathbf{R}$ so that

$$\begin{aligned} \delta\theta(s^{-1}\sigma(s)) = s^{-1}\sigma(s) \ &\Leftrightarrow\ (\delta\theta - 1)(s^{-1}\sigma(s)) = 1 \\ &\Leftrightarrow\ (\delta\theta - 1)(s) = \sigma((\delta\theta - 1)(s)) \\ &\Leftrightarrow\ (\delta\theta - 1)(s) \in S(\mathbf{R}) \end{aligned}$$

It is easily verified that the map $s \mapsto a_s$ is a homomorphism with kernel $S(\mathbf{R})$. This homomorphism passes to an isomorphism from the kernel of $S/S(\mathbf{R})S^{\delta\theta} \xrightarrow{\delta\theta-1} S/S(\mathbf{R})$ to the image of the connecting homomorphism

$$D_0 : ((\delta\theta - 1)S)(\mathbf{R}) \to H^1(\Gamma, S^{\delta\theta}).$$

In consequence, the fibres of our canonical surjection have an alternative description as the image of D_0. The map D_0 is the connecting homomorphism in the long exact sequence of cohomology

$$\cdots \to S(\mathbf{R}) \xrightarrow{\delta\theta-1} ((\delta\theta - 1)S)(\mathbf{R}) \xrightarrow{D_0} H^1(\Gamma, S^{\delta\theta}) \to H^1(\Gamma, S) \to \cdots \tag{68}$$

induced by the exact sequence of Γ-modules

$$1 \to S^{\delta\theta} \to S \xrightarrow{\delta\theta - 1} (\delta\theta - 1)S \to 1.$$

The identity component $((\delta\theta - 1)S)(\mathbf{R})^0$ in the real manifold topology is contained in the image of

$$S(\mathbf{R}) \xrightarrow{\delta\theta - 1} ((\delta\theta - 1)S)(\mathbf{R}).$$

Indeed, by Corollary 5.3.3 (i) [Spr98]

$$\dim(\delta\theta - 1)S = \dim S - \dim S^{\delta\theta}$$

and each of the groups in this equation is defined over $\mathbf{R}$. Therefore,

$$\begin{aligned} \dim_{\mathbf{R}}((\delta\theta - 1)S)(\mathbf{R}) &= \dim(\delta\theta - 1)S \\ &= \dim S - \dim S^{\delta\theta} \\ &= \dim_{\mathbf{R}} S(\mathbf{R}) - \dim_{\mathbf{R}} S^{\delta\theta}(\mathbf{R}) \\ &= \dim_{\mathbf{R}}(\delta\theta - 1)(S(\mathbf{R})). \end{aligned}$$

We now know that $(\delta\theta - 1)(S(\mathbf{R}))$ is open (and closed) in the Lie group $((\delta\theta-1)S)(\mathbf{R})$, and of necessity contains $((\delta\theta-1)S)(\mathbf{R})^0$. We conclude from the exactness of (68) that $((\delta\theta - 1)S)(\mathbf{R})^0$ lies in the kernel of D_0.

We may decompose $((\delta\theta - 1)S)(\mathbf{R})$ in terms of $((\delta\theta - 1)S)(\mathbf{R})^0$ in the following manner. As S is an elliptic torus, its split component is equal to A_G. Since $\delta\theta - 1$ is defined over $\mathbf{R}$, it maps the split and anisotropic components of S to those of the torus $(\delta\theta - 1)S$. In particular, the split component of $(\delta\theta - 1)S$ is $(\delta\theta - 1)A_G$. The group of real points of this split component decomposes as a product of real multiplicative groups and so decomposes as direct product

$$((\delta\theta - 1)A_G)(\mathbf{R}) = ((\delta\theta - 1)A_G)(\mathbf{R})^0 F, \tag{69}$$

where F is an elementary 2-group. An application of Theorem 14.4 [BT65] now yields

$$\begin{aligned} ((\delta\theta - 1)S)(\mathbf{R}) &= ((\delta\theta - 1)S)(\mathbf{R})^0 \ ((\delta\theta - 1)A_G)(\mathbf{R}) \\ &= ((\delta\theta - 1)S)(\mathbf{R})^0 \ F \end{aligned}$$

As a result

$$D_0(((\delta\theta - 1)S)(\mathbf{R})) = D_0(((\delta\theta - 1)A_G)(\mathbf{R})) = D_0(F).$$

The second assertion concerning the fibres of our canonical surjection now follows from the isomorphism between the image of D_0 and the kernel of $S/S(\mathbf{R})S^{\delta\theta} \overset{\delta\theta-1}{\longrightarrow} S/S(\mathbf{R})$ outlined above.■

Lemma 15 parametrizes some data pertinent geometric transfer with the set $S^{\delta\theta}\backslash(N_G(S)/N_{G(\mathbf{R})}(S))^{\delta\theta}$ and Proposition 2 describes its relation to $(\Omega(G,S)/\Omega_{\mathbf{R}}(G,S))^{\delta\theta}$. The following lemma shows that the set $(\Omega(G,S)/\Omega_{\mathbf{R}}(G,S))^{\delta\theta}$ parametrizes data pertinent to spectral transfer. Before stating it, let us review the form of the L-packet Π_φ. Setting $\Lambda = \Lambda(\mu_0, \lambda)$ as in (21) we have $\Pi_\varphi = \{\pi_{w^{-1}\Lambda} : w \in \Omega(G,S)/\Omega_{\mathbf{R}}(G,S)\}$. We may assume $\pi = \pi_\Lambda$ so that the differential of Λ is positive and regular. Recalling the groundwork of §5.3.1, it becomes apparent that the restriction of the differential of Λ to $\mathfrak{s}_{\mathrm{der}} = \mathfrak{g}_{\mathrm{der}} \cap \mathfrak{s}$ is $\Lambda_1 \in \mathfrak{s}_{\mathrm{der}}^*$, and the restriction of Λ to $Z_G(\mathbf{R})$ is χ_π. We may therefore write $\varpi_1 = \varpi_{\Lambda_1}$ and (27) is subsumed by

$$(70) \quad \pi_{w^{-1}\Lambda} \cong \mathrm{ind}_{Z_G(\mathbf{R})G_{\mathrm{der}}(\mathbf{R})^0}^{G(\mathbf{R})}(\chi_\pi \otimes \varpi_{w^{-1}\Lambda_1}), \ w \in \Omega(G,S)/\Omega_{\mathbf{R}}(G,S).$$

Lemma 16 *Suppose $w \in \Omega(G,S)$ is a representative of a coset in $\Omega(G,S)/\Omega_{\mathbf{R}}(G,S)$. Then*

$$\pi_{w^{-1}\Lambda} \cong \omega \otimes \pi_{w^{-1}\Lambda}^\theta$$

if and only if w satisfies (64). In particular, the subset of representations $\pi' \in \Pi_\varphi$ satisfying $\pi' \cong \omega \otimes (\pi')^\theta$ is

$$\{\pi_{w^{-1}\Lambda} : \ w \in (\Omega(G,S)/\Omega_{\mathbf{R}}(G,S))^{\delta\theta}\}.$$

Proof. It is a simple exercise to verify that if U' is an intertwining operator satisfying

$$\mathsf{U}' \circ (\omega^{-1} \otimes \pi') = (\pi')^\theta \circ \mathsf{U}',$$

for some $\pi' \in \Pi_\varphi$, then $\pi'(\delta)\mathsf{U}'$ is an intertwining operator satisfying

$$\pi'(\delta)\mathsf{U}' \circ (\omega^{-1} \otimes \pi') = (\pi')^{\delta\theta} \circ \pi'(\delta)\mathsf{U}'.$$

Similarly if U' intertwines $\omega^{-1} \otimes \pi'$ with $(\pi')^{\delta\theta}$ then $\pi'(\delta)^{-1}\mathsf{U}'$ intertwines $\omega^{-1} \otimes \pi'$ with $(\pi')^\theta$. Proving the lemma is therefore equivalent to proving that $\omega^{-1} \otimes \pi_{w^{-1}\Lambda} \cong (\pi_{w^{-1}\Lambda})^{\delta\theta}$ if and only if w satisfies (64).

Using (70) we compute that

$$(\pi_{w^{-1}\Lambda})^{\delta\theta} \cong \mathrm{ind}_{Z_G(\mathbf{R})G_{\mathrm{der}}(\mathbf{R})^0}^{G(\mathbf{R})}(\chi_\pi^\theta \otimes \varpi_{w^{-1}\Lambda_1}^{\delta\theta}). \tag{71}$$

The representation $\varpi_{w^{-1}\Lambda_1}^{\delta\theta}$ is determined up to equivalence by its character values on $\mathfrak{s}_{\mathrm{der}}$ (§14 [HC65a], Theorem 12.6 [Kna86]). By Theorem 12.7 (a) [Kna86] and $\mathrm{Int}(\delta)\theta(\Lambda_1) = \Lambda_1$ (Lemma 6), these character values are determined by the homomorphism

$$\exp(iw^{-1}\Lambda_1(\mathrm{Int}(\delta)\theta(\cdot))) = \exp(i(\mathrm{Int}(\delta)\theta)^{-1}(w^{-1})\Lambda_1(\cdot))$$

from $\mathfrak{s}_{\mathrm{der}}$ to $\mathbf{C}^\times$. This equation implies that $\varpi_{w^{-1}\Lambda_1}^{\delta\theta} \cong \varpi_{(\mathrm{Int}(\delta)\theta)^{-1}(w^{-1})\Lambda_1}$. By making this substitution in the right-hand side of (71) and the leftmost equivalence of (33) we find that

$$(\pi_{w^{-1}\Lambda})^{\delta\theta} \cong \omega^{-1} \otimes \pi_{(\mathrm{Int}(\delta)\theta)^{-1}(w^{-1})\Lambda}.$$

The representation $\omega^{-1} \otimes \pi_{(\mathrm{Int}(\delta)\theta)^{-1}(w^{-1})\Lambda}$ is equivalent to $\omega^{-1} \otimes \pi_{w^{-1}\Lambda}$ if and only if

$$(\mathrm{Int}(\delta)\theta)^{-1}(w)\,\Omega_{\mathbf{R}}(G,S) = w\,\Omega_{\mathbf{R}}(G,S)$$

([HC66]). As the automorphism $\mathrm{Int}(\delta)\theta$ is defined over $\mathbf{R}$. Applying $\mathrm{Int}(\delta)\theta$ to this equation yields

$$w\,\Omega_{\mathbf{R}}(G,S) = (\mathrm{Int}(\delta)\theta)(w)\,\Omega_{\mathbf{R}}(G,S),$$

and this equation holds if and only if w satisfies (64).■

6.2 Geometric transfer factors

Recall from §3.4 that we are assuming the existence of functions which satisfy a matching of orbital integrals (16). The matching identity contains geometric transfer factors. Under the assumptions of section 6, the geometric transfer factors that interest us, are the form $\Delta(\eta_1(x)\gamma_1, x\delta)$, where $x \in \mathcal{V} \subset S^{\delta\theta}(\mathbf{R})^0$ and $\eta_1(x) \in H_1(\mathbf{R})$ is an element whose image under the surjection $H_1(\mathbf{R}) \to H(\mathbf{R})$ (see (6)) is $\eta(x) \in T_H(\mathbf{R})$. We wish to highlight certain properties of these particular transfer factors by referring to their definition in §§4-5 [KS99].

We set forth by fixing $\gamma_1^0 \in H_1(\mathbf{R})$ and strongly θ-regular $\delta^0 \in G(\mathbf{R})$ such that γ_1^0 is a norm of δ^0. One may choose $\Delta(\gamma_1^0, \delta^0)$ arbitrarily in $\mathbf{C}^\times$ and then set (see (5.1.1) [KS99])

$$\Delta(\bar{\gamma}_1, \bar{\delta}) = \Delta(\bar{\gamma}_1, \bar{\delta}; \gamma_1^0, \delta^0)\ \Delta(\gamma_1^0, \delta^0) \tag{72}$$

for any strongly θ-regular $\bar{\delta} \in G(\mathbf{R})$ with norm $\bar{\gamma}_1 \in H_1(\mathbf{R})$. The term $\Delta(\bar{\gamma}_1, \bar{\delta}; \gamma_1^0, \delta^0)$ is the product of four scalars $\Delta_I(\bar{\gamma}_1, \bar{\delta}; \gamma_1^0, \delta^0), \ldots, \Delta_{IV}(\bar{\gamma}_1, \bar{\delta}; \gamma_1^0, \delta^0)$. Other than $\Delta_{III}(\bar{\gamma}_1, \bar{\delta}; \gamma_1^0, \delta^0)$, these scalars are quotients of the form

$$\Delta_j(\bar{\gamma}_1, \bar{\delta}; \gamma_1^0, \delta^0) = \Delta_j(\bar{\gamma}_1, \bar{\delta})/\Delta_j(\gamma_1^0, \delta^0),\ j = I,\ II,\ IV.$$

We choose

$$\Delta(\gamma_1^0, \delta^0) = \Delta_I(\gamma_1^0, \delta^0)\ \Delta_{II}(\gamma_1^0, \delta^0)\ \Delta_{IV}(\gamma_1^0, \delta^0)$$

so that (72) becomes

$$\Delta(\bar{\gamma}_1, \bar{\delta}) = \Delta_I(\bar{\gamma}_1, \bar{\delta})\ \Delta_{II}(\bar{\gamma}_1, \bar{\delta})\ \Delta_{III}(\bar{\gamma}_1, \bar{\delta}; \gamma_1^0, \delta^0)\ \Delta_{IV}(\bar{\gamma}_1, \bar{\delta}).$$

In the case that $\bar{\gamma}_1 = \eta_1(x)\gamma_1$ and $\bar{\delta} = x\delta$ the transfer factor $\Delta(\eta_1(x)\gamma_1, x\delta)$ is equal to

$$\Delta_I(\eta_1(x)\gamma_1, x\delta)\, \Delta_{II}(\eta_1(x)\gamma_1, x\delta)\, \Delta_{III}(\eta_1(x)\gamma_1, x\delta; \gamma_1^0, \delta^0)\, \Delta_{IV}(\eta_1(x)\gamma_1, x\delta) \tag{73}$$

We shall mention a few features of each of these four terms in order to demonstrate how they depend on $x \in S^{\delta\theta}(\mathbf{R})^0$. The first term $\Delta_I(\eta_1(x)\gamma_1, x\delta)$ depends on $(\eta_1(x)\gamma_1, x\delta)$ only through the maximal torus $T' \subset G^*$ defined in §3.3 (see §4.2 [KS99]). To be more precise, the torus T' is the centralizer in G^* of $g_{T'} m(x\delta)\theta^*(g_{T'})^{-1}$ (see (13)). As we are assuming $x\delta$ to be θ-regular in G and x belongs to $G^{\delta\theta}$, it follows from (14) that T' is the centralizer of $g_{T'} m(\delta)\theta^*(g_{T'})^{-1}$ and depends on δ alone. We may therefore write

$$\Delta_I(\eta_1(x)\gamma_1, x\delta) = \Delta_I(\gamma_1, \delta) = \Delta_I(\gamma, \delta),$$

where the right-most term follows the convention of §4.2 [KS99].

The second and fourth terms of (73) are defined in terms of comparable data. The definition of $\Delta_{II}(\eta_1(x)\gamma_1, x\delta)$ depends on a choice of a-data and a choice of χ-data. The definitions of these data are given in (2.2) and (2.5) [LS87] respectively (see also §1.3 [KS99], §9 [She08]). Under our assumptions, all of the roots in

$$R_{\mathrm{res}}(G^*, T') = \{\alpha_{|((T')^{\theta^*})^0} : \alpha \in R(G^*, T')\}$$

(§1.3 [KS99]) and $R(H, T_H)$ are imaginary (see (17)). We leave it to the reader to verify that under this assumption valid choices of a-data are given by

$$a_\alpha = \begin{cases} -i, & \alpha > 0 \\ i, & \alpha < 0 \end{cases}.$$

Here, α is a root in $R_{\text{res}}(G^*, T')$ or $R(H, T_H)$. The positive system for $R(H, T_H)$ is the one determined by Borel subgroup B_H and the positive system for $R_{\text{res}}(G^*, T')$ is inherited from the positive system of $R(G^*, T')$ determined by the Borel subgroup B' (see section 3.3). We also leave it to the reader to verify that valid choices of χ-data are given by

$$\chi_\alpha(z) = \begin{cases} |z|/z, & \alpha > 0 \\ z/|z|, & \alpha < 0 \end{cases} \quad z \in \mathbf{C}^\times \tag{74}$$

(*cf.* §9 [She08]). As before, α is a root in $R_{\text{res}}(G^*, T')$ or $R(H, T_H)$.

The term $\Delta_{II}(\eta_1(x)\gamma_1, x\delta)$ is a quotient (§4.3 [KS99]). Using the above choice of a- and χ-data, the numerator of this quotient may be written as

$$(-i)^{\dim \mathfrak{u}_{(G^*)^{\theta^*}}} \frac{\left|\prod_{\alpha_{\text{res}}<0,\, \text{type}\, R_1,R_2} N\alpha((x\delta)^*) - 1 \prod_{\alpha_{\text{res}}<0,\, \text{type}\, R_3} N\alpha((x\delta)^*) + 1\right|}{\prod_{\alpha_{\text{res}}<0,\, \text{type}\, R_1,R_2} N\alpha((x\delta)^*) - 1 \prod_{\alpha_{\text{res}}<0,\, \text{type}\, R_3} N\alpha((x\delta)^*) + 1}. \tag{75}$$

To justify this expression we must refer to §1.3 [KS99]. Our products are taken over negative roots α_{res} in $R_{\text{res}}(G^*, T')$. Since all roots are assumed to be imaginary, this is equivalent to taking products over Galois orbits of roots as in §1.3 [KS99]. The root system $\mathrm{R}_{\text{res}}(G^*, T')$ is not necessarily reduced and so the roots α_{res} may be categorized into three types, R_1-R_3, depending on whether $2\alpha_{\text{res}}$ or $\frac{1}{2}\alpha_{\text{res}}$ are also roots ((1.3.4) [KS99]). The roots of type R_1 and R_2 are the indivisible roots which coincide with the root system $\mathrm{R}(((G^*)^{\theta^*})^0, ((T')^{\theta^*})^0)$ ((1.3.4) [KS99]). The positive root spaces of type R_1 and R_2 therefore generate a Borel subalgebra $\mathfrak{u}_{(G^*)^{\theta^*}}$ in the complex Lie algebra of $(G^*)^{\theta^*}$. This fact and our choice of a-data account for the term $(-i)^{\dim \mathfrak{u}_{(G^*)^{\theta^*}}}$ in (75).

For the remaining terms, we define the regular element $(x\delta)^* \in T'$ as $g_{T'}\, m(x\delta)\, \theta^*(g_{T'})^{-1}$, in accordance with (13). The character $N\alpha$ is defined on page 16 [KS99] as $\sum_{j=0}^{l_\alpha - 1} \theta^*\alpha$ (with the convention of additive notation), where $\alpha_{\text{res}} = \alpha_{|(T')^{\theta^*}}$ and l_α is the cardinality of the θ^*-orbit of α.

As intimated in §4.5 [KS99], one may express

$$\prod_{\alpha_{\mathrm{res}}<0,\,\mathrm{type}\,R_1,R_2} N\alpha((x\delta)^*) - 1 \prod_{\alpha_{\mathrm{res}}<0,\,\mathrm{type}\,R_3} N\alpha((x\delta)^*) + 1$$

as $(-1)^{\dim \mathfrak{u}_{(G^*)^{\theta^*}}} \det(1-\mathrm{Ad}((x\delta)^*)\theta^*)_{|\bar{\mathfrak{u}}_{G^*}}$. Indeed, the exponent $\dim \mathfrak{u}_{(G^*)^{\theta^*}}$ of -1 appears for the same reasons it did as an exponent of $-i$ in (75). Furthermore, taking the union of the sets of root vectors $\{X_\alpha, \theta^*(X_\alpha), \ldots, (\theta^*)^{l_\alpha-1}(X_\alpha)\}$, as α runs over a set of representatives of θ^*-orbits of negative (resp. positive) roots in $R(G^*, T')$, yields a basis for a Borel subalgebra $\bar{\mathfrak{u}}_{G^*}$ (resp. $\mathfrak{u}_{G^*}$) of the complex Lie algebra of G^*. Referring to (1.3.5)-(1.3.7) [KS99], one may compute that

$$\det(1-\mathrm{Ad}((x\delta)^*)\theta^*)_{|\bar{\mathfrak{u}}_{G^*}} = \prod_{\alpha_{\mathrm{res}}<0,\,\mathrm{type}\,R_1,R_2} 1-N\alpha((x\delta)^*) \prod_{\alpha_{\mathrm{res}}<0,\,\mathrm{type}\,R_3} N\alpha((x\delta)^*)+1.$$

By following this description of the numerator, one sees that denominator of $\Delta_{II}(\eta_1(x)\gamma_1, x\delta)$ has an analogous, but simpler form, namely

$$i^{\dim \mathfrak{u}_H} \frac{\left|\det(1-\mathrm{Ad}(\eta(x)\gamma))_{|\bar{\mathfrak{u}}_H}\right|}{\det(1-\mathrm{Ad}(\eta(x)\gamma))_{|\bar{\mathfrak{u}}_H}}.$$

We conclude our exposition of $\Delta_{II}(\eta_1(x)\gamma_1, x\delta)$ by writing it as $\Delta_{II}(\eta(x)\gamma, x\delta)$ and noting that it is equal to

$$(76)\quad i^{\dim \mathfrak{u}_{(G^*)^{\theta^*}} - \dim \mathfrak{u}_H} \frac{\det(1-\mathrm{Ad}(\eta(x)\gamma))_{|\bar{\mathfrak{u}}_H} \left|\det(1-\mathrm{Ad}((x\delta)^*)\theta^*)_{|\bar{\mathfrak{u}}_{G^*}}\right|}{\det(1-\mathrm{Ad}((x\delta)^*)\theta^*)_{|\bar{\mathfrak{u}}_{G^*}} \left|\det(1-\mathrm{Ad}(\eta(x)\gamma))_{|\bar{\mathfrak{u}}_H}\right|}$$

for our choice of a- and χ-data.

The fourth term $\Delta_{IV}(\eta_1(x)\gamma_1, x\delta)$ is defined in §4.5 [KS99], where it may be seen that it is equal to

$$(77)\qquad \Delta_{IV}(\eta(x)\gamma, x\delta) = \frac{\left|\det(1-\mathrm{Ad}((x\delta)^*\theta^*)_{|\mathfrak{u}_{G^*}+\bar{\mathfrak{u}}_{G^*}}\right|^{1/2}}{\left|\det(1-\mathrm{Ad}(\eta(x)\gamma))_{|\mathfrak{u}_H+\bar{\mathfrak{u}}_H}\right|^{1/2}}.$$

As in §4.5 [KS99], we adopt the notation $D_{G^*\theta^*}((x\delta)^*)$ and $D_H(\eta(x)\gamma)$ for the numerator and denominator in (77). Before moving to the third term in (73), we state an identity involving Δ_{II} and Δ_{IV}.

Lemma 17 *Suppose $x \in S^{\delta\theta}(\mathbf{R})^0$, such that $x\delta$ is strongly θ-regular and $\eta_1(x)\gamma_1$ is a norm of $x\delta$. Then, with the above choice of a- and χ-data,*

$$\frac{\Delta_{II}(\eta(x)\gamma, x\delta)\,\Delta_{IV}(\eta(x)\gamma, x\delta)\,D_H(\eta(x)\gamma)^2}{\det(1-\mathrm{Ad}(\eta(x)\gamma))_{|\bar{\mathfrak{u}}_H}}$$

is equal to

$$i^{\dim \mathfrak{u}_{(G^*)^{\theta^*}} - \dim \mathfrak{u}_H}\ \frac{D_{G\theta}(x\delta)^2}{\det(1-\mathrm{Ad}(x\delta)\theta)_{|\bar{\mathfrak{u}}}}.$$

Proof. According to Corollary 3, the group $T_H(\mathbf{R})/Z_H(\mathbf{R})$ is compact. Replacing G with H and S with T_H in the discussion preceding (29), we find that $T_H(\mathbf{R})/Z_H(\mathbf{R})$ is connected. It follows that, $|\det(\mathrm{Ad})_{|\mathfrak{u}_H}|$, as a homomorphism from $T_H(\mathbf{R})/Z_H(\mathbf{R})$ to $\mathbf{R}^\times$, is trivial. Therefore,

$$\begin{aligned} |\det(1-\mathrm{Ad})_{|\mathfrak{u}_H}| &= |(-1)^{\dim \mathfrak{u}_H} \det(1-\mathrm{Ad}^{-1})_{|\mathfrak{u}_H} \det(\mathrm{Ad})_{|\mathfrak{u}_H}| \\ &= |\det(1-\mathrm{Ad})_{|\bar{\mathfrak{u}}_H}|. \end{aligned}$$

This implies that

$$|\det(1-\mathrm{Ad})_{|\mathfrak{u}_H}| = |\det(1-\mathrm{Ad})_{|\mathfrak{u}_H}|^{1/2} |\det(1-\mathrm{Ad})_{|\bar{\mathfrak{u}}_H}|^{1/2} = D_H.$$

Combining these facts with (76) and (77), the first expression of the lemma is seen to be equal to

$$i^{\dim \mathfrak{u}_{(G^*)^{\theta^*}} - \dim \mathfrak{u}_H} \frac{D_{G^*\theta^*}((x\delta)^*)^2\ D_H(\eta(x)\gamma)^2}{\det(1-\mathrm{Ad}((x\delta)^*)\theta^*)_{|\bar{\mathfrak{u}}_{G^*}}\ D_H(\eta(x)\gamma)^2}$$

Transport via isomorphism (14) yields $D_{G^*\theta^*}((x\delta)^*) = D_{G\theta}(x\delta)$ and

$$\det(1-\mathrm{Ad}((x\delta)^*\theta^*)_{|\bar{\mathfrak{u}}_{G^*}} = \det(1-\mathrm{Ad}(x\delta)\theta)_{|\bar{\mathfrak{u}}}.\blacksquare$$

We move on to the third term in (73). It is a consequence of Lemma 5.1.A. [KS99], that

(78) $\Delta_{III}(\eta_1(x)\gamma_1, x\delta; \gamma_1^0, \delta^0) = \Delta_{III}(\eta_1(x)\gamma_1, x\delta; \gamma_1, \delta)\ \Delta_{III}(\gamma_1, \delta; \gamma_1^0, \delta^0)$.

We shall trace the definition of $\Delta_{III}(\eta_1(x)\gamma_1, x\delta; \gamma_1, \delta)$ given in §4.4 [KS99]. Afterwards, we shall relate our findings to characters on $T_H(\mathbf{R})$ and $S^{\delta\theta}(\mathbf{R})$.

$\Delta_{III}(\eta_1(x)\gamma_1, x\delta; \gamma_1, \delta)$ is defined in terms of a fibre product of two maximal tori. The first torus is $T' \subset G^*$. Denote by $N : T' \to T_H$ the homomorphism which is the composition of the coset map $T' \to T'_{\theta^*}$ with the isomorphism $T'_{\theta^*} \cong T_H$ of §3.3. The second torus is T_{H_1}. Let T_1 be the fibre product

$$\{(t', t) \in T' \times T_{H_1} : N(t') = p_H(t)\}. \tag{79}$$

The projection of T_1 to its first coordinate produces a surjection

$$p_1 : T_1 \to T' \tag{80}$$

whose kernel is isomorphic to Z_1. By definition,

$$\Delta_{III}(\eta_1(x)\gamma_1, x\delta; \gamma_1, \delta) = \langle \mathbf{V}_1, \mathbf{A}_1 \rangle, \tag{81}$$

where $\mathbf{V}_1$ is a class in a hypercohomology group $H^1(\Gamma, U \to S_1)$, $\mathbf{A}_1$ is a class in hypercohomology group $H^1(W_{\mathbf{R}}, \hat{S}_1 \to \hat{U})$, and $\langle \cdot, \cdot \rangle$ is a pairing given in A.3 [KS99]. We shall describe both $\mathbf{V}_1$ and $\mathbf{A}_1$, and then turn to computing $\langle \mathbf{V}_1, \mathbf{A}_1 \rangle$. In our description, it shall be important to realize that both elements, $x\delta$ and δ, are mapped under (13) to respective elements $(x\delta)^*$ and δ^* in the *same torus*, namely T'.

As seen on p. 42 [KS99], the torus S_1 in $H^1(\Gamma, U \to S_1)$ is the quotient of $T_1 \times T_1$ whose rational characters are

$$X^*(S_1) = \{(\mu_1, \mu_2) \in X^*(T_1) \times X^*(T_1) : \mu_1 - \mu_2 \in X^*(T'_{\mathrm{ad}})\}.$$

In fact, the quotient is taken with respect to group of elements $\{(z, z^{-1})\}$, where z lies in the inverse image under (80) of the centre $Z_{G^*} = Z_G$. It is an instructive exercise to prove that $S_1 \cong T_1 \times T'_{\mathrm{ad}}$. The isomorphism is given by the map which sends each pair $(t, t_{H_1}), (\bar{t}, \bar{t}_{H_1}) \in T_1$ to $((t\bar{t}, t_{H_1}\bar{t}_{H_1}), \tilde{t})$, the element $\tilde{t}$ being the image of t under the covering map $G^* \to G^*_{\mathrm{ad}}$.

In the same vein, the other torus U, appearing in $H^1(\Gamma, U \to S_1)$, is defined (p. 37 [KS99]) to be the quotient

$$(T'_{\mathrm{sc}} \times T'_{\mathrm{sc}})/\{(z, z^{-1}) : z \in Z_{G_{\mathrm{sc}}}\}.$$

The map which sends each pair $t, \bar{t} \in T'_{\mathrm{sc}}$ to the element $(t\bar{t}, \tilde{t})$ in $T'_{\mathrm{sc}} \times T'_{\mathrm{ad}}$, $\tilde{t}$ being the image of t under the covering map $G^*_{\mathrm{sc}} \to G_{\mathrm{ad}}$, induces an isomorphism $U \cong T'_{\mathrm{sc}} \times T'_{\mathrm{ad}}$.

The class $\mathbf{V}_1 \in H^1(\Gamma, U \to S_1)$ is defined by the 1-hypercocycle (V, D_1). In the first entry is a 1-cocycle $V \in Z^1(\Gamma, U)$. It is the image of (v^{-1}, v) in U of a 1-cochain $v \in C^1(\Gamma, T'_{\mathrm{sc}})$ defined by

$$v(\sigma) = g_{T'} u_\sigma \sigma(g_{T'})^{-1}$$

(see (3), (13) and p. 38 [KS99]). In the second entry is a 0-cochain $D_1 \in S_1$. It is equal to the image in S_1 of the pair $(((x\delta)^*, \eta_1(x)\gamma_1), (\delta^*, \gamma_1)^{-1})$ in $T' \times T'$. A straightforward computation shows that $(x\delta)^*$ may be written as $x^*\delta^*$, where $x^* \in T'(\mathbf{R})$ (abusively) denotes the image of x under (14). This allows us to decompose D_1 into a product $D_\delta D_x$, where D_δ and D_x are the respective cosets of $((\delta^*, \gamma_1), (\delta^*, \gamma_1)^{-1})$ and $((x^*, \eta_1(x)), 1)$ in S_1. One may then decompose (V, D_1) into a product $(V, D_\delta)(1, D_x)$ of 1-hypercochains. By Lemma 4.4.A [KS99], (V, D_δ) is a 1-hypercocycle. Since $(x^*, \eta_1(x))$ belongs to the real torus $T'(\mathbf{R}) \times T_H(\mathbf{R})$, the element $(\sigma(x^*)(x^*)^{-1}, \sigma(\eta_1(x))\eta_1(x)^{-1})$ is trivial and $(1, D_x)$ is also a 1-hypercocycle. Writing $\mathbf{V}_\delta$ and $\mathbf{V}_x$ for the respective classes in $H^1(\Gamma, U \to S_1)$ of (V, D_δ) and $(1, D_x)$, we obtain $\mathbf{V}_1 = \mathbf{V}_\delta \mathbf{V}_x$. This is analogous to the decomposition given in [Shea].

We may further decompose $\mathbf{V}_\delta$ and $\mathbf{V}_x$ according to the isomorphisms $S_1 \cong T_1 \times T'_{\mathrm{ad}}$ and $U \cong T'_{\mathrm{sc}} \times T'_{\mathrm{ad}}$ given above. Through these isomorphisms, we may identify $D_\delta \in S_1$ with the element $(1, \tilde{\delta}^*) \in T_1 \times T'_{\mathrm{ad}}$, and the element $D_x \in S_1$ with $((x^*, \eta_1(x)), \tilde{x}^*) \in T_1 \times T'_{\mathrm{ad}}$. Similarly, the 1-cochain $V \in C^1(\Gamma, U)$ is identified with $(1, \tilde{v}) \in C^1(\Gamma, T'_{\mathrm{sc}} \times T'_{\mathrm{ad}})$, where $\tilde{v} \in C^1(\Gamma, T'_{\mathrm{ad}})$ is taken to be the composition of v with the covering map $G^* \to G^*_{\mathrm{ad}}$.

