Tempered spectral transfer in the twisted endoscopy of real groups*

Paul Mezo†

1 Introduction

Let G be a connected reductive algebraic group which is defined over the real numbers $\mathbf{R}$. The notion of spectral transfer combines two seemingly dissimilar areas. The word "spectral" has its roots in harmonic analysis, specifically in the spectrum of the regular representation. By contrast, the word "transfer" pertains to the theory of endoscopy, which is motivated by number theory. The theory of twisted endoscopy ascribes to G, and an automorphism θ thereof, a collection of so-called *endoscopic* groups. The goal of the present article is to transfer information about representations of the real group $G(\mathbf{R})$ to representations of its endoscopic groups.

Let us dwell on the representations of $G(\mathbf{R})$ for a while. We are only interested in representations up to equivalence, and thus restrict our attention to their distribution characters. In harmonic analysis, the trace or Plancherel formulae affirm that these characters are dual to conjugacy classes. This is to say that there is a duality between the values of characters on the one hand, and orbital integrals of functions on $G(\mathbf{R})$ on the other. We may picture this duality as

$$\boxed{\text{conjugacy classes, orbital integrals}} \leftrightsquigarrow \boxed{\text{representations, characters}} \tag{1.1}$$

or more coarsely under the traditional headings

$$\boxed{\text{geometric}} \leftrightsquigarrow \boxed{\text{spectral}}.$$

Pursuing number-theoretic goals, Langlands connected representations of $G(\mathbf{R})$ to entirely different objects called *L-parameters* ([Lan89], [Bor79]). This is known as the Local Langlands Correspondence. It is a bijection between L-parameters φ and finite sets Π_φ of irreducible representations of $G(\mathbf{R})$ (up to

*MSC 22E45, 22E47, 11S37

†The School of Mathematics and Statistics, Carleton University, Ottawa, ON, Canada. mezo@math.carleton.ca, 613-520-2600. This work supported by National Sciences and Engineering Research Council grant RGPIN 293148-2010

equivalence). The sets Π_φ are called *L-packets*. Combining this bijection with the earlier duality (1.1), we may conjure up a picture

$$\boxed{\text{stable orbital integrals}} \leftrightsquigarrow \boxed{L\text{-packets}} \leftrightarrow \boxed{L\text{-parameters}} \tag{1.2}$$

in which the packaging of the representations in the middle translates into a packaging of conjugacy classes and orbital integrals called *stable* conjugacy classes and *stable* orbital integrals.

From this height it is difficult to see where endoscopic groups enter the picture. However, anyone familiar with the details of the Local Langlands Correspondence is able to appreciate the inherent path from an L-parameter of G to another real reductive algebraic group and another L-parameter thereof (*p.* 24 [KS99], §1 [Mez12]). The set of such groups, the *endoscopic groups* of G, has been axiomatized (§1.2 [LS87], §2 [KS99]).

Let us fix an endoscopic group[1] H. There is a correspondence between conjugacy classes of G and conjugacy classes of H (§1.3 [LS87], §3 [KS99]). There is also a correspondence between L-parameters, and hence L-packets, of G and H (§2 [She10]). Looking back to (1.2), these correspondences evoke the following picture

$$\begin{tikzcd}
\boxed{\text{stable orbital integrals for } G} \arrow[r, leftrightsquigarrow] \arrow[d, leftrightarrow, dotted] & \boxed{L\text{-packets for } G} \arrow[d, leftrightarrow, dotted] \\
\boxed{\text{stable orbital integrals for } H} \arrow[r, leftrightsquigarrow] & \boxed{L\text{-packets for } H}
\end{tikzcd} \tag{1.3}$$

The vertical arrows require some interpretation. One could interpret the vertical arrow on the right to simply be the above correspondence of L-packets. However, this would not address the duality implicit in the horizontal arrows. To incorporate the duality we must match functions on $G(\mathbf{R})$ and $H(\mathbf{R})$ through orbital integrals or characters. This matching was achieved by Shelstad in the case that θ is trivial, *i.e.* in ordinary endoscopy ([She82], [She08a], [She10]).

In this case, one may interpret the vertical arrow on the left in (1.3) as *geometric transfer*, which states that for certain functions f on $G(\mathbf{R})$ there exist functions f_H on $H(\mathbf{R})$ such that

$$\sum_{\gamma} \mathcal{O}_\gamma(f_H) = \sum_{\delta} \Delta(\gamma, \delta)\, \mathcal{O}_\delta(f). \tag{1.4}$$

Here, γ and δ are corresponding conjugacy classes and the sums run over representatives in a stable conjugacy class. The summands on the left are (suitably normalized) orbital integrals. The sum on the left is a stable orbital integral by definition. The sum on the right is not quite a stable orbital integral as it is modified by constants $\Delta(\gamma, \delta)$ called *geometric transfer factors*. These transfer

[1] We shall ignore the technicalities of z-extensions in the introduction.

factors force us not to take too literal an attitude towards the picture (1.3). They are also responsible for much of the hard work in proving geometric transfer.

In the ordinary case, the vertical arrow on the right in (1.3) is called *spectral transfer* and takes the shape

$$\sum_{\pi_H \in \Pi_{\varphi_H}} \Theta_{\pi_H}(f_H) = \sum_{\pi \in \Pi_{\varphi}} \Delta(\boldsymbol{\varphi_H}, \pi)\, \Theta_{\pi}(f). \tag{1.5}$$

Here, $\boldsymbol{\varphi_H}$ is a tempered L-parameter of H which corresponds to the L-parameter $\boldsymbol{\varphi}$ of G, and f and f_H are as in the statement of geometric transfer (1.4). The sum on the left is a sum of character values from the L-packet Π_{φ_H} of tempered representations. The matching sum on the right has been modified by constants $\Delta(\boldsymbol{\varphi_H}, \pi)$ called *spectral transfer factors.*

So far we have left the matter of twisting in the background. How does a non-trivial $\mathbf{R}$-automorphism θ of G enter into picture (1.3)? A convenient way of answering this question is to consider the connected component $G\theta$ of the group $G \rtimes \langle\theta\rangle$. One may then replace G-conjugacy classes of elements in G with G-conjugacy classes of elements in $G\theta$. This replacement defines the twisted *θ-conjugacy classes* and *twisted orbital integrals* on the geometric side of (1.1). These twisted orbital integrals enter (1.4) in the general case.

On the spectral side of (1.1), one only considers representations of $G(\mathbf{R})$ which are equivalent to their composition under θ. These are the *θ-stable* representations. The character of a θ-stable representation π may be twisted by introducing an intertwining operator T_π exhibiting the above equivalence. The resulting twisted characters enter equation (1.5) in the general case.

In spirit, and sometimes in practice, the alterations introduced by twisting amount to the shifting of the harmonic analysis of $G(\mathbf{R})$ to the harmonic analysis of the non-identity component $G(\mathbf{R})\theta$.