We now proceed to the class $\mathbf{A}_1 \in H^1(W_{\mathbf{R}}, \hat{S}_1 \to \hat{U})$, with which $\mathbf{V}_1$ is paired. The class $\mathbf{A}_1$ is defined (p. 45 [KS99]) by a 1-hypercocycle

$$(A^{-1}, \mathsf{s}_U) \in Z^1(W_{\mathbf{R}}, \hat{T}_1 \times \hat{T}'_{\mathrm{sc}} \to \hat{U}).$$

This definition makes use of the isomorphism $\hat{S}_1 \cong \hat{T}_1 \times \hat{T}'_{\mathrm{sc}}$, which is dual to $S_1 \cong T_1 \times T'_{\mathrm{ad}}$ (see p. 38 and p. 42 [KS99]). We shall ignore the definition of s_U and merely state that under the isomorphism $\hat{U} \cong \hat{T}'_{\mathrm{ad}} \times \hat{T}'_{\mathrm{sc}}$ (p. 38 [KS99]) it may be identified with an element $(\mathsf{s}_{\mathrm{ad}}, 1)$ defined in terms of the endoscopic datum s (p. 41 [KS99]). The term A is a 1-cocycle of the form $(a_{T'}, 1) \in Z^1(W_{\mathbf{R}}, \hat{T}_1 \times \hat{T}'_{\mathrm{sc}})$. We shall have much more to say about the definition of $a_{T'} \in Z^1(W_{\mathbf{R}}, \hat{T}_1)$ shortly. For the time being, we recognize that the Local Langlands Correspondence for tori (Theorem 1 [Lan97], 9.1 [Bor79]) attaches to the class of $a_{T'}$ in $H^1(W_{\mathbf{R}}, \hat{T}_1)$ a quasicharacter of $T_1(\mathbf{R})$. We shall denote this quasicharacter, somewhat abusively, by $\langle \cdot, a_{T'} \rangle$.

Lemma 18 *The geometric transfer factor $\Delta_{III}(\eta_1(x)\gamma_1, x\delta; \gamma_1, \delta)$ is equal to $\langle (x^*, \eta_1(x)), a_{T'} \rangle$.*

Proof. Recalling (81), we compute each factor in the product

$$\langle \mathbf{V}_1, \mathbf{A}_1 \rangle = \langle \mathbf{V}_\delta, \mathbf{A}_1 \rangle \langle \mathbf{V}_x, \mathbf{A}_1 \rangle.$$

We may write

$$\begin{aligned}\langle \mathbf{V}_\delta, \mathbf{A}_1\rangle &= \langle (V, D_\delta), (A^{-1}, \mathsf{s}_U)\rangle \\ &= \langle (((1,\tilde{v}), (1, \tilde{\delta}^*)), ((a_{T'}^{-1}, 1), (\mathsf{s}_{\mathrm{ad}}, 1))\rangle.\end{aligned}$$

To compute this pairing, we employ isomorphism (A.3.4) [KS99]. The inverse of this isomorphism maps the 1-cocycle $(1,\tilde{v}) \in Z^1(\Gamma, T'_{\mathrm{sc}} \times T'_{\mathrm{ad}})$ to a 0-cycle in $C_0(W_{\mathbf{R}}, X_*(T'_{\mathrm{sc}}) \times X_*(T'_{\mathrm{ad}}))$ which we will abusively also denote by $(1,\tilde{v}) \in X_*(T'_{\mathrm{sc}}) \times X_*(T'_{\mathrm{ad}})$ (see (A.3.5) [KS99]). The inverse of (A.3.4) [KS99] maps $(1,\tilde{\delta}^*)$ to a 1-cycle in $C_1(W_{\mathbf{R}}, X_*(T_1) \times T'_{\mathrm{ad}})$ which we again abusively denote by $(1,\tilde{\delta}^*)$. According to p. 135 [KS99], the pairing $\langle \mathbf{V}_\delta, \mathbf{A}_1\rangle$ is equal to the product of the value of $(1,\tilde{v}) \in X_*(T'_{\mathrm{sc}}) \times X_*(T'_{\mathrm{ad}})$ at $(\mathsf{s}_{\mathrm{ad}}, 1) \in \hat{T}'_{\mathrm{ad}} \times \hat{T}'_{\mathrm{sc}}$, which is one, with

$$\prod_{w \in W_{\mathbf{R}}} 1(a_{T'}(w))\, \tilde{\delta}^*_w(1) = \prod_{w \in W_{\mathbf{R}}} (1)(1) = 1.$$

This proves that $\langle \mathbf{V}_\delta, \mathbf{A}_1\rangle = 1$.

We use the same method to compute

$$\begin{aligned}\langle \mathbf{V}_x, \mathbf{A}_1\rangle &= \langle (1, D_x), (A^{-1}, \mathsf{s}_U)\rangle \\ &= \langle (1, ((x^*, \eta_1(x)), \tilde{x}^*)), ((a_{T'}^{-1}, 1), (\mathsf{s}_{\mathrm{ad}}, 1))\rangle.\end{aligned}$$

Following the same procedure as for $\langle \mathbf{V}_\delta, \mathbf{A}_1\rangle$, this pairing is a product of three terms: the value of the identity in $X_*(T'_{\mathrm{sc}}) \times X_*(T'_{\mathrm{ad}})$ at $(\mathsf{s}_{\mathrm{ad}}, 1)$, which is one; the product $\prod_{w \in W_{\mathbf{R}}} \tilde{x}^*_w(1) = 1$; and a pairing of $a_{T'}^{-1} \in Z^1(W_{\mathbf{R}}, \hat{T}_1)$ with $(x^*, \eta_1(x))$. In this final pairing we identify $(x^*, \eta_1(x)) \in T_1(\mathbf{R})$ with a 1-chain in $C_1(W_{\mathbf{R}}, X_*(T_1))$. This identification is identical to that formulated by Langlands in [Lan97] (p. 131 [KS99]). Consequently, the final pairing equals the product $\prod_{w \in W_{\mathbf{R}}} (x^*, \eta_1(x))_w (a_{T'}(w))$, and this is $\langle (x^*, \eta_1(x)), a_{T'}\rangle$ (see (5) 9.2 [Bor79]).$\blacksquare$

Lemma 18 records how Δ_{III} is affected when multiplying the pair (γ_1, δ^*) by the pair $(\eta_1(x), x^*)$. In a similar fashion, Kottwitz and Shelstad record how Δ_{III} is affected when multiplying the pair (γ_1, δ^*) by a pair $(z_1, z) \in Z_1(\mathbf{R}) \times Z_G(\mathbf{R})$ in the inverse image of $Z_G(\mathbf{R})$ under (80) (see pp. 53-54 [KS99]). They denote this inverse image by C and record this change in terms of a quasicharacter Λ_C of C. In particular,

$$\Delta_{III}(z_1\gamma_1, z\delta; \gamma_1, \delta) = \Lambda_C(z_1, z)^{-1}.$$

In our discussion of Δ_{III} it is harmless to replace $(\eta_1(x), x)$ with (z_1, z) as above and arrive at

$$\Delta_{III}(z_1\gamma_1, z\delta; \gamma_1, \delta) = \langle (z, z_1), a_{T'} \rangle. \tag{82}$$

In this way we see that λ_C^{-1} is the restriction of the quasicharacter $\langle \cdot, a_{T'} \rangle$ to the subgroup C of $T_1(\mathbf{R})$.

6.2.1 Δ_{III} and quasicharacters of $T_{H_1}(\mathbf{R})$ and $S^{\delta\theta}(\mathbf{R})$

Recall from the beginning of section 6 that our aim is to compare characters of representations in the L-packet $\Pi_{\varphi_{H_1}}$ with those in the L-packet Π_{φ^*}. The purpose of this section is to show that the quasicharacters of $T_{H_1}(\mathbf{R})$ attached to an admissible homomorphism $\varphi_{H_1} \in \boldsymbol{\varphi_{H_1}}$ (see (20)) are comparable to quasicharacters of $S^{\delta\theta}(\mathbf{R})$, which are attached to $\varphi^* \in \boldsymbol{\varphi^*}$. The comparison will be made through the torus T_1, which connects T_{H_1} to $T' \cong S$, and the 1-cocycle $a_{T'} \in Z^1(W_\mathbf{R}, \hat{T}_1)$ appearing in the Δ_{III}-term. For the comparison, we also employ the Local Langlands Correspondence for tori ([Lan97], 9 [Bor79]), which connects characters of real tori to certain 1-cocycles and thence to admissible homomorphisms of $W_\mathbf{R}$.

We begin with a description of the 1-cocycle $a_{T'}$ given on page 45 [KS99]. The χ-data fixed in (74) determine an admissible embedding ${}^LT_H \to {}^LH$ as shown in §2.6 [LS87]. The composition of this embedding with the inclusion ${}^LH \hookrightarrow {}^LH_1$ (see (6)) produces an admissible embedding $\xi_{T_H} : {}^LT_H \to {}^LH_1$. The restriction of ξ_{T_H} to $W_\mathbf{R}$ is an admissible homomorphism into ${}^LT_{H_1}$. Recall from §3.2 the L-homomorphism $\xi_{H_1} : \mathcal{H} \to {}^LH_1$. As shown in §2.2 [KS99], for every $w \in W_\mathbf{R}$ there exist an element $u(w) \in \mathcal{H}$ which preserves the pair $(\mathcal{B}_H, \mathcal{T}_H)$ under conjugation and projects to w under the surjection $\mathcal{H} \to W_\mathbf{R}$. For every $w \in W_\mathbf{R}$ there exists a unique element $t_1(w) \in \hat{T}_H \cong \mathcal{T}_H$ such that

$$\xi_{T_H}(1, w) = t_1(w)\ \xi_{H_1}(u(w)),\ w \in W_\mathbf{R}. \tag{83}$$

This is half of $a_{T'}$.

The other half is an element $t(w) \in \hat{T}' \cong \mathcal{T}$. This element is again defined through a comparison of L-homomorphisms. On the one hand, the map $\xi : \mathcal{H} \to {}^LG^*$ is an L-homomorphism (see 4b §3.2) which maps $u(w)$ to an element in $\mathcal{T} \rtimes W_\mathbf{R}$. On the other hand, the χ-data of (74) may be used to define an injective L-homomorphism $\xi_{T'_{\theta^*}} : {}^LT'_{\theta^*} \to {}^LG^*$ (p. 40 [KS99]). We shall say more about this soon. For the moment, let us use $\xi_{T'_{\theta^*}}$ to define an injective L-homomorphism $\xi_1 : {}^LT_H \rightarrow {}^LG^*$. Isomorphism (12) induces an L-isomorphism ${}^LT_H \cong {}^LT'_{\theta^*}$ (§2.1 [Bor79]), which we may compose with $\xi_{T'_{\theta^*}}$ to embed into ${}^LG^*$. The resulting embedding is defined as ξ_1, and it restricts to an admissible homomorphism of $W_{\mathbf{R}}$. As a result, there is an element $t(w)$ as above such that

$$\xi_1(1, w) = t(w)\ \xi(u(w)),\ \ w \in W_{\mathbf{R}}. \tag{84}$$

It is shown on pages 44-45 [KS99] that the pair (t_1, t^{-1}) defines the cocycle $a_{T'} \in Z^1(W_{\mathbf{R}}, \hat{T}_1)$ and that it is independent of our choice of splitting $u(w) \in \mathcal{H}$ (see also §11 [She08]).

A description of $a_{T'}$ being given, we turn to admissible homomorphisms $\varphi_{H_1} : W_{\mathbf{R}} \rightarrow {}^LH_1$ and $\varphi^* : W_{\mathbf{R}} \rightarrow {}^LG$ which respectively represent the L-packets $\Pi_{\varphi_{H_1}}$ and Π_{φ^*}. Looking back to section 4.1, we see that we may identify φ_{H_1} with a pair $\mu_{H_1}, \lambda_{H_1} \in X_*(\hat{T}_{H_1}) \otimes \mathbf{C}$, and identify φ^* with a pair

$$\mu, \lambda \in X_*(((\hat{T}')^{\theta^*})^0) \otimes \mathbf{C} \cong X^*(T'_{\theta^*}) \otimes \mathbf{C}. \tag{85}$$

Furthermore, these two pairs correspond to characters of $T_{H_1}(\mathbf{R})$ and $T'_{\theta^*}(\mathbf{R})$ respectively (see (18)). The Local Langlands Correspondence for tori attaches admissible homomorphisms $\varphi_{T_{H_1}} : W_{\mathbf{R}} \rightarrow {}^LT_{H_1}$ and $\varphi_{T'_{\theta^*}} : W_{\mathbf{R}} \rightarrow {}^LT'_{\theta^*}$ to these two characters.

It is a direct consequence of the definitions that the pair attached to $\varphi_{T_{H_1}}$ is $\mu_{H_1} - \iota_H, \lambda_{H_1} \in X_*(\hat{T}_{H_1}) \otimes \mathbf{C}$. Here, $\iota_H \in X^*(T_{H_1}) \otimes \mathbf{C} \cong X_*(\hat{T}_{H_1}) \otimes \mathbf{C}$ is the image of the half-sum of the roots of $R(H, T_H)$ under (7). In addition, by §7 [She10] and §2.6 [LS87], we have

$$\varphi_{H_1} = \xi_{T_{H_1}} \circ \varphi_{T_{H_1}}. \tag{86}$$

To describe the pair attached to $\varphi_{T'_{\theta^*}}$ we work in the identity components of $(\hat{G}^*)^{\hat{\theta}^*}$ and $(\hat{T}')^{\hat{\theta}^*}$. Denote by $\iota_{G^*\mathrm{res}}$ the half-sum of the roots defined by duals of these groups (see (1.3.4) [KS99]). The pair attached to $\varphi_{T'_{\theta^*}}$ is $\mu - \iota_{G^*\mathrm{res}}, \lambda \in X^*(T'_{\theta^*}) \otimes \mathbf{C}$. Furthermore, the χ-data of (74) may be transferred to the roots above and yield a homomorphism ${}^LT'_{\theta^*} \rightarrow (\hat{G}^*)^{\hat{\theta}^*} \rtimes W_{\mathbf{R}}$ (§2.6 [LS87]), which we compose with the inclusion $(\hat{G}^*)^{\hat{\theta}^*} \rtimes W_{\mathbf{R}} \hookrightarrow {}^LG^*$ to define an L-homomorphism

$$\xi_{T'_{\theta^*}} : {}^LT'_{\theta^*} \rightarrow {}^LG^*.$$

Reasoning as before, we also have

$$\varphi^* = \xi_{T'_{\theta^*}} \circ \varphi_{T'_{\theta^*}}. \tag{87}$$

All the maps required for the comparison are in place.

Lemma 19 *Let* $a_{T'_{\theta^*}} \in Z^1(W_{\mathbf{R}}, (\hat{T}')^{\hat{\theta}^*})$ *and* $a_{T_{H_1}} \in Z^1(W_{\mathbf{R}}, \hat{T}_{H_1})$ *denote the 1-cocycles which satisfy*

$$\varphi_{T'_{\theta^*}}(w) = (a_{T'_{\theta^*}}(w), w), \ w \in W_{\mathbf{R}}$$

and

$$\varphi_{T_{H_1}}(w) = (a_{T_{H_1}}(w), w), \ w \in W_{\mathbf{R}}.$$

Identifying the torus $\hat{T}_H$ *with* $\mathcal{T}_H$*, and the torus* $\hat{T}'$ *with* $\mathcal{T}$ *(see (10)), the following identity holds*

$$a_{T'_{\theta^*}} = t^{-1}(\xi \circ t_1)\,(\xi \circ a_{T_{H_1}}). \tag{88}$$

Proof. Suppose $w \in W_{\mathbf{R}}$. Bearing in mind (57), (83) and (86), we compute that

$$\begin{aligned}
\xi_{H_1}^{-1} \circ \varphi_{T_{H_1}}(w) &= \xi_{H_1}^{-1} \circ \xi_{T_{H_1}} \circ \varphi_{H_1}(w) \\
&= \xi_{H_1}^{-1}(a_{T_{H_1}}(w), 1)\ \xi_{H_1}^{-1} \circ \xi_{T_{H_1}}(1, w) \\
&= \xi_{H_1}^{-1}(a_{T_{H_1}}(w), 1)\ \xi_{H_1}^{-1}(t_1(w), 1)\, u(w) \\
&= \xi_{H_1}^{-1}(a_{T_{H_1}}(w)\, t_1(w), 1)\ u(w).
\end{aligned}$$

Since the L-homomorphism $\xi_{H_1} : \mathcal{H} \to {}^L H_1$ extends the inclusion $\hat{H} \hookrightarrow \hat{H}_1$ (Lemma 2.2.A [KS99]) and $a_{T_{H_1}}(w)\, t_1(w)$ lies in the image of ξ_{H_1}, it follows that $a_{T_{H_1}}(w)\, t_1(w)$ lies in $\hat{T}_H$ and

$$\xi_{H_1}^{-1} \circ \varphi_{H_1}(w) = (a_{T_{H_1}}(w)\, t_1(w), 1)\ u(w)$$

Applying the L-homomorphism ξ to this equation and recalling (84), we find

$$\xi \circ \xi_{H_1}^{-1} \circ \varphi_{H_1}(w) = (t^{-1}(w)(\xi \circ t_1)(w)\ (\xi \circ a_{T_{H_1}})(w), 1)\ \xi_1(1, w). \tag{89}$$

By (87) and the definition of φ^*, the left-hand side of (89) is equal to $\xi_{T'_{\theta^*}} \circ \varphi_{T'_{\theta^*}}(w)$. As for the right-hand side, it follows from the definition of ξ_1 that $\xi_1(1, w) = \xi_{T'_{\theta^*}}(1, w)$. Furthermore, under the identification of $\hat{T}'$ with $\mathcal{T}$, the map $\xi_{T'_{\theta^*}}$ is the identity map on the identity component of $\mathcal{T}^{\hat{\theta}^*}$ ((i) §2.6 [LS87]). Therefore, equation (89) may be written as

$$\xi_{T'_{\theta^*}} \circ \varphi_{T'_{\theta^*}}(w) = \xi_{T'_{\theta^*}}(t^{-1}(w)\,(\xi \circ t_1)(w)\,(\xi \circ a_{T_{H_1}})(w), w).$$

As $\xi_{T'_{\theta^*}}$ is injective, the lemma follows from this equation.■

It is now time to transplant (88) to the torus T_1. We shall do this by using the injection $p_1^* : \hat{T}' \to \hat{T}_1$ induced by surjective projection (80). An application of p_1^* to (88) produces a 1-cocycle in $Z^1(W_{\mathbf{R}}, \hat{T}_1)$. In order to obtain a clearer picture of this cocycle, it is worth reexamining $\hat{T}_1$. Restriction to each coordinate of T_1 produces a injection $T_1 \to T' \times T_{H_1}$, whose dual is a surjection

$$\hat{T}' \times \hat{T}_{H_1} \to \hat{T}_1. \tag{90}$$

It is left as an exercise to show that the kernel of this surjection is isomorphic to $\hat{T}_H$, and that $\hat{T}_1$ is isomorphic to the quotient

$$\hat{T}' \times \hat{T}_{H_1} / \{(\xi \circ \alpha, \alpha^{-1}) : \alpha \in \hat{T}_H\}$$

(we are identifying $\hat{T}_H$ with its image in $\hat{T}_{H_1}$ under (7)). Viewing $\hat{T}_1$ as this quotient, a representative for the 1-cocycle $p_1^* \circ a_{T'_{\theta^*}}$ of (88) is of the form

$$\begin{aligned} p_1^* \circ (t^{-1}(\xi \circ t_1)\,(\xi \circ a_{T_{H_1}})) &= (t^{-1}(\xi \circ t_1), 1)\,((\xi \circ a_{T_{H_1}}), 1) \\ &= (t^{-1}, t_1)\,(1, a_{T_{H_1}}) \\ &= a_{T'}\,(1, a_{T_{H_1}}). \end{aligned} \tag{91}$$

We are ready to make the comparison indicated at the beginning of this section. The quasicharacter of $T_{H_1}(\mathbf{R})$ attached to the admissible homomorphism φ_{H_1} is $\Lambda(\mu_{H_1} - \iota_H, \lambda_{H_1})$ (see (20)). The 1-cocycle attached to $\Lambda(\mu_{H_1} - \iota, \lambda_{H_1})$ through the Local Langlands Correspondence is $a_{T_{H_1}}$ (see (86)), and so we may somewhat abusively write

$$\Lambda(\mu_{H_1} - \iota_H, \lambda_{H_1})(\eta_1(x)) = \langle \eta_1(x), a_{T_{H_1}} \rangle. \tag{92}$$

In a similar fashion we see that the quasicharacter of $T'(\mathbf{R})$ attached to φ^* is $\Lambda(\mu - \iota_{G^*}, \lambda)$, where ι_{G^*} denotes the half-sum of the positive roots of $R(G^*, T')$. Since θ^* preserves the positive roots (Lemma 3.3.B [KS99]), it preserves ι_{G^*}. Regarding ι_{G^*} as an element in $X_*((\hat{T}')^{\hat{\theta}^*}) \otimes \mathbf{C}$, it follows from (85) that $(1 - \theta^*)T'$ lies in the kernel of $\Lambda(\mu - \iota_{G^*}, \lambda)$. This being the case, we may identify $\Lambda(\mu - \iota_{G^*}, \lambda)$ with a quasicharacter of $T'_{\theta^*}(\mathbf{R})$. Under this identification ι_{G^*} is equal to $\iota_{G^*\mathrm{res}}$ (§1.3 [KS99]), and the 1-cocycle attached to $\Lambda(\mu - \iota_{G^*\mathrm{res}}, \lambda)$ through the Local Langlands Correspondence is $a_{T'_{\theta^*}}$. As before, we write

$$\Lambda(\mu - \iota_{G^*\mathrm{res}}, \lambda)(x^*\delta^*) = \langle x^*\delta^*, a_{T'_{\theta^*}} \rangle. \tag{93}$$

Proposition 3 *Suppose $x \in \mathcal{V}$. Then*

$$\Delta_{III}(\eta_1(x)\gamma_1, x\delta; \gamma_1, \delta)\ \Lambda(\mu_{H_1} - \iota_H, \lambda_{H_1})(\eta_1(x)) = \Lambda(\mu - \iota_{G^*\mathrm{res}}, \lambda)(x^*)$$

Proof. As in (91), let us write $(1, a_{T_{H_1}})$ for the composition of $a_{T_{H_1}}$ under (90). Then, by (91) and (92), we have

$$\begin{aligned}
\Lambda(\mu_{H_1} - \iota_H, \lambda_{H_1})(\eta_1(x)) &= \langle (x^*, \eta_1(x)), (1, a_{T_{H_1}}) \rangle \\
&= \langle (x^*, \eta_1(x)), a_{T'}^{-1}\, p_1^* \circ a_{T'_{\theta^*}} \rangle \\
&= \langle (x^*, \eta_1(x)), a_{T'}^{-1} \rangle\ \langle x^*, a_{T'_{\theta^*}} \rangle.
\end{aligned}$$

The proposition now follows from Lemma 18 and (93).■

Corollary 4 *Suppose $x \in \mathcal{V}$ and $w_1 \in \Omega(H, T_H)$. Then*

$$\Delta_{III}(\eta_1(x)\gamma_1, x\delta; \gamma_1, \delta)\ \Lambda(\mu_{H_1} - \iota_H, \lambda_{H_1})(w_1\eta_1(x)w_1^{-1}) = \Lambda(\mu - \iota_{G^*\mathrm{res}}, \lambda)(x^*).$$

Proof. The element $w_1\eta_1(x)\gamma_1 w_1^{-1} = w_1\eta_1(x)w_1^{-1}w_1\gamma_1 w_1^{-1}$ is stably conjugate to $\eta_1(x)\gamma_1$ (Lemma 6.4.1 [Lab08]). The corollary follows from the proof of Lemma 5.1.B [KS99], where it is explained how $w_1\eta_1(x)\gamma_1 w_1^{-1}$ is a norm of $x\delta$, and

$$\Delta_{III}(\eta_1(x)\gamma_1, x\delta; \gamma_1, \delta) = \Delta_{III}(w_1\eta_1(x)\gamma_1 w_1^{-1}, x\delta; \gamma_1, \delta). \blacksquare \tag{94}$$

We remark that (94) may be interpreted through Lemma 18 as

$$\langle (x^*, \eta_1(x)), a_{T'} \rangle = \langle (x^*, w_1\eta_1(x)w_1^{-1}), a_{T'} \rangle$$

for x in a open set of $S^{\delta\theta}(\mathbf{R})$. As a result, the character $\langle \cdot, a_{T'} \rangle$ of T_1 is invariant under this action of $\Omega(H, T_H)$, and for any $w_1 \in \Omega(H, T_H)$ we have that

$$\langle (\delta^*, \gamma_1), a_{T'} \rangle = \langle (\delta^*, w_1\gamma_1 w_1^{-1}), a_{T'} \rangle.$$

As in the proof of Proposition 3, we see

$$\begin{aligned}
\langle (\delta^*, \gamma_1), a_{T'} \rangle\ \Lambda(\mu_{H_1} - \iota_H, \lambda_{H_1})(w_1\gamma_1 w_1^{-1})) &= \langle (\delta^*, w_1\gamma_1 w_1^{-1}), p_1^* \circ a_{T'_{\theta^*}} \rangle \\
&= \Lambda(\mu - \iota_{G^*}, \lambda)(\delta^*).
\end{aligned} \tag{95}$$

Before departing from the quasicharacter on $T_1(\mathbf{R})$ determined by $a_{T'}$, let us illustrate how $a_{T'}$ also determines a linear form $\lambda_{a_{T'}} : (\mathfrak{t}')^{\theta^*} \to \mathbf{C}$ on the Lie algebra of $(T')^{\theta^*}(\mathbf{R})$. One may take differentials of the maps of the fibre product (79) to define the fibre product of $\mathfrak{t}'$ and $\mathfrak{t}_{H_1}$. This fibre product is equal to $\mathfrak{t}_1$, and is isomorphic to a fibre product of $\mathfrak{t}'$ with $\mathfrak{z}_1 \oplus \mathfrak{t}_H$, according to the splitting of (63). The latter product is isomorphic to the direct sum of $\mathfrak{z}_1$ with a fibre product of $\mathfrak{t}'$ with $\mathfrak{t}_H$ (in which elements of $\mathfrak{t}'$ map to $\mathfrak{t}_H$ under the differential of N). The decomposition

$$\mathfrak{t}' = (\mathfrak{t}')^{\theta^*} \oplus (1-\theta^*)\mathfrak{t}' \tag{96}$$

yields an isomorphism between the Lie algebra of $T'_{\theta^*}(\mathbf{R})$ and $(\mathfrak{t}')^{\theta^*}$. This allows one to identify $(\mathfrak{t}')^{\theta^*}$ with a subspace of the fibre product of $\mathfrak{t}'$ and $\mathfrak{t}_H$, and ultimately allows one to identify $(\mathfrak{t}')^{\theta^*}$ with a subspace of $\mathfrak{t}_1$. We define the linear form $\lambda_{a_{T'}}$ to be restriction to $(\mathfrak{t}')^{\theta^*}$ of the differential of the quasicharacter of $T_1(\mathbf{R})$ determined by $a_{T'}$. In essence, and making identifications as necessary, one may regard $\lambda_{a_{T'}}$ as being equal to the difference of the differentials of $\Lambda(\mu - \iota_{G^*_{\mathrm{res}}}, \lambda)$ and $\Lambda(\mu_{H_1} - \iota_H, \lambda_{H_1})$.

It shall be valuable to note that by the admissible embedding (12), one may also regard $\lambda_{a_{T'}}$ as a linear form on $\mathfrak{t}_H$. Regarded in this way, the $\Omega(H, T_H)$-invariance of $\Delta_{III}(\eta_1(x)\gamma_1, x\delta; \gamma_1, \delta)$ (see the proof of Lemma 5.1.B [KS99]) and Lemma 18 tell us that $\lambda_{a_{T'}}$ too is invariant under the action of $\Omega(H, T_H)$.

6.3 A spectral comparison with small support

Recall from the discussion following (62) that there is an open subset of the identity $\mathcal{V} \subset S^{\delta\theta}(\mathbf{R})^0$ such that η maps $\mathcal{V}$ homeomorphically onto $\eta(\mathcal{V}) \subset T_H(\mathbf{R})^0$. Suppose $f \in C_c^\infty(G(\mathbf{R})\theta)$ has support in the $G(\mathbf{R})$-conjugates of $Z_G(\mathbf{R})^0 \mathcal{V}\delta\theta$. We shall assume that there is a function $f_{H_1} \in C^\infty(H_1(\mathbf{R}))$ as in §3.4, i.e. a function whose orbital integrals match those of f (see (16)). We wish to show that there is a corresponding matching (60) of (twisted) characters between representations in the packets $\Pi_{\varphi_{H_1}}$ and Π_φ.

Our first step towards showing such a matching is to define the twisted characters appearing on the right-hand side of (60). If π is equivalent to $\omega \otimes \pi^\theta$ then the distributions $\Theta_{\pi, \mathsf{U}_\pi}$ are twisted characters (see (34)) where U_π is obtained as follows. If $\pi = \pi_\Lambda$, as in the discussion surrounding (70), then U_π is obtained as in section 5.4 through the extension of ϖ_1 afforded by the operator $\tau_0 \tilde{\rho}_1^{-1} \otimes S_{\Lambda_1}(\delta\theta)$. Otherwise $\pi \in \Pi_\varphi$ is of the form $\pi_{w^{-1}\Lambda}$ as in Lemma 16 and U_π is determined in the same manner by the operator $(\tau_0(\tilde{\rho}_1)^{-1})^{\dot{w}^{-1}} \otimes S_{w^{-1}\Lambda_1}(\dot{w}^{-1}\delta\theta\dot{w})$, where $\dot{w} \in G$ is a representative of w satisfying (64) (see

Lemma 15 and Proposition 2). Although U_π, and hence $\Theta_{\pi,\mathsf{U}_\pi}$ are only determined up to a root of unity, we shall eventually see that the product $\Delta(\boldsymbol{\varphi_{H_1}}, \pi)\, \Theta_{\pi,\mathsf{U}_\pi}(x\delta\theta)$ is well-defined.

We commence our proof of (60) from the integral on the left-hand side. Taking the integral over the quotient with $Z_1(\mathbf{R})$ is justified by the fact that the integrand is invariant under $Z_1(\mathbf{R})$ (see (15) and (56)). By the usual Weyl integration formula, this integral becomes

$$\frac{1}{|\Omega(H(\mathbf{R}), T_H(\mathbf{R}))|} \int_{T_H(\mathbf{R})} \sum_{\pi_{H_1} \in \Pi_{\varphi_{H_1}}} \Theta_{\pi_{H_1}}(t)\, \mathcal{O}_t(f_{H_1})\, D_H(t)^2\, dt. \tag{97}$$

In this integral we are identifying $T_H(\mathbf{R})$ with the quotient $T_{H_1}(\mathbf{R})/Z_1(\mathbf{R})$ (see §6.2). According to Lemma 16 (or Proposition 7.1.1 [Lab08]), we have

$$\sum_{\pi_{H_1} \in \Pi_1} \Theta_{\pi_{H_1}}(t) = \sum_{w_1 \in \Omega(H,T_H)/\Omega_{\mathbf{R}}(H,T_H)} \Theta_{\pi'_{H_1}}(\dot{w}_1 t \dot{w}_1^{-1})$$

for any $\pi'_{H_1} \in \Pi_{\varphi_{H_1}}$ and representatives $\dot{w}$ of w. In addition, as $T_H(\mathbf{R})$ is elliptic (Corollary 3), the Weyl groups $\Omega_{\mathbf{R}}(H, T_H)$ and $\Omega(H(\mathbf{R}), T_H(\mathbf{R}))$ are isomorphic (Lemma 6.4.1 [Lab08]). We may consequently rewrite (97) as

$$\frac{1}{|\Omega_{\mathbf{R}}(H, T_H)|} \int_{T_H(\mathbf{R})} \sum_{w_1 \in \Omega(H,T_H)/\Omega_{\mathbf{R}}(H,T_H)} \Theta_{\pi_{H_1}}(\dot{w}_1 t \dot{w}_1^{-1})\, \mathcal{O}_t(f_{H_1})\, D_H(t)^2\, dt, \tag{98}$$

where π_{H_1} is some representation in $\Pi_{\varphi_{H_1}}$. The change of variable

$$t \mapsto \dot{w} t \dot{w}^{-1},\ w \in \Omega(H, T_H)/\Omega_{\mathbf{R}}(H, T_H)$$

has the sole effect of replacing $\mathcal{O}_t(f_{H_1})$ by $\mathcal{O}_{\dot{w}t\dot{w}^{-1}}(f_{H_1})$ in the integrand of (98). Making this change of variable for every $w \in \Omega(H, T_H)/\Omega_{\mathbf{R}}(H, T_H)$, we see that (98) is equal to

$$\frac{1}{|\Omega(H, T_H)|} \int_{T_H(\mathbf{R})} \sum_{w_1 \in \Omega(H,T_H)/\Omega_{\mathbf{R}}(H,T_H)} \Theta_{\pi_{H_1}}(\dot{w}_1 t \dot{w}_1^{-1})\, \mathcal{SO}_t(f_{H_1})\, D_H(t)^2\, dt, \tag{99}$$

where

$$\mathcal{SO}_t(f_{H_1}) = \sum_{w \in \Omega(H,T_H)/\Omega_{\mathbf{R}}(H,T_H)} \mathcal{O}_{\dot{w}t\dot{w}^{-1}}(f_{H_1}).$$

By Lemma 16 (or Proposition 6.4.2 [Lab08]), the sum $\mathcal{SO}_t(f_{H_1})$ of orbital integrals is equal to the left-hand side of (16) for t in a dense subset of $T_H(\mathbf{R})^0\gamma$. In fact, for almost every $t \in T_H(\mathbf{R})^0$, the element $t\gamma$ is a the image under η of a strongly θ-regular element in $S^{\delta\theta}(\mathbf{R})^0\delta$ and is a norm of this element. Recall that the η is a local isomorphism. The geometric matching identity (16) and Lemma 15 therefore allow us to replace $\mathcal{SO}_t(f_{H_1})$ in (99) by

$$\sum_{w \in (S^{\delta\theta} \backslash (N_G(S)/N_{G(\mathbf{R})})^{\delta\theta}} \Delta(t\gamma, \dot{w}^{-1}\eta^{-1}(t)\delta\theta(\dot{w}))\, O_{\dot{w}^{-1}\eta^{-1}(t)\delta\theta\dot{w}}(f),$$

where $\dot{w} \in G(\mathbf{R})$ is a representative for $w \in (S^{\delta\theta} \backslash (N_G(S)/N_{G(\mathbf{R})})^{\delta\theta}$ satisfying (65). Alternatively, we may replace (99) with

$$\begin{aligned} &\frac{1}{|\Omega(H, T_H)|} \int_{S^{\delta\theta}(\mathbf{R})^0} \sum_{w_1} \Theta_{\pi_{H_1}}(\dot{w}_1 \eta_1(x)\gamma_1 \dot{w}_1^{-1}) \\ &\quad \times \sum_{w} \Delta(\eta(x)\gamma, \dot{w}^{-1}x\delta\theta(\dot{w}))\, \mathcal{O}_{\dot{w}^{-1}x\delta\theta\dot{w}}(f)\, D_H(\eta(x)\gamma)^2\, dx, \end{aligned} \tag{100}$$

where the first and second sums are taken over $\Omega(H, T_H)/\Omega_{\mathbf{R}}(H, T_H)$ and $(S^{\delta\theta} \backslash (N_G(S)/N_{G(\mathbf{R})})^{\delta\theta}$ respectively.

Let us now turn our attention to the character values $\Theta_{\pi_{H_1}}(\dot{w}_1\eta(x)\gamma\dot{w}_1^{-1})$ in (100). The work of Harish-Chandra (see [Var77]) expresses the values of $\Theta_{\pi_{H_1}}$ at regular elements $t \in T_{H_1}(\mathbf{R})$ as

$$\operatorname{sgn}(H) \sum_{w' \in \Omega(H_1(\mathbf{R}), T_{H_1}(\mathbf{R}))} \frac{\det(w')\, \Lambda(w'\mu_{H_1} - \iota_H, \lambda_{H_1})(t)}{\prod_{\alpha>0} 1 - \alpha^{-1}(t)}. \tag{101}$$

Here, $\Lambda(\mu_{H_1} - \iota_H, \lambda_{H_1})$ is a quasicharacter of $T_{H_1}(\mathbf{R})$ given in (92). Its differential after restriction to $T_{H_1}(\mathbf{R})^0$ is a regular element in $\mathfrak{t}^*_{H_1} \otimes \mathbf{C}$. In the language of §5.3.2, this element is equal to $i\Lambda_{H_1} - \rho_H$, where ρ_H is the half-sum of the positive roots of the Borel subalgebra B_H. Given the freedom in our choice of $\pi_{H_1} \in \Pi_{\varphi_{H_1}}$, we may assume that this is the Borel subalgebra determined by the regular element Λ_{H_1}. The product in the denominator is taken over the set of positive roots of (B_H, T_H). The term $\operatorname{sgn}(H)$ is a sign which is a product of terms (see (25)-(27) [Var77]) depending only on H.

We wish to substitute (101) into (100). By Lemma 6.4.1 [Lab08], the sum in (101) may be taken over

$$\Omega_{\mathbf{R}}(H, T_H) \cong \Omega(H(\mathbf{R}), T_H(\mathbf{R})) \cong \Omega(H_1(\mathbf{R}), T_{H_1}(\mathbf{R})).$$

Although $\eta(x)\gamma$ in (100) belongs to $T_H(\mathbf{R})^0$ and not $T_{H_1}(\mathbf{R})$, the invariance of the integrand under $Z_1(\mathbf{R})$ alluded to earlier justifies the abuse of writing

$$\begin{aligned}
&\sum_{w_1\in\Omega(H,T_H)/\Omega_{\mathbf{R}}(H,T_H)} \Theta_{\pi_{H_1}}(\dot{w}_1\eta(x)\gamma\dot{w}_1^{-1}) \\
&= \operatorname{sgn}(H)\sum_{w_1}\sum_{w'\in\Omega_{\mathbf{R}}(H,T_H)} \frac{\det(w_1w')\,\Lambda(w_1w'\mu_{H_1}-\iota_H,\lambda_{H_1})(\eta(x)\gamma)}{\prod_{\alpha>0}1-\alpha^{-1}(\eta(x)\gamma)} \\
&= \operatorname{sgn}(H)\sum_{w_1\in\Omega(H,T_H)} \frac{\det(w_1)\,\Lambda(w_1\mu_{H_1}-\iota_H,\lambda_{H_1})(\eta(x)\gamma)}{\prod_{\alpha>0}1-\alpha^{-1}(\eta(x)\gamma)} \\
&= \operatorname{sgn}(H)\sum_{w_1\in\Omega(H,T_H)} \frac{\det(w_1)\,\Lambda(w_1\mu_{H_1}-\iota_H,\lambda_{H_1})(\eta(x)\gamma)}{\det(1-\mathrm{Ad}\eta(x)\gamma)_{|\bar{\mathfrak{u}}_H}} \\
(102) &= \operatorname{sgn}(H)\sum_{w_1\in\Omega(H,T_H)} \frac{\Lambda(\mu_{H_1}-\iota_H,\lambda_{H_1})(\dot{w}_1\eta(x)\gamma\dot{w}_1^{-1})}{\det(1-\mathrm{Ad}\eta(x)\gamma)_{|\bar{\mathfrak{u}}_H}}.
\end{aligned}$$

In these equations we have combined the actions from $\Omega(H,T_H)/\Omega_{\mathbf{R}}(H,T_H)$ and $\Omega_{\mathbf{R}}(H,T_H)$ into a single action from $\Omega(H,T_H)$ using (23) [Var77].

Before we substitute (102) into (100), let us explain how it is to be converted into a character on $S^{\delta\theta}(\mathbf{R})^0$. The character expansion (102) is linked to the geometric transfer factor $\Delta(\eta_1(x)\gamma_1, w^{-1}x\delta\theta(w))$, by first applying the expansion

$$\Delta(\eta(x)\gamma,\dot{w}^{-1}x\delta\theta(\dot{w})) = \langle \mathrm{inv}(x\delta,\dot{w}^{-1}x\delta\theta(\dot{w})),\kappa_\delta\rangle\;\Delta(\eta(x)\gamma,x\delta\theta)$$

from Lemma 5.1.B and Theorem 5.1.D [KS99]. The definition of $\mathrm{inv}(x\delta,\dot{w}^{-1}x\delta\theta(\dot{w}))$ depends only on w and is independent of x (page 54 [KS99]), so we may write

$$\Delta(\eta(x)\gamma,\dot{w}^{-1}x\delta\theta(\dot{w})) = \langle \mathrm{inv}(\delta,w^{-1}\delta\theta(w)),\kappa_\delta\rangle\;\Delta(\eta(x)\gamma,x\delta\theta).$$

Then, expanding $\Delta(\eta_1(x)\gamma_1,x\delta\theta)$ according to (73) and (78), we may apply Corollary 4. Corollary 4 delivers a character on $T'_{\theta^*}(\mathbf{R})$ which we may transport locally to $S^{\delta\theta}(\mathbf{R})^0$ via isomorphisms (14) and (61). Isomorphism (14) also applies to (95), thereby accounting for the character values at γ and δ. Finally, the denominator of (102) may be combined with the transfer factors Δ_{II} and Δ_{IV} as in Lemma 17. The net effect of this course of action is to replace (102) with

$$\operatorname{sgn}(H)\sum_{w_1\in\Omega(H,T_H)}\frac{\Lambda(\mu-\iota_{G^*},\lambda)(x^*\delta^*)}{\det(1-\mathrm{Ad}x\delta\theta)_{|\bar{\mathfrak{u}}}} = \operatorname{sgn}(H)|\Omega(H,T_H)|\frac{\Lambda(\mu-\iota_{G^*},\lambda)(x^*\delta^*)}{\det(1-\mathrm{Ad}x\delta\theta)_{|\bar{\mathfrak{u}}}}$$

and express (100) as

$$\operatorname{sgn}(H) i^{\dim \mathfrak{u}_{G^\theta} - \dim \mathfrak{u}_H} \Delta_I(\gamma,\delta)\ \Delta_{III}(\gamma_1,\delta;\gamma_1^0,\delta^0)\ \langle(\delta^*,\gamma_1), a_{T'}^{-1}\rangle$$

$$(103)\quad \int_{S^{\delta\theta}(\mathbf{R})^0} \frac{\Lambda(\mu - \iota_{G^*},\lambda)(x^*\delta^*)}{\det(1-\mathrm{Ad}x\delta\theta)_{|\bar{\mathfrak{u}}}}$$

$$\times \sum_{w \in S^{\delta\theta}\backslash (N_G(S)/N_{G(\mathbf{R})}(S))^{\delta\theta}} \langle \mathrm{inv}(\delta, \dot{w}^{-1}\delta\theta(\dot{w})), \kappa_\delta\rangle\ \mathcal{O}_{\dot{w}^{-1}x\delta\theta\dot{w}}(f)\ D_{G^\theta}(x\delta)^2\ dx.$$

Given our assumption on the support of f, we may replace the integral over $S^{\delta\theta}(\mathbf{R})^0$ by an integral over $S^{\delta\theta}(\mathbf{R}) = S^{\delta\theta}_{\mathrm{der}}(\mathbf{R}) Z_G^{\delta\theta}(\mathbf{R})$ (see (29)) to obtain

$$\operatorname{sgn}(H) i^{\dim \mathfrak{u}_{G^\theta} - \dim \mathfrak{u}_H} \Delta_I(\gamma,\delta)\ \Delta_{III}(\gamma_1,\delta;\gamma_1^0,\delta^0)\ \langle(\delta^*,\gamma_1), a_{T'}^{-1}\rangle$$

$$(104)\quad \times \int_{S^{\delta\theta}_{\mathrm{der}}(\mathbf{R})} \int_{Z_G^{\delta\theta}(\mathbf{R})} \frac{\Lambda(\mu - \iota_{G^*},\lambda)(x^*\delta^*)}{\det(1-\mathrm{Ad}x\delta\theta)_{|\bar{\mathfrak{u}}}}\ \chi_\pi(z)$$

$$\times \sum_{w \in S^{\delta\theta}\backslash (N_G(S)/N_{G(\mathbf{R})}(S))^{\delta\theta}} \langle \mathrm{inv}(\delta, \dot{w}^{-1}\delta\theta(\dot{w})), \kappa_\delta\rangle\ \mathcal{O}_{\dot{w}^{-1}zx\delta\theta\dot{w}}(f)\ D_{G^\theta}(x\delta)^2\ dz\, dx.$$

In order to recover the character value (47), we shall make a change of variable

$$(105)\qquad x \mapsto \dot{w}_1 x \delta\theta(\dot{w}_1^{-1})\delta^{-1}$$

for representatives $\dot{w}_1 \in G_{\mathrm{der}}(\mathbf{R})^0$ of elements in $w_1 \in \Omega(G_{\mathrm{der}}(\mathbf{R})^0, S_{\mathrm{der}}(\mathbf{R}))^{\delta\theta}$ and take the sum over $\Omega(G_{\mathrm{der}}(\mathbf{R})^0, S_{\mathrm{der}}(\mathbf{R}))^{\delta\theta}$ in the integral of (104). Each change of variable has no effect on the summand

$$\langle \mathrm{inv}(\delta, \dot{w}^{-1}\delta\theta(\dot{w})), \kappa_\delta\rangle\ \mathcal{O}_{\dot{w}^{-1}zx\delta\theta\dot{w}}(f)\ D_{G^\theta}(x\delta)^2.$$

by Theorem 5.1.D (2) [KS99] (see also page 71 [KS99]). To describe the effect of the changes of variable in the denominator $\det(1-\mathrm{Ad}x\delta\theta)_{|\bar{\mathfrak{u}}}$ we recall the definition of $e(\dot{w}_1) \in S_{\mathrm{der}}(\mathbf{R})$ in (45).

Lemma 20 *Suppose* $\dot{w}_1 \in G_{\mathrm{der}}(\mathbf{R})^0$ *is a representative of an element in* $w_1 \in \Omega(G_{\mathrm{der}}(\mathbf{R})^0, S_{\mathrm{der}}(\mathbf{R}))^{\delta\theta}$. *Then* $\det(1-\mathrm{Ad}\dot{w}_1 x\delta\theta\dot{w}_1^{-1})_{|\bar{\mathfrak{u}}}$ *is equal to*

$$\det(w_1)\,(\iota_G - w_1\iota_G)(x)\ \iota_G^{-1}(e(\dot{w}_1))\ \det(1-\mathrm{Ad}x\delta\theta)_{|\bar{\mathfrak{u}}}.$$

Moreover, this expression is independent of the choice of representative.

Proof. As in §1.3 [KS99], set $\alpha_{\mathrm{res}} = \alpha_{|S^{\delta\theta}}$ for any root $\alpha \in \mathrm{R}(G, S)$. For any negative root $\alpha \in R(G, S)$, let $\mathfrak{g}^{\alpha_{\mathrm{res}}}$ denote the subspace in the Lie algebra $\mathfrak{g}$ of G generated by the $\delta\theta$-orbit of the root space of α (see Lemma 3), and $\mathfrak{g}^{\alpha}$ denote the root space of α. We further adopt the notation of §1.3 [KS99] by setting l_α equal to the cardinality of the $\delta\theta$-orbit of α and $N\alpha = \sum_{i=0}^{l_\alpha - 1} (\delta\theta)^i \alpha$ (in additive notation). Then, performing the type of computation indicated in §4.5 [KS99], we see that

$$\begin{aligned}
\det(1 - \mathrm{Ad}\dot{w}_1 x \delta\theta \dot{w}_1^{-1})_{|\bar{\mathfrak{u}}} &= \det(1 - \mathrm{Ad}\dot{w}_1 x \dot{w}_1^{-1} e(\dot{w}_1)\delta\theta)_{|\bar{\mathfrak{u}}} \\
&= \prod_{\alpha_{\mathrm{res}}<0} \det(1 - \mathrm{Ad}\dot{w}_1 x \dot{w}_1^{-1} e(\dot{w}_1)\delta\theta)_{|\mathfrak{g}^{\alpha_{\mathrm{res}}}} \\
&= \prod_{\alpha_{\mathrm{res}}<0} 1 - N w_1\alpha(x)\, N\alpha(e(\dot{w}_1))\, (\mathrm{Ad}\delta\theta)^{l_\alpha}_{|\mathfrak{g}^{\alpha}}
\end{aligned}$$

Now

$$\begin{aligned}
N\alpha(e(\dot{w}_1)) &= \alpha(\prod_{i=0}^{l_\alpha - 1} (\mathrm{Int}(\delta)\theta)^i (e(\dot{w}_1))) \\
&= \mathrm{Ad}((e(\dot{w}_1)\delta\theta)^{l_\alpha} (\delta\theta)^{-l_\alpha})_{|\mathfrak{g}^{\alpha}} \\
&= \mathrm{Ad}(\dot{w}_1 (\delta\theta)^{l_\alpha} \dot{w}_1^{-1})_{|\mathfrak{g}^{\alpha}}\, (\mathrm{Ad}\delta\theta)^{-l_\alpha}_{|\mathfrak{g}^{\alpha}}
\end{aligned}$$

so that

$$\begin{aligned}
&\det(1 - \mathrm{Ad} w_1 x \delta\theta w_1^{-1})_{|\bar{\mathfrak{u}}} \\
&= \prod_{\alpha_{\mathrm{res}}<0} 1 - N w_1\alpha(x)\, \mathrm{Ad}(\dot{w}_1 (\delta\theta)^{l_\alpha} \dot{w}_1^{-1})_{|\mathfrak{g}^{\alpha}} \\
&= \prod_{\alpha_{\mathrm{res}}<0} 1 - N\alpha(x)\, \mathrm{Ad}((\delta\theta)^{l_\alpha})_{|\mathfrak{g}^{w_1\alpha}} \\
&= \prod_{\alpha_{\mathrm{res}}<0} \det(1 - \mathrm{Ad}x\delta\theta)_{|\mathfrak{g}^{w_1\alpha_{\mathrm{res}}}} \\
&= \prod_{\alpha_{\mathrm{res}}<0,\, w_1\alpha_{\mathrm{res}}<0} \det(1 - \mathrm{Ad}x\delta\theta)_{|\mathfrak{g}^{w_1\alpha_{\mathrm{res}}}} \prod_{\alpha_{\mathrm{res}}<0,\, w_1\alpha_{\mathrm{res}}>0} \det(1 - \mathrm{Ad}x\delta\theta)_{|\mathfrak{g}^{w_1\alpha_{\mathrm{res}}}}
\end{aligned}$$

The second product of this last expression is equal to

$$\begin{aligned}
&\prod_{\alpha_{\mathrm{res}}<0,\, w_1\alpha_{\mathrm{res}}>0} \det(\mathrm{Ad}x\delta\theta)_{|\mathfrak{g}^{w_1\alpha_{\mathrm{res}}}}\, (-1)^{l_\alpha}\, \det(1 - \mathrm{Ad}(x\delta\theta)^{-1})_{\mathfrak{g}^{w_1\alpha_{\mathrm{res}}}} \\
&= \det(w_1) \prod_{\alpha_{\mathrm{res}}<0,\, w_1\alpha_{\mathrm{res}}>0} \det(\mathrm{Ad}x\delta\theta)_{|\mathfrak{g}^{w_1\alpha_{\mathrm{res}}}}\, \det(1 - \mathrm{Ad}x\delta\theta)_{\mathfrak{g}^{-w_1\alpha_{\mathrm{res}}}}
\end{aligned}$$

(see p. 395 [Ren97]). Substituting back, we arrive at

$$\begin{aligned}
&\det(1-\mathrm{Ad}w_1x\delta\theta w_1^{-1})_{|\bar{\mathfrak{u}}}\\
&= \det(1-\mathrm{Ad}x\delta\theta)_{|\bar{\mathfrak{u}}}\,\det(w_1)\prod_{\alpha_{\mathrm{res}}<0,\,w_1\alpha_{\mathrm{res}}>0}\det(\mathrm{Ad}x\delta\theta)_{|\mathfrak{g}^{w_1\alpha_{\mathrm{res}}}}.
\end{aligned}$$

To derive the first assertion of the lemma, we compute

$$\begin{aligned}
&\prod_{\alpha_{\mathrm{res}}<0,\,w_1\alpha_{\mathrm{res}}>0}\det(\mathrm{Ad}x\delta\theta)_{|\mathfrak{g}^{w_1\alpha_{\mathrm{res}}}}\\
&= \prod_{\alpha_{\mathrm{res}}<0,\,w_1\alpha_{\mathrm{res}}>0}\det(\mathrm{Ad}\dot{w}_1x\delta\theta\dot{w}_1^{-1})_{|\mathfrak{g}^{\alpha_{\mathrm{res}}}}\\
&= \prod_{\alpha_{\mathrm{res}}>0}\det(\mathrm{Ad}x\delta\theta)^{1/2}_{|\mathfrak{g}^{\alpha_{\mathrm{res}}}}\,\det(\mathrm{Ad}\dot{w}_1x\delta\theta\dot{w}_1^{-1})^{-1/2}_{|\mathfrak{g}^{\alpha_{\mathrm{res}}}}\\
&= (\iota_G-w_1\iota_G)(x)\,\det(\mathrm{Ad}\delta\theta)^{1/2}_{|\mathfrak{u}}\,\det(\mathrm{Ad}\dot{w}_1\delta\theta\dot{w}_1^{-1})^{-1/2}_{|\mathfrak{u}}\\
&= (\iota_G-w_1\iota_G)(x)\ \det(\mathrm{Ad}\dot{w}_1\delta\theta\dot{w}_1^{-1}(\delta\theta)^{-1})^{-1/2}_{|\mathfrak{u}}\\
&= (\iota_G-w_1\iota_G)(x)\ \iota_G^{-1}(e(\dot{w}_1)).
\end{aligned}$$

This value is independent of the choice of representative $\dot{w}_1$, since, for any choice of $s\in S_{\mathrm{der}}(\mathbf{R})$, we have

$$\begin{aligned}
\det(1-\mathrm{Ad}s\dot{w}_1x\delta\theta\dot{w}_1^{-1}s^{-1})_{|\bar{\mathfrak{u}}} &= \det(\mathrm{Ad}s)_{|\bar{\mathfrak{u}}}\det(1-\mathrm{Ad}\dot{w}_1x\delta\theta\dot{w}_1^{-1})_{|\bar{\mathfrak{u}}}\det(\mathrm{Ad}s^{-1})_{|\bar{\mathfrak{u}}}\\
&= \det(1-\mathrm{Ad}\dot{w}_1x\delta\theta\dot{w}_1^{-1})_{|\bar{\mathfrak{u}}}.\blacksquare
\end{aligned}$$

Under transport via isomorphism (14), the effect of the change of variable (105) on the numerator $\Lambda(\mu-\iota_{G^*},\lambda)(x^*\delta^*)$ in (104) may be coupled with Lemma 20 and equations (48) and (43) to yield

$$\begin{aligned}
&\frac{\Lambda(\mu-\iota_{G^*},\lambda)(\dot{w}_1x^*\dot{w}_1^{-1}\dot{w}_1\delta^*\theta^*(\dot{w}_1^{-1})(\delta^*)^{-1}\delta^*)}{\det(1-\mathrm{Ad}\dot{w}_1x\delta\theta\dot{w}_1^{-1})_{|\bar{\mathfrak{u}}}}\\
&= \frac{\Lambda(\mu-\iota_{G^*},\lambda)(\dot{w}_1x^*\dot{w}_1^{-1})\ \Lambda(\mu-\iota_{G^*},\lambda)(e(\dot{w}_1))\ \Lambda(\mu-\iota_{G^*},\lambda)(\delta^*)}{\det(w_1)\,(\iota_{G^*}-w_1\iota_{G^*})(x^*)\,\iota_{G^*}^{-1}(e(\dot{w}_1))\ \det(1-\mathrm{Ad}x\delta\theta)_{|\bar{\mathfrak{u}}}}\\
&= \frac{\det(w_1)\,\Lambda(w_1\mu-\iota_{G^*},\lambda)(x^*)\ \Lambda(\mu,\lambda)(e(w_1))\ \bar{\tau}_0(\delta\theta)/\zeta}{\det(1-\mathrm{Ad}x\delta\theta)_{|\bar{\mathfrak{u}}}}.
\end{aligned}$$

Comparing this expression with $\Theta_{\varpi_1,\mathsf{U}_1}(x\delta\theta)$ as in (47), and performing the desired changes of variable in (104) results in the product of the scalar

$$\frac{\mathrm{sgn}(H)i^{\dim\mathfrak{u}_{G^\theta}-\dim\mathfrak{u}_H}\Delta_I(\gamma,\delta)\ \Delta_{III}(\gamma_1,\delta;\gamma_1^0,\delta^0)\ \langle(\delta^*,\gamma_1),a_{T'}^{-1}\rangle}{|\Omega(G_{\mathrm{der}}(\mathbf{R})^0,S_{\mathrm{der}}(\mathbf{R}))^{\delta\theta}|\ (-1)^{q-\Lambda_1}\zeta}$$

with the sum

$$\sum_{w\in S^{\delta\theta}\backslash(N_G(S)/N_{G(\mathbf{R})}(S))^{\delta\theta}} \langle \mathrm{inv}(\delta, \dot{w}^{-1}\delta\theta(\dot{w})), \kappa_\delta\rangle \int_{S^{\delta\theta}_{\mathrm{der}}(\mathbf{R})} \Theta_{\varpi_1,\mathsf{U}_1}(x\delta\theta) \times \int_{Z^{\delta\theta}_G(\mathbf{R})} \chi_\pi(z)\ \mathcal{O}_{\dot{w}^{-1}zx\delta\theta\dot{w}}(f)\, dz\, D_{G\theta}(x\delta)^2\, dx. \tag{106}$$

There are two steps left in placing this expression into the desired form. The first step is to "move" conjugation by w from the orbital integral to the character. The second step is to apply the Weyl integration formula.

Towards the first step, let us determine a subset of $S^{\delta\theta}\backslash(N_G(S)/N_{G(\mathbf{R})}(S))^{\delta\theta}$ outside of which the summands of (106) vanish. By Proposition 2, we may express (106) as

$$\sum_{w\in(\Omega(G,S)/\Omega_{\mathbf{R}}(G,S))^{\delta\theta}} \sum_{z_1} \langle \mathrm{inv}(\delta, (z_1\dot{w})^{-1}\delta\theta(z_1\dot{w})), \kappa_\delta\rangle \times \int_{S^{\delta\theta}_{\mathrm{der}}(\mathbf{R})} \Theta_{\varpi_1,\mathsf{U}_1}(x\delta\theta) \int_{Z^{\delta\theta}_G(\mathbf{R})} \chi_\pi(z)\ \mathcal{O}_{(\theta-1)(z_1)z\dot{w}^{-1}x\delta\theta\dot{w}}(f)\, dz\, D_{G\theta}(x\delta)^2\, dx, \tag{107}$$

where $z_1 \in A_G$ is in the split component of the centre of G, and $(\theta-1)(z_1)$ belongs to $F \subset A_G(\mathbf{R})$, an elementary 2-group as in (69).

The support of f ensures that $\mathcal{O}_{(\theta-1)(z_1)zw^{-1}s\delta\theta w}(f)$ vanishes unless $(\theta-1)(z_1)zw^{-1}s\delta\theta w$ lies in $Z_G(\mathbf{R})^0 S^{\delta\theta}\delta\theta$. For this to hold the element $(\theta-1)(z_1)$ must belong to

$$((\theta-1)A_G)(\mathbf{R}) \cap Z_G(\mathbf{R})^0 \cap F = ((\theta-1)A_G)(\mathbf{R})^0 \cap F = \{1\}.$$

The sum in (107) consequently reduces to a positive integer multiple of

$$\sum_w \langle \mathrm{inv}(\delta, \dot{w}^{-1}\delta\theta(\dot{w})), \kappa_\delta\rangle \times \int_{S^{\delta\theta}_{\mathrm{der}}(\mathbf{R})} \Theta_{\varpi_1,\mathsf{U}_1}(x\delta\theta) \int_{Z^{\delta\theta}_G(\mathbf{R})} \chi_\pi(z)\ \mathcal{O}_{z\dot{w}^{-1}x\delta\theta\dot{w}}(f)\, dz\, D_{G\theta}(x\delta)^2\, dx \tag{108}$$

in which the sum is taken over those $w \in (\Omega(G,S)/\Omega_{\mathbf{R}}(G,S))^{\delta\theta}$ for which there is a representative $\dot{w} \in G$ such that

$$\dot{w}^{-1} S^{\delta\theta}_{\mathrm{der}}(\mathbf{R})\delta\theta\dot{w} \subset Z_G(\mathbf{R})^0\, S^{\delta\theta}_{\mathrm{der}}(\mathbf{R})\delta\theta. \tag{109}$$

Specializing to the identity element in $S^{\delta\theta}_{\mathrm{der}}(\mathbf{R})$ in the left-hand side of (109) we see that

$$\dot{w}^{-1}\delta\theta\dot{w}(\delta\theta)^{-1} \in Z_G(\mathbf{R})^0\, S^{\delta\theta}_{\mathrm{der}}(\mathbf{R}) \subset S(\mathbf{R}). \tag{110}$$

Under this assumption, $\dot{w}$ is a representative of a coset in $\Omega(G,S)^{\delta\theta}$. It follows that the sum in (108) reduces to a sum over the subset

$$(\Omega(G,S)/\Omega_{\mathbf{R}}(G,S))^{\delta\theta} \cap \Omega(G,S)^{\delta\theta}\Omega_{\mathbf{R}}(G,S)/\Omega_{\mathbf{R}}(G,S). \tag{111}$$

It is easily verified that this subset is in canonical bijection with $\Omega(G,S)^{\delta\theta}_{\mathbf{R}}/\Omega_{\mathbf{R}}(G,S)^{\delta\theta}$, where $\Omega(G,S)^{\delta\theta}_{\mathbf{R}}$ is defined to be the subgroup of elements $w \in \Omega(G,S)^{\delta\theta}$, which have a representative $\dot{w} \in G$ satisfying $w^{-1}\delta\theta w(\delta\theta)^{-1} \in S(\mathbf{R})$.

We may now reduce (107) to the sum

$$\begin{aligned} &\frac{n_\theta}{|\Omega_{\mathbf{R}}(G,S)^{\delta\theta}|} \sum_{w \in \Omega(G,S)^{\delta\theta}_{\mathbf{R}}} \langle \mathrm{inv}(\delta, \dot{w}^{-1}\delta\theta(\dot{w})), \kappa_\delta\rangle \\ &\times \int_{S^{\delta\theta}_{\mathrm{der}}(\mathbf{R})} \Theta_{\varpi_1,\mathsf{U}_1}(x\delta\theta) \int_{Z^{\delta\theta}_G(\mathbf{R})} \chi_\pi(z)\ \mathcal{O}_{\dot{w}^{-1}zx\delta\theta\dot{w}}(f)\, dz\, D_{G\theta}(x\delta)^2\, dx, \end{aligned} \tag{112}$$

in which we have included the sum over $\Omega_{\mathbf{R}}(G,S)^{\delta\theta}$ and n_θ is the number of representatives $z_1 \in A_G$ in (107) such that $(\theta-1)(z_1) = 1$.