A precise conjecture for twisted geometric transfer, that is the twisted analogue of (1.4), was given in §5.5 [KS99]. Shelstad has recently proved twisted geometric transfer in complete generality ([She12]). Specific examples of spectral transfer appearing in the theory of base change were proven in [Clo82], [Bou89] and [Clo11]. The main theorem of this paper is Theorem 8.5, which is the twisted analogue of (1.5). It is the twisted spectral transfer theorem dual to the twisted geometric transfer theorem of [She12], save for three notable restrictions. We shall call attention to the three restrictions in the discussion of the proof. An alternative proof of spectral transfer, without these restrictions, is given in the recent preprint of Waldspurger ([Wal13]). The techniques used there rely on Paley-Wiener theorems which bypass the elucidation of spectral transfer factors.

Before discussing the proof, let us mention some anticipated consequences of twisted spectral transfer. As in the case of ordinary endoscopy, one expects to be able to invert the spectral transfer formulae for a fixed L-packet Π_φ relative to a set of endoscopic groups (§5.4 [She82], [She08b]). In so doing, one expects to pair Π_φ with a group-theoretic structure fine enough to isolate individual representations (§6 [Art08]). Such a pairing is of fundamental importance to twisted trace formula comparisons. This is evidenced by Arthur's recent work in classifying automorphic representations of symplectic and orthogonal groups (see the remarks following Theorem 2.2.1 [Art]). Happily, neither of the three restrictions alluded to above are relevant to the twisted groups considered in Arthur's work (§1.2 [Art]).

There is however, an important technical matter which remains to be worked out before the results here become fully compatible with the contemporary theory. That is the matter of proving that the spectral transfer factors are *canonical*. Indeed, there are certain choices made in the definition of these transfer factors (section 4.5, §6.3 [Mez12]) and one wishes to show that the transfer factors are independent of these choices. This type of canonicity holds for geometric transfer factors (§4.6 [KS99]), and the analogous canonicity for spectral transfer factors is heralded by the preprint [She] (see also §12 [She10]).

Let us now discuss the proof of twisted spectral transfer. Section 3 gives an outline of twisted endoscopic groups, correspondences of conjugacy classes, and the twisted geometric transfer statement. All of this may be found in [KS99]. We assume that θ acts semisimply on the centre of G.

The first of the three restrictions in our twisted spectral transfer theorem is the technical statement (3.9), concerning the Galois-equivariance of the correspondence of conjugacy classes. It is satisfied for quasisplit groups when θ fixes an $\mathbf{R}$-splitting. All twisted groups in §1.2 [Art] have these properties.

In section 3 we have chosen to include the additional twisting datum of a quasicharacter ω of $G(\mathbf{R})$. This quasicharacter falls away after section 4. However, it is a part of the bigger picture included in twisted geometric transfer ([She12])), and should be compatible with the later sections of this work after some results in ω-equivariant harmonic analysis have been established (see §6.4 [Mez12] and §1.6 [Wal13]). This being so, the second restriction on our twisted spectral transfer theorem is that the quasicharacter ω is trivial. This is the case for the twisted groups in §1.2 [Art].

Section 4 provides a proof of twisted spectral transfer for fundamental series representations (Theorem 4.22). To understand why the class of fundamental series representations has been chosen, we should contrast the structure of proof of ordinary spectral transfer with that of twisted spectral transfer. In ordinary spectral transfer it suffices to prove spectral transfer for (limits of) discrete series and then use parabolic induction to finesse a proof for the tempered representations (§14 [She10]). Following this template, a proof of twisted spectral transfer for (limits of) discrete series was given in [Mez12]. Alas, there is an obstruction in the twisted case in using parabolic induction to pass to spectral transfer for tempered representations. The obstruction is that the automorphism θ might not preserve any parabolic subgroups available in the induction argument. This already appears in the example of induction from a non-Borel subgroup of $\mathrm{SL}(3, \mathbf{R})$ under twisting by an outer automorphism and is. Thankfully, this obstruction may be circumvented by inducing from the broader class of (limits of) fundamental series representations.

The proof of twisted spectral transfer for fundamental series representations follows the proof for the discrete series representations (§6 [Mez12]) and rests on the work of Duflo ([Duf82]) and Bouaziz ([Bou87]). A very particular case of this approach appears in §2.5 [CC09]. For an overview of our methods we refer the reader to the introduction of [Mez12]. The new results needed in section 4 are concerned with the passage from elliptic tori, which are basic to discrete series representations, to fundamental tori, which are basic to fundamental series representations. In particular, we establish structural properties (Corollary 4.3), parameterize stable data (section 4.2), and define spectral transfer factors (section 4.5).

In section 5 we extend twisted spectral transfer to a class of representations which we call *limit of fundamental series* representations (Theorem 5.4). These are the representations obtained from fundamental series representations using Zuckerman tensoring. This parallels the proof of twisted spectral transfer for limits of discrete series representations in §7 [Mez12]. The proof in section 5 diverges from the proof in [Mez12] this time only slightly due to the fact that fundamental tori appear in place of elliptic tori.

The reader will notice in both Theorem 4.22 and Theorem 5.4 that θ is assumed to have finite order. This finiteness condition is too severe for the passage to tempered representations. Section 6 is devoted to mitigating this restriction to θ merely having finite order on the centre of G. This is the third restriction on our spectral transfer theorem. All automorphisms of the twisted groups of §1.2 [Art] have finite order on the centre.

The topic of the final two sections is the passage from limits of fundamental series representations to tempered representations via parabolic induction. A list of hypotheses sufficient for this passage to succeed is assembled in section 7. In section 8 these hypotheses are shown to hold. This culminates in the main theorem, Theorem 8.5. As a final thought, we wish to point out that the twisted spectral transfer theorems are only of interest when endoscopic correspondences of conjugacy classes are sufficiently abundant (section 7.1), and the L-parameters of H pass to L-parameters of G (§7.3 [Mez12]). This is why we prove twisted spectral transfer under these hypotheses.

2 Notation

In this section only G is a real Lie group which acts upon a non-empty set J. We set

$$N_G(J) = \{g \in G : g \cdot J \subset J\},$$

$$Z_G(J) = \{g \in G : g \cdot j = j \text{ for all } j \in J\}.$$

In the sequel, the set $N_G(J_1)$ always forms a group. We set $\Omega(G, J)$ equal to the resulting factor group $N_G(J)/Z_G(J)$.

For an automorphism θ of G we set $\langle\theta\rangle$ equal to the group of automorphism generated by θ. There is a corresponding semidirect product $G \rtimes \langle\theta\rangle$. When elements of G are written side-by-side with elements in $\langle\theta\rangle$ we consider them to belong to this semidirect product.

The inner automorphism of an element $\delta \in G$ is defined by

$$\mathrm{Int}(\delta)(x) = \delta x \delta^{-1},\ x \in G.$$

It shall be convenient to denote the fixed-point set of $\mathrm{Int}(\delta) \circ \theta$ in G by $G^{\delta\theta}$. We shall abbreviate the notation $\mathrm{Int}(\delta) \circ \theta$ to $\mathrm{Int}(\delta)\theta$ or $\delta\theta$ habitually.