Writing $\varpi_1 = \varpi_{\Lambda_1}$ as in (70) and noting that the intertwining operator U_1 corresponds to a unique choice of τ_0 as in section 5.4, it is valid to write $\Theta_{\varpi_{\Lambda_1},\tau_0}$ in place of $\Theta_{\varpi_1,\mathsf{U}_1}$. Exchanging the summation over $\Omega(G,S)^{\delta\theta}_{\mathbf{R}}$ in (112) with the summation over $\Omega(G_{\mathrm{der}}(\mathbf{R}), S_{\mathrm{der}}(\mathbf{R}))^{\delta\theta}$ in (47) and noting that for all $w_1 \in \Omega(G_{\mathrm{der}}(\mathbf{R}), S_{\mathrm{der}}(\mathbf{R}))$

$$\mathrm{inv}(\delta, (w_1ww_1^{-1})^{-1}\delta\theta(w_1ww_1^{-1})) = \mathrm{inv}(\delta, w^{-1}\delta\theta(w))$$

(page 54 [KS99], Proposition 2 [Tit66]) it may be computed that (112) is equal to

$$\begin{aligned} &\frac{n_\theta}{|\Omega_{\mathbf{R}}(G,S)^{\delta\theta}|} \sum_{w \in \Omega(G,S)^{\delta\theta}_{\mathbf{R}}} \langle \mathrm{inv}(\delta, \dot{w}^{-1}\delta\theta(\dot{w})), \kappa_\delta\rangle \int_{S^{\delta\theta}_{\mathrm{der}}(\mathbf{R})} \Theta_{\varpi_{w^{-1}\Lambda_1},\tau_0^{\dot{w}^{-1}}}(\dot{w}^{-1}x\dot{w}\dot{w}^{-1}\delta\theta\dot{w}) \\ &\quad\times \int_{Z^{\delta\theta}_G(\mathbf{R})} \chi_\pi(z)\ \mathcal{O}_{z\dot{w}^{-1}x\dot{w}\dot{w}^{-1}\delta\theta\dot{w}}(f)\, dz\, D_{G\theta}(x\delta)^2\, dx \\ &= n_\theta \sum_{w \in \Omega(G,S)^{\delta\theta}_{\mathbf{R}}/\Omega_{\mathbf{R}}(G,S)^{\delta\theta}} \langle \mathrm{inv}(\delta, \dot{w}^{-1}\delta\theta(\dot{w})), \kappa_\delta\rangle \int_{S^{\delta\theta}_{\mathrm{der}}(\mathbf{R})} \Theta_{\varpi_{w^{-1}\Lambda_1},\tau_0^{\dot{w}^{-1}}}(x\dot{w}^{-1}\delta\theta\dot{w}) \\ &\quad \int_{Z^{\delta\theta}_G(\mathbf{R})} \chi_\pi(z)\ \mathcal{O}_{zx\dot{w}^{-1}\delta\theta\dot{w}}(f)\, dz\, D_{G\theta}(x\delta)^2\, dx. \end{aligned} \tag{113}$$

It follows from Corollary 1 that (113) is equal to the product of

$$\frac{n_\theta |\Omega(G_{\mathrm{der}}(\mathbf{R})^0, S^{\delta\theta}_{\mathrm{der}}(\mathbf{R})^0\delta\theta)|}{|\det(1-\mathrm{Ad}(\delta\theta))_{|\mathfrak{s}/\mathfrak{s}^{\delta\theta}\otimes\mathbf{C}}|}$$

with

$$(114) \qquad \sum_{w\in\Omega(G,S)^{\delta\theta}_{\mathbf{R}}/\Omega_{\mathbf{R}}(G,S)^{\delta\theta}} \langle \mathrm{inv}(\delta, \dot{w}^{-1}\delta\theta(\dot{w})), \kappa_\delta\rangle \ \Theta_{\pi_{w^{-1}\Lambda}, \mathsf{U}_{\pi_{w^{-1}\Lambda}}}(f)$$

(see (110)).

We must account for the twisted characters missing from this sum. The following lemma shows that the missing twisted characters are zero.

Lemma 21 *Suppose $w \in (\Omega(G,S)/\Omega_{\mathbf{R}}(G,S))^{\delta\theta}$ does not belong to the subset (111) and that w has a representative $\dot{w} \in N_G(S)$ satisfying $\dot{w}^{-1}\delta\theta\dot{w}(\delta\theta)^{-1} \in G(\mathbf{R})$. Then $\Theta_{\pi_{w^{-1}\Lambda}, \mathsf{U}_{\pi_{w^{-1}\Lambda}}}(f) = 0$.*

Proof. Arguing as in Lemma 5, one finds that $\Theta_{\pi_{w^{-1}\Lambda}, \mathsf{U}_{\pi_{w^{-1}\Lambda}}}(f)$ is equal to $(\chi_\pi \otimes \Theta_{\varpi_{w^{-1}\Lambda_1, \tau_0^{\dot{w}^{-1}}}})(f')$, where $f' \in C_c^\infty(Z_G(\mathbf{R})G_{\mathrm{der}}(\mathbf{R})^0\dot{w}^{-1}\delta\theta\dot{w})$ is defined by

$$f'(x\dot{w}^{-1}\delta\theta\dot{w}) = \sum_{r=1}^{k} \omega(\delta_r)\, f(\delta_r^{-1}x\dot{w}^{-1}\delta\theta\dot{w}\delta_r), \ x \in Z_G(\mathbf{R})G_{\mathrm{der}}(\mathbf{R})^0.$$

Suppose $f'(x\dot{w}^{-1}\delta\theta\dot{w}) \neq 0$. Then by our assumption on the support of f we know that $x\dot{w}^{-1}\delta\theta\dot{w}$ is $G(\mathbf{R})$-conjugate to an element of $Z_G(\mathbf{R})^0 S^{\delta\theta}_{\mathrm{der}}(\mathbf{R})\delta\theta$, that is there exist $g_1 \in G(\mathbf{R})$ and $s \in Z_G(\mathbf{R})^0 S^{\delta\theta}_{\mathrm{der}}(\mathbf{R})$ such that

$$x\dot{w}^{-1}\delta\theta\dot{w} = g_1^{-1}s\delta\theta g_1.$$

We will now show that $(\chi_\pi \otimes \Theta_{\varpi_{w^{-1}\Lambda_1, \tau_0^{\dot{w}^{-1}}}})(g_1^{-1}s\delta\theta g_1)$ vanishes when $s\delta$ is strongly θ-regular, thereby proving the lemma. According to Theorem 5.5.3 (i) [Bou87], the character value $(\chi_\pi \otimes \Theta_{\varpi_{w^{-1}\Lambda_1, \tau_0^{\dot{w}^{-1}}}})(g_1^{-1}s\delta\theta g_1)$ vanishes if there is no element $g_2 \in Z_G(\mathbf{R})G_{\mathrm{der}}(\mathbf{R})^0$ such that

$$g_2\dot{w}^{-1} \cdot \Lambda_1 \in (\mathfrak{s}^*_{\mathrm{der}})^{g_1^{-1}s\delta\theta g_1}.$$

By way of contradiction, let us suppose there is an element $g_2 \in Z_G(\mathbf{R})G_{\mathrm{der}}(\mathbf{R})^0$ as above. Then, setting $g = g_1g_2$, we deduce that

$$g\dot{w}^{-1} \cdot \Lambda_1 \in (\mathfrak{s}^*_{\mathrm{der}})^{\delta\theta}.$$

Recall from section 5.3.1 that $\Lambda_1 \in \mathfrak{s}^*_{\mathrm{der}}$ is regular. The fact that the coadjoint action of $g \in G(\mathbf{R})$ on the regular element $\dot{w}^{-1} \cdot \Lambda_1 \in \mathfrak{s}^*_{\mathrm{der}}$ belongs again to $\mathfrak{s}^*_{\mathrm{der}}$ implies that g lies in $N_{G(\mathbf{R})}(S)$. Obviously $g\dot{w}^{-1}$ lies in $N_G(S)$ and the previous containment implies

$$(\delta\theta)^{-1} \cdot g\dot{w}^{-1} \cdot \Lambda_1 = g\dot{w}^{-1} \cdot \Lambda_1.$$

This equation is equivalent to

$$(g\dot{w}^{-1})^{-1}(\delta\theta)^{-1}g\dot{w}^{-1}\delta\theta \cdot \Lambda_1 = \Lambda_1$$

by Lemma 6, and so

$$(g\dot{w}^{-1})^{-1}(\delta\theta)^{-1}g\dot{w}^{-1}\delta\theta \in S$$

(Theorem 7.101 [Kna96]). Since both $g\dot{w}^{-1}$ and $\delta\theta$ normalize S, this containment is equivalent to

$$g\dot{w}^{-1}\ \delta\theta\ (g\dot{w}^{-1})^{-1}\ (\delta\theta)^{-1} \in S.$$

The element on the left is equal to

$$g(\dot{w}^{-1}\delta\theta\dot{w}(\delta\theta)^{-1})g^{-1}(g\delta\theta g^{-1}(\delta\theta)^{-1}) \in G(\mathbf{R})$$

and so we conclude that

$$(g\dot{w}^{-1})^{-1}(\delta\theta)^{-1}g\dot{w}^{-1}\delta\theta \in S(\mathbf{R}).$$

However, this means that wg^{-1} belongs to $\Omega(G,S)^{\delta\theta}_{\mathbf{R}}$, or equivalently that w belongs to the intersection (111). This contradicts the hypothesis of the lemma.■

Lemma 21 allows us to replace the sum over $\Omega(G,S)^{\delta\theta}_{\mathbf{R}}/\Omega_{\mathbf{R}}(G,S)^{\delta\theta}$ in (114) by the sum over the entire set $(\Omega(G,S)/\Omega_{\mathbf{R}}(G,S))^{\delta\theta}$. At last, we conclude that the left-hand side of (60) is equal to the product of the three scalars

$$\frac{n_\theta|\Omega(G_{\mathrm{der}}(\mathbf{R})^0,(S^{\delta\theta}_{\mathrm{der}}(\mathbf{R}))^0\delta\theta)|}{|\Omega(G_{\mathrm{der}}(\mathbf{R})^0,S_{\mathrm{der}}(\mathbf{R}))^{\delta\theta}||\det(1-\mathrm{Ad}(\delta\theta))_{|\mathfrak{s}/\mathfrak{s}^{\delta\theta}\otimes\mathbf{C}}|}, \tag{115}$$

$$i^{\dim\mathfrak{u}_{G^\theta}-\dim\mathfrak{u}_H}\Delta_I(\gamma,\delta)\ \Delta_{III}(\gamma_1,\delta;\gamma_1^0,\delta^0)\ \langle(\delta^*,\gamma_1),a_{T'}^{-1}\rangle, \tag{116}$$

$$\frac{\mathrm{sgn}(H)}{(-1)^{q^{-\Lambda_1}}\zeta} \tag{117}$$

and the linear combination of twisted character values

$$\sum_{w \in (\Omega(G,S)/\Omega_{\mathbf{R}}(G,S))^{\delta\theta}} \langle \mathrm{inv}(\delta, \dot{w}^{-1}\delta\theta(\dot{w})), \kappa_\delta \rangle \int_{G(\mathbf{R})} f(x\delta\theta)\ \Theta_{\pi_{w^{-1}\Lambda}, \mathsf{U}_{\pi_{w^{-1}\Lambda}}}(x\delta\theta)\, dx.$$

The bijection of Lemma 16 associates to each $w \in (\Omega(G,S)/\Omega_{\mathbf{R}}(G,S))^{\delta\theta}$ a unique $\pi \in \Pi_\varphi$. Using this bijection, we define $\Delta(\boldsymbol{\varphi_{H_1}}, \pi)$ to be equal to the product of $\langle \mathrm{inv}(\delta, w^{-1}\delta\theta(w)), \kappa_\delta \rangle$ with the three scalars (115)-(117). With this definition of $\Delta(\boldsymbol{\varphi_{H_1}}, \pi)$, it is immediate that identity (60) holds.

6.3.1 Spectral comparisons with small support at other points

The spectral transfer factors $\Delta(\boldsymbol{\varphi_{H_1}}, \pi)$ we have recently defined, appear to depend on our choice of $\delta \in G(\mathbf{R})$. We shall show that, in fact, the spectral transfer factors do not depend on this choice. In consequence of this, the spectral transfer factors are well-defined when some θ-elliptic element of $G(\mathbf{R})$ has a norm in $H_1(\mathbf{R})$.

Suppose that $\bar{\delta} \in G(\mathbf{R})$ shares the same properties as $\delta \in G(\mathbf{R})$, namely, that $\bar{\delta}$ is strongly θ-regular, θ-elliptic and has norm $\bar{\gamma}_1 \in H_1(\mathbf{R})$ (*cf.* earlier section 6). The unique maximal torus of G containing $G^{\bar{\delta}\theta}$ is elliptic over $\mathbf{R}$ and conjugate to the torus S under some $g \in G(\mathbf{R})$ (Proposition 6.61 [Kna96]). Let us suppose initially that g is trivial so that S contains $G^{\bar{\delta}\theta}$.

Lemma 22 *The compact torus $S_{\mathrm{der}}(\mathbf{R})$ is equal to the product of $S_{\mathrm{der}}^{\delta\theta}(\mathbf{R})$ and $(1-\delta\theta)(S_{\mathrm{der}}(\mathbf{R}))$.*

Proof. The compact torus $S_{\mathrm{der}}(\mathbf{R})$ is a product of circle groups and therefore it is equal to the image of its Lie algebra $\mathfrak{s}_{\mathrm{der}}$ under exponentiation. Since the restriction of $\delta\theta$ to S_{der} is finite, it is a semisimple automorphism defined over $\mathbf{R}$. Consequently, the decomposition

$$\mathfrak{s}_{\mathrm{der}} = \mathfrak{s}_{\mathrm{der}}^{\delta\theta} \oplus (1-\delta\theta)\mathfrak{s}_{\mathrm{der}}$$

holds and implies that $S_{\mathrm{der}}(\mathbf{R}) = S_{\mathrm{der}}^{\delta\theta}(\mathbf{R})\ (1-\delta\theta)(S_{\mathrm{der}}(\mathbf{R}))$.■

The following lemma exhibits the strict relationship between $\bar{\delta}$ and δ.

Lemma 23 *There exists a representative $\dot{w} \in G$ of a unique element in $(\Omega(G,S)/\Omega_{\mathbf{R}}(G,S))^{\delta\theta}$, an element $z \in Z_G^0(\mathbf{R})$ and an element $s \in S_{\mathrm{der}}^{\delta\theta}(\mathbf{R})$ such that $\bar{\delta} = z\dot{w}^{-1}s\delta\theta(\dot{w})$.*

Proof. As in the proof of Corollary 2, there exist quasicharacters $\bar{\Lambda}$ and Λ of $S(\mathbf{R})$ such that

$$\bar{\Lambda}\circ\mathrm{Int}(\bar{\delta}\theta)=\omega_{|S(\mathbf{R})}\bar{\Lambda},\ \ \Lambda\circ\mathrm{Int}(\delta\theta)=\omega_{|S(\mathbf{R})}\Lambda,$$

$\pi_{\bar{\Lambda}}\cong\omega\otimes\pi_{\bar{\Lambda}}^{\theta}$ and $\pi_{\Lambda}\cong\omega\otimes\pi_{\Lambda}^{\theta}$. Lemma 16 informs us that $\bar{\Lambda}=w^{-1}\Lambda$ for a unique element w in $(\Omega(G,S)/\Omega_{\mathbf{R}}(G,S))^{\delta\theta}$. By Proposition 2 the representative $\dot{w}\in N_G(S)$ may be chosen so that $\dot{w}^{-1}\delta\theta\dot{w}(\delta\theta)^{-1}\in G(\mathbf{R})$ (*cf.* (65)). It follows that $\dot{w}^{-1}\delta\theta(\dot{w})\in G(\mathbf{R})$. The earlier equations now entail

$$\begin{aligned}&w^{-1}\Lambda\circ\mathrm{Int}(\bar{\delta}\theta)=\omega_{|S(\mathbf{R})}\,w^{-1}\Lambda\\ \Leftrightarrow\quad&\dot{w}(\bar{\delta}\theta)^{-1}\dot{w}^{-1}\cdot\Lambda=\omega_{|S(\mathbf{R})}\,\Lambda\\ \Leftrightarrow\quad&\dot{w}(\bar{\delta}\theta)^{-1}\dot{w}^{-1}\cdot\Lambda=(\delta\theta)^{-1}\cdot\Lambda\\ \Leftrightarrow\quad&(\delta\theta)\dot{w}(\bar{\delta}\theta)^{-1}\dot{w}^{-1}\cdot\Lambda=\Lambda.\end{aligned}$$

As the differential of Λ is a regular element in $\mathfrak{s}^*\otimes\mathbf{C}$ (section 4.1 and Lemma 3.3 [Lan89]), the final equation implies that

$$\delta\,\theta(\dot{w})\,\bar{\delta}^{-1}\dot{w}^{-1}=(\delta\theta)\dot{w}(\bar{\delta}\theta)^{-1}\dot{w}^{-1}\in S$$

(Lemma B §10.3 [Hum94]). We may rewrite this as $\dot{w}^{-1}\delta\,\theta(\dot{w})\,\bar{\delta}\in S$, and by the choice of $\dot{w}$ above, we actually have $\dot{w}^{-1}\delta\,\theta(\dot{w})\,\bar{\delta}^{-1}\in S(\mathbf{R})$. Consequently, $\bar{\delta}\in\dot{w}^{-1}S(\mathbf{R})\delta\,\theta(\dot{w})$ (Lemma 6.4.1 [Lab08]), so that $\bar{\delta}=\dot{w}^{-1}s'\delta\,\theta(\dot{w})$ for some $s'\in S(\mathbf{R})$. By (29) and Lemma 22, the element s' is equal to $zs_1s\mathrm{Int}(\delta)(\theta(s_1))$ for some $z\in Z_G^0(\mathbf{R})$, $s_1\in S_{\mathrm{der}}(\mathbf{R})$ and $s\in S_{\mathrm{der}}^{\delta\theta}(\mathbf{R})$. In conclusion, $\bar{\delta}$ is equal to $z(s_1\dot{w})^{-1}s\delta\,\theta(s_1\dot{w})$ where $s_1\dot{w}$ remains a representative for w.■

Let us now return to the definition of $\Delta(\boldsymbol{\varphi_{H_1}},\pi)$ in section 6.3 and record the changes in each of its terms when δ is replaced by $\bar{\delta}=\dot{w}^{-1}s\delta\,\theta(\dot{w})$ as in Lemma 23. Obviously (117) is left unchanged by the replacement. Making use of the identities $\dot{w}^{-1}s\delta\theta(\dot{w})\theta=\dot{w}^{-1}s\delta\theta\dot{w}$ and $S^{\dot{w}^{-1}s\delta\theta\dot{w}}=\dot{w}^{-1}S^{\delta\theta}\dot{w}$, one may easily verify that (115) is also unaffected by the replacement.

It remains to compute the product of the remaining two terms, namely

$$\frac{\Delta_I(\bar{\gamma},\bar{\delta})\ \Delta_{III}(\bar{\gamma}_1,\bar{\delta};\gamma_1^0,\delta^0)\ \ \langle\mathrm{inv}(\bar{\delta},\dot{w}_1^{-1}\bar{\delta}\theta(\dot{w}_1)),\kappa_{\bar{\delta}}\rangle}{\langle(\bar{\delta}^*,\bar{\gamma}_1),a_{T'}\rangle},\tag{118}$$

where $\pi=\pi_{w_1^{-1}\bar{\Lambda}}$ for some $w_1\in(\Omega(G,S)/\Omega_{\mathbf{R}}(G,S))^{\bar{\delta}\theta}$ (Lemma 16) and $\dot{w}_1\in G$ is a representative for w_1 satisfying (65) (Proposition 2). This expression relies on the isomorphism $\mathrm{Int}(g_{T'}\psi(\dot{w}))\psi : S^{\dot{w}^{-1}s\delta\theta\dot{w}} \to (T')^{\theta^*}$ in place of (14). This being so, one may compute that $\bar{\delta}^*$ is equal to $s^*\delta^*$ (where * in the latter case is computed with respect to $\mathrm{Int}(g_{T'})\psi$ as of old), and $\bar{\gamma}_1 = \eta_1(s)\gamma_1$. Therefore, the denominator of (118) is equal to

$$\langle (z, z_1), a_{T'}\rangle \ \langle (s^*, \eta_1(s)), a_{T'}\rangle \ \langle (\delta^*, \gamma_1), a_{T'}\rangle,$$

where the pair (z_1, z) belongs to the inverse image of $Z_G(\mathbf{R})$ under (80). There is some cancellation here with $\Delta_{III}(\bar{\gamma}_1, \bar{\delta}; \gamma_1^0, \delta^0)$, for according to Lemma 5.1.A [KS99], the proof of 5.1.D [KS99], Lemma 18 and (82), it is equal to

$$\begin{aligned}
&\Delta_{III}(\bar{\gamma}_1, \bar{\delta}; \gamma_1, \delta)\ \Delta_{III}(\gamma_1, \delta; \gamma_1^0, \delta^0)\\
&\quad = \ \Delta_{III}(z_1\eta_1(s)\gamma_1, z\dot{w}^{-1}s\delta\theta(\dot{w}); \gamma_1, \delta)\ \Delta_{III}(\gamma_1, \delta; \gamma_1^0, \delta^0)\\
&\quad = \ \langle \mathrm{inv}(zs\delta, \dot{w}^{-1}zs\delta\theta(\dot{w})), \kappa_{zs\delta}\rangle\ \Delta_{III}(z_1\eta_1(s)\gamma_1, zs\delta; \gamma_1, \delta)\ \Delta_{III}(\gamma_1, \delta; \gamma_1^0, \delta^0)\\
&\quad = \ \langle \mathrm{inv}(\delta, \dot{w}^{-1}\delta\theta(\dot{w})), \kappa_{\delta}\rangle\ \Delta_{III}(z_1\eta_1(s)\gamma_1, zs\delta; \gamma_1, \delta)\ \Delta_{III}(\gamma_1, \delta; \gamma_1^0, \delta^0)\\
&\quad = \ \langle \mathrm{inv}(\delta, \dot{w}^{-1}\delta\theta(\dot{w})), \kappa_{\delta}\rangle\ \langle (z, z_1), a_{T'}\rangle\ \langle (s^*, \eta_1(s)), a_{T'}\rangle\ \Delta_{III}(\gamma_1, \delta; \gamma_1^0, \delta^0).
\end{aligned}$$

At this point we see that (118) is equal to

$$\frac{\Delta_I(\bar{\gamma}, \bar{\delta})\ \Delta_{III}(\gamma_1, \delta; \gamma_1^0, \delta^0)\ \ \langle \mathrm{inv}(\delta, \dot{w}^{-1}\delta\theta(\dot{w})), \kappa_{\delta}\rangle\ \langle \mathrm{inv}(\bar{\delta}, \dot{w}_1^{-1}\bar{\delta}\theta(\dot{w}_1)), \kappa_{\bar{\delta}}\rangle}{\langle (\delta^*, \gamma_1), a_{T'}\rangle}. \tag{119}$$

The geometric factor $\Delta_I(\gamma, \delta)$ is defined in terms of the a-data, the torus T', the automorphism θ^* and the endoscopic datum s (§4.1 [KS99]). As none of these data are altered in replacing δ with $\bar{\delta}$, we conclude that $\Delta_I(\bar{\gamma}, \bar{\delta}) = \Delta_I(\gamma, \delta)$.

To deal with the last two terms in the numerator of (119), we compute that

$$\begin{aligned}
&w_1 \in (\Omega(G,S)/\Omega_{\mathbf{R}}(G,S)))^{\bar{\delta}\theta} = (\Omega(G,S)/\Omega_{\mathbf{R}}(G,S)))^{\dot{w}^{-1}s\delta\theta\dot{w}}\\
&\quad \Leftrightarrow \ w_1^{-1}w^{-1}(\delta\theta)ww_1w^{-1}(\delta\theta)^{-1}w \in \Omega_{\mathbf{R}}(G,S)\\
&\quad \Leftrightarrow \ (ww_1)^{-1}\,(\delta\theta)\,(ww_1)\,(\delta\theta)^{-1}(w^{-1}\delta\theta w(\delta\theta)^{-1})^{-1} \in \Omega_{\mathbf{R}}(G,S)\\
&\quad \Leftrightarrow \ (ww_1)^{-1}\,(\delta\theta)\,(ww_1)\,(\delta\theta)^{-1} \in \Omega_{\mathbf{R}}(G,S)\\
&\quad \Leftrightarrow \ ww_1 \in (\Omega(G,S)/\Omega_{\mathbf{R}}(G,S)))^{\delta\theta}.
\end{aligned}$$

Moreover, we compute that

$$w_1^{-1}\cdot\bar{\Lambda} = w^{-1}w^{-1}\cdot\Lambda = (ww_1)^{-1}\cdot\Lambda,$$

using $\bar{\Lambda} = w^{-1} \cdot \Lambda$ from the proof of Lemma 23. The last step in showing that (119) is equal to the spectral transfer factor $\Delta(\boldsymbol{\varphi_{H_1}}, \pi_{(ww_1)^{-1}\Lambda})$ computed with respect to δ, is to show that

$$\langle \mathrm{inv}(\delta, \dot{w}^{-1}\delta\theta(\dot{w})), \kappa_\delta\rangle \, \langle \mathrm{inv}(\bar{\delta}, \dot{w}_1^{-1}\bar{\delta}\theta(\dot{w}_1)), \kappa_{\bar{\delta}}\rangle = \langle \mathrm{inv}(\delta, (\dot{w}\dot{w}_1)^{-1}\delta\theta(\dot{w}\dot{w}_1)), \kappa_\delta\rangle. \tag{120}$$

For this, we point out that the definition of κ_δ relies on $\mathrm{Int}(g_{T'})\psi$, whereas the definition of $\kappa_{\bar{\delta}}$ relies on $\mathrm{Int}(g_{T'}\psi(\dot{w}))\psi = \mathrm{Int}(g_{T'})\psi\mathrm{Int}(\dot{w})$ (page 55 [KS99]). In a complementary fashion, the first component of $\mathrm{inv}(\delta, \dot{w}^{-1}\delta\theta(\dot{w})))$ lies in the simply connected covering group of $S^{\delta\theta}$, and the first component of $\mathrm{inv}(\bar{\delta}, \dot{w}_1^{-1}\bar{\delta}\theta(\dot{w}_1))$ lies in the simply connected covering group of $S^{\dot{w}^{-1}zs\delta\theta\dot{w}} = \dot{w}^{-1}S^{\delta\theta}\dot{w}$ (page 54 [KS99]). The second components of the "inv" terms lie in the centre of G and are unaffected by conjugation with $\dot{w}$. These observations amount to the identity

$$\langle \mathrm{inv}(\bar{\delta}, \dot{w}_1^{-1}\bar{\delta}\theta(\dot{w}_1)), \kappa_{\bar{\delta}}\rangle = \langle \dot{w}\, \mathrm{inv}(\bar{\delta}, \dot{w}_1^{-1}\bar{\delta}\theta(\dot{w}_1))\dot{w}^{-1}, \kappa_\delta\rangle.$$

The desired equation (120) now follows from this identity and the fact that the projection of the first component of

$$\mathrm{inv}(\delta, \dot{w}^{-1}\delta\theta(\dot{w})) \;\dot{w}\, \mathrm{inv}(\bar{\delta}, \dot{w}_1^{-1}\bar{\delta}\theta(\dot{w}_1))\dot{w}^{-1}$$

to $S^{\delta\theta}$ is

$$\sigma(\dot{w})\dot{w}^{-1}\dot{w}\sigma(\dot{w}_1)\dot{w}_1^{-1}\dot{w}^{-1} = \sigma(\dot{w}\dot{w}_1)\,(\dot{w}\dot{w}_1)^{-1},$$

the projection to $S^{\delta\theta}$ of the first component of $\mathrm{inv}(\delta, (\dot{w}\dot{w}_1)^{-1}\delta\theta(\dot{w}\dot{w}_1))$.

We have now proven that $\Delta(\boldsymbol{\varphi_{H_1}}, \pi)$ is independent of the choice of θ-elliptic element $\bar{\delta}$ with norm in $H_1(\mathbf{R})$ when $G^{\bar{\delta}\theta}$ lies in S. As mentioned earlier, there is always an element $g \in G(\mathbf{R})$ such that $G^{g\bar{\delta}\theta g^{-1}} = gG^{\bar{\delta}\theta}g^{-1} \subset S$. If g is non-trivial then we may superficially revise the arguments above to obtain $\bar{\delta} = (\dot{w}g)^{-1}s\delta\theta(\dot{w}g)$ and use $\mathrm{Int}(g_{T'}\psi(\dot{w}g))\psi : G^{\bar{\delta}\theta} \to (T')^{\theta^*}$ in place of (14) to obtain the same conclusion.

6.4 A spectral comparison with compact support

Our next objective is to prove identity (60) without any restriction on the support of $f \in C_c^\infty(G(\mathbf{R})\theta)$. For convenience let us set

$$\sum_{\pi_{H_1} \in \Pi_{\varphi_{H_1}}} \Theta_{\pi_{H_1}}(f_{H_1}) = \int_{H_1(\mathbf{R})/Z_1(\mathbf{R})} f_{H_1}(h) \sum_{\pi_{H_1} \in \Pi_{\varphi_{H_1}}} \Theta_{\pi_{H_1}}(h)\, dh$$

for all $f \in C_c^\infty(G(\mathbf{R})\theta)$. Our objective then is to prove that the distribution Θ on $G(\mathbf{R})\theta$, defined by

$$\Theta(f) = \sum_{\pi_{H_1} \in \Pi_{\varphi_{H_1}}} \Theta_{\pi_{H_1}}(f_{H_1}) - \sum_{\pi \in \Pi_\varphi} \Delta(\boldsymbol{\varphi_{H_1}}, \pi)\, \Theta_{\pi, \mathsf{U}_\pi}(f), \; f \in C_c^\infty(G(\mathbf{R})\theta)$$

(see §§3.4 and 5.2), is the zero distribution. We shall first prove this under the assumption that ω is trivial. Under this assumption we shall explain how Θ is a $G(\mathbf{R})$-invariant eigendistribution. Once this is established, we will have a fairly concrete description of the values of Θ on the θ-regular elements of $S^{\delta\theta}(\mathbf{R})^0$. These values will be seen to equal zero in view of section 6.3. Finally, an extension of Harish-Chandra's Uniqueness Theorem (Theorem I.7.13 [Var77]) allows us to conclude that Θ is zero everywhere. These steps imitate those followed in §15 [She08]. However, they rely on extensions of Harish-Chandra's methods given by Bouaziz and Renard. The proof that $\Theta = 0$ when ω is non-trivial will be effected through a bit of surgery in Theorem 1.

To say that Θ is $G(\mathbf{R})$-invariant is to say that $\Theta(f^y) = \Theta(f)$ for all $y \in G(\mathbf{R})$ and $f \in C_c^\infty(G(\mathbf{R})\theta)$, where f^y is defined by

$$f^y(x\theta) = f(y^{-1}x\theta y) = f(y^{-1}x\theta(y)\,\theta), \; x \in G(\mathbf{R}).$$

An obvious change of variable in (34) results in

$$\begin{aligned}
\Theta_{\pi,\mathsf{U}}(f^y) &= \operatorname{tr} \int_{G(\mathbf{R})} f^y(x\theta)\, \pi(x)\mathsf{U}\, dx \\
&= \operatorname{tr} \int_{G(\mathbf{R})} f(y^{-1}x\theta(y)\theta)\, \pi(x)\mathsf{U}\, dx \\
&= \operatorname{tr} \int_{G(\mathbf{R})} f(x\theta)\, \pi(y)\pi(x)\, \pi^\theta(y^{-1})\mathsf{U}\, dx \\
&= \omega(y) \operatorname{tr} \pi(y) \left(\int_{G(\mathbf{R})} f(x\theta)\, \pi(x)\mathsf{U}\, dx \right) \pi(y)^{-1} \\
&= \omega(y)\, \Theta_{\pi,\mathsf{U}}(f)
\end{aligned}$$

Here, we have used equation (24) and the invariance of the trace map under conjugation. This makes it clear that $\Theta_{\pi,\mathsf{U}}$ is $G(\mathbf{R})$-invariant if ω is trivial. A parallel change of variable in the twisted orbital integrals shows that $\mathcal{O}_{x\theta}(f^y) = \omega(y)\, \mathcal{O}_{x\theta}(f)$ for any θ-regular and θ-semisimple $x \in G(\mathbf{R})$.

Identity (16) therefore makes it plain that the function $(f^y)_{H_1}$ may be taken as

(121) $$(f^y)_{H_1} = \omega(y)\, f_{H_1}.$$

It follows that $\Theta(f^y) = \omega(y)\,\Theta(f)$ so that Θ is $G(\mathbf{R})$-invariant when ω is trivial.

To say what it means for Θ to be an eigendistribution, we set $\mathcal{Z}(\mathfrak{g} \otimes \mathbf{C})$ equal to the centre of the universal enveloping algebra of the complexified Lie algebra of $G(\mathbf{R})$. The algebra $\mathcal{Z}(\mathfrak{g} \otimes \mathbf{C})$ acts on $C^{\infty}(G(\mathbf{R})\theta)$ by differential operators (cf. III.1 [Kna86]). By definition, Θ is an eigendistribution if there is an algebra homomorphism $\chi_{\Theta} : \mathcal{Z}(\mathfrak{g} \otimes \mathbf{C}) \to \mathbf{C}$ such that

$$z\Theta(f) = \chi_{\Theta}(z)\,\Theta(f),\ z \in \mathcal{Z}(\mathfrak{g} \otimes \mathbf{C}),\ f \in C_c^{\infty}(G(\mathbf{R})\theta).$$

The action of $\mathcal{Z}(\mathfrak{g}\otimes\mathbf{C})$ on Θ is defined by $z\Theta(f) = \Theta(z^{\mathrm{tr}} f)$, where the adjoint map $z \mapsto z^{\mathrm{tr}}$ is an involution which is trivial on scalars and negates elements of $\mathfrak{g} \otimes \mathbf{C}$ (§5 X [Kna86]). Thus, to better understand $z\Theta$, we require an understanding of the function $(zf)_{H_1}$ on $H_1(\mathbf{R})$ which matches zf through identity (16). We shall show that there exists an algebra homomorphism $z \mapsto z_{H_1}$ from $\mathcal{Z}(\mathfrak{g} \otimes \mathbf{C})$ to $\mathcal{Z}(\mathfrak{h}_1 \otimes \mathbf{C})$ such that $(zf)_{H_1}$ may be taken to equal $z_{H_1} f_{H_1}$. Some patience is required, as this homomorphism shall be constructed by way of at least seven different homomorphisms.