Unless otherwise mentioned, we denote the real Lie algebra of a Lie group using Gothic script. For example the real Lie group of G is denoted by $\mathfrak{g}$. Suppose that J is Cartan subgroup of a reductive group G. Then the pair of complex Lie algebras $(\mathfrak{g} \otimes \mathbf{C}, \mathfrak{j} \otimes \mathbf{C})$ determines a root system which we denote by $\mathrm{R}(\mathfrak{g} \otimes \mathbf{C}, \mathfrak{j} \otimes \mathbf{C})$. We denote the Lie algebra dual to $\mathfrak{g}$ by $\mathfrak{g}^*$. The differential of the inner automorphism $\mathrm{Int}(\delta)$ is the adjoint automorphism $\mathrm{Ad}(\delta)$ on $\mathfrak{g}$. The adjoint automorphism induces an automorphism on $\mathfrak{g}^*$ in the usual way. Often, it shall be convenient to write $\delta \cdot X$ in place of $\mathrm{Ad}(\delta)(X)$ for $X \in \mathfrak{g}$. Similarly, we write $\theta \cdot X$ to mean the differential of θ acting on $X \in \mathfrak{g}$. We extend this slightly abusive notation to the dual spaces, writing $\delta \cdot \lambda$ or even simply $\delta\lambda$ in place of the coadjoint action of δ on $\lambda \in \mathfrak{g}^*$.

Finally, if we take H to be an algebraic group defined over $\mathbf{R}$, we denote its identity component by H^0. The group of real points of H is denoted by $H(\mathbf{R})$. This is a real Lie group and we denote the identity component of $H(\mathbf{R})$ in the real manifold topology by $H(\mathbf{R})^0$.

3 The foundations of real twisted endoscopy

This section is a digest of some early material in [KS99], in the special case that the field of definition is equal to $\mathbf{R}$. It is essentially a reproduction of chapter 3 [Mez12] and is included for convenience and completeness.

3.1 Groups and automorphisms

Let G be a connected reductive algebraic group defined over $\mathbf{R}$. We take θ to be an algebraic automorphism of G defined over $\mathbf{R}$ and assume additionally that it acts semisimply on the centre Z_G of G. Set $G(\mathbf{R})$ to be the group of real points of G. Let Γ be the Galois group of $\mathbf{C}/\mathbf{R}$ and σ be its non-trivial element.

Let us fix a triple

$$(B_T, T, \{X\}) \tag{3.1}$$

in which B_T is a Borel subgroup of G, $T \subset B$ is a maximal torus of G, and $\{X\}$ is a collection of root vectors corresponding to the simple roots determined by B_T and T . Such triples are called *splittings* of G. If $(B_T, T, \{X\})$ is preserved by Γ then it is called an $\mathbf{R}$-*splitting*. We may assume that T is defined over $\mathbf{R}$ and that it contains a maximally $\mathbf{R}$-split torus of G (see page 257 [Spr98]).

There is a unique quasisplit group G^* of which G is an inner form (Lemma 16.4.8 [Spr98]). This is to say that there is an isomorphism $\psi : G \to G^*$ and $\psi\sigma\psi^{-1}\sigma^{-1} = \mathrm{Int}(u')$ for some $u' \in G^*$. We shall choose u_σ in the simply connected covering group G^*_{sc} of the derived group G^*_{der} of G^* so that its image under the covering map is u'. We shall then abuse notation slightly by identifying u_σ with u' in equations such as

$$\psi\sigma\psi^{-1}\sigma^{-1} = \mathrm{Int}(u_\sigma).$$

As G^* is quasisplit, there is a Borel subgroup B^* defined over $\mathbf{R}$. Applying Theorem 7.5 [Ste97] to B^* and σ, we obtain an $\mathbf{R}$-splitting $(B^*, T^*, \{X^*\})$. Following the convention made for $u_\sigma \in G^*_{\mathrm{sc}}$, we may choose $g_\theta \in G^*_{\mathrm{sc}}$ so that the automorphism

$$\theta^* = \mathrm{Int}(g_\theta)\psi\theta\psi^{-1} \tag{3.2}$$

preserves $(B^*, T^*, \{X^*\})$ (Theorems 6.2.7 and 6.4.1 [Spr98], §16.5 [Hum94]). Since

$$\sigma(\theta^*) = \sigma\theta^*\sigma^{-1} = \mathrm{Int}(\sigma(g_\theta u_\sigma)g_\theta^{-1}\theta^*(u_\sigma))\theta^*$$

preserves $(B^*, T^*, \{X^*\})$, and the only inner automorphisms which do so are trivial, it follows in turn that $\mathrm{Int}(\sigma(g_\theta u_\sigma)g_\theta^{-1}\theta^*(u_\sigma))$ is trivial and $\sigma(\theta^*) = \theta^*$. This means that the automorphism θ^* is defined over $\mathbf{R}$.

We wish to describe the action of θ induced on the L-group of G. The splitting $(B, T, \{X\})$ determines a based root datum (Proposition 7.4.6 [Spr98]) and an action of Γ on the Dynkin diagram of G (§1.3 [Bor79]). To the dual based root datum there is attached a dual group $\hat{G}$ defined over $\mathbf{C}$, a Borel subgroup $\mathcal{B} \subset \hat{G}$ and a maximal torus $\mathcal{T} \subset \mathcal{B}$ (2.12 [Spr79]). Let us fix a splitting

$$(\mathcal{B}, \mathcal{T}, \{\mathcal{X}\})$$

of $\hat{G}$. This allows us to transfer the action of Γ from the Dynkin diagram of $\hat{G}$ to an algebraic action of $\hat{G}$ (Proposition 2.13 [Spr79]). This action may be extended trivially to the Weil group $W_{\mathbf{R}}$, which as a set we write as $\mathbf{C}^\times \cup \sigma\mathbf{C}^\times$ (§9.4 [Bor79]). The L-group ${}^LG$ is defined by the resulting semidirect product ${}^LG = \hat{G} \rtimes W_{\mathbf{R}}$.

In a parallel fashion, θ induces an automorphism of the Dynkin diagram of G, which then transfers to an automorphism $\hat{\theta}$ on $\hat{G}$. We define ${}^L\theta$ to be the automorphism of ${}^LG$ equal to $\hat{\theta} \times 1_{W_{\mathbf{R}}}$. By definition, the automorphism $\hat{\theta}$ preserves $(\mathcal{B}, \mathcal{T}, \{\mathcal{X}\})$.

We close this section with some remarks concerning Weyl groups. Let us assume for the moment that B and T are preserved by θ, and T^1 is the identity component of $T^\theta \subset T$. The torus T^1 contains strongly regular elements (*pp.* 227-228 [Art88]), so its centralizer in G is the maximal torus T. Setting the identity component of G^θ equal to G^1 and the Weyl group of G^1 relative to T^1 equal to $\Omega(G^1, T^1)$, we see that we have an embedding

$$\Omega(G^1, T^1) \to \Omega(G, T)^\theta$$

into the θ-fixed elements of the Weyl group $\Omega(G, T)$. In fact, this embedding is an isomorphism (Lemma II.1.2 [Lab04]).

3.2 Endoscopic data and z-pairs

Endoscopic data are defined in terms of the group G, the automorphism θ, and a cohomology class $\mathbf{a} \in H^1(W_{\mathbf{R}}, Z_{\hat{G}})$, where $Z_{\hat{G}}$ denotes the centre of $\hat{G}$. Let ω be the quasicharacter of $G(\mathbf{R})$ determined by $\mathbf{a}$ (*pp.* 122-123 [Lan89]), and let us fix a one-cocycle a in the class $\mathbf{a}$. By definition (*pp.* 17-18 [KS99]), *endoscopic data* for $(G, \theta, \mathbf{a})$ consist of

1. a quasisplit group H defined over $\mathbf{R}$
2. a split topological group extension
$$1 \to \hat{H} \to \mathcal{H} \overset{c}{\leftrightarrows} W_{\mathbf{R}} \to 1,$$
whose corresponding action of $W_{\mathbf{R}}$ on $\hat{H}$ coincides with the action given by the L-group ${}^L H = \hat{H} \rtimes W_{\mathbf{R}}$
3. an element $\mathsf{s} \in \hat{G}$ such that $\mathrm{Int}(\mathsf{s})\hat{\theta}$ is a semisimple automorphism (§7 [Ste97])
4. an L-homomorphism (*p.* 18 [KS99]) $\xi : \mathcal{H} \to {}^L G$ satisfying
   (a) $\mathrm{Int}(\mathsf{s})\, {}^L\theta \circ \xi = a' \cdot \xi$ (8.5 [Bor79]) for some one-cocycle a' in the class $\mathbf{a}$
   (b) ξ maps $\hat{H}$ isomorphically onto the identity component of $\hat{G}^{\mathsf{s}\hat{\theta}}$, the group of fixed points of $\hat{G}$ under the automorphism $\mathrm{Int}(\mathsf{s})\hat{\theta}$.

Despite requirement 2 of this definition, it might not be possible to define an isomorphism between $\mathcal{H}$ and ${}^L H$ which extends the identity map on $\hat{H}$. One therefore introduces a z-extension (§2.2 [KS99], [Lan79])

$$1 \to Z_1 \to H_1 \overset{p_H}{\to} H \to 1 \tag{3.3}$$

in which H_1 is a connected reductive group containing a central torus Z_1. The surjection p_H restricts to a surjection $H_1(\mathbf{R}) \to H(\mathbf{R})$.

Dual to (3.3) is the extension

$$1 \to \hat{H} \to \hat{H}_1 \to \hat{Z}_1 \to 1. \tag{3.4}$$

Regarding $\hat{H}$ as a subgroup of $\hat{H}_1$, we may assume that ${}^L H$ embeds into ${}^L H_1$ and that $\hat{H}_1 \to \hat{Z}_1$ extends to an L-homomorphism

$$p : {}^L H_1 \to {}^L Z_1.$$

According to Lemma 2.2.A [KS99], there is an L-homomorphism $\xi_{H_1} : \mathcal{H} \to {}^L H_1$ which extends the inclusion of $\hat{H} \to \hat{H}_1$ and defines a topological isomorphism between $\mathcal{H}$ and $\xi_{H_1}(\mathcal{H})$. Kottwitz and Shelstad call (H_1, ξ_{H_1}) a *z-pair* for $\mathcal{H}$.

Observe that the composition

$$W_{\mathbf{R}} \overset{c}{\to} \mathcal{H} \overset{\xi_{H_1}}{\to} {}^L H_1 \overset{p}{\to} {}^L Z_1 \tag{3.5}$$

determines a quasicharacter λ_{Z_1} of $Z_1(\mathbf{R})$ via the Local Langlands Correspondence (§9 [Bor79]).

3.3 Norm mappings

Our goal here is to fix endoscopic data $(H, \mathcal{H}, \mathsf{s}, \xi)$ as defined in the previous section and to describe a map from the semisimple conjugacy classes of the endoscopic group H to the semisimple θ-conjugacy classes of G. The map uses the quasisplit form G^* as an intermediary. The basic reference for this section is chapter 3 [KS99].

Since we are interested in *semisimple* conjugacy classes, and semisimple elements lie in tori, we shall begin by defining maps between the tori of H and G^*. Suppose B_H is a Borel subgroup of H containing a maximal torus T_H and $(\mathcal{B}_H, \mathcal{T}_H, \{\mathcal{X}_H\})$ is the splitting of $\hat{H}$ used in the definition of ${}^L H$ (section 3.1). Suppose further that B' is a Borel subgroup of G^* containing a maximal torus T', and that both are preserved by θ^*.[2] We may assume that $\mathsf{s} \in \mathcal{T}$, $\xi(\mathcal{T}_H) = (\mathcal{T}^{\hat{\theta}})^0$ and $\xi(\mathcal{B}_H) \subset \mathcal{B}$. The pairs $(\hat{B}_H, \hat{T}_H)$ and $(\mathcal{B}_H, \mathcal{T}_H)$ determine an isomorphism $\hat{T}_H \cong \mathcal{T}_H$. Similarly, through the pairs $(\hat{B}', \hat{T}')$ and $(\mathcal{B}, \mathcal{T})$, we conclude that $\hat{T}' \cong \mathcal{T}$. We may combine the former isomorphism with requirement 4b of §3.2 for the endoscopic map ξ to obtain isomorphisms

$$\hat{T}_H \cong \mathcal{T}_H \overset{\xi}{\cong} (\mathcal{T}^{\hat{\theta}})^0.$$

To connect $(\mathcal{T}^{\hat{\theta}})^0$ with T', we define $T'_{\theta^*} = T'/(1-\theta^*)T'$ and leave it as an exercise to prove that $((\hat{T}')^{\hat{\theta}})^0 \cong \widehat{T'_{\theta^*}}$. Combining this isomorphism with the earlier ones, we obtain in turn that

$$\hat{T}_H \cong \mathcal{T}_H \overset{\xi}{\cong} (\mathcal{T}^{\hat{\theta}})^0 \cong ((\hat{T}')^{\hat{\theta}})^0 \cong \widehat{T'_{\theta^*}}, \tag{3.6}$$

and $T_H \cong T'_{\theta^*}$.

The isomorphic groups T_H and T'_{θ^*} are related to the conjugacy classes, which we now define. The θ^*-*conjugacy* class of an element $\delta \in G^*$ is defined as $\{g^{-1}\delta\theta^*(g) : g \in G^*\}$. The element δ is called θ^*-*semisimple* if the automorphism $\mathrm{Int}(\delta)\theta^*$ preserves a Borel subgroup of G^* and maximal torus thereof. A θ^*-*semisimple* θ^*-*conjugacy class* is a θ^*-conjugacy class of a θ^*-semisimple element. Let $Cl(G^*, \theta^*)$ be the set of all θ^*-conjugacy classes and $Cl_{\mathrm{ss}}(G^*, \theta^*)$ be the subset of θ^*-semisimple θ^*-conjugacy classes. With this notation in hand, we look to Lemma 3.2.A [KS99], which tells us that there is a bijection

$$Cl_{\mathrm{ss}}(G^*, \theta^*) \to T'_{\theta^*} / \Omega(G^*, T')^{\theta^*},$$

given by taking the coset of the intersection of a θ^*-conjugacy class with T'.