Recall that the Harish-Chandra isomorphism (§5 VIII [Kna86])

$$\beta_{H_1} : \mathcal{Z}(\mathfrak{t}_{H_1} \otimes \mathbf{C}) \to \mathcal{S}(\mathfrak{t}_{H_1} \otimes \mathbf{C})^{\Omega(H_1, T_{H_1})}$$

which takes values in the $\Omega(H_1, T_{H_1})$-invariant elements of the symmetric algebra of $\mathfrak{t}_{H_1} \otimes \mathbf{C}$. A review of the final paragraph of section 3.1 and the maps in section 3.3 makes it clear that

$$\Omega(H_1, T_{H_1}) \cong \Omega(H, T_H) \cong \Omega(\hat{H}, \hat{T}_H) \cong \Omega(\xi(\hat{H}), \xi(\mathcal{T}_H)) \hookrightarrow \Omega(\hat{G}^*, \hat{T}')^{\hat{\theta}^*} \cong \Omega(G^*, T')^{\theta^*}.$$

By (12) and (96), we also have an isomorphism $\mathfrak{t}_H \cong (\mathfrak{t}')^{\theta^*}$. Combining this isomorphism with the previous embedding of Weyl groups produces an injection

(122) $$\mathcal{S}((\mathfrak{t}')^{\theta^*} \otimes \mathbf{C})^{\Omega(G^*, T')^{\theta^*}} \hookrightarrow \mathcal{S}(\mathfrak{t}_H \otimes \mathbf{C})^{\Omega(H, T_H)}.$$

The decomposition $\mathfrak{t}_{H_1} \cong \mathfrak{t}_H \oplus \mathfrak{z}_1$ of (63) produces an algebra isomorphism

$$\mathcal{S}(\mathfrak{t}_{H_1} \otimes \mathbf{C})^{\Omega(H_1, T_{H_1})} \cong \mathcal{S}(\mathfrak{t}_H \otimes \mathbf{C})^{\Omega(H, T_H)}\, \mathcal{S}(\mathfrak{z}_1 \otimes \mathbf{C}).$$

The algebra $\mathcal{S}(\mathfrak{t}_H \otimes \mathbf{C})^{\Omega(H,T_H)}$ embeds into the right-hand side of this isomorphism, yielding an injection

$$\mathcal{S}(\mathfrak{t}_H \otimes \mathbf{C})^{\Omega(H,T_H)} \hookrightarrow \mathcal{S}(\mathfrak{t}_{H_1} \otimes \mathbf{C})^{\Omega(H_1,T_{H_1})}. \tag{123}$$

Taking $x = \delta^*\theta^*$ in §2.4 [Bou87], we obtain an algebra homomorphism

$$\phi : \mathcal{S}(\mathfrak{t}' \otimes \mathbf{C})^{\Omega(G^*,T')} \to \mathcal{S}((\mathfrak{t}')^{\theta^*} \otimes \mathbf{C})^{\Omega(G^*,T')^{\theta^*}}$$

induced by the projection $\mathfrak{t}'$ onto $(\mathfrak{t}')^{\theta^*}$ (see (96)).

The final homomorphism necessary for the definition of $z \mapsto z_{H_1}$ is given by the linear form $\lambda_{a_{T'}} : (\mathfrak{t}')^{\theta^*} \to \mathbf{C}$, defined at the end of section 6.2.1. As noted there, we may also regard $\lambda_{a_{T'}}$ as a $\Omega(H,T_H)$-invariant linear form on $\mathfrak{t}_H$. Our admissible embedding (12) allows us to transfer $\iota_{G^*_{\mathrm{res}}}$ to a linear form on $\mathfrak{t}_H$. We may therefore define the linear form λ^* on $\mathfrak{t}_H$ as

$$\lambda^* = \lambda_{a_{T'}} + \iota_{G^*_{\mathrm{res}}} - \iota_H. \tag{124}$$

The linear form λ^* remains $\Omega(H,T_H)$-invariant, as the positive roots "outside of H" are merely permuted by $\Omega(H,T_H)$ (Lemma B §10.2 [Hum94]). One may define the algebra automorphism of $\mathcal{S}(\mathfrak{t}_H \otimes \mathbf{C})^{\Omega(H,T_H)}$ by extending the map

$$X \mapsto X - \lambda^*(X),\ X \in \mathfrak{t}_H^{\Omega(H,T_H)}$$

to $\mathcal{S}(\mathfrak{t}_H \otimes \mathbf{C})^{\Omega(H,T_H)}$ (Proposition 3.1 [Kna86]). We denote this automorphism by $I_{-\lambda^*}$.

We define the algebra homomorphism $z \mapsto z_{H_1}$ from $\mathcal{Z}(\mathfrak{g} \otimes \mathbf{C})$ to $\mathcal{Z}(\mathfrak{h}_1 \otimes \mathbf{C})$ as the composition

$$\begin{array}{rl} \mathcal{Z}(\mathfrak{g} \otimes \mathbf{C}) & \to \mathcal{Z}(\mathfrak{g} \otimes \mathbf{C}) \to \mathcal{S}(\mathfrak{t}' \otimes \mathbf{C})^{\Omega(G^*,T')} \to \mathcal{S}((\mathfrak{t}')^{\theta^*} \otimes \mathbf{C})^{\Omega(G^*,T')^{\theta^*}} \\ \to & \mathcal{S}(\mathfrak{t}_H \otimes \mathbf{C})^{\Omega(H,T_H)} \to \mathcal{S}(\mathfrak{t}_H \otimes \mathbf{C})^{\Omega(H,T_H)} \\ \to & \mathcal{S}(\mathfrak{t}_{H_1} \otimes \mathbf{C})^{\Omega(H_1,T_{H_1})} \to \mathcal{Z}(\mathfrak{h}_1 \otimes \mathbf{C}), \end{array} \tag{125}$$

where the maps, from left to right, are given by (14), β_{G^*}, ϕ, (122), $I_{-\lambda^*}$, (123) and $\beta_{H_1}^{-1}$ respectively. By the very definition of $\mathcal{Z}(\mathfrak{g} \otimes \mathbf{C})$ and $\mathcal{Z}(\mathfrak{h}_1 \otimes \mathbf{C})$, it follows that this homomorphism is invariant under conjugation. As a result, this homomorphism is independent of $g_{T'}$ and the choices of tori, T' and T_{H_1}. Nevertheless, the individual maps in (125) do depend on these choices and are flexible enough to accommodate them.

Lemma 24 *Suppose $f \in C_c^\infty(G(\mathbf{R})\theta)$ and $z \in \mathcal{Z}(\mathfrak{g} \otimes \mathbf{C})$. Let*

$$\phi' : \mathcal{S}(\mathfrak{s} \otimes \mathbf{C})^{\Omega(G,S)} \rightarrow \mathcal{S}(\mathfrak{s}^{\delta\theta})^{\Omega(G,S)^{\delta\theta}}$$

be the transfer of ϕ to $S(\mathbf{R})$ via (14). Then

$$D_{G\theta}(x\delta)\ \mathcal{O}_{x\delta\theta}(zf) = \phi' \circ \beta_G(z)\ D_{G\theta}(x\delta)\, \mathcal{O}_{x\delta\theta}(f)$$

for all $\delta\theta$-regular $x \in S^{\delta\theta}(\mathbf{R})$.

Proof. We follow the proof of Proposition II.10.4. [Var77]. Fix $\delta\theta$-regular $y \in S^{\delta\theta}(\mathbf{R})$. It suffices to prove the lemma on a small open neighbourhood of $y\delta\theta$. According to §6 and §10 [Ren97], there exists a small $G^{\delta\theta}(\mathbf{R})$-invariant neighbourhood $\mathcal{V}_y \subset S^{\delta\theta}(\mathbf{R})$ of y such that restriction to $\mathcal{V}_y\delta\theta$ is an isomorphism from the space of $G(\mathbf{R})$-invariant smooth functions supported on the union of $G(\mathbf{R})$-conjugates of $\mathcal{V}_y\delta\theta$ to the space of smooth $G^{\delta\theta}(\mathbf{R})$-invariant functions on $\mathcal{V}_y\delta\theta$. Given any function $\tilde{f}$ in the former space, and any function $\tilde{f}_1$ compactly supported in union of $G(\mathbf{R})$-conjugates of $\mathcal{V}_y\delta\theta$, the Weyl integration formula (Proposition 1) tells us that up to a constant the integral $\int_{G(\mathbf{R})} \tilde{f}(g\delta\theta)\tilde{f}_1(g\delta\theta)\, dg$ is equal to

$$\int_{\mathcal{V}_y} \tilde{f}_{|\mathcal{V}_y\delta\theta}(x) D_{G\theta}(x\delta)^2\, \mathcal{O}_{x\delta\theta}(\tilde{f}_1)\, dx.$$

Applying, Corollary 2.4.11 [Bou87] to the distribution

$$f_1 \mapsto \int_{G(\mathbf{R})} \tilde{f}(g\delta\theta)\tilde{f}_1(g\delta\theta)\, dg$$

results in the identity

$$\begin{aligned}
&\int_{\mathcal{V}_y} \tilde{f}_{|\mathcal{V}_y\delta\theta}(x\delta\theta) D_{G\theta}(x\delta)^2\, \mathcal{O}_{x\delta\theta}(z\tilde{f}_1)\, dx \\
&\quad = \int_{\mathcal{V}_y} \tilde{f}_{|\mathcal{V}_y\delta\theta}(x\delta\theta) D_{G\theta}(x\delta)\, \phi' \circ \beta_G(z) D_{G\theta}(x\delta)\mathcal{O}_{x\delta\theta}(\tilde{f}_1)\, dx.
\end{aligned}$$

As this identity holds for all $\tilde{f}$, we conclude that

$$D_{G\theta}(x\delta)\, \mathcal{O}_{x\delta\theta}(z\tilde{f}_1) = \phi' \circ \beta_G(z) D_{G\theta}(x\delta)\mathcal{O}_{x\delta\theta}(\tilde{f}_1), \tag{126}$$

for all $\delta\theta$-regular x in a open neighbourhood of y. By Corollary 8.5 [Ren97], there exists a smooth $G(\mathbf{R})$-invariant function f_y such that $\tilde{f}_1$ may be replaced with $f_y f$, and f_y is equal to one on a neighbourhood of $y\delta\theta$. In consequence, identity (126) holds with f in place of $\tilde{f}_1$.■

Lemma 25 *Suppose $f \in C_c^\infty(G(\mathbf{R})\theta)$ and $f_{H_1} \in C^\infty(H_1(\mathbf{R}))$ are functions which satisfy geometric transfer identity (16). Then, for any $z \in \mathcal{Z}(\mathfrak{g} \otimes \mathbf{C})$, the functions $zf \in C_c^\infty(G(\mathbf{R})\theta)$ and $z_{H_1} f_{H_1} \in C^\infty(H_1(\mathbf{R}))$ also satisfy (16).*

Proof. We shall verify the desired identity first at elements of $S^{\delta\theta}(\mathbf{R})\delta\theta$. Suppose $x \in S^{\delta\theta}(\mathbf{R})$ and $x\delta$ is strongly θ-regular. By Lemma 24 we have

$$D_{G\theta}(x\delta)\ \mathcal{O}_{x\delta\theta}(zf) = (\mathrm{Int}(g_{T'})\psi)^{-1} \circ \phi \circ (\mathrm{Int}(g_{T'})\psi) \circ \beta_G(z)\ D_{G\theta}(x\delta)\, \mathcal{O}_{x\delta\theta}(zf)$$

Here, we are identifying the isomorphism $\mathrm{Int}(g_{T'})\psi : G \rightarrow G^*$ with its differential. It is the isomorphism used in (14) and in (125). Using $\mathrm{Int}(g_{T'})\psi$ we obtain

$$D_{G\theta}(x\delta) = D_{G^*\theta^*}(x^*\delta^*).$$

We may also rearrange the differential operator as

$$(\mathrm{Int}(g_{T'})\psi)^{-1} \circ \phi \circ \mathrm{Int}(g_{T'})\psi \circ \beta_G(z) = (\mathrm{Int}(g_{T'})\psi)^{-1} \circ \phi \circ \beta_{G^*} \circ \mathrm{Int}(g_{T'})\psi(z).$$

Observe that the composition of the three maps $\phi \circ \beta_{G*} \circ \mathrm{Int}(g_{T'})\psi$ on the right-hand side coincides with the composition of the first three maps of (125). Observe also, that for any representative $w \in G(\mathbf{R})$ of an element in $(\Omega(G,S)/\Omega_{\mathbf{R}}(G,S))^{\delta\theta}$ we have

$$\begin{aligned}
&D_{G\theta}(w^{-1}x\delta\theta(w))\, \mathcal{O}_{wx\delta\theta w^{-1}}(zf)\\
&= (\mathrm{Int}(g_{T'})\psi)^{-1} \circ \phi \circ \beta_{G^*} \circ \mathrm{Int}(g_{T'})\psi(z)\, D_{G\theta}(w^{-1}x\delta\theta(w))\ \mathcal{O}_{w^{-1}x\delta\theta w}(f)\\
&= (\mathrm{Int}(g_{T'})\psi)^{-1} \circ \phi \circ \beta_{G^*} \circ \mathrm{Int}(g_{T'})\psi(z)\, D_{G\theta}(x\delta)\ \mathcal{O}_{w^{-1}x\delta\theta w}(f).
\end{aligned}$$

Looking back to Lemma 15, we see that to prove the lemma we must prove the identity

$$\begin{aligned}
&\beta_{H_1}(z_{H_1})\frac{D_{H_1}(x_1\gamma_1)}{D_{G^*\theta^*}(x^*\delta^*)}\Delta(x_1\gamma_1, x\delta)\\
(127) \quad &= \frac{D_{H_1}(x_1\gamma_1)}{D_{G^*\theta^*}(x^*\delta^*)}\Delta(x_1\gamma_1, x\delta)\ (\mathrm{Int}(g_{T'})\psi)^{-1} \circ \phi \circ \beta_{G^*} \circ \mathrm{Int}(g_{T'})\psi(z),
\end{aligned}$$

where $x_1 \in T_{H_1}(\mathbf{R})$ is obtained from x through (14), the coset map $T' \rightarrow T'/(1-\theta^*)T'$, the admissible embedding (12) and a local lifting given by (63). Indeed, this is to be interpreted as a local identity, in the sense that it suffices to prove it on a small open set in $S^{\delta\theta}(\mathbf{R})$. One may replace $\eta_1(x)$ with x_1 in (73), (77) and (78), so that the identity to be proven reads as

$$\begin{aligned} & \beta_{H_1}(z_{H_1})\Delta_I(x_1\gamma_1, x\delta)\,\Delta_{II}(x_1\gamma_1, x\delta)\,\Delta_{III}(x_1\gamma_1, x\delta; \gamma_1, \delta) \\ = \;& \Delta_I(x_1\gamma_1, x\delta)\,\Delta_{II}(x_1\gamma_1, x\delta)\,\Delta_{III}(x_1\gamma_1, x\delta; \gamma_1, \delta) \\ \times \;& (\mathrm{Int}(g_{T'})\psi)^{-1} \circ \phi \circ \beta_{G^*} \circ \mathrm{Int}(g_{T'})\psi(z). \end{aligned}$$

It is clear from the definition of $\Delta_I(x_1\gamma_1, x\delta)$ (§4.2 [KS99]) that it is a constant independent of x. Therefore the above identity simplifies to

$$\begin{aligned} & \beta_{H_1}(z_{H_1})\Delta_{II}(x_1\gamma_1, x\delta)\,\Delta_{III}(x_1\gamma_1, x\delta; \gamma_1, \delta) \\ = \;& \Delta_{II}(x_1\gamma_1, x\delta)\,\Delta_{III}(x_1\gamma_1, x\delta; \gamma_1, \delta)\;(\mathrm{Int}(g_{T'})\psi)^{-1} \circ \phi \circ \beta_{G^*} \circ \mathrm{Int}(g_{T'})\psi(z). \end{aligned} \tag{128}$$

We wish to simplify this required identity further and relate it to $I_{-\lambda^*}$ appearing in (125). The term $\Delta_{II}(x_1\gamma_1, x\delta)$ is a quotient whose numerator is of the form (75). Using the arguments of Lemma 20, each expression $N\alpha(x^*\delta^*) \pm 1$ in this numerator is seen to be equal to

$$1 - N\alpha(x^*)\,\mathrm{Ad}((\delta^*\theta^*)^{l_\alpha})_{|\mathfrak{g}^\alpha} = \det(1 - \mathrm{Ad}x^*\delta^*\theta^*)_{|\mathfrak{g}^{\alpha_{\mathrm{res}}}}$$

up to multiplication by ± 1. The square of the numerator is therefore equal to the product over all negative $\alpha_{\mathrm{res}} \in R_{\mathrm{res}}(G^*, T')$ of

$$\begin{aligned} \frac{|\det(1 - \mathrm{Ad}x^*\delta^*\theta^*)_{|\mathfrak{g}^{\alpha_{\mathrm{res}}}}|^2}{\det(1 - \mathrm{Ad}x^*\delta^*\theta^*)^2_{|\mathfrak{g}^{\alpha_{\mathrm{res}}}}} &= \frac{\det(1 - \mathrm{Ad}x^*\delta^*\theta^*)_{|\mathfrak{g}^{\alpha_{\mathrm{res}}}}\overline{\det(1 - \mathrm{Ad}x^*\delta^*\theta^*)_{|\mathfrak{g}^{\alpha_{\mathrm{res}}}}}}{\det(1 - \mathrm{Ad}x^*\delta^*\theta^*)^2_{|\mathfrak{g}^{\alpha_{\mathrm{res}}}}} \\ &= \frac{\det(1 - \mathrm{Ad}x^*\delta^*\theta^*)_{|\mathfrak{g}^{-\alpha_{\mathrm{res}}}}}{\det(1 - \mathrm{Ad}x^*\delta^*\theta^*)_{|\mathfrak{g}^{\alpha_{\mathrm{res}}}}} \\ &= (-1)^{\dim \mathfrak{g}^{\alpha_{\mathrm{res}}}} N\alpha(x^*\delta^*) \end{aligned}$$

up to multiplication by ± 1. Therefore, the square of this product is equal to

$$\pm \prod_{\alpha_{\mathrm{res}} < 0} N\alpha(x^*\delta^*) = \pm \prod_{\alpha_{\mathrm{res}} > 0} N\alpha(x^*\delta^*) = \pm \iota^2_{G^*_{\mathrm{res}}}(x^*)\iota^2_{G^*}(\delta^*).$$

The same type of arguments hold true for the denominator of $\Delta_{II}(x_1\gamma_1, x\delta)$. We conclude that $\Delta_{II}(x_1\gamma_1, x\delta)$ is equal to $\iota_{G^*_{\mathrm{res}}}(x^*) - \iota_H(x_1)$ in additive notation and up to multiplication by a constant independent of x. Now, by Proposition 3 we see that

$$\Delta_{III}(x_1\gamma_1, x\delta; \gamma_1, \delta) = \Lambda(-\mu_{H_1} + \iota_H, -\lambda_{H_1})(x_1)\;\Lambda(\mu - \iota_{G^*\mathrm{res}}, \lambda)(x^*).$$

By the definition (124), we may interpret the map

$$x \mapsto \Delta_{II}(x_1\gamma_1, x\delta)\, \Delta_{III}(x_1\gamma_1, x\delta; \gamma_1, \delta),\ x \in S^{\delta\theta}(\mathbf{R})$$

locally to be given by a constant multiple of e^{λ^*}. That is to say, on a small open subset of $S^{\delta\theta}(\mathbf{R})$, this map may be lifted to λ^* on a small open subset of $\mathfrak{t}_H$ with the identification of the isomorphic Lie algebras $\mathfrak{s}^{\delta\theta} \cong (\mathfrak{t}')^{\theta^*} \cong \mathfrak{t}_H$ ((12), (14)). With this interpretation, the required identity (128) reads as

$$\beta_{H_1}(z_{H_1})\, e^{\lambda^*} = e^{\lambda^*}\ \phi \circ \beta_{G^*} \circ \mathrm{Int}(g_{T'})\psi(z),$$

on a small open set of $\mathfrak{t}_H \subset \mathfrak{t}_{H_1}$. To derive this identity, one may apply the product rule to show that the right-hand side is equal to

$$I_{-\lambda^*} \circ \phi \circ \beta_{G^*} \circ \mathrm{Int}(g_{T'})\psi(z)\, e^{\lambda^*}$$

and then compare with (125). This concludes the proof of matching (16) at θ-regular elements which preserve an elliptic torus S.

To prove the matching in general, suppose $\delta' \in G(\mathbf{R})$ is θ-semisimple and strongly θ-regular and that $\gamma_1' \in H_1(\mathbf{R})$ is a norm of δ'. Then $G^{\delta'\theta}$ is equal to the fixed-point set of a maximal torus torus S_M which is defined over $\mathbf{R}$. Let M be the centralizer in G of the maximally split subtorus of $S_M(\mathbf{R})$. It is a Levi subgroup of G which is defined over $\mathbf{R}$ and $S_M(\mathbf{R})$ is elliptic in $M(\mathbf{R})$. For details of this construction see the appendix. The identity to prove in this context is an analogue of (127), namely

$$\begin{aligned}
&\beta_{H_1}(z_{H_1}) \frac{D_{H_1}(x_1\gamma_1)}{D_{G^*\theta^*}(x^*\delta^*)} \Delta(x_1\gamma_1, x\delta)\\
= \quad & \frac{D_{H_1}(x_1\gamma_1)}{D_{G^*\theta^*}(x^*\delta^*)} \Delta(x_1\gamma_1, x\delta)\ (\mathrm{Int}(g_{T_M'})\psi)^{-1} \circ \phi \circ \beta_{G^*} \circ \mathrm{Int}(g_{T_M'})\psi(z).
\end{aligned}$$

Following section 3.3 we have replaced T' with a pertinent torus $T_M' \cong S_M$, β_{G^*} is to be taken as the Harish-Chandra isomorphisms onto $\mathcal{S}(\mathfrak{t}_M' \otimes \mathbf{C})^{\Omega(G^*, T_M')}$, and x is in a small neighbourhood of the identity in $S_M^{\delta'\theta}(\mathbf{R}) \subset M(\mathbf{R})$. The elements $x^*, \delta^* \in T_M'$ follow the formalism laid out in section 3.3. We may rewrite this identity as

$$\beta_{H_1}(z_{H_1}) \frac{\Delta_M(x_1\gamma_1', x\delta')}{\Delta_{M,IV}(x_1\gamma_1', x\delta')} = \frac{\Delta_M(x_1\gamma_1', x\delta')}{\Delta_{M,IV}(x_1\gamma_1', x\delta')}\ \psi_M^{-1} \circ \phi_{M^*} \circ \beta_{M^*} \circ \psi_M(z_M),$$

in which we adopt the notation of the appendix and apply (150). The maps ϕ_{M^*} and β_{M^*} are the analogues of ϕ and β with G replaced by M, T' replaced by T'_M, etc. and z_M is the image of z under the unique algebra monomorphism $\mathcal{Z}(\mathfrak{g}\otimes\mathbf{C}) \to \mathcal{Z}(\mathfrak{m}\otimes\mathbf{C})$ (p. 52 [Var77]). Since this identity is of the same form and is stated under the same essential assumptions as (127) we have reduced the proof of the lemma to the case of elliptic tori, which has already been proved.■

Lemma 26 *The distribution* Θ *is an eigendistribution.*

Proof. According to Lemma 25

$$\begin{aligned} z\Theta(f) &= \Theta(z^{\mathrm{tr}}f) \\ &= \sum_{\pi_{H_1}\in\Pi_{\varphi_{H_1}}} \Theta_{\pi_{H_1}}((z^{\mathrm{tr}})_{H_1} f_{H_1}) - \sum_{\pi\in\Pi_\varphi} \Delta(\boldsymbol{\varphi_{H_1}},\pi)\,\Theta_{\pi,\mathsf{U}_\pi}(z^{\mathrm{tr}}f), \end{aligned}$$

for every $z \in \mathcal{Z}(\mathfrak{g}\otimes\mathbf{C})$ and $f \in C_c^\infty(G(\mathbf{R})\theta)$. It is important to realize that $(z^{\mathrm{tr}})_{H_1}$ is not equal to $(z_{H_1})^{\mathrm{tr}}$. Looking back to the maps in the definition of z_{H_1} (125) and §5 X [Kna86], we see that the only map which does not commute with the adjoint is $I_{-\lambda^*}$. It is converted to its inverse, namely $I_{\lambda*}$. Consequently, the definition of $(z^{\mathrm{tr}})_{H_1}$ differs from the definition of $(z_{H_1})^{\mathrm{tr}}$ only through replacing $-\lambda^*$ with λ^* in (124).

The representations in Π_φ are all of the form (70), they share a common infinitesimal character $\chi_{\lambda_\varphi} : \mathcal{Z}(\mathfrak{g}\otimes\mathbf{C}) \to \mathbf{C}$ determined by the linear form $\lambda_\varphi \in \mathfrak{s}^*\otimes\mathbf{C}$ which is equal to the sum of the differential of $\Lambda(\mu-\iota_G,\lambda)$ and ι_G (see (85), §§VIII 5-6 and Theorem 9.20 [Kna86]). We may regard the linear form λ_φ as being defined on $\mathfrak{t}_H\otimes\mathbf{C}$ upon identifying $X_*(((\hat{T}')^{\theta^*})^0)\otimes\mathbf{C}$ with $((\mathfrak{t}')^{\theta^*})^*\otimes\mathbf{C}$ (see §9.1 [Bor79]), and identifying $\mathfrak{t}_H \cong (\mathfrak{t}')^{\theta^*} \cong \mathfrak{s}^{\delta\theta}$ via (12) and (14). Similarly, the representations in $\Pi_{\varphi_{H_1}}$ share a common infinitesimal character $\chi_{\lambda_{\varphi_{H_1}}}$, where $\lambda_{\varphi_{H_1}} \in \mathfrak{t}^*_{H_1}\otimes\mathbf{C}$ is the sum of the differential of $\Lambda(\mu_{H_1}-\iota_H,\lambda_{H_1})$ and ι_H. The value $\chi_{\lambda_{\varphi_{H_1}}}(z_{H_1})$ depends only on the restriction of $\lambda_{\varphi_{H_1}}$ to $\mathfrak{t}^*_H\otimes\mathbf{C}$ by virtue of the penultimate map (123) given in the definition of z_{H_1} (125). This restriction differs from λ_φ by λ^* (see (124) and the definition of $\lambda_{a_{T'}}$ in section 6.2.1). The upshot of these observations is that $\chi_{\lambda_\varphi}(z) = \chi_{\lambda_{\varphi_{H_1}}}(((z^{\mathrm{tr}})_{H_1})^{\mathrm{tr}})$ so that

$$\begin{aligned} z\Theta(f) &= \chi_{\lambda_{\varphi_{H_1}}}(((z^{\mathrm{tr}})_{H_1})^{\mathrm{tr}}) \sum_{\pi_{H_1}\in\Pi_{\varphi_{H_1}}} \Theta_{\pi_{H_1}}(f_{H_1}) - \chi_{\lambda_\varphi}(z)\sum_{\pi\in\Pi_\varphi} \Delta(\boldsymbol{\varphi_{H_1}},\pi)\,\Theta_{\pi,\mathsf{U}_\pi}(f) \\ &= \chi_{\lambda_\varphi}(z)\,\Theta(f). \blacksquare \end{aligned}$$

Proposition 4 *Suppose ω is trivial. Then Θ is the zero distribution.*

Proof. Lemma 26 and the computations at the beginning of this section tell us that Θ is a $G(\mathbf{R})$-invariant eigendistribution. We wish to show that Θ is a tempered distribution, i.e. that Θ defines a continuous linear form with respect to the topology on $C_c^\infty(G(\mathbf{R})\theta)$ given by the seminorms of §3.5 [Bou87]. This may be seen by first noting that each $\pi \in \Pi_\varphi$ extends to a representation of $G \rtimes \langle\theta\rangle$ through the designation $\pi(\theta) = \mathsf{U}_\pi$. Indeed, this kind of extension was detailed at the beginning of section 5.4. According to Lemma 3.5.1 [Bou87], the character of the resulting extension of π is tempered. Therefore the distribution $\Theta_{\pi,\mathsf{U}_\pi}$, which is the restriction of the character of the extension to $G(\mathbf{R})\theta$, is also tempered. We are assuming that $f \mapsto f_{H_1}$ is continuous, and we know that $\Theta_{\pi_{H_1}}$ is tempered for each $\pi_{H_1} \in \Pi_{\varphi_{H_1}}$. It follows that Θ is tempered.

Next, we argue that Θ has regular infinitesimal character. Recall from the proof of Lemma 26 that the infinitesimal character of Θ is given by a linear form $\lambda_\varphi \in \mathfrak{s}^* \otimes \mathbf{C}$. As π is essentially square-integrable, so too is each irreducible subrepresentation of its restriction to $G_{\mathrm{der}}(\mathbf{R})^0$. The infinitesimal character of each of these subrepresentations is given by the restriction of λ_φ to $\mathfrak{s}_{\mathrm{der}} \otimes \mathbf{C}$ and is regular (Theorem 9.20 [Kna86]). This implies that λ_φ is regular.

Now, since Θ is a $G(\mathbf{R})$-invariant tempered eigendistribution Θ (supported on $G(\mathbf{R})\theta$), with regular infinitesimal character, it follows from §3.6 [Bou87] that on any connected open set $\mathcal{V}' \subset \mathfrak{s}^{\delta\theta}$ such that $\exp(X)\delta$ is θ-regular for every $X \in \mathcal{V}'$ there exist a finite number of constants C_j and linear forms $\mu_j \in (\mathfrak{s}^{\delta\theta})^* \otimes \mathbf{C}$ which satisfy

$$\Theta(\exp(X)\delta\theta) = \frac{\sum_j C_j e^{\mu_j(X)}}{D_{G\theta}(\exp(X))},\ X \in \mathcal{V}' \tag{129}$$

(see p. 34 [Bou87] and Proposition 12.3 (b) [Duc02]). It was proven in section 6.3 that this expansion is zero when $\mathcal{V}'$ is taken to be the logarithmic image of $\mathcal{V} \subset S^{\delta\theta}(\mathbf{R})^0$. It follows that Θ vanishes on the component of the strongly θ-regular elements of $S^{\delta\theta}(\mathbf{R})^0\delta$ containing δ. If $s \in S^{\delta\theta}(\mathbf{R})^0$ and $s\delta$ is strongly θ-regular, the properties of $s\delta$ and δ are the same as far as the results of section 6.3 are concerned (see section 6.3.1). We may therefore replace δ replaced with $s\delta$ as above to conclude that Θ vanishes on any θ-regular component of $S^{\delta\theta}(\mathbf{R})^0\delta\theta$.

We may now apply Proposition 3.6.1 [Bou87] and Theorem 15.1 [Ren97], the latter being a twisted version of Harish-Chandra's Uniqueness Theorem (I.7.13 [Var77]). The result is that Θ vanishes on the set of $G(\mathbf{R})$-conjugates of $Z_G(\mathbf{R})G(\mathbf{R})^0\delta\theta$, and by section 6.3.1 this is true for any strongly θ-regular θ-elliptic element $\delta \in G(\mathbf{R})$ which has a norm in $H_1(\mathbf{R})$.