The aforementioned map specializes to give the bijections on either end of

$$Cl_{\mathrm{ss}}(H) \leftrightarrow T_H/\Omega(H, T_H) \to T'_{\theta^*}/\Omega(G^*, T')^{\theta^*} \leftrightarrow Cl_{\mathrm{ss}}(G^*, \theta^*). \tag{3.7}$$

To describe the remaining map in the middle of (3.7), recall from (3.6) that the isomorphism between T_H and T'_{θ^*} is obtained by way of ξ. Using these ingredients and the closing remarks of §3.1, we obtain maps

$$\Omega(H, T_H) \cong \Omega(\hat{H}, \hat{T}_H) \cong \Omega(\hat{H}, \mathcal{T}_H) \to \Omega(\hat{G}^*, \mathcal{T})^{\hat{\theta}} \cong \Omega(G^*, T')^{\theta^*}.$$

[2] Readers of [KS99] should note that we write T' for the torus T occurring there.

This completes the description of the map from $Cl_{\rm ss}(H)$ to $Cl_{\rm ss}(G^*, \theta^*)$.

We proceed by describing the map from $Cl_{\rm ss}(G^*, \theta^*)$ to $Cl_{\rm ss}(G, \theta)$. The function $m : G \rightarrow G^*$ defined by

$$m(\delta) = \psi(\delta) g_\theta^{-1}, \ \delta \in G \tag{3.8}$$

passes to a bijection from $Cl(G, \theta)$ to $Cl(G^*, \theta^*)$, since

$$m(g^{-1}\delta\theta(g)) = \psi(g)^{-1}\, m(\delta)\, \theta^*(\psi(g)).$$

We abusively denote this map on θ^*-conjugacy classes by m as well. It is pointed out in §3.1 [KS99] that this bijection need not be equivariant under the action of Γ. One of our key assumptions is that the element g_θ of (3.2) may be chosen so that

$$g_\theta u_\sigma \sigma(g_\theta^{-1}) \theta^*(u_\sigma)^{-1} \in (1 - \theta^*) Z_{G^*_{\rm sc}}. \tag{3.9}$$

Under this assumption m is Γ-equivariant ((3) Lemma 3.1.A [KS99])). Finally, we may combine this bijection with (3.7) to obtain a map

$$\mathcal{A}_{H \backslash G} : Cl_{\rm ss}(H) \rightarrow Cl_{\rm ss}(G, \theta).$$

In keeping with §3.3 [KS99], we define an element $\delta \in G$ to be *θ-regular* if the identity component of $G^{\delta\theta}$ is a torus. It is said to be *strongly θ-regular* if $G^{\delta\theta}$ itself is abelian. An element $\gamma \in H$ is said to be (*strongly*) *G-regular* if the elements in the image of its conjugacy class under $\mathcal{A}_{H\backslash G}$ are (strongly) regular. An element $\gamma \in H(\mathbf{R})$ is called a *norm* of an element $\delta \in G(\mathbf{R})$ if the θ-conjugacy class of δ equals the image of the conjugacy class of γ under $\mathcal{A}_{H\backslash G}$. It is possible for $\mathcal{A}_{H\backslash G}(\gamma)$ to be a θ-conjugacy class which contains no points in $G(\mathbf{R})$ even though $\gamma \in H(\mathbf{R})$. In this case one says that γ is *not a norm*. These definitions are carried to the z-extension H_1 in an obvious manner. For example, we say that $\gamma_1 \in H_1(\mathbf{R})$ is a norm of $\delta \in G(\mathbf{R})$ if the image of γ_1 in $H(\mathbf{R})$ under (3.3) is a norm of δ.

As in §3.3 [KS99], we conclude with a portrayal of the situation when a strongly regular element $\gamma \in H(\mathbf{R})$ is the norm of a strongly θ-regular element $\delta \in G(\mathbf{R})$. We may let $T_H = H^\gamma$ as γ is strongly regular. The maximal torus T_H is defined over $\mathbf{R}$ since γ lies in $H(\mathbf{R})$. Lemma 3.3.B [KS99] allows us to choose B_H, B' and T' as above so that $\theta^*(B') = B'$, and both T' and the isomorphism $T_H \cong T'_{\theta^*}$ are defined over $\mathbf{R}$. The resulting isomorphism

$$T_H(\mathbf{R}) \cong T'_{\theta^*}(\mathbf{R}) \tag{3.10}$$

is called an *admissible embedding* in §3.3 [KS99]. The image of γ under this admissible embedding defines a coset in $T'/\Omega(G^*, T')^{\theta^*}$. This coset corresponds to the θ^*-conjugacy class of $m(\delta)$. In fact, by Lemma 3.2.A [KS99] there exists some $g_{T'} \in G^*_{\rm sc}$ such that (after $g_{T'}$ has been identified with its image in G^*), this coset equals $g_{T'} m(\delta) \theta^*(g_{T'})^{-1} \Omega(G^*, T')^{\theta^*}$. The element

$$\delta^* = g_{T'}\, m(\delta)\, \theta^*(g_{T'})^{-1} \tag{3.11}$$

belongs to T' and it is an exercise to show that $\mathrm{Int}(g_{T'}) \circ \psi$ furnishes an isomorphism between $G^{\delta\theta}$ and $(G^*)^{\delta^*\theta^*}$. Since $\mathrm{Int}(\delta^*) \circ \theta^*$ preserves (B', T'), the torus $(G^*)^{\delta^*\theta^*}$ contains strongly G-regular elements of T' (*pp.* 227-228 [Art88]) so we see in turn that the centralizer of $(G^*)^{\delta^*\theta^*}$ in G^* is T', and $(G^*)^{\delta^*\theta^*} = (T')^{\theta^*}$. By (3.3.6) [KS99], the resulting isomorphism

$$G^{\delta\theta} \overset{\mathrm{Int}(g_{T'})\psi}{\longrightarrow} (T')^{\theta^*} \tag{3.12}$$

is defined over $\mathbf{R}$.

3.4 Twisted geometric transfer

The underlying assumption of this work is twisted geometric transfer, which is laid out generally in §5.5 [KS99]. For real groups, it has been proven in near generality in [She12]. It shall be convenient for us to state our version of this assumption in the framework of orbital integrals on the component $G(\mathbf{R})\theta$ of the group $G(\mathbf{R}) \rtimes \langle\theta\rangle$. Let $\delta \in G(\mathbf{R})$ be θ-semisimple and strongly θ-regular, and assume that the quasicharacter ω is trivial on $G^{\delta\theta}(\mathbf{R})$. Let $C_c^\infty(G(\mathbf{R})\theta)$ be the space of smooth compactly supported functions on the component $G(\mathbf{R})\theta$. Define the *twisted orbital integral* of $f \in C_c^\infty(G(\mathbf{R})\theta)$ at $\delta\theta \in G(\mathbf{R})\theta$ to be

$$\mathcal{O}_{\delta\theta}(f) = \int_{G^{\delta\theta}(\mathbf{R})\backslash G(\mathbf{R})} \omega(g)\, f(g^{-1}\delta\theta g)\, dg.$$

This integral depends on a choice of quotient measure dg.

We wish to match functions in $C_c^\infty(G(\mathbf{R})\theta)$ with functions on the z-extension H_1. Specifically, let $C_c^\infty(H_1(\mathbf{R}), \lambda_{Z_1})$ be the space of smooth functions f_{H_1} on $H_1(\mathbf{R})$ whose support is compact modulo $Z_1(\mathbf{R})$ and which satisfy

$$f_{H_1}(zh) = \lambda_{Z_1}(z)^{-1} f_{H_1}(h),\ z \in Z_1(\mathbf{R}),\ h \in H_1(\mathbf{R}) \tag{3.13}$$

(see the end of §3.2). The definition of orbital integrals easily carries over to functions of this type at semisimple regular elements.

Suppose $\gamma_1 \in H_1(\mathbf{R})$ is a norm of a θ-semisimple strongly θ-regular element $\delta \in G(\mathbf{R})$. Our geometric transfer assumption is that for every $f \in C_c^\infty(G(\mathbf{R})\theta)$ there exists a function $f_{H_1} \in C^\infty(H_1(\mathbf{R}), \lambda_{Z_1})$ as above such that

$$\sum_{\gamma_1'} \mathcal{O}_{\gamma_1'}(f_{H_1}) = \sum_{\delta'} \Delta(\gamma_1, \delta')\, \mathcal{O}_{\delta'\theta}(f). \tag{3.14}$$

The sum on the left is taken over representatives in $H_1(\mathbf{R})$ of $H_1(\mathbf{R})$-conjugacy classes contained in the H_1-conjugacy class of γ_1. The sum on the right is taken over representatives in $G(\mathbf{R})$ of θ-conjugacy classes under $G(\mathbf{R})$ contained in the θ-conjugacy class of δ. The terms $\Delta(\gamma_1, \delta')$ are geometric transfer factors and are defined in chapter 4 [KS99]. We will come to the geometric transfer factors again in section 4.5. Normalization is required for the measures in the orbital integrals to be compatible (*p.* 71 [KS99]). We also assume that the map $f \mapsto f_{H_1}$ induces a map from stably invariant distributions on $H_1(\mathbf{R})$ to distributions on $G(\mathbf{R})$ as in ordinary endoscopy (Remark 2 §6 [Bou94]).

4 Spectral transfer for the fundamental series

Our goal here is to prove spectral transfer under assumptions that produce representations in the "fundamental series" as presented in III [Duf82]. The presentation in [Duf82] is not given in the language of algebraic groups, so part of our goal will be to match the objects in the realm of algebraic groups with those of [Duf82].

We arrange the setting we are to work in, making six notable assumptions along the way. A list of the six assumptions will be given again at the end. The quadruple $(H, \mathcal{H}, \mathfrak{s}, \xi)$ is a fixed set of endoscopic data together with a z-pair (H_1, ξ_{H_1}). We take an L-parameter $\boldsymbol{\varphi_{H_1}}$ which is the $\hat{H}_1$-conjugacy class of an admissible homomorphism $\varphi_{H_1} : W_{\mathbf{R}} \rightarrow {}^L H_1$ (§8.2 [Bor79]). We suppose that the composition of φ_{H_1} with ${}^L H_1 \rightarrow {}^L Z_1$ corresponds to the quasicharacter $\lambda_{Z_1} : Z_1(\mathbf{R}) \rightarrow \mathbf{C}^\times$ of (3.5) under the Local Langlands Correspondence. The endoscopic Langlands parameter $\boldsymbol{\varphi_{H_1}}$ corresponds to a Langlands parameter $\boldsymbol{\varphi^*}$ of the quasisplit form G^* (§6 [Mez12]). Our first assumption is that φ_{H_1} is not contained in a proper parabolic subgroup of ${}^L H_1$. This is equivalent to the assertion that the L-packet $\Pi_{\varphi_{H_1}}$ consists of essentially square-integrable representations ((3) §10.3 [Bor79]).

Our second assumption is that there exists a strongly θ-regular element $\delta \in G(\mathbf{R})$ which has a *norm* $\gamma \in H(\mathbf{R})$ (section 3.3), and $(G^{\delta\theta}/Z_G^\theta)(\mathbf{R})$ is compact. A strongly θ-regular element in $G(\mathbf{R})$ satisfying the latter compactness condition is called *θ-elliptic* (*p.* 5 [KS99]). The compactness condition passes to a condition on a maximal torus. We say that a maximal torus S in G, which is defined over $\mathbf{R}$, is *fundamental* if $R(G, S)$ has no real roots. This is equivalent to $S(\mathbf{R})$ being a maximally compact Cartan subgroup in $G(\mathbf{R})$ (Lemma 2.3.5 [Wal88]). Similarly, on the level of Lie algebras, one says that $\mathfrak{s}$ is *fundamental* if $R(\mathfrak{g} \otimes \mathbf{C}, \mathfrak{s}, \otimes \mathbf{C})$ has no real roots.

Lemma 4.1. *The element $\delta \in G(\mathbf{R})$ fixes a unique maximal torus S of G which contains $G^{\delta\theta}$. Moreover, the torus S is defined over $\mathbf{R}$ and is fundamental.*

Proof. By definition of strongly θ-regular, $G^{\delta\theta}$ is an abelian group. It contains strongly G-regular elements (*pp.* 227-228 [Art88]), so that the identity component of $Z_G(G^{\delta\theta})$ is a maximal torus of G, which is uniquely determined by δ. Suppose first that Z_G is trivial. Then $G^{\delta\theta}(\mathbf{R})$ is compact, for δ is θ-elliptic. The Lie algebra of $G^{\delta\theta}(\mathbf{R})$ is therefore contained in a Cartan subalgebra of the Lie algebra of a maximally compact subgroup of $G(\mathbf{R})$. The centralizer of this Cartan subalgebra in $\mathfrak{g}$ is a fundamental Cartan subalgebra $\mathfrak{s}$ of $\mathfrak{g}$ (Proposition 6.60 [Kna96]). The exponential of $\mathfrak{s} \otimes \mathbf{C}$ is a maximal torus S in G (Corollary 15.3 [Hum94]). By construction, the torus S is defined over $\mathbf{R}$ and $S(\mathbf{R})$ is maximally compact. Furthermore, S contains $G^{\delta\theta}$ so that S is equal the uniquely determined torus mentioned above.

Now we remove the assumption that Z_G is trivial and observe that there is a canonical bijection between the set of maximal tori of G and the set of maximal tori of the semisimple algebraic group G/Z_G, which is induced by the quotient map. The quotient map is defined over $\mathbf{R}$ (Theorem 12.2.1 [Spr98]).

This bijection therefore passes to a bijection of maximal $\mathbf{R}$-tori. In addition, the quotient map sends δ to an element of $(G/Z_G)(\mathbf{R})$, and it is immediate that this element retains the analogues of the properties of strong θ-regularity and θ-ellipticity. By our earlier argument, we obtain a maximal torus of G/Z_G. It is left to the reader to verify that its pre-image under the quotient map is a maximal torus in G with the desired properties. $\square$

The torus S of Lemma 4.1 has a maximally split subtorus S_d and a maximally anisotropic subtorus S_a such that $S = S_d S_a$ (Proposition 8.15 [Bor91]). The centralizer $M = Z_G(S_d)$ is a Levi subgroup of G which is defined over $\mathbf{R}$ (Proposition 20.4 [Bor91]). By construction, $Z_M \supset S_d$ and it therefore follows that S is elliptic in M. The torus S_d is also the split component of the centre of M (Proposition 20.6 [Bor91]). The usual notation for the latter is A_M. Observe that since $\mathrm{Int}(\delta)\theta$ is defined over $\mathbf{R}$ and preserves S, it also preserves $S_d = A_M$ and M.