We must show that Θ vanishes on any remaining subsets of $G(\mathbf{R})$. By Lemma 1.6.1 [Bou87] any maximal compact subgroup of $G(\mathbf{R})$ has non-trivial intersection with any component of $G(\mathbf{R})$ (in the manifold topology). It then follows from Lemma 1.5.1 and Lemma 1.6.2 (ii) [Bou87] that every component of $G(\mathbf{R})$ contains a strongly θ-regular and θ-elliptic element of $G(\mathbf{R})$. If such an element has a norm in $H_1(\mathbf{R})$ we know from the above that Θ vanishes on the component. The other possibility is that a component of $G(\mathbf{R})$ is equal to $G(\mathbf{R})^0\bar{\delta}$ where $\bar{\delta} \in G(\mathbf{R})$ is strongly θ-regular and θ-elliptic, and no strongly θ-regular θ-elliptic element of $Z_G(\mathbf{R})G(\mathbf{R})^0\bar{\delta}$ has a norm in $H_1(\mathbf{R})$. In this case, for any function $f \in C_c^\infty(G(\mathbf{R})\theta)$ with small elliptic support about $\bar{\delta}\theta$ we may assume that $f_{H_1} = 0$ (see (16) so that the left-hand side of (60) vanishes. We must show that $\Theta_{\pi,\mathsf{U}_\pi}(f) = 0$ for every $\pi \in \Pi_\varphi$, and we do so by contradiction.

Suppose first that $\pi = \pi_\Lambda$, where $\Lambda \in \mathfrak{s}^*$ is as in section 6.1 and $\Theta_{\pi,\mathsf{U}_\pi}(f) \neq 0$. Lemma 5 tells us that $\Theta_{\pi,\mathsf{U}_\pi}(f)$ vanishes if the support of f lies outside of the $G(\mathbf{R})$-conjugates of $Z_G(\mathbf{R})G_{\mathrm{der}}(\mathbf{R})^0\delta\theta$. Together with the density results, Lemma 1.5.1 and Lemma 1.6.2 [Bou87], we deduce that there are elements $x \in Z_G(\mathbf{R})G_{\mathrm{der}}(\mathbf{R})^0$ and $g \in G(\mathbf{R})$ such that $g^{-1}x\bar{\delta}\theta(g)$ is equal to a strongly θ-regular and θ-elliptic element $zs\delta$, where $z \in Z_G(\mathbf{R})$ and $s \in S_{\mathrm{der}}(\mathbf{R})$. Taking Lemma 22 into consideration, we may assume that s actually belongs to $S_{\mathrm{der}}^{\delta\theta}(\mathbf{R})$. Now, by assumption δ has a norm and by virtue of (62) the element $s\delta$ has a norm as well. Let $z_1 \in Z_{H_1}(\mathbf{R})$ be any element such that $p_1(z, z_1) = z$ (see (80). Then the product of z_1 and the norm of $s\delta$ is a norm of $zs\delta$. Composing (14) with $\mathrm{Int}(g)$ yields a map which produces a norm of $x\bar{\delta}$ equal to that of $zs\delta$ (*cf.* §6). However, this contradicts the assumption that no strongly θ-regular and θ-elliptic element of $Z_G(\mathbf{R})G(\mathbf{R})^0\bar{\delta}$ has a norm. We conclude that $\Theta_{\pi,\mathsf{U}_\pi}(f) = 0$.

The remaining representations $\pi \in \Pi_\varphi$ are of the form $\pi = \pi_{w^{-1}\Lambda}$ where $w \in (\Omega(G,S)/\Omega_{\mathbf{R}}(G,S))^{\delta\theta}$ (Lemma 16). The previous argument applies to these representations with δ replaced by $\dot{w}^{-1}\delta\theta(\dot{w}) \in G(\mathbf{R})$, where $\dot{w}G$ is a representative for w obtained from Proposition 2. We conclude that $\Theta_{\pi,\mathsf{U}_\pi}(f)$ vanishes for all $\pi \in \Pi_\varphi$ so that Θ vanishes on $Z_G(\mathbf{R})G(\mathbf{R})^0\bar{\delta}\theta$ (Theorem 15.1 [Ren97]).■ In our main theorem, we remove the hypothesis that the quasicharacter ω is trivial.

Theorem 1 *Suppose* $f \in C_c^\infty(G(\mathbf{R})\theta)$. *Then*

$$\sum_{\pi_{H_1} \in \Pi_{\varphi_{H_1}}} \Theta_{\pi_{H_1}}(f_{H_1}) = \sum_{\pi \in \Pi_\varphi} \Delta(\boldsymbol{\varphi_{H_1}}, \pi)\, \Theta_{\pi, \mathsf{U}_\pi}(f).$$

Proof. It follows from Proposition 7.3.1 (i) and Corollary 8.1.6 [Spr98], that the homomorphism

$$Z_G^0(\mathbf{R}) \times G_{\mathrm{der}}(\mathbf{R}) \to G(\mathbf{R})$$

given by multiplication is surjective with finite kernel isomorphic to $Z_G^0(\mathbf{R}) \cap G_{\mathrm{der}}(\mathbf{R})$. One may further decompose $Z_G^0(\mathbf{R})$ using the isomorphisms

$$\begin{aligned} Z_G^0(\mathbf{R})/(Z_G^0)^\theta(\mathbf{R}) &\overset{1-\theta}{\to} (1-\theta)Z_G^0(\mathbf{R})(Z_G^0)^\theta(\mathbf{R})/(Z_G^0)^\theta(\mathbf{R}) \\ &\cong (1-\theta)Z_G^0(\mathbf{R})/(1-\theta)Z_G^0(\mathbf{R}) \cap (Z_G^0)^\theta(\mathbf{R}). \end{aligned}$$

An argument given in the proof of Lemma 4.11 [Ren03] shows that the intersection $(1-\theta)Z_G^0(\mathbf{R}) \cap (Z_G^0)^\theta(\mathbf{R})$ finite. Accordingly, the following commutative diagram

$$\begin{tikzcd} (1-\theta)Z_G^0(\mathbf{R}) \times (Z_G^0)^\theta(\mathbf{R}) \times G_{\mathrm{der}}(\mathbf{R}) \arrow[r] \arrow[d] & Z_G^0(\mathbf{R}) \times G_{\mathrm{der}}(\mathbf{R}) \arrow[d] \\ (1-\theta)Z_G^0(\mathbf{R}) \times (Z_G^0)^\theta(\mathbf{R})G_{\mathrm{der}}(\mathbf{R}) \arrow[r] & G(\mathbf{R}) \end{tikzcd} \tag{130}$$

has finite fibres when moving on the top right, or down. Therefore the lower homomorphism also has finite fibres. In other words, the kernel of the lower homomorphism is isomorphic to the *finite* intersection

$$(1-\theta)Z_G^0(\mathbf{R}) \cap (Z_G^0)^\theta(\mathbf{R})G_{\mathrm{der}}(\mathbf{R}). \tag{131}$$

Suppose first that this kernel is trivial. Then the linear combinations of products of functions in $C_c^\infty((1-\theta)Z_G^0(\mathbf{R}))$ and $C_c^\infty((Z_G^0)^\theta(\mathbf{R})G_{\mathrm{der}}(\mathbf{R})\theta)$ form a dense subspace of $C_c^\infty(G(\mathbf{R})\theta)$ (Proposition 1 §4.8 [Hor66]) Let us fix $h \in C_c^\infty((1-\theta)Z_G^0(\mathbf{R}))$ for a moment and define a distribution Θ_h on $(Z_G^0)^\theta(\mathbf{R})G_{\mathrm{der}}(\mathbf{R})\theta$ by $\Theta_h(\tilde{f}) = \Theta(h\tilde{f})$. It is left as an exercise to the reader to verify that Θ_h is $(Z_G^0)^\theta(\mathbf{R})G_{\mathrm{der}}(\mathbf{R})$-invariant (see (25)) and is an eigendistribution of $\mathcal{Z}((\mathfrak{z}_G + \mathfrak{g}_{\mathrm{der}}) \otimes \mathbf{C})$. In fact, the distribution Θ_h satisfies all of the hypotheses required in the proof of Proposition 4, and so Θ_h vanishes.

Since this is true for any $h \in C_c^\infty((1-\theta)Z_G^0(\mathbf{R}))$, it follows that for $f = h\tilde{f}$, we have

$$\Theta(f) = \Theta(h\tilde{f}) = \Theta_h(\tilde{f}) = 0.$$

By the density statement above and since Θ is tempered, this implies that Θ vanishes.

In the case that (131) is non-trivial, the argument becomes more delicate. Define $f_1 \in C_c^\infty(G(\mathbf{R})\theta)$ by setting

$$f_1(x\theta) = \sum_z \chi_\pi(z)\, f(zx\theta),\ x \in G(\mathbf{R}), \tag{132}$$

where the sum is taken over the finite number of elements z belonging to (131). Clearly, the function f_1 is equivariant under (131) in the sense that

$$f_1(zx\theta) = \chi_\pi^{-1}(z)\, f_1(x\theta),\ x \in G(\mathbf{R}),$$

for all z belonging to (131). Given that such z are of the form $y_j\theta(y_j^{-1})$ for $y_j \in Z_G^0(\mathbf{R})$, we may also write

$$f_1(x\theta) = \sum_j \chi_\pi(y_j\theta(y_j^{-1}))\, f(y_j\theta(y_j^{-1})x\theta) = \sum_j \omega(y_j)\, f^{y_j^{-1}}(x\theta),\ x \in G(\mathbf{R})$$

(see (25)). Recalling the observations made at the beginning of this section, we see that

$$\Theta(f_1) = \sum_j \omega(y_j)\,\Theta(f^{y_j^{-1}}) = \sum_j \omega(y_j)\,\omega(y_j^{-1})\,\Theta(f) = \sum_j \Theta(f).$$

For the sake of simplicity let $F_1 = (1-\theta)Z_G^0(\mathbf{R})$ and $F_2 = (Z_G^0)^\theta(\mathbf{R})G_{\mathrm{der}}(\mathbf{R})$. As $G(\mathbf{R})$ is isomorphic to $F_1 \times F_2/F_1 \cap F_2$, we have $C_c^\infty(G(\mathbf{R})\theta) \cong C_c^\infty(F_1 \times F_2/F_1 \cap F_2)$. We denote the isomorphic $F_1 \cap F_2$-equivariant subspaces by $C_c^\infty(G(\mathbf{R})\theta, \chi_\pi^{-1})$ and $C_c^\infty(F_1 \times F_2/F_1 \cap F_2, \chi_\pi^{-1})$. If we denote by $C_c^\infty(F_1 \times F_2, \chi_\pi^{-1})$ the smooth, compactly supported functions on $F_1 \times F_2$, which are $F_1 \cap F_2$-equivariant in each coordinate then we have an isomorphism

$$C_c^\infty(F_1 \times F_2, \chi_\pi^{-1}) \to C_c^\infty(F_1 \times F_2/F_1 \cap F_2, \chi_\pi^{-1})$$

defined by the averaging over $\{(z, z^{-1}) : z \in F_1 \cap F_2\}$. The inverse of this isomorphism is induced by the quotient map $F_1 \times F_2 \to F_1 \times F_2/F_1 \cap F_2$, as the group $F_1 \cap F_2$ is finite. Together with our earlier observation concerning the density of the tensor product, we obtain the commutative diagram

$$\begin{tikzcd} C_c^\infty(F_1, \chi_\pi^{-1}) \otimes C_c^\infty(F_2, \chi_\pi^{-1}) \arrow[rr] \arrow[d] & & C_c^\infty(F_1 \times F_2, \chi_\pi^{-1}) \arrow[d, "\cong"] \\ C_c^\infty(F_1, \chi_\pi^{-1}) \otimes_{\mathbf{C}[F_1 \cap F_2]} C_c^\infty(F_2, \chi_\pi^{-1}) \arrow[r] & C_c^\infty(F_1 \times F_2 / F_1 \cap F_2, \chi_\pi^{-1}) \cong & C_c^\infty(G(\mathbf{R})\theta, \chi_\pi^{-1}). \end{tikzcd} \tag{133}$$

In this diagram the horizontal maps are injective with dense image and the lower tensor product is taken over the complex group algebra of $F_1 \cap F_2$. It follows that the function $f_1 \in C_c^\infty(G(\mathbf{R})\theta, \chi_\pi^{-1})$ may be approximated by linear combinations of products $h\tilde{f}$, where $h \in C_c^\infty((1-\theta)Z_G^0(\mathbf{R}), \chi_\pi^{-1})$ and $\tilde{f} \in C_c^\infty((Z_G^0)^\theta(\mathbf{R})G_{\mathrm{der}}(\mathbf{R}))\theta, \chi_\pi^{-1})$. We may now imitate the earlier strategy by fixing h and defining Θ_h on $(Z_G^0)^\theta(\mathbf{R})G_{\mathrm{der}}(\mathbf{R})\theta$ to conclude that

$$\Theta(h\tilde{f}) = \Theta_h(\tilde{f}) = 0.$$

The continuity of Θ and the density of the tensor product imply that

$$\sum_j \Theta(f) = \Theta(f_1) = 0$$

and so Θ vanishes.■

6.5 The case of no norms

The last case to consider in proving (60) for essentially square-integrable representations is the case in which there are no strongly θ-regular θ-elliptic elements in $G(\mathbf{R})$ which have norm in $H(\mathbf{R})$. Let us assume that this is so. In this case we set the spectral transfer factors $\Delta(\boldsymbol{\varphi_{H_1}}, \pi) = 0$ for all $\pi \in \Pi_\varphi$. Our task then is to prove that

$$\int_{H_1(\mathbf{R})/Z_1(\mathbf{R})} f_{H_1}(h) \sum_{\pi_{H_1} \in \Pi_{\varphi_{H_1}}} \Theta_{\pi_{H_1}}(h)\, dh = 0,\ f \in C_c^\infty(G(\mathbf{R})\theta).$$

In other words, we must show that the distribution Θ, defined by

$$\Theta(f) = \int_{H_1(\mathbf{R})/Z_1(\mathbf{R})} f_{H_1}(h) \sum_{\pi_{H_1} \in \Pi_{\varphi_{H_1}}} \Theta_{\pi_{H_1}}(h)\, dh,\ f \in C_c^\infty(G(\mathbf{R})\theta),$$

is the zero distribution. Our tactics are essentially the same as in the case where norms exist and our proof that Θ vanishes consists of pointing to the relevant facts already present in that case. We first assume that ω is trivial and establish that Θ is given by a locally integrable function on the θ-regular set of $G(\mathbf{R})\theta$. We then show that this locally integrable function vanishes on the θ-elliptic subset and deduce that Θ is zero by applying the twisted version of the Harish-Chandra Uniqueness Theorem. The vanishing of Θ for non-trivial ω follows using the same arguments employed in the proof of Theorem 1.

Suppose therefore that ω is trivial. Then (121) implies that Θ is $G(\mathbf{R})$-invariant. Lemma 26 is not affected by $\Delta(\boldsymbol{\varphi_{H_1}}, \pi)$ being zero, so Θ remains an eigendistribution. As Θ is a $G(\mathbf{R})$-invariant eigendistribution, it is given by a locally integrable function which is analytic on the regular set (Theorem 2.1.1 [Bou87]). Recall that $\Theta_{\pi_{H_1}}$ is tempered for all $\pi_{H_1} \in \Pi_{\varphi_{H_1}}$ and the map $f \mapsto f_{H_1}$ is assumed to be continuous, so that Θ is also tempered.

Now suppose δ is a θ-regular and θ-elliptic element of $G(\mathbf{R})$ and $f \in C_c^\infty(G(\mathbf{R})\theta)$ has small support about $\delta\theta$ as in section 6.3. Then the support of f lies in the θ-regular θ-elliptic subset of $G(\mathbf{R})\theta$. As elements in this subset are assumed to have no norm in $H(\mathbf{R})$ we may take $f_{H_1} = 0$ (*cf.* (16)). As a result, $\Theta(f)$ vanishes for all such f and so Θ vanishes at $\delta\theta$. This proves that Θ is zero on the θ-regular θ-elliptic set. Since Θ vanishes on the θ-regular θ-elliptic set, it vanishes everywhere by Theorem 15.1 [Ren97] and Proposition 3.6.1 [Bou87].

7 Spectral transfer for limits of discrete series

In this section we wish to generalize the character identity of Theorem 1 to include the possibility that Π_φ consists of limits of discrete series (§8 XII [Kna86], §5 [SJ80]). We shall work with the same general framework as given in section 6, except that we shall loosen assumptions 1-3. In this section we shall assume that $\Pi_{\varphi_{H_1}}$ is an essentially square-integrable L-packet (cf. Lemma 12). However, we shall weaken the assumption that Π_{φ^*} is an essentially square-integrable L-packet to Π_{φ^*} containing an "essential" limit of discrete series representation π which is equivalent to $\omega \otimes \pi^{\theta^*}$. The meaning of this assumption is the topic of the next subsection.

This weaker assumption allows for the possibility that $\varphi^*(W_{\mathbf{R}})$ *is* contained in a proper parabolic subgroup of $\hat{G}^* = \hat{G}$, and it may be possible for this parabolic subgroup *not* to be relevant (§3.3 [Bor79]). When this parabolic subgroup is not relevant the L-parameter $\boldsymbol{\varphi}$is not defined.

This being said, our approach will be to first prove an identity of the form

$$\sum_{\pi_{H_1} \in \Pi_{\varphi_{H_1}}} \Theta_{\pi_{H_1}}(f_{H_1}) = \sum_{\pi \in \Pi_{\varphi^*}} \Delta(\boldsymbol{\varphi_{H_1}}, \pi)\, \Theta_{\pi, \mathsf{U}_\pi}(f)$$

in the case that G is quasisplit, that is $G = G^*$. Thereafter we shall treat the case when G is not quasisplit, considering the possibilities of relevant and non-relevant parabolic subgroups in turn.

7.1 Essential limits of discrete series

The theory of limits of discrete series has its roots in the work of Zuckerman, Schmid, Hecht and Knapp ([Zuc77], [HS75], [KZ84])). An extension of this theory to a class of Lie groups which includes connected real reductive algebraic groups is given in §5 [SJ80]. Let us state how limits of discrete series appear in the assumptions of this section. Assume for the time being that φ is any relevant L-parameter for G and $\mu, \lambda \in X_*(\mathcal{T}) \otimes \mathbf{C}$ is a defining pair (section 4). Suppose $\langle \mu, \alpha \rangle \geq 0$ for all roots $\alpha \in R(\mathcal{B}, \mathcal{T})$. Note that the inequalities here are weak, so that μ may be singular. We assume that there is some $\nu \in X_*(\mathcal{T}) \otimes \mathbf{C}$ such that the pair $\mu + \nu, \lambda$ defines an admissible homomorphism $\varphi^\nu : W_{\mathbf{R}} \to \hat{G}$ whose L-packet Π_{φ^ν} consists of tempered essentially square-integrable representations. In other words, we are assuming that φ^ν satisfies the conditions of section 4.1 with φ replaced by φ^ν. In addition, we assume that equation (18) is satisfied for *both* $\mu_0 = \mu - \iota$ and $\mu_0 = \mu + \nu - \iota$. Recall that this assumption allots quasi-characters $\Lambda(\mu - \iota, \lambda)$ and $\Lambda(\mu + \nu - \iota, \lambda)$ of an elliptic maximal torus $S(\mathbf{R})$ to φ and φ^ν respectively. As a result

$$\Lambda(\nu) = \Lambda(\mu + \nu - \iota, \lambda)\ \Lambda(\mu - \iota, \lambda)^{-1} \qquad (134)$$

is a character of $S(\mathbf{R})$ (i.e. ν is analytically integral in the parlance of [Kna86]). Moreover, μ and λ may be identified with elements of $X_*(\hat{S}) \otimes \mathbf{C}$.

Let π_ν be the representation in (an equivalence class of) Π_{φ^ν} attached to $\Lambda(\mu + \nu - \iota, \lambda)$. The group $G(\mathbf{R})$ has a Langlands decomposition $G(\mathbf{R}) = {}^0G(\mathbf{R})\, A_G(\mathbf{R})^0$ in which A_G is the split component of the centre of G and ${}^0G(\mathbf{R})$ is the intersection of kernels of all homomorphisms of $G(\mathbf{R})$ into $(\mathbf{R}^\times)^0$

(§§5.1, 5.11-5.12 [Spr79]). The representation π_ν is a tensor product of the quasicharacter given by restriction of $\Lambda(\mu+\nu-\iota,\lambda)$ to $A_G(\mathbf{R})^0$, and a square-integrable (discrete series) representation ${}^0\pi_\nu$ of ${}^0G(\mathbf{R})$. We define the limit of discrete series character $\Theta_{{}^0\pi}$ as on page 397 [KZ84] (or Lemma 5.7 [SJ80]). It is possible for this character to be zero. When it is non-zero it is the character of an irreducible tempered representation ${}^0\pi$ which is independent of our choice of ν as above (Theorem 1.1 [KZ84]). For non-zero $\Theta_{{}^0\pi}$ one may define an irreducible representation π of $G(\mathbf{R})$ by setting it to be equal to the tensor product of ${}^0\pi$ with the quasicharacter given by the restriction of $\Lambda(\mu-\iota,\lambda)$ to $A_G(\mathbf{R})^0$. The representation π is tempered if and only if this quasicharacter is unitary. We shall call representations of this form *essential limits of discrete series*. We emphasize that an essential limit of discrete series representation carries the assumption that the distribution $\Theta_{{}^0\pi}$ is not zero. We denote the character of π by $\Theta(\mu,\lambda,\mathcal{B})$. If the distribution $\Theta_{{}^0\pi}$ vanishes we define $\Theta(\mu,\lambda,\mathcal{B})$ to be zero.

Given an essential limit of discrete series representation π as above, the set of characters of the representations in Π_φ is equal to the subset of non-zero distributions in

$$\{\Theta(w^{-1}\mu, w^{-1}\lambda, w\mathcal{B}w^{-1}) : w \in \Omega(\hat{G},\mathcal{T})\}$$

((4.3.4) [She82], §14 [She08]). It is immediate that Π_φ consists entirely of essential limits of discrete series.

In this section we will make the assumption that π is an essential limit of discrete series representation as above which is equivalent to $\omega\otimes\pi^\theta$. Thankfully, the setup of section 5.1 applies to the present context. We adopt the objects defined in that section, e.g. the elliptic maximal torus S, and the representatives $\delta_1,\ldots,\delta_k$ for $G(\mathbf{R})/Z_G(\mathbf{R})G_{\mathrm{der}}(\mathbf{R})^0$. This presupposes a choice of positive system for the roots of $(\mathfrak{k}\otimes\mathbf{C},\mathfrak{s}_{\mathrm{der}}\otimes\mathbf{C})$. This choice is fixed by by the unique Borel subgroup B containing S such that the pair $(\hat{B},\hat{S})$ is conjugate to $(\mathcal{B},\mathcal{T})$, which in turn fixes a positive system for the roots of $(\mathfrak{g}\otimes\mathbf{C},\mathfrak{s}\otimes\mathbf{C})$ and fixes a positive system for the roots $(\mathfrak{k}\otimes\mathbf{C},\mathfrak{s}_{\mathrm{der}}\otimes\mathbf{C})$ by restriction.

It is equally fortuitous that the representation ${}^0\pi$ of ${}^0G(\mathbf{R})$ is obtained via finite induction (p. 397 [KZ84]) from the subgroup

$${}^0G(\mathbf{R})^0\,{}^0Z_G(\mathbf{R}) = {}^0G_{\mathrm{der}}(\mathbf{R})^0\,{}^0Z_G(\mathbf{R}).$$

The representation being induced may be written as a tensor product of the central character of ${}^0\pi$ with a representation ϖ_1 in the *limit* of discrete series of ${}^0G_{\mathrm{der}}(\mathbf{R})^0$. With these modifications Lemma 4 carries over to the present situation. As in earlier sections, we set $\delta = \delta_m$ where $1 \leq m \leq k$ is the unique integer such that $\varpi_1^{\delta_m \theta}$ is equivalent to ϖ_1 (31). It follows that $\pi \cong \omega \otimes \pi^{\delta\theta}$ and, after identifying $(\hat{B}, \hat{S})$ with $(\mathcal{B}, \mathcal{T})$, that $\Theta(\mu, \lambda, \hat{B}) = \omega\, \Theta(\delta\theta \cdot \mu, \delta\theta \cdot \lambda, \delta\theta \cdot \hat{B})$. The latter identity and Theorem 1.1 (c) [KZ84] imply that there exists $w \in \Omega_{\mathbf{R}}(G, S)$ such that $w\delta\theta$ fixes B. Since $\Omega_{\mathbf{R}}(G, S) \cong \Omega(\mathfrak{k} \otimes \mathbf{C}, \mathfrak{s}_{\mathrm{der}} \otimes \mathbf{C})$ (Lemma 5.18 [Spr79], Theorem 4.41 [Kna86]) and $\delta\theta$ fixes the positive system of $(\mathfrak{k} \otimes \mathbf{C}, \mathfrak{s}_{\mathrm{der}} \otimes \mathbf{C})$ (Lemma 3) determined by B, we deduce that w is trivial. Consequently both B and S are stable under the action of $\delta\theta$. It further follows from Theorem 1.1 (c) [KZ84] that $\Lambda(\delta\theta \cdot \mu - \iota, \delta\theta \cdot \lambda)$ is a quasicharacter of $S(\mathbf{R})$ satisfying

$$\omega_{|S(\mathbf{R})}\, \Lambda(\delta\theta \cdot \mu - \iota, \delta\theta \cdot \lambda) = \Lambda(\mu - \iota, \lambda). \tag{135}$$

Here, we have identified ι with the half-sum of the positive roots $R(\hat{B}, \hat{S})$.

7.2 The quasisplit case

In this section we assume that $G = G^*$, i.e. that G is quasisplit. Our remaining assumptions are as follows. We assume that φ_{H_1} is an L-parameter as in section 6 such that $\Pi_{\varphi_{H_1}}$ is a essentially square-integrable L-packet (cf. Lemma 12). We assume that $\varphi^* = \varphi$ is the L-parameter for G as in section 6 and that π is a tempered essential limit of discrete series representation of $G^*(\mathbf{R})$ contained in Π_{φ^*} satisfying $\pi \cong \omega \otimes \pi^{\delta\theta}$. It is implicit in these last assumptions that we are adopting the framework of section 7.1 and its modified relation to section 5.1. This framework attaches to $\varphi^* = \varphi$ a defining pair $\mu, \lambda \in X_*(\hat{S}) \otimes \mathbf{C}$ (§§3.2, 4).

As stated in section 7.1, we are free to choose $\nu \in X_*(\hat{S}_{\mathrm{der}}) \otimes \mathbf{C}$ such that it lies in the positive chamber determined by B, and $\mu - \iota$ and $\mu + \nu - \iota$ satisfy equation (18). After possibly averaging over the finite action of $\delta\theta$ on $X_*(\hat{S}_{\mathrm{der}})$ (Lemma 3.1), we may assume that ν is fixed by $\delta\theta$. With this assumption in place we see from (134) and (135) that

$$\begin{aligned}
\Lambda(\delta\theta \cdot (\mu + \nu) - \iota, \delta\theta \cdot \lambda) &= \Lambda(\delta\theta \cdot \mu - \iota + \nu, \delta\theta \cdot \lambda) \\
&= \Lambda(\delta\theta \cdot \mu - \iota, \delta\theta \cdot \lambda)\, \Lambda(\nu) \\
&= \omega_{|S(\mathbf{R})}^{-1}\, \Lambda(\mu - \iota, \lambda)\, \Lambda(\nu) \\
&= \omega_{|S(\mathbf{R})}^{-1}\, \Lambda(\mu + \nu - \iota, \lambda).
\end{aligned} \tag{136}$$

This means in turn that $\omega \otimes \pi_\nu^{\delta\theta} \cong \pi_\nu$, $\omega \otimes \pi_\nu^{\theta} \cong \pi_\nu$ and $\omega \otimes \Pi_{\varphi^\nu} = \Pi_{\varphi^\nu}$.

Let us look back to the three assumptions enumerated in section 6 and see to what extent φ^ν and π_ν fit in with them. The first assumption is satisfied by φ^ν since π_ν is a tempered essentially square-integrable representation (Lemma 3.4 [Lan89], §4.1, 12.3 [Bor79]). We have just shown that the second assumption is satisfied by π_ν. However, the third assumption, namely that there exists $\gamma_1 \in H_1(\mathbf{R})$ which is the norm of θ-elliptic δ does not follow from the preceding assumptions. We make this our final assumption so that we may appeal to Theorem 1 for φ^ν when the time is ripe.

7.2.1 A minimal Levi subgroup containing $\varphi(W_{\mathbf{R}})$

In order to understand the (twisted) characters of the L-packet Π_φ. One must find a Levi subgroup M of G such that $\varphi(W_{\mathbf{R}}) \subset \hat{M} \rtimes W_{\mathbf{R}}$ and $\varphi(W_{\mathbf{R}})$ is not contained in any proper Levi subgroup of $\hat{M} \rtimes W_{\mathbf{R}}$ (§4.2). A procedure for finding such M in the case of ordinary endoscopy is given in §4.3 [She82] and §14 [She08]. We shall follow this procedure partway, and continue with a result of Borel (Proposition 3.6 [Bor79]). This procedure uses a root system $R(\varphi) \subset R(\hat{G}, \mathcal{T})$ to describe a torus in $\hat{G}_{\mathrm{der}}$. The centralizer of this torus in ${}^LG$ will be a Levi subgroup and it pins down the Levi subgroup M of sought for in G.

For the next few paragraphs it shall be easier to return to the view that μ and λ are elements of $X_*(\mathcal{T}) \otimes \mathbf{C}$, for we shall be using the endoscopic data given in section 3.2. Recall that $\hat{H} \overset{\xi}{\cong} (\hat{G}^{\mathsf{s}\hat{\theta}})^0$ and $\varphi = \xi \circ \xi_{H_1}^{-1} \circ \varphi_{H_1}$ so that $\varphi(W_{\mathbf{R}}) \subset (\hat{G}^{\mathsf{s}\hat{\theta}})^0$. As in the proof of Lemma 2.2.A [KS99], we may assume that $s \in \mathcal{T}$. This means that μ actually belongs to $X_*((\mathcal{T}^{\hat{\theta}})^0) \otimes \mathbf{C}$.

The set $R(\varphi) = \{\alpha \in R(\hat{G}, \mathcal{T}) : \langle \mu, \alpha \rangle = 0\}$ is easily shown to satisfy the axioms of a root system. This root system is related to the root system $R(\hat{H}_1, \hat{T}_{H_1})$. To explain this relationship, consider the following diagram

$$R(\hat{H}_1, \hat{T}_{H_1}) \leftrightarrow R(\hat{H}, \mathcal{T}_H) \leftrightarrow R((\hat{G}^{\mathsf{s}\hat{\theta}})^0, (\mathcal{T}^{\hat{\theta}})^0) \hookrightarrow R_{\mathrm{res}}(\hat{G}, \mathcal{T}). \tag{137}$$

The leftmost map is a bijection (actually an isomorphism as in §9.2 [Hum94]) which is induced by conjugation (see section 3.3) and the inclusion $\hat{H} \to \hat{H}_1$ (see (7)). The middle map is a bijection obtained from ξ (see condition 4 of section 3.2). The object on the right a root system defined as

$$\mathrm{R}_{\mathrm{res}}(\hat{G}, \mathcal{T}) = \{\alpha_{\mathrm{res}} = \alpha_{|(\mathcal{T}^{\hat{\theta}})^0} : \alpha \in R(\hat{G}, \mathcal{T})\}$$

(§1.3 [KS99]). The map on the right is an injection described by (1.3.5) [KS99][2], which tells us that α_{res} belongs to $R((\hat{G}^{\mathsf{s}\hat{\theta}})^0, (\mathcal{T}^{\hat{\theta}})^0)$ if and only if $N\alpha(\mathsf{s}) = \sum_{j=0}^{l_\alpha - 1} \hat{\theta}^j \alpha(\mathsf{s}) = 1$. Here, l_α is the cardinality of the $\hat{\theta}$-orbit of $\alpha \in R(\hat{G}, \mathcal{T})$. Making identifications according to the (137), we may state this as

$$\alpha_{\mathrm{res}} \in R(\hat{H}_1, \hat{T}_{H_1}) \Leftrightarrow N\alpha(\mathsf{s}) = 1. \tag{138}$$

An immediate consequence of this equivalence is that $N\alpha(\mathsf{s}^2) = 1$ when $\alpha_{\mathrm{res}} \in R(\hat{H}_1, \hat{T}_{H_1})$. The following lemma shows that this is true in general.

Lemma 27 *The square* s^2 *of the endoscopic datum belongs to* $Z_{\hat{G}}$.