Our third assumption is that $\boldsymbol{\varphi^*}$ has a representative homomorphism φ^* whose image is minimally contained in a parabolic subgroup of ${}^L G$, and that this parabolic subgroup is dual, in the sense of §3.3 (2) [Bor79], to an $\mathbf{R}$-parabolic subgroup P of G with Levi component M. In the language of §8.2 [Bor79], this translates as the the parabolic subgroup of ${}^L G$ being *relevant*, and φ^* being *admissible* with respect to G. Under this assumption, we set $\boldsymbol{\varphi} = \boldsymbol{\varphi^*}$ with the intention that $\boldsymbol{\varphi}$ be regarded as a Langlands parameter of G.

We choose a Levi subgroup $\mathcal{M}$ of $\hat{G}$ and an admissible homomorphism $\varphi \in \boldsymbol{\varphi}$ such that $\mathcal{M} \cong \hat{M}$ and $\mathcal{M} \rtimes W_{\mathbf{R}}$ is a standard Levi subgroup of ${}^L G$ which contains $\varphi(W_{\mathbf{R}})$ minimally (§3.4 [Bor79], 4.1 [Mez12]). We may thus regard φ as an admissible homomorphism into $\mathcal{M} \rtimes W_{\mathbf{R}}$ and derive from it an L-packet $\Pi_{\varphi,M}$ of essentially square-integrable representations of $M(\mathbf{R})$ (§10.3 (3) and §11.3 [Bor79]).

Our fourth assumption is that the representations in $\Pi_{\varphi,M}$ have unitary central character. From this, the Local Langlands Correspondence prescribes that the representations in Π_{φ} are the irreducible subrepresentations of the representations induced from those in $\Pi_{\varphi,M}$ (§11.3 [Bor79]).

Before making our fifth assumption we must recall some facts about the homomorphism φ and the L-packet $\Pi_{\varphi,M}$. The homomorphism φ is determined by a pair $\mu, \lambda \in X_*(\hat{S}) \otimes \mathbf{C}$ (§3 [Lan89], §4 [Mez12]). One may regard the elements in this pair as elements in the dual of the complex Lie algebra of S via the isomorphisms $X_*(\hat{S}) \cong X^*(S)$ and

$$X^*(S) \otimes \mathbf{C} \cong \mathfrak{s}^* \otimes \mathbf{C}. \tag{4.1}$$

To be more precise isomorphism (4.1) is an isomorphism of $\mathbf{R}[\Gamma]$-modules, given that Γ acts on both $X^*(S)$ and $\mathbf{C}$ in the usual way (*cf.* §9.4 [Bor79]). In other words, isomorphism (4.1) rests upon an isomorphism

$$(X^*(S_a) \otimes i\mathbf{R}) \oplus (X^*(S_d) \otimes \mathbf{R}) \cong \mathfrak{s}^* \tag{4.2}$$

of $\mathbf{R}$-vector spaces. The pair may be lifted to a quasicharacter of $S(\mathbf{R})$ in the following manner. The element μ is $\hat{M}$-regular and so determines a positive system on $R(M,S)$ (Lemma 3.3 [Lan89]). Let $\iota_M \in X_*(\hat{S}) \otimes \mathbf{C}$ be the half-sum of the positive roots of $R(M,S)$. The pair $(\mu - \iota_M, \lambda)$ corresponds to a linear form on $\mathfrak{s}$, and satisfies a condition which allows one to lift to a quasicharacter $\Lambda = \Lambda(\mu - \iota_M, \lambda)$ of $S(\mathbf{R})$ (*p.* 132 [Lan89], §4.1 [Mez12]).

By the work of Harish-Chandra, the quasicharacter Λ corresponds to an essentially square-integrable representation of $Z_M(\mathbf{R})M_{\mathrm{der}}(\mathbf{R})^0$ ([HC66]). Inducing this representation to $M(\mathbf{R})$ produces an irreducible representation $\varpi_\Lambda \in \Pi_{\varphi,M}$ (*p.* 134 [Lan89]). The remaining representations of $\Pi_{\varphi,M}$ are obtained by replacing Λ by $w^{-1}\Lambda = \Lambda(w^{-1}\cdot(\mu - \iota_M), \lambda)$, where $w \in \Omega(M,S)/\Omega_{\mathbf{R}}(M,S)$ (see §4.1 [Mez12]).

Let us consider the differential of the quasicharacter Λ. The differential only records the behaviour of Λ on the identity component $S(\mathbf{R})^0$ and this behaviour is given precisely by $\mu - \iota_M$ (§4.1 [She81]). The infinitesimal character of ϖ_Λ corresponds to μ and the restriction of this infinitesimal character to $\mathfrak{s} \cap [\mathfrak{m},\mathfrak{m}] \subset \mathfrak{s}_a$ is equal to the Harish-Chandra parameter of the underlying representation of $M_{\mathrm{der}}(\mathbf{R})^0$ (*p.* 310 [Kna86]).

Our fifth assumption is really two separate regularity assumptions. The first regularity assumption is that μ is $\hat{G}$-regular, that is

$$\langle \mu, \alpha \rangle \neq 0, \ \alpha \in R(\hat{G}, \hat{S}).$$

The second regularity assumption pertains to Duflo's characterization of fundamental series representations, and this depends on the behaviour of μ on the anisotropic part $S_a(\mathbf{R})$ of $S(\mathbf{R})^0$ ((ii) III.1 [Duf82]). By identifying μ with a linear form in $\mathfrak{s}^* \otimes \mathbf{C}$ under (4.1), the second regularity assumption reads as

$$\left\langle \mu_{|\mathfrak{s}_a}, \alpha \right\rangle \neq 0, \ \alpha \in R(\hat{G}, \hat{S}).$$

Holding this view, the second regularity assumption is equivalent to the $\mathfrak{g} \otimes \mathbf{C}$-regularity of the $\mathfrak{s}_a^* \otimes \mathbf{C}$-component of μ. Alternatively, the $\mu_{|\mathfrak{s}_a}$ may be regarded as the restriction to S_a of $\mu \in X^*(S_a) \otimes \mathbf{C}$.

We come to our sixth and final assumption. In order for twisted spectral transfer to have any content, we assume that Π_φ is stable under twisting, that is

$$\Pi_\varphi = \omega \otimes (\Pi_\varphi \circ \theta)$$

(see §4.3 [Mez12]).

We list the six assumptions of this section for convenience.

Assumption 1 φ_{H_1} is not contained in a proper parabolic subgroup of ${}^L H_1$.

Assumption 2 There exists a strongly θ-regular and θ-elliptic element $\delta \in G(\mathbf{R})$ which has a norm $\gamma \in H(\mathbf{R})$.