Proof. Suppose $a \in \hat{G}$ satisfies $\varphi^\nu(\sigma) = (a, \sigma) \in {}^LG$. The element $\varphi^\nu(\sigma)$ acts on $\mathcal{T}/Z_{\hat{G}}$ by conjugation and the resulting action is equal to inversion (see section 4.1 and 9.4 [Bor79]). From this we compute that

$$\mathsf{s}Z_{\hat{G}} = a\,\sigma(\mathsf{s}^{-1}a)Z_{\hat{G}}.$$

On the other hand, condition 4 of section 3.2 dictates that

$$\mathsf{s}\,{}^L\theta(\varphi^\nu(\sigma))\mathsf{s}^{-1}Z_{\hat{G}} = a'(\sigma)\,\varphi^\nu(\sigma)Z_{\hat{G}} = \varphi^\nu(\sigma)Z_{\hat{G}}$$

whence we compute

$$\mathsf{s}Z_{\hat{G}} = a\sigma(\mathsf{s}\hat{\theta}(a))Z_{\hat{G}}.$$

Equating terms in the above two computations, we arrive at

$$\mathsf{s}^2 Z_{\hat{G}} = \hat{\theta}(a)\,a^{-1}Z_{\hat{G}}.$$

We shall now prove that $\hat{\theta}(a)\,a^{-1}$ lies in $Z_{\hat{G}}$ by using the splitting $(\mathcal{B}, \mathcal{T}, \{\mathcal{X}\})$, which is preserved by both σ and $\hat{\theta}$ (§3.1). The action of $\varphi^\nu(\sigma)^2$ on $\hat{G}$ is trivial, and our assumption that Π_{φ^ν} is essentially square-integrable ensures equation (17). Consequently,

$$\mathrm{Int}(a)\sigma \cdot (\mathcal{B}, \mathcal{T}, \{\mathcal{X}\}) = (\bar{\mathcal{B}}, \mathcal{T}, \{\pm\bar{\mathcal{X}}\}),$$

where $\bar{\mathcal{B}}$ is the Borel subgroup opposite to $\mathcal{B}$ and $\{\bar{\mathcal{X}}\}$ is the set of root vectors given by $\{\mathcal{X}\}$ and Proposition 8.3 (f) [Hum94]. We may apply $\hat{\theta}$

[2] Note that $R(\hat{H}, \mathcal{T}_H)$ is reduced (Lemma 7.4.4 [Spr98]) so that (1.3.6) and (1.3.7) [KS99] do not apply.

to the left-hand side of this equation, without affecting the right-hand side. This in turn implies

$$\mathrm{Int}(a)\sigma \cdot (\mathcal{B}, \mathcal{T}, \{\mathcal{X}\}) = \mathrm{Int}(\hat{\theta}(a))\sigma \cdot (\mathcal{B}, \mathcal{T}, \{\mathcal{X}\}),$$

$\mathrm{Int}(a^{-1}\hat{\theta}(a)) \cdot (\mathcal{B}, \mathcal{T}, \{\mathcal{X}\}) = (\mathcal{B}, \mathcal{T}, \{\mathcal{X}\})$, and $a^{-1}\hat{\theta}(a) \in Z_{\hat{G}}$. The trivial action of $\varphi^{\nu}(\sigma)^2$ on $\hat{G}$ implies that $\mathrm{Int}(a\sigma(a))$ is trivial. We therefore conclude that $a\sigma(a) \in Z_{\hat{G}}$ and

$$(\hat{\theta}(a)\, a^{-1})^{-1} Z_{\hat{G}} = a\, \hat{\theta}(a^{-1}) Z_{\hat{G}} = \sigma(a^{-1})\, \hat{\theta}(\sigma(a)) Z_{\hat{G}} = \sigma(a^{-1}\hat{\theta}(a)) Z_{\hat{G}} = Z_{\hat{G}}.\blacksquare$$

Lemma 27 strengthens (138) to the statement that $\alpha_{\mathrm{res}} \in R(\hat{H}_1, \hat{T}_{H_1})$ if and only if $N\alpha(\mathsf{s}) = 1$ and $\alpha_{\mathrm{res}} \notin R(\hat{H}_1, \hat{T}_{H_1})$ if and only if $N\alpha(\mathsf{s}) = -1$ (*cf.* proof of Proposition 4.4.7 [She82]). This is an explicit connection between $N\alpha$ and $\alpha_{\mathrm{res}} \in R(\hat{H}_1, \hat{T}_{H_1})$. There is a further connection between $R(\hat{H}_1, \hat{T}_{H_1})$ and $R(\varphi)$.

Lemma 28 *Suppose $\alpha \in R(\varphi)$. Then $\alpha_{\mathrm{res}} \notin R(\hat{H}_1, \hat{T}_{H_1})$.*

Proof. We prove the contrapositive assertion. Suppose that $\alpha_{\mathrm{res}} \in R(\hat{H}_1, \hat{T}_{H_1})$. We are making some identifications here (*cf.* (137)), and to make them apparent we may write $\alpha_{\mathrm{res}} = \xi \circ \xi_{H_1}^{-1} \cdot \beta$ for some $\beta \in R(\hat{H}_1, \hat{T}_{H_1})$. Let $\mu_{H_1}, \lambda_{H_1} \in X_*(\hat{T}_{H_1}) \otimes \mathbf{C}$ be a defining pair for φ_{H_1} (as in section 4.1). As we are assuming that $\Pi_{\varphi_{H_1}}$ is an essentially square-integrable L-packet we have $\langle \mu_{H_1}, \beta \rangle \neq 0$ (Lemma 3.3 [Lan89]). Unwrapping the terms, we deduce that

$$\langle \mu, \alpha \rangle = \langle \mu, \alpha_{\mathrm{res}} \rangle = \langle \xi \circ \xi_{H_1}^{-1} \cdot \mu_{H_1}, \xi \circ \xi_{H_1}^{-1} \cdot \beta \rangle = \langle \mu_{H_1}, \beta \rangle \neq 0$$

so that $\alpha \notin R(\varphi)$.$\blacksquare$

The next lemma is Proposition 4.4.7 [She82] in the case of ordinary endoscopy. Our proof is a mild paraphrase of Shelstad's.

Lemma 29 *The root system $R(\varphi)$ is of type $A_1 \times A_1 \times \cdots \times A_1$.*

Proof. The assertion of the lemma is equivalent to $\alpha + \beta \notin R(\varphi)$ for all $\alpha, \beta \in R(\varphi)$. Suppose $\alpha, \beta \in R(\varphi)$. Then Lemma 28 tells us that $\alpha_{\mathrm{res}}, \beta_{\mathrm{res}} \notin R(\hat{H}_1, \hat{T}_{H_1})$. By the observation following Lemma 27, we know that $N\alpha(\mathsf{s}) = N\beta(\mathsf{s}) = -1$. In additive notation, this implies that $N(\alpha+\beta)(\mathsf{s}) = 1$, so that $\alpha_{\mathrm{res}} + \beta_{\mathrm{res}} \in R(\hat{H}_1, \hat{T}_{H_1})$ (see (138)). Finally, Lemma 28 tells us again that $\alpha + \beta \notin R(\varphi)$.$\blacksquare$

Having established the mutual orthogonality of the roots in $R(\varphi)$, we may use the construction on page 407 [She82] by defining for each positive $\alpha \in R(\varphi)$ an element

$$\mathbf{s}_\alpha = \begin{cases} \exp\left(\frac{\pi}{4}(\mathcal{X}_\alpha - \mathcal{X}_{-\alpha})\right), & \text{if } \varphi(\sigma)\cdot\mathcal{X}_\alpha = \mathcal{X}_{-\alpha} \\ \exp\left(\frac{i\pi}{4}(\mathcal{X}_\alpha + \mathcal{X}_{-\alpha})\right), & \text{if } \varphi(\sigma)\cdot\mathcal{X}_\alpha = -\mathcal{X}_{-\alpha} \end{cases}$$

Here, the elements $\mathcal{X}_\alpha$ are chosen from $\{\mathcal{X}\}$ (fixed in (5)) and the $\mathcal{X}_{-\alpha}$ are defined so that $\mathcal{X}_\alpha$, $\mathcal{X}_{-\alpha}$, $\mathcal{H}_\alpha = [\mathcal{X}_\alpha, \mathcal{X}_{-\alpha}]$ map to the usual basis for $\mathfrak{sl}(2,\mathbf{C})$ under an isomorphism (Proposition 8.3 (f) [Hum94]). Orthogonality allows us to define $\mathbf{s} = \prod_\alpha \mathbf{s}_\alpha$. It is easy to calculate that $\varphi(\sigma)\cdot\mathbf{s}_\alpha = \mathbf{s}_\alpha^{-1}$, so that

$$\varphi(\sigma)\cdot\mathbf{s} = \mathbf{s}^{-1}. \tag{139}$$

Lemma 30 *Suppose $\alpha \in R(\varphi)$ is a positive root and $w_\alpha \in \Omega(\hat{G},\mathcal{T})$ is the reflection generated by α. Then $\mathrm{Int}(\mathbf{s}_\alpha^2)$ acts on $\mathcal{T}$ as w_α.*

Proof. Suppose $\mathbf{s}_\alpha = \exp\left(\frac{\pi}{4}(\mathcal{X}_\alpha - \mathcal{X}_{-\alpha})\right)$. Then we compute that

$$\begin{aligned}
\mathrm{Ad}(\mathbf{s}_\alpha^2)H_\alpha &= \mathrm{Ad}\left(\exp\left(\frac{\pi}{2}(\mathcal{X}_\alpha - \mathcal{X}_{-\alpha})\right)\right)H_\alpha \\
&= H_\alpha - \frac{\pi}{2}(2(\mathcal{X}_\alpha + \mathcal{X}_{-\alpha})) - \frac{(\pi/2)^2}{2!}H_\alpha + \cdots \\
&= \sum_{j=0}^{\infty}(-1)^j\frac{\pi^{2j}}{2j!}H_\alpha + \sum_{r=0}^{\infty}(-1)^{r+1}\frac{\pi^{2r+1}}{(2r+1)!}(\mathcal{X}_\alpha + \mathcal{X}_{-\alpha}) \\
&= \cos(\pi)H_\alpha - \sin(\pi)(\mathcal{X}_\alpha + \mathcal{X}_{-\alpha}) \\
&= w_\alpha \cdot H_\alpha.
\end{aligned}$$

Now suppose $\mathbf{s}_\alpha = \exp\left(\frac{i\pi}{4}(\mathcal{X}_\alpha + \mathcal{X}_{-\alpha})\right)$. A similar computation yields

$$\mathrm{Ad}(\mathbf{s}_\alpha^2)H_\alpha = \cos(\pi)H_\alpha - i\sin(\pi)(\mathcal{X}_\alpha - \mathcal{X}_{-\alpha}) = w_\alpha \cdot H_\alpha$$

from which the lemma follows.■

Lemma 31 *The element $\varphi(\sigma)$ normalizes the maximal torus $\mathbf{s}\mathcal{T}\mathbf{s}^{-1}$ of $\hat{G}$ and fixes each root $\mathbf{s}\cdot\alpha \in R(\hat{G},\mathbf{s}\mathcal{T}\mathbf{s}^{-1})$ for every $\alpha \in R(\varphi)$.*

Proof. Applying (139) and Lemma 30, we obtain

$$\varphi(\sigma)\cdot\mathbf{s}\mathcal{T}\mathbf{s}^{-1} = \mathbf{s}^{-1}(\varphi(\sigma)\cdot\mathcal{T})\mathbf{s}$$

$$
\begin{aligned}
&= \mathbf{s}\mathbf{s}^{-2}(\mathcal{T})\mathbf{s}^2\mathbf{s}^{-1} \\
&= \mathbf{s}\left(\prod_{\alpha\in R(\varphi),\,\alpha>0} w_\alpha\cdot\mathcal{T}\right)\mathbf{s}^{-1} \\
&= \mathbf{s}\mathcal{T}\mathbf{s}^{-1}.
\end{aligned}
$$

This proves the first assertion. To prove the second assertion, recall from section 7.1 and (17) that $\varphi(\sigma)$ negates the roots of $R(\hat{G},\mathcal{T})$. Therefore, as in the computation above we see that for any $\beta\in R(\varphi)$ we have

$$\varphi(\sigma)\cdot(\mathbf{s}\cdot\beta) = \mathbf{s}\cdot(\prod w_\alpha\varphi(\sigma)\cdot\beta) = \mathbf{s}\cdot(-w_\beta\cdot\beta) = \mathbf{s}\cdot\beta.\blacksquare$$

We will now follow the procedure given in the proof of Proposition 3.6 [Bor79] to produce the Levi subgroup M. We compute the identity component of the centralizer of $\varphi(W_{\mathbf{R}})$ in $\hat{G}_{\mathrm{der}}$, which we denote by $Z_{\hat{G}_{\mathrm{der}}}(\varphi(W_{\mathbf{R}}))^0$. It is equal to the intersection of the identity components of the respective centralizers $Z_{\hat{G}_{\mathrm{der}}}(\varphi(\mathbf{C}^\times))^0$ and $Z_{\hat{G}_{\mathrm{der}}}(\varphi(\sigma))^0$ of $\varphi(\mathbf{C}^\times)$ and $\varphi(\sigma)$. The former group depends only on $\mu\in X_*(\mathcal{T})\otimes\mathbf{C}$ (§4). Observe that $\mathbf{s}\cdot\mu=\mu$, so that μ also belongs to $X_*(\mathbf{s}\mathcal{T}\mathbf{s}^{-1})\otimes\mathbf{C}$. Let us identify $X_*(\mathbf{s}\mathcal{T}\mathbf{s}^{-1})\otimes\mathbf{C}$ with the Lie algebra of $\mathbf{s}\mathcal{T}\mathbf{s}^{-1}$, and consider the root space decomposition of the Lie algebra of $\hat{G}$ with respect to $\mathbf{s}\mathcal{T}\mathbf{s}^{-1}$. We then compute that the Lie algebra of $Z_{\hat{G}_{\mathrm{der}}}(\varphi(\mathbf{C}^\times))$ is generated by the elements $\mathbf{s}\cdot\mathcal{X}_\alpha$, $\mathbf{s}\cdot\mathcal{X}_{-\alpha}$, $\mathbf{s}\cdot\mathcal{H}_\alpha$, where α runs through the positive roots in $R(\varphi)$, $\mathcal{X}_\alpha$ is chosen from $\{\mathcal{X}\}$ (fixed in (5)), and $\mathcal{X}_{-\alpha}$ is chosen so that $\mathcal{H}_\alpha=[\mathcal{X}_\alpha,\mathcal{X}_{-\alpha}]$ (Proposition 8.3 (f) [Hum94]). By Lemma 29 this Lie algebra is isomorphic to a direct sum of copies of $\mathfrak{sl}(2,\mathbf{C})$. By Lemma 31 the element $\mathbf{s}\cdot\mathcal{H}_\alpha$ is contained in the Lie algebra of $Z_{\hat{G}_{\mathrm{der}}}(\varphi(\sigma))$ for every $\alpha\in R(\varphi)$. As a result, the complex span of $\{\mathbf{s}\cdot\mathcal{H}_\alpha : \alpha\in R(\varphi),\ \alpha>0\}$ is the Lie algebra of a maximal torus of $Z_{\hat{G}_{\mathrm{der}}}(\varphi(W_{\mathbf{R}}))^0$. This torus is clearly a subtorus of $\mathbf{s}\mathcal{T}\mathbf{s}^{-1}$.

Clearly, the centralizer in ${}^LG$ of this torus contains $Z_{\hat{G}_{\mathrm{der}}}(\varphi(W_{\mathbf{R}}))^0$, which itself contains $\varphi(W_{\mathbf{R}})$. The projection of $\varphi(W_{\mathbf{R}})$ to $W_{\mathbf{R}}$ is surjective. By Lemma 3.5 [Bor79] and the proof of Proposition 3.6 [Bor79] this centralizer is a Levi subgroup of ${}^LG$ which contains $\varphi(W_{\mathbf{R}})$ minimally. According to §§3.3-3.4 [Bor79], there exists $m\in\hat{G}$ and a Levi subgroup M of G defined over $\mathbf{R}$ such that the previous Levi subgroup is m-conjugate to $\hat{M}\rtimes W_{\mathbf{R}}$. In this way $m\mathbf{s}\mathcal{T}\mathbf{s}^{-1}m^{-1}$ is a maximal torus of $\hat{M}$ and by Lemma 3.3 [Lan89], it is the unique maximal torus of $\hat{M}$ normalized by $m\varphi(W_{\mathbf{R}})m^{-1}$. By Lemma 3.1 [Lan89] there is a maximal torus S_M of M defined over $\mathbf{R}$ and elliptic in

$M(\mathbf{R})$. As in section section 4.1, we identify $m\mathbf{s}\mathcal{T}\mathbf{s}^{-1}m^{-1}$ with $\hat{S}_M$. Set

$$\varphi_M(w) = m\varphi(w)m^{-1},\ w \in W_{\mathbf{R}}.$$

It is immediate from the definitions that $\varphi_M \in \varphi$. The roots $\alpha \in R(m\mathbf{s} \cdot \mathcal{B}, \hat{S}_M)$ correspond to the coroots $\alpha^\vee \in R(G, S_M)$ (§7.3 [Spr98]), which fix a unique Borel subgroup B_M of G containing S_M (§8.2 [Spr98]). This completes our goal of finding the desired Levi subgroup M of G.

We close this subsection by describing how $S_M(\mathbf{R})$ may be converted into an elliptic torus of $G(\mathbf{R})$ through the application of Cayley transforms. By Lemma 29, we may enumerate the positive roots in $m\mathbf{s}\cdot R(\varphi)$ as $\{\alpha_1, \ldots \alpha_r\} \subset R(m\mathbf{s} \cdot \mathcal{B}, \hat{S}_M)$, and thereby enumerate the positive roots of $R(G, S_M)$.

Lemma 32 *The positive real roots (p. 349 [Kna86]) in* $\mathrm{R}(G, S_M)$ *are* $\alpha_1^\vee, \ldots, \alpha_r^\vee$ *and each of these roots is simple.*

Proof. Lemma 31 implies that $\varphi_M(\sigma) \cdot \alpha_j = \alpha_j$ for every $1 \leq j \leq r$ and

$$\alpha_j \in X^*(m\mathbf{s}\mathcal{T}\mathbf{s}^{-1}m^{-1}) \cong X^*(\hat{S}_M) \cong X_*(S_M),\ 1 \leq j \leq r.$$

Therefore in the L-group ${}^L S_M$, the element σ, which acts as $\varphi_M(\sigma)$ does on $m\mathbf{s}\mathcal{T}\mathbf{s}^{-1}m^{-1}$, fixes pointwise the subtorus of S_M generated by the images of the cocharacters $\alpha_1, \ldots, \alpha_r$. In particular, this subtorus is defined over $\mathbf{R}$ and split in $S_M(\mathbf{R})$ (§9.4 [Bor79]). It follows from (17) that the corresponding coroots (§7.3 [Spr98]) $\alpha_1^\vee, \ldots, \alpha_r^\vee \in \mathrm{R}(G, S_M)$ vanish on the maximally compact subtorus of $S_M(\mathbf{R})$ and are therefore *real* roots (p. 349 [Kna86]).

It is immediate from the definitions that if $\beta \in R(\hat{G}, \hat{S}_M)$ is not in $\{\pm\alpha_1, \ldots, \pm\alpha_r\}$ then $\langle m \cdot \mu, \beta\rangle \neq 0$ and (17) implies

$$\langle m \cdot \mu, \varphi_M(\sigma) \cdot \beta\rangle = \langle \varphi_M(\sigma) \cdot (m \cdot \mu), \beta\rangle = \langle m \cdot (\varphi(\sigma) \cdot \mu), \beta\rangle = -\langle m \cdot \mu, \beta\rangle.$$

Consequently, $\varphi_M(\sigma)\cdot\beta \neq \beta$ and, reasoning as above, the root $\beta^\vee \in R(G, S_M)$ is not real. This proves that the only real roots of $R(\hat{G}, \hat{S}_M)$ are $\{\pm\alpha_1^\vee, \ldots, \pm\alpha_r^\vee\}$.

Finally, let us illustrate by way of an example why the roots $\alpha_1, \ldots, \alpha_r$ are simple in the set of positive roots $R(m\mathbf{s} \cdot \mathcal{B}, \hat{S}_M)$. For instance, if α_1 is decomposable as a sum of two simple roots β_1, β_2 then $\langle m \cdot \mu, \beta_1\rangle = \langle m \cdot \mu, \beta_2\rangle = 0$ and so $\beta_1, \beta_2 \in \{\alpha_1, \ldots, \alpha_r\}$. However, this contradicts Lemma 29. The simplicity of $\alpha_1^\vee, \ldots, \alpha_r^\vee$ is equivalent to that of $\alpha_1, \ldots, \alpha_r$.$\blacksquare$

Lemma 32 makes it possible for us to apply Cayley transforms $\mathbf{d}_{\alpha_1^\vee}$, $\ldots$,$\mathbf{d}_{\alpha_r^\vee}$ to S_M (§XI.6 [Kna86]). These transforms commute with one another thanks to Lemma 29, and the torus $\mathbf{d}_{\alpha_1^\vee} \circ \cdots \circ \mathbf{d}_{\alpha_r^\vee}(S_M)$ is defined over $\mathbf{R}$ and elliptic (Proposition 11.16 (a) [Kna86] and Lemma 32). All elliptic tori in $G(\mathbf{R})$ being conjugate in $G(\mathbf{R})$ (Proposition 6.61 [Kna96]), we lose no generality in assuming that the torus S of section 7.1 is equal to $\mathbf{d}_{\alpha_1^\vee} \circ \cdots \circ \mathbf{d}_{\alpha_r^\vee}(S_M)$ and $B = \mathbf{d}_{\alpha_1^\vee} \circ \cdots \circ \mathbf{d}_{\alpha_r^\vee}(B_M)$. We shall make this assumption. However, in doing so, the reader is cautioned that the identification between the two pairs $(\hat{B}, \hat{S})$ and $(\mathcal{B}, \mathcal{T})$ made in section 7.1, abusively identifies μ with $m\mathbf{s} \cdot \mu = m \cdot \mu$ and λ with $m\mathbf{s} \cdot \lambda$.

The map $\mathbf{d}_{\alpha_1^\vee} \circ \cdots \circ \mathbf{d}_{\alpha_r^\vee}$ induces an embedding of $R(\hat{M}, \hat{S}_M)$ as a root subsystem of $R(\hat{G}, \hat{S})$. To see this, we apply Lemma 31 and (17) to obtain

$$(140) \langle \alpha_j, \beta^\vee \rangle = \langle \sigma \cdot \alpha_j, \sigma \cdot \beta^\vee \rangle = -\langle \alpha_j, \beta^\vee \rangle,\ 1 \leq j \leq r,\ \beta \in R(\hat{M}, \hat{S}_M).$$

As a result, α_j is orthogonal to β and $\beta = \mathbf{d}_{\alpha_1^\vee} \circ \cdots \circ \mathbf{d}_{\alpha_r^\vee} \cdot \beta$ for every $\beta \in R(\hat{M}, \hat{S}_M)$. The root on the right belongs to $R(\hat{G}, \hat{S})$.

It is noteworthy, that orthogonality to $\{\alpha_1, \ldots, \alpha_r\}$ in $R(\hat{G}, \hat{S}_M)$ characterizes $R(\hat{M}, \hat{S}_M)$. For $\beta \in R(\hat{G}, \hat{S}_M)$ being orthogonal to $\{\alpha_1, \ldots, \alpha_r\}$ is equivalent to the root space of β belonging to the (Lie algebra of the) centralizer of $(\cap_{j=1}^r \ker \alpha_j)^0 = \hat{M}$ (Proposition 8.4 (e) and §9.4 [Hum94]). Similar consequences may be drawn between $R(M, S_M)$ and $R(G, S)$.

7.2.2 L-packets for essential limits of discrete series under twisting

According to section 4.2, (21), and section 7.2.1, there exists a tempered essentially square-integrable representation π_Λ in $\Pi_{\varphi_M, M}$ such that Π_φ is equal to the (equivalence classes of) irreducible subrepresentations of

$$(141) \qquad \bigoplus_{w \in \Omega(M, S_M)/\Omega_{\mathbf{R}}(M, S_M)} \operatorname{ind}_{P(\mathbf{R})}^{G(\mathbf{R})} \pi_{w^{-1}\Lambda}.$$

Here, P is the $\mathbf{R}$-parabolic subgroup of G dual to a parabolic subgroup in ${}^L G$, whose Levi subgroup is $\hat{M} \rtimes W_{\mathbf{R}}$ (§3.3 [Bor79]) and contains $m\mathbf{s} \cdot \mathcal{B}$. In addition, $\Lambda = \Lambda(m \cdot \mu - \iota_M, \lambda)$ is a character of $S_M(\mathbf{R})$ which is positive relative to B_M and ι_M is the half-sum of the positive roots in $R(\hat{M}, \hat{S}_M)$.

On page 408 [She82], it is shown that the characters of the irreducible subrepresentations of (141) are $\Theta(w^{-1} \cdot \mu, \lambda, w^{-1} \cdot \hat{B})$ where w runs through

$$(142) \qquad \langle w_{\alpha_1^\vee}, \ldots, w_{\alpha_r^\vee} \rangle\, \Omega(M, S_M) \Omega_{\mathbf{R}}(G, S)/\Omega_{\mathbf{R}}(G, S).$$

In this quotient, $w_{\alpha_j^\vee}$ denotes the reflection generated by the root $\mathbf{d}_{\alpha_j^\vee} \cdot \alpha_j^\vee = \mathbf{d}_{\alpha_1^\vee} \circ \cdots \circ \mathbf{d}_{\alpha_r^\vee} \cdot \alpha_j^\vee \in R(G, S)$. Also, we are identifying $\Omega(M, S_M)$ by way of the embedding of $R(M, S_M)$ in $R(G, S)$ given in section 7.2.1.

As we shall be making a comparison of *twisted* characters, not every $\theta(w^{-1}\cdot\mu,\lambda,w^{-1}\cdot\hat{B})$ is necessarily pertinent to the comparison. The characters that are pertinent are ascertained using the following lemma.

Lemma 33 *Suppose $w\in\Omega(G,S)$ and the essential limit of discrete series character $\Theta(w^{-1}\cdot\mu,\lambda,w^{-1}\cdot\hat{B})$ is non-zero. Then $\Theta(w^{-1}\cdot\mu,\lambda,w^{-1}\cdot\hat{B})$ is equal to $\omega\,\Theta(\theta\cdot w^{-1}\cdot\mu,\theta\cdot\lambda,\theta\cdot w^{-1}\cdot\hat{B})$ if and only if $w\in\Omega_{\mathbf{R}}(G,S)$ belongs to $(\Omega(G,S)/\Omega_{\mathbf{R}}(G,S))^{\delta\theta}$.*

Proof. Since characters are invariant under conjugation by $G(\mathbf{R})$, we may replace $\omega\,\Theta(\theta\cdot w^{-1}\cdot\mu,\theta\cdot\lambda,\theta\cdot w^{-1}\cdot\hat{B})$ in the assertion by $\omega\,\Theta(\delta\theta\cdot w^{-1}\cdot\mu,\delta\theta\cdot\lambda,\delta\theta\cdot w^{-1}\cdot\hat{B})$ without changing the content of the assertion. Suppose first that $\Theta(w^{-1}\cdot\mu,\lambda,w^{-1}\cdot\hat{B})$ is equal to $\omega\,\Theta(\delta\theta\cdot w^{-1}\cdot\mu,\delta\theta\cdot\lambda,\delta\theta\cdot w^{-1}\cdot\hat{B})$. Then there exists $w_1\in\Omega_{\mathbf{R}}(G,S)$ such that $w_1\delta\theta w^{-1}\cdot B=w^{-1}\cdot B$ (Theorem 1.1 (c) [KZ84]). As noted in section 7.1, the action of $\delta\theta$ on G preserves B, so the previous identity is equivalent to $w_1\delta\theta w^{-1}(\delta\theta)^{-1}\cdot B=w^{-1}\cdot B$. Since $\delta\theta$ preserves $\Omega(G,S)$ (Lemma 3) the last identity implies that $ww_1\delta\theta w^{-1}(\delta\theta)^{-1}$ is trivial in $\Omega(G,S)$ and this means that $w\Omega_{\mathbf{R}}(G,S)$ belongs to $(\Omega(G,S)/\Omega_{\mathbf{R}}(G,S))^{\delta\theta}$ (definition (64)).

Conversely, suppose that $w_1=w^{-1}\delta\theta w(\delta\theta)^{-1}w$ belongs to $\Omega_{\mathbf{R}}(G,S)$. Then, using equation (135), we have the following list of equivalent identities between quasicharacters of $S(\mathbf{R})$:

$$\Lambda(\delta\theta w(\delta\theta)^{-1}w^{-1}w_1\cdot(\mu-\iota),\lambda)=\Lambda(\mu-\iota,\lambda),$$

$$\begin{aligned}\Lambda(w(\delta\theta)^{-1}w^{-1}w_1\cdot(\mu-\iota),(\delta\theta)^{-1}\cdot\lambda) &= \Lambda((\delta\theta)^{-1}\cdot(\mu-\iota),(\delta\theta)^{-1}\cdot\lambda)\\ &= \omega_{|S(\mathbf{R})}^{(\delta\theta)^{-1}}\,\Lambda(\mu-\iota,\lambda),\end{aligned}$$

$$\Lambda(w_1\cdot(\mu-\iota),\lambda)=\omega_{|S(\mathbf{R})}\,\Lambda(w\delta\theta w^{-1}\cdot(\mu-\iota),\delta\theta\cdot\lambda).$$

The final identity and Theorem 1.1 (c) [KZ84] imply that

$$\Theta(\mu,\lambda,\hat{B})=\Theta(w_1^{-1}\cdot\mu,\lambda,w_1^{-1}\cdot\hat{B})=\omega\,\Theta(w\delta\theta w^{-1}\cdot(\mu-\iota),\delta\theta\cdot\lambda,w\delta\theta w^{-1}\cdot\hat{B})$$

from which it follows that

$$\Theta(w^{-1}\cdot\mu,\lambda,w^{-1}\cdot\hat{B})=\omega\,\Theta(\delta\theta\cdot w^{-1}\cdot(\mu-\iota),\delta\theta\cdot\lambda,\delta\theta\cdot w^{-1}\cdot\hat{B}).\blacksquare$$

Corollary 5 *Suppose* $\nu \in X_*((\hat{S}^{\delta\theta})^0) \otimes \mathbf{C}$ *as in section 7.2,* $w \in \Omega(G,S)$ *is a representative of an element in* $(\Omega(G,S)/\Omega_{\mathbf{R}}(G,S))^{\delta\theta}$*, and the character* $\Theta(w^{-1}\cdot\mu, \lambda, w^{-1}\cdot\hat{B})$ *is non-zero. Then the essentially square-integrable character* $\Theta(w^{-1}\cdot(\mu+\nu), \lambda, w^{-1}\cdot\hat{B})$ *is equal to* $\omega\,\Theta(\delta\theta\cdot w^{-1}\cdot(\mu+\nu), \delta\theta\cdot\lambda, w^{-1}\cdot\hat{B})$*.*

Proof. The corollary follows from the $\delta\theta$-stability of $\hat{B}$ and replacing μ with $w^{-1}\cdot\mu$ in equation (136).■

The characters $\Theta(w^{-1}\cdot\mu, \lambda, w^{-1}\cdot\hat{B})$ of the L-packet Π_φ which are pertinent to a twisted character comparison are those which satisfy the hypotheses of Lemma 33, namely those characters for which w belongs to the intersection of $(\Omega(G,S)/\Omega_{\mathbf{R}}(G,S))^{\delta\theta}$ and (142). We denote this intersection by $(\langle w_{\alpha_1^\vee}, \ldots, w_{\alpha_r^\vee}\rangle\,\Omega(M,S_M)\Omega_{\mathbf{R}}(G,S)/\Omega_{\mathbf{R}}(G,S))^{\delta\theta}$.

Lemma 34 *The action of* $\delta\theta$ *on* $R(\hat{G},\hat{S})$ *preserves* $\{\pm\mathbf{d}_{\alpha_1^\vee}\cdot\alpha_1, \ldots, \pm\mathbf{d}_{\alpha_r^\vee}\cdot\alpha_r\}$ *and* $R(\hat{M},\hat{S}_M)$*.*

Proof. Recall that the defining property of the roots $\alpha_1,\ldots,\alpha_r$ is that they are orthogonal to μ (actually, orthogonal to $m\cdot\mu$ if one removes the identification of section 7.1). It suffices to show that $\langle\delta\theta\cdot\mathbf{d}_{\alpha_j^\vee}\cdot\alpha_j^\vee, \mu\rangle = 0$. Consider equation (135). The quasicharacter $\omega_{|S(\mathbf{R})}$ therein is trivial on $S_{\mathrm{der}}(\mathbf{R})$ and may therefore be identified with a quasicharacter of Z_G (see (29)) and represented by a pair of elements in $X_*(\hat{Z}_G)\otimes\mathbf{C}$ (§9 [Bor79]). Equation (135) implies that $\delta\theta\cdot\mu = \mu$ modulo $X_*(Z_{\hat{G}})\otimes\mathbf{C} \cong X_*(\hat{Z}_G)\otimes\mathbf{C}$, and consequently

$$\langle\delta\theta\cdot\mathbf{d}_{\alpha_j^\vee}\cdot\alpha_j^\vee, \mu\rangle = \langle\alpha_j^\vee, \mathbf{d}_{\alpha_j^\vee}^{-1}\cdot(\delta\theta)^{-1}\cdot\mu\rangle = \langle\alpha_j^\vee, \mu\rangle = 0,\ 1\leq j\leq r.$$

This proves the first assertion. The second assertion follows from the fact that $R(\hat{M},\hat{S}_M)$ is the set of roots in $R(\hat{G},\hat{S})$ which are orthogonal to $\{\mathbf{d}_{\alpha_1^\vee}\cdot\alpha_1, \ldots, \mathbf{d}_{\alpha_r^\vee}\cdot\alpha_r\}$ (see the end of section 7.2.1).■

Lemma 35 *Suppose* $w_1 \in \langle w_{\alpha_1^\vee}, \ldots, w_{\alpha_r^\vee}\rangle$ *and* $w_2 \in \Omega(M,S_M)$*. Then the coset* $w_1w_2\Omega_{\mathbf{R}}(G,S)$ *belongs to* $(\Omega(G,S)/\Omega_{\mathbf{R}}(G,S))^{\delta\theta}$ *if and only if* $w_1\Omega_{\mathbf{R}}(G,S)$ *and* $w_2\Omega_{\mathbf{R}}(G,S)$ *belong to* $(\Omega(G,S)/\Omega_{\mathbf{R}}(G,S))^{\delta\theta}$*.*

Proof. To say that $w_1w_2\Omega_{\mathbf{R}}(G,S)$ belongs to $(\Omega(G,S)/\Omega_{\mathbf{R}}(G,S))^{\delta\theta}$ is the saying that

$$(w_1w_2)^{-1}\delta\theta(w_1w_2)(\delta\theta)^{-1} \in \Omega_{\mathbf{R}}(G,S).$$

Recall from the end of section 7.2.1 that each $\beta \in \Omega(M, S_M)$ is orthogonal to each $\alpha_j^\vee$, $1 \leq j \leq r$. Thus, the elements w_1 and w_2 commute (Lemma 9.2 [Hum94]). We may therefore rewrite the above membership statement as

$$w_1^{-1}(w_2^{-1}\delta\theta w_2(\delta\theta)^{-1})\delta\theta w_1(\delta\theta)^{-1} \in \Omega_{\mathbf{R}}(G, S).$$

From Lemma 34 it is evident that $w_2^{-1}\delta\theta w_2(\delta\theta)^{-1} \in \Omega(M, S_M)$ and the orthogonality relations discussed in section 7.2.1 imply that this membership is equivalent to

$$(w_2^{-1}\delta\theta w_2(\delta\theta)^{-1})\,(w_1^{-1}\delta\theta w_1(\delta\theta)^{-1}) \in \Omega_{\mathbf{R}}(G, S).$$

Lemma 34 also makes it clear that $w_1^{-1}\delta\theta w_1(\delta\theta)^{-1} \in \langle w_{\alpha_1^\vee}, \ldots w_{\alpha_r^\vee}\rangle$. Since σ negates all roots in $R(G, S)$ (*cf.* (17)), it preserves $\Omega(M, S_M)$ and $\langle w_{\alpha_1^\vee}, \ldots w_{\alpha_r^\vee}\rangle$. The previous statement is therefore equivalent to

$$\sigma(w_2^{-1}\delta\theta w_2(\delta\theta)^{-1})(w_2^{-1}\delta\theta w_2(\delta\theta)^{-1})^{-1} = (w_1^{-1}\delta\theta w_1(\delta\theta)^{-1})\sigma(w_1^{-1}\delta\theta w_1(\delta\theta)^{-1})^{-1}.$$

The left-hand side lies in the subgroup $\Omega(M, S_M)$ and the right-hand side lies in the subgroup $\langle w_{\alpha_1^\vee}, \ldots w_{\alpha_r^\vee}\rangle$, and these subgroups have trivial intersection. In conclusion, this identity is equivalent to

$$w_1^{-1}\delta\theta w_1(\delta\theta)^{-1},\ w_2^{-1}\delta\theta w_2(\delta\theta)^{-1} \in \Omega_{\mathbf{R}}(G, S).\blacksquare$$

Proposition 5 *The irreducible characters of* Π_φ *which satisfy the conditions of Lemma 33 are of the form*

$$\Theta(w_1^{-1}w_2^{-1} \cdot \mu, \lambda, w_1^{-1}w_2^{-1} \cdot \hat{B})$$

where w_1 *runs through* $(\langle w_{\alpha_1^\vee}, \ldots w_{\alpha_r^\vee}\rangle\Omega_{\mathbf{R}}(G, S)/\Omega_{\mathbf{R}}(G, S))^{\delta\theta}$ *and* w_2 *runs through* $(\Omega(M, S_M)/\Omega_{\mathbf{R}}(M, S_M))^{\delta\theta}$*. In particular, the representations in* Π_φ *whose characters satisfy the conditions of Lemma 33 are subrepresentations of*

$$\mathrm{ind}_{P(\mathbf{R})}^{G(\mathbf{R})}\,\pi_{w^{-1}\Lambda},\ \ w \in (\Omega(M, S_M)/\Omega_{\mathbf{R}}(M, S_M))^{\delta\theta}.$$

Proof. The first assertion is an obvious result of Lemma 35. The second assertion follows from the manner in which the Hecht-Schmid identities are used in giving the original decomposition of Π_φ (p. 408 [She82]).$\blacksquare$

7.2.3 Twisted coherent continuation and the spectral comparison

Coherent continuation is a means of passing from a character of a Harish-Chandra module with infinitesimal character to a character of a new Harish-Chandra module whose infinitesimal character is a modification of the previous one by the weight of a finite-dimensional representation (§5 [SJ80]). On the level of Harish-Chandra modules themselves, this passage is sometimes called Zuckerman tensoring (§X.9 [Kna86]). These equivalent methods form the foundation of the theory of limits of discrete series discussed in section 7.1.

J.-Y. Ducloux (§12 [Duc02]) has extended this theory to a class of Lie groups which includes groups of the form $G(\mathbf{R}) \rtimes \langle\theta\rangle$, when $G(\mathbf{R})$ is connected as a Lie group. We shall sketch how his extension applies to our twisted characters.

We may apply the machinery of section 5.1 to the essentially square integrable representation π_ν of section 7.2 to obtain an irreducible square-integrable representation representation ϖ_ν of $G_{\mathrm{der}}(\mathbf{R})^0$. We are free to choose ϖ_ν so that the limit of discrete series representation ϖ_1 of section 7.2 is obtained from ϖ_ν through Zuckerman tensoring with a finite-dimensional irreducible representation whose lowest weight is $-\Lambda(\nu)_{|S_{\mathrm{der}}(\mathbf{R})}$ (in additive notation). It follows from the $\delta\theta$-stability of ϖ_1 and ν that ϖ_ν is also $\delta\theta$-stable. The representation ϖ_ν extends, by the methods of section 5.4, to a unique irreducible representation $\bar{\varpi}_\nu$ of $G_{\mathrm{der}}(\mathbf{R})^0 \rtimes \langle\delta\theta\rangle$, upon making a choice of intertwining operator U_ν satisfying

$$\mathsf{U}_\nu \circ \omega^{-1}(x)\,\pi_\nu(x) = \pi_\nu^\theta(x) \circ \mathsf{U}_\nu,\ x \in G(\mathbf{R})$$

(*cf.* (24)).

The representation space of ϖ_1 is obtained from that of ϖ_ν through Zuckerman tensoring. According to Proposition 12.3 (c) [Duc02], this representation space is a $G_{\mathrm{der}}(\mathbf{R})$-module which is stable under the action of $\delta\theta$ and thus, may be regarded as the representation space of a representation $\bar{\varpi}_1$ of $G_{\mathrm{der}}(\mathbf{R})^0 \rtimes \langle\delta\theta\rangle$. By definition, ϖ_1 is the restriction of $\bar{\varpi}_1$ to $G_{\mathrm{der}}(\mathbf{R})^0$. We obtain a intertwining operator U between π and $\omega \otimes \pi^\theta$ by defining $\mathsf{U} = \pi(\delta)^{-1}\bar{\varpi}_1(\delta\theta)$ on the space of ϖ_1 (*cf.* (31)), and the twisted character identity of Lemma 5 holds with $\mathsf{U}_1 = \bar{\varpi}_1(\delta\theta)$.

More can be said about these twisted characters. Proposition 12.3 (b) [Duc02] tells us that the twisted character $\Theta_{\varpi_\nu, \bar{\varpi}_\nu(\delta\theta)}$ has an expansion of the form (129) on θ-regular components of $\mathfrak{s}_{\mathrm{der}}^{\delta\theta}$. Proposition 12.3 (c) [Duc02] tells us that the twisted character $\Theta_{\varpi_1,\mathsf{U}_1}$ is obtained from $\Theta_{\varpi_\nu,\bar{\varpi}_\nu(\delta\theta)}$ by coherent continuation (p. 260 [SJ80]). In other words, the twisted character $\Theta_{\varpi_1,\mathsf{U}_1}$ satisfies the same form of expansion (129) as $\Theta_{\varpi_\nu,\bar{\varpi}_\nu(\delta\theta)}$, only shifted by the weight of the finite-dimensional representation attached to the lowest weight $-\Lambda(\nu)_{|S_{\mathrm{der}}(\mathbf{R})}$.[3]

The process and relationships we have just described are valid if the initial data $\mu, \lambda, \hat{B}, \nu$ attached to π are replaced by $w^{-1}\cdot\mu, \lambda, w^{-1}\cdot\hat{B}, w^{-1}\cdot\nu$, with $w \in (\Omega(G,S)/\Omega_{\mathbf{R}}(G,S))^{\delta\theta}$ as in Lemma 33 and its corollary. However, in the case that $\Theta(w^{-1}\cdot\mu, \lambda, w^{-1}\cdot\hat{B})$ vanishes the statements are vacuous. To be more precise, the $G_{\mathrm{der}}(\mathbf{R})^0$-module obtained by Zuckerman tensoring from the square-integrable representation $\varpi_{w^{-1}\cdot\nu}$ might vanish, and in this case one defines the character $\Theta(w^{-1}\cdot\mu, \lambda, w^{-1}\cdot\hat{B})$ and its twisted analogue to be zero. We may write these matters more succinctly by denoting the twisted character by $\Theta_{\mathsf{U}}(w^{-1}\cdot\mu, \lambda, w^{-1}\cdot\hat{B})$, and coherent continuation of twisted characters from $\mu+\nu$ to μ by $\Psi_{\mu}^{\mu+\nu}$. With this notation in hand we have

$$\Psi_{w^{-1}\cdot\mu}^{w^{-1}\cdot(\mu+\nu)}(\Theta_{\pi_{w^{-1}\cdot\nu},\mathsf{U}_\nu}) = \Theta_{\mathsf{U}}(w^{-1}\cdot\mu, \lambda, w^{-1}\cdot\hat{B}), \tag{143}$$

for all $w \in (\Omega(G,S)/\Omega_{\mathbf{R}}(G,S))^{\delta\theta}$. On pages 408-409 [She82], Shelstad shows that $\Theta(w^{-1}\cdot\mu, \lambda, w^{-1}\cdot\hat{B})$ vanishes for $w \notin \langle w_{\alpha_1^\vee}, \ldots w_{\alpha_r^\vee}\rangle\, \Omega(M,S_M)\Omega_{\mathbf{R}}(G,S)/\Omega_{\mathbf{R}}(G,S)$. As we have noted, this means that the right-hand side of (143) vanishes for w outside of $\langle w_{\alpha_1^\vee}, \ldots w_{\alpha_r^\vee}\rangle\, \Omega(M,S_M)\Omega_{\mathbf{R}}(G,S)/\Omega_{\mathbf{R}}(G,S)$.

At last, we are ready to return the twisted character comparison of Theorem 1. Replace φ_{H_1} by $\varphi_{H_1}^{\nu}$, and π by π_ν on the right-hand side of the identity of Theorem 1. If one applies $\Psi_{w^{-1}\cdot\mu}^{w^{-1}\cdot(\mu+\nu)}$ to each of the twisted characters on the right one finds that

$$\sum_{w\in(\Omega(G,S)/\Omega_{\mathbf{R}}(G,S))^{\delta\theta}} \Delta(\boldsymbol{\varphi}_{\boldsymbol{H_1}}^{\boldsymbol{\nu}}, \pi_{w^{-1}\cdot\nu})\, \Psi_{w^{-1}\cdot\mu}^{w^{-1}\cdot(\mu+\nu)}\Theta_{\pi_{w^{-1}\cdot\nu},\mathsf{U}_\nu} \tag{144}$$

is equal to the distribution

$$\sum_{w_1,w_2} \Delta(\boldsymbol{\varphi}_{\boldsymbol{H_1}}^{\boldsymbol{\nu}}, \pi_{(w_1w_2)^{-1}\cdot\nu})\ \Theta_{\mathsf{U}}((w_1w_2)^{-1}\cdot\mu, \lambda, (w_1w_2)^{-1}\cdot\hat{B}) \tag{145}$$

where $\varphi_{H_1}^{\nu} : W_{\mathbf{R}} \rightarrow {}^{L}H_1$ is an admissible homomorphism which maps to φ^{ν} (Lemma 11), w_1 runs through $(\langle w_{\alpha_1^\vee}, \ldots w_{\alpha_r^\vee}\rangle\Omega_{\mathbf{R}}(G,S)/\Omega_{\mathbf{R}}(G,S))^{\delta\theta}$ and w_2 runs through $(\Omega(M,S_M)/\Omega_{\mathbf{R}}(M,S_M))^{\delta\theta}$ as in Proposition 5.

[3]This amounts to a twisted version of Lemma 5.5 [SJ80]. Analogous expansions hold on the other Cartan subspaces (Definition 7.1 [Ren97]) of $G_{\mathrm{der}}(\mathbf{R})^0\delta\theta$.

To complete the twisted character comparison for the essential limits of discrete series, we investigate the behaviour of coherent continuation on left-hand side of the identity in Theorem 1. For this we need the requisite notation. Let $\nu_{H_1} \in X_*(\hat{T}_{H_1}) \otimes \mathbf{C}$ be the image of $\nu \in X_*((\hat{S}^{\delta\theta})^0 \otimes \mathbf{C})$ by way of the maps (14), (10) and (7). Assuming that $\varphi_{H_1} \in \varphi_{H_1}$ is determined by the pair $\mu_{H_1}, \lambda_{H_1} \in X_*(\hat{T}_{H_1}) \otimes \mathbf{C}$, one may verify that $\varphi^{\nu}_{H_1}$ is determined by the pair $\mu_{H_1} + \nu_{H_1}, \lambda_{H_1}$. Let $\pi_{\nu_{H_1}} \in \Pi_{\varphi^{\nu}_{H_1}}$ denote the unique essentially square-integrable representation whose character on $T_{H_1}(\mathbf{R})$ is positive with respect to the system determined by the regular element $\mu_{H_1} + \nu_{H_1}$ (see section 4.1). We denote coherent continuation $\Psi^{\mu_{H_1}+\nu_{H_1}}_{\mu_{H_1}}$ as before and observe that $\Psi^{\mu_{H_1}+\nu_{H_1}}_{\mu_{H_1}} \Theta_{\pi_{\nu_{H_1}}}$ is the character of an essentially square integrable representation in $\Pi_{\varphi_{H_1}}$ (Theorem 5.2 [SJ80]). We deduce from (21) that

$$\sum_{w \in \Omega(H_1, T_{H_1})/\Omega_{\mathbf{R}}(H_1, T_{H_1})} \Psi^{w^{-1}\cdot(\mu_{H_1}+\nu_{H_1})}_{w^{-1}\cdot\mu_{H_1}} \Theta_{\pi_{w^{-1}\cdot\nu_{H_1}}} = \sum_{\pi_{H_1} \in \Pi_{\varphi_{H_1}}} \Theta_{\pi_{H_1}}. \tag{146}$$

Theorem 2 *The distribution*

$$f \mapsto \sum_{\pi_{H_1} \in \Pi_{\varphi_{H_1}}} \Theta_{\pi_{H_1}}(f_{H_1}),\ f \in C_c^{\infty}(G(\mathbf{R})\theta) \tag{147}$$

is equal to (145).

Proof. Suppose first that ω is trivial. Then, as may be seen from the proof of Lemma 26 and Proposition 4, the distribution (147) is a tempered, $G(\mathbf{R})$-invariant, eigendistribution. Hence, according to the twisted version of Harish-Chandra's uniqueness theorem (Theorem 15.1 [Ren97]), distribution (147) is completely determined by its expansion on $S^{\delta\theta}(\mathbf{R})^0\delta\theta$. Looking back to (100), (101) and taking the twisted Weyl integration formula (Proposition 1) into consideration, we see that on $S(\mathbf{R})^0\delta\theta$ the character expansion of (147) is of the form (129), where the μ_j are Weyl conjugates of the linear form determined by μ, the "lift" of μ_{H_1} via ξ. According to the procedure of coherent continuation (Theorem 12.3 [Duc02]), one may shift each of these linear forms, by the consonant Weyl conjugate of the weight the finite-dimensional representation determined by ν. Alternatively one may first shift the linear forms attached to μ_{H_1} by the weights of the finite-dimensional representations determined by ν_{H_1} and then "lift" via ξ. These two procedures have the same effect and this is the same as saying that coherent continuation commutes with the spectral transfer of the distribution $\sum_{\pi^{\nu}_{H_1}} \Theta_{\pi_{H_1}}$. Applying this commutativity in conjunction with Theorem 1 and (146), we obtain

$$\begin{aligned}
&\sum_{\pi_{H_1}\in\Pi_{\varphi_{H_1}}} \Theta_{\pi_{H_1}(f_{H_1})} \\
&= \sum_{w\in\Omega(H_1,T_{H_1})/\Omega_{\mathbf{R}}(H_1,T_{H_1})} \Psi^{w^{-1}\cdot(\mu_{H_1}+\nu_{H_1})}_{w^{-1}\cdot\mu_{H_1}} \Theta_{\pi_{w^{-1}\cdot\nu_{H_1}}}(f_{H_1}) \\
&= \sum_{w\in(\Omega(G,S)/\Omega_{\mathbf{R}}(G,S))^{\delta\theta}} \Delta(\boldsymbol{\varphi}^{\boldsymbol{\nu}}_{\boldsymbol{H_1}}, \pi_{w^{-1}\cdot\nu})\, \Psi^{w^{-1}\cdot(\mu+\nu)}_{w^{-1}\cdot\mu} \Theta_{\pi_{w^{-1}\cdot\nu,\mathsf{U}_\nu}}(f) \\
&= \sum_{w_1,w_2} \Delta(\boldsymbol{\varphi}^{\boldsymbol{\nu}}_{\boldsymbol{H_1}}, \pi_{(w_1w_2)^{-1}\cdot\nu})\ \Theta_{\mathsf{U}}((w_1w_2)^{-1}\cdot\nu, \lambda, (w_1w_2)^{-1}\cdot\hat{B})(f).
\end{aligned}$$

This prove the theorem in the case that ω is trivial. To complete the theorem, we allow ω to to non-trivial and apply the techniques used in the proof of Theorem 1, setting Θ to be the difference of (145) and (147).■

We conclude by inviting the reader to review section 6.2 and thereby realize that the spectral transfer factors $\Delta(\boldsymbol{\varphi}^{\boldsymbol{\nu}}_{\boldsymbol{H_1}}, \pi_{(w_1w_2)^{-1}\cdot\nu})$ in (145) do not depend on the choice of ν. This justifies the definition

$$\Delta(\boldsymbol{\varphi}_{\boldsymbol{H_1}}, \pi(w^{-1}\cdot\mu, \lambda, w^{-1}\cdot\hat{B})) = \begin{cases} \Delta(\boldsymbol{\varphi}^{\boldsymbol{\nu}}_{\boldsymbol{H_1}}, \pi_{w^{-1}\cdot\nu}), & w = w_1w_2 \text{ as in Prop. 5} \\ 0, & \text{otherwise} \end{cases}$$

for essential limit of discrete series representations $\pi(w^{-1}\cdot\mu, \lambda, w^{-1}\cdot\hat{B})$ with character $\Theta(w^{-1}\cdot\mu, \lambda, w^{-1}\cdot\hat{B})$. With this notation Theorem 2 reads as

$$\sum_{\pi_{H_1}\in\Pi_{\varphi_{H_1}}} \Theta_{\pi_{H_1}}(f_{H_1}) = \sum_{\pi\in\Pi_{\varphi}} \Delta(\boldsymbol{\varphi}_{\boldsymbol{H_1}}, \pi)\, \Theta_{\pi,\mathsf{U}_\pi}(f) \tag{148}$$

(see (141), (142)).

7.3 The remaining cases

We now remove the hypothesis that $G = G^*$ so that we must revisit the definition of φ from φ^* (section 6). The admissible homomorphism φ is defined (and equal to φ^*) only if the Levi subgroup $\hat{M} \rtimes W_{\mathbf{R}} \subset {}^LG$ defined in section 7.2.1 is relevant (§3.3 [Bor79]), i.e. only if there is a Levi subgroup M

in G defined over $\mathbf{R}$ such that $\hat{M} \rtimes W_{\mathbf{R}}$ contains $\varphi_M(W_{\mathbf{R}})$ minimally. If this is true then we may proceed as in section 7.2, as long as one bears in mind that the objects related to M, such as S_M, B_M, S, P, are objects related to the *inner form* G of G^* and not necessarily G^* itself. In this way, one recovers (148) for G.

The image of the admissible homomorphism φ^{ν} of section 7.1, being attached to an essentially square-integrable L-packet, is not contained in a proper subgroup of ${}^LG$, and ${}^LG$ is relevant. We therefore have a maximal torus S of G defined over $\mathbf{R}$ and elliptic and it makes sense to consider the distributions $\Theta(w^{-1} \cdot \mu, \lambda, w^{-1} \cdot \hat{B})$ for any $w \in \Omega(G^*, S)/\Omega_{\mathbf{R}}(G^*, S)$. If $\hat{M} \rtimes W_{\mathbf{R}} \subset {}^LG$ is not relevant then Shelstad has proven that $\Theta(w^{-1} \cdot \mu, \lambda, w^{-1} \cdot \hat{B}) = 0$ for all $w \in \Omega(G, S)/\Omega_{\mathbf{R}}(G, S)$ ((4.3.3) [She82]). This implies that the $G_{\mathrm{der}}(\mathbf{R})^0$-module obtained by Zuckerman tensoring from ϖ_{ν} in section 7.2.3 is also zero. As a result the twisted character $\Theta_{\mathsf{U}}(w^{-1} \cdot \mu, \lambda, w^{-1} \cdot \hat{B})$ also vanishes for every $w \in (\Omega(G, S)/\Omega_{\mathbf{R}}(G, S))^{\delta\theta}$. In conclusion, if $\hat{M} \rtimes W_{\mathbf{R}}$ is not relevant then φ is undefined and the twisted characters obtained by coherent continuation vanish.

Finally, we consider the case that there is no θ-elliptic element of $G(\mathbf{R})$ which has a norm in $H_1(\mathbf{R})$. In this case the arguments of section 6.5 may be applied to (60) with π replaced by π_{ν} as in section 7.2. Both sides of (60) vanish in this case and coherent continuation, as applied above, does not change matters. In other words, the character identity (60) holds with all spectral transfer factors set to equal zero.

A Parabolic descent for geometric transfer factors

In this section we assume that we are given endoscopic data $(H, \mathcal{H}, \mathsf{s}, \xi)$ for $(G, \theta, \mathbf{a})$ and a z-extension H_1 of H (see section 3.2). We wish to produce compatible endoscopic data for some Levi subgroups of G and to normalize geometric transfer factors for these Levi subgroups. All in all, this is an application of §11 [Sheb].

Suppose that $k \in G(\mathbf{R})$ and M is a $k\theta$-stable Levi subgroup of G which is defined over $\mathbf{R}$ and that $M(\mathbf{R})$ is cuspidal. Furthermore, suppose that $\delta \in M(\mathbf{R})$ is a $k\theta$-semisimple, strongly $k\theta$-regular element. We know that $G^{\delta k\theta}$ contains G-regular elements. We assume that some of these G-regular elements lie in M. Then the centralizer of $M^{\delta k\theta}$ is a maximal torus S_M of M is contained in the centralizer of $G^{\delta k\theta}$ in G, which is a maximal torus in G. Since the maximal tori of M are maximal in G, the torus S_M is equal to the centralizer of $G^{\delta k\theta}$ and $S_M^{\delta k\theta} = G^{\delta k\theta}$. We assume that $S_M(\mathbf{R})$ is elliptic in $M(\mathbf{R})$. Observe that δ is (strongly) $k\theta$-regular if and only if δk is (strongly) θ-regular (section 3.1).

Our next assumption is that there exists $\gamma_1 \in H_1(\mathbf{R})$ which is a norm of $\delta k \in G(\mathbf{R})$. In §11 [Sheb] Shelstad introduces the notion of a z-norm for specific elements $z \in Z_G$. In the case at hand, Shelstad specifies an element $z_k \in Z_G$ such that γ_1 is a z_k-norm of δ with respect to the algebraic $\mathbf{R}$-automorphism $\mathrm{Int}(k)\theta$ of G. Attached to this z_k-norm is a geometric transfer factor $\Delta_{z_k}(\gamma_1, \delta)$ which one may assume to be normalized so that it is equal to $\Delta(\gamma_1, \delta k)$.

The centralizer of γ_1 in H_1 is a maximal torus T_{H_1} which is defined over $\mathbf{R}$. All of the conveniences of section 3.3 are available to us, with δk in place of δ, so that we have homomorphisms

$$T_{H_1} \to T_H \cong T'_{\theta^*}$$

(cf. 12) and

$$G^{\delta k\theta} = S_M^{\delta k\theta} \cong (T')^{\theta^*} \to T'_{\theta^*}$$

(cf. (14)), all being defined over $\mathbf{R}$. Recall that the isomorphism between $S^{\delta k\theta}$ and $(T')^{\theta^*}$ is of the form $\mathrm{Int}(g_{T'})\psi$. It extends to an $\mathbf{R}$-isomorphism of the respective centralizers

$$\mathrm{Int}(g_{T'})\psi : S_M \cong T', \tag{149}$$

as the commutator of σ and $\mathrm{Int}(g_{T'})\psi$ lies in $\mathrm{Int}(T')$ ((3.3.6 [KS99]). We may use this $\mathbf{R}$-isomorphism to define a Levi subgroup M^* in G^* in the following fashion. The maximal compact and split subtori of $S_M(\mathbf{R})$ are $S_M^{-\sigma}(\mathbf{R})$ and $S_M^{\sigma}(\mathbf{R})$ respectively (§9.4 [Bor79]). As S_M is elliptic, we know that

$$S_M(\mathbf{R})/Z_M(\mathbf{R}) = S_M^{-\sigma}(\mathbf{R}) S_M^{\sigma}(\mathbf{R})/Z_M(\mathbf{R})$$

is compact. Therefore the subtorus $S_M^{\sigma}(\mathbf{R})$ is contained in $Z_M(\mathbf{R})$, S_M^{σ} is a maximal $\mathbf{R}$-split torus in the centre of M, and $M = Z_G(S_M^{\sigma})$ (§3.6 [Spr98]). The $\mathbf{R}$-isomorphism $\mathrm{Int}(g_{T'})\psi$ sends $S_M^{-\sigma}$ to $(T')^{-\sigma}$, S_M^{σ} to $(T')^{\sigma}$, and $Z_G(S_M^{\sigma})$ to $Z_{G^*}((T')^{\sigma})$. This being said, we set $M^* = Z_{G^*}((T')^{\sigma})$ and see that M^* is a Levi subgroup of G^* (§3.6) and $T'(\mathbf{R})$ is an elliptic torus therein. We define ψ_M to be the restriction of $\mathrm{Int}(g_{T'})\psi$ to M. Clearly, $\psi_M : M \to M^*$ is an inner twisting.

Lemma 36 *The Levi subgroup M^* is stable under θ^*.*

Proof. As we are assuming that $\delta \in M(\mathbf{R})$ and M is preserved by the $\mathbf{R}$-automorphism $k\theta$, the automorphism $\delta k\theta$ preserves M, is defined over $\mathbf{R}$ and so preserves S_M^σ. It is left to the reader to compute that

$$\mathrm{Int}(\delta^*)\theta^*\mathrm{Int}(g_{T'})\psi = \mathrm{Int}(g_{T'})\psi\mathrm{Int}(\delta)\theta.$$

Applying this equation to S_M^σ, and recalling that $\delta^* \in T'$ one sees that $\theta^*((T')^\sigma) = (T')^\sigma$. The θ^*-stability of M^* now follows from the definition of M^* as the centralizer $Z_{G^*}((T')^\sigma)$.$\blacksquare$

We turn to defining endoscopic data on the level of M. If one fixes an $\mathbf{R}$-torus of G which is maximally $\mathbf{R}$-split, one has the construct of a *standard* Levi ($\mathbf{R}$-)subgroup (§§3.5-3.6 [Spr79]). Standard Levi subgroups are parameterized by subsets of a fixed base, and every Levi subgroup of G^* is $G^*(\mathbf{R})$-conjugate to a unique standard Levi subgroup (Theorem 15.4.6 [Spr98]). Hence, we may associate to M^* a unique subset of a base. The dual of this base corresponds to a Levi subgroup of ${}^LG$ (§3.3 [Bor79]) and, after possibly conjugating this Levi subgroup by an element of $\hat{G}$, one may assume that it is standard with respect to the torus $\mathcal{T}$ and base defined by $\mathcal{B}$, i.e. of the form $\mathcal{M} \rtimes W_{\mathbf{R}}$, where $\mathcal{M}$ is a standard Levi subgroup in $\hat{G}$.

As in section 3.3, we assume that the endoscopic datum s belongs to $\mathcal{T}$. The θ^*-stability of M^* is equivalent to the $\hat{\theta}$-stability of $\mathcal{M}$ (§1.2 [KS99]) and the $k\theta$-stability of M. We let θ_M equal the restriction of $\mathrm{Int}(k)\theta$ to M. We set about defining endoscopic data $(H_M, \mathcal{H}_M, \mathsf{s}_M, \xi_M)$ for the triple $(M, \theta_M, \mathbf{a})$ by putting $\mathsf{s}_M = \mathsf{s}$. Since $\mathcal{M}$ is standard, it is determined by a set simple roots of $R(\mathcal{B}, \mathcal{T})$. The subgroup $\xi^{-1}((\mathcal{M}^{\mathsf{s}\hat{\theta}})^0) \subset \hat{H}$ is determined in the same way by the restriction of the corresponding roots to $(\mathcal{T}^{\hat{\theta}})^0 \cong \mathcal{T}_H$ (see §1.3 [KS99]). In particular, it is a standard Levi subgroup $\hat{M}_H$ of $\hat{H}$ with respect to the pair $\mathcal{B}_H \supset \mathcal{T}_H$ of section 3.3. We let $\mathcal{H}_M$ equal the subgroup $\hat{M}_H \rtimes_c W_{\mathbf{R}}$ of $\mathcal{H}$ (see 2. in section 3.2)and set $\xi_M = \xi_{|\mathcal{H}_M}$. The action of $W_{\mathbf{R}}$ on $\hat{M}_H$ in $\mathcal{H}$ coincides with the action in ${}^LH$. Therefore, in accordance with the notation, $\hat{M}_H$ truly corresponds to the dual of a Levi subgroup M_H in H which is defined over $\mathbf{R}$ (§3.3 [Bor79]). One may now verify that this construction produces the desired endoscopic data. Additionally, the subgroup $M_{H_1} = p_H^{-1}(M_H) \subset H_1$ is a z-extension of H (*cf.* (6)).

Our final assumption is that the norm γ_1 of δk lies in $M_H(\mathbf{R})$. In §11 [Sheb], Shelstad produces an element $z^\dagger \in Z_M$ such that γ_1 is a $z^\dagger$-norm of

$\delta \in M(\mathbf{R})$ with respect to the endoscopic data $(M_H, \mathcal{H}_M, \mathsf{s}, \xi_M)$ of $(M, \theta_M, \mathbf{a})$. Using this $z^\dagger$-norm, one may define twisted geometric transfer factors Δ_M as in §4 [KS99]. By Lemma 11.4 [Sheb] we have

$$\Delta_M(\gamma_1, \delta) = \frac{\Delta_{M,IV}(\gamma_1, \delta)}{\Delta_{IV}(\gamma_1, \delta k)} \Delta(\gamma_1, \delta k). \tag{150}$$

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