Assumption 3 $\boldsymbol{\varphi^*}$ has a representative φ^* whose image is minimally contained in a parabolic subgroup of ${}^L G$ which is dual to an $\mathbf{R}$-parabolic subgroup P with Levi component M.

Assumption 4 The representations in $\Pi_{\varphi,M}$ have unitary central character.

Assumption 5 The elements μ and $\mu_{|\mathfrak{s}_a}$ in $X_*(\hat{S}) \otimes \mathbf{C}$ are $\hat{G}$-regular.

Assumption 6 $\Pi_\varphi = \omega \otimes (\Pi_\varphi \circ \theta)$.

The main goal of this section is to prove an identity of the shape

$$
\begin{aligned}
(4.3) \qquad & \int_{H_1(\mathbf{R})/Z_1(\mathbf{R})} f_{H_1}(h) \sum_{\pi_{H_1} \in \Pi_{\varphi_{H_1}}} \Theta_{\pi_{H_1}}(h)\, dh \\
& = \sum_{\pi \in \Pi_\varphi} \Delta(\boldsymbol{\varphi_{H_1}}, \pi)\, \Theta_{\pi, \mathsf{U}_\pi}(f)
\end{aligned}
$$

for all $f \in C_c^\infty(G(\mathbf{R})\theta)$. This requires us to define the (twisted) characters $\Theta_{\pi_{H_1}}$ and $\Theta_{\pi,\mathsf{U}_\pi}$, and to define the *spectral transfer factors* $\Delta(\boldsymbol{\varphi_{H_1}}, \pi)$. In the special case that S is elliptic in G, ω is trivial and θ is of finite order, these definitions were made and the identity was proven in §6 [Mez12]. To complete these tasks for fundamental S we shall follow more or less the same path, indicating where additional arguments to [Mez12] are needed.

Our first step on this path is an exposition of twisted characters. We shall give their definitions and reduce them to an explicit formula. In the next step we parametrize L-packets in identity (4.3). We will follow this by definitions of the spectral transfer factors, so that the identity is intelligible. The identity is then proven in two steps: first locally about θ-elliptic elements, then globally using the theory of eigendistributions.

4.1 Fundamental series representations

The purpose of this section is to portray the representations in Π_φ with the theory developed by Harish-Chandra and Duflo ([Duf82]). By definition, the representations in Π_φ are (equivalence classes of) irreducible subrepresentations of $\mathrm{ind}_{P(\mathbf{R})}^{G(\mathbf{R})} \varpi$, where $\varpi \in \Pi_{\varphi, M}$ (§11.3 [Bor79], parabolic induction throughout is normalized). The possible disconnectedness of $G(\mathbf{R})$ in the manifold topology complicates the description of $\mathrm{ind}_{P(\mathbf{R})}^{G(\mathbf{R})} \varpi$. The relationship of each representation $\mathrm{ind}_{P(\mathbf{R})}^{G(\mathbf{R})} \varpi$ to the identity component $G(\mathbf{R})^0$ is given in §III.5 [Duf82]. Duflo notes that this relationship is simplified enormously when the component structure of the real torus $S(\mathbf{R})$ is governed by $Z_G(\mathbf{R})$ (Remark 2 *p.* 172 [Duf82]). Fortunately, this is the case for fundamental S and our first task is to make this clear.

Proposition 4.2. *Suppose G is semisimple. Then $S(\mathbf{R})$ is connected as a real manifold.*

Proof. If one identifies S with $X_*(S) \otimes \mathbf{C}/2\pi i X_*(S)$ through the exponential map (§9 [Bor79]), the component group $S(\mathbf{R})/S(\mathbf{R})^0$ is identified with

$$\{\exp(\pi i \lambda) : \lambda \in X_*(S)^\Gamma\} / \{\exp(\pi i \lambda) : \lambda \in (1+\sigma) X_*(S)\}$$

(§4.1 [She81]). The latter group is isomorphic to

$$\left(\pi i X_*(S)^\Gamma / 2\pi i X_*(S)^\Gamma\right) / \left(\pi i (1+\sigma) X_*(S) / 2\pi i (1+\sigma) X_*(S)\right)$$

and this reduces to $X_*(S)^\Gamma/(1+\sigma)X_*(S)$ since $2X_*(S)^\Gamma \subset (1+\sigma)X_*(S)$. We may choose a base for the root system $R(G,S)^\vee$ as in VI §8 [Kna96]. For instance, one may fix ordered bases of each of the two eigenspaces of σ on $\mathfrak{s}\otimes\mathbf{C}$ and then take a lexicographic ordering of the roots by taking the -1-eigenspace before the $+1$-eigenspace. This produces a positive system $R^+(G,S)^\vee$ which is stable under $-\sigma$, so that the positive simple roots are permuted by $-\sigma$ amongst themselves.

Let $\lambda_1^\vee, \ldots, \lambda_m^\vee$ be the (dual) fundamental dominant weights corresponding to the set of simple roots in $R^+(G,S)^\vee$ (§31.1 [Hum75]). We may assume that the first $0 \leq k \leq m$ of these weights are those fixed by $-\sigma$ and write the fundamental dominant weights as

$$\lambda_1^\vee, \ldots, \lambda_k^\vee, \lambda_{k+1}^\vee, \ldots, \lambda_{k+r}^\vee, -\sigma(\lambda_{k+1}^\vee), \ldots, -\sigma(\lambda_{k+r}^\vee).$$

These weights form a $\mathbf{Z}$-basis of a free $\mathbf{Z}$-module containing $X_*(S)$ (§1.8 [Spr79]). Now suppose $\lambda \in X_*(S)^\Gamma$ and

$$\lambda = \sum_{j=1}^{k} c_j\, \lambda_j^\vee + \sum_{s=k+1}^{k+r} c_s\, \lambda_s^\vee + c_s'\, \sigma(\lambda_s^\vee)$$

where $c_1, \ldots, c_{k+r}, c_{k+1}', \ldots, c_{k+r}' \in \mathbf{Z}$. Then $2\lambda = (1+\sigma)\lambda$ implies in turn that

$$\sum_{j=1}^{k} 2c_j \lambda_j^\vee + \sum_{s=k+1}^{k+r} 2c_s\, \lambda_s^\vee + 2c_s'\, \sigma(\lambda_s^\vee) = \sum_{s=k+1}^{k+r} (c_s + c_s')\lambda_s^\vee + (c_s + c_s')\sigma(\lambda_s^\vee),$$

$c_1 = \cdots = c_k = 0$, $c_{k+1} = c_{k+1}'$, $\ldots$, $c_{k+r} = c_{k+r}'$, and

$$\lambda = \sum_{s=k+1}^{k+r} c_s (1+\sigma)(\lambda_s^\vee).$$

This proves that $X_*(S)^\Gamma = (1+\sigma)X_*(S)$. We conclude that

$$S(\mathbf{R})/S(\mathbf{R})^0 \cong X_*(S)^\Gamma/(1+\sigma)X_*(S)$$

is trivial and $S(\mathbf{R}) = S(\mathbf{R})^0$. □

Let G_{der} be the derived subgroup of G and $S_{\mathrm{der}} = S \cap G_{\mathrm{der}}$. The group G_{der} is semisimple (Corollary 8.1.6 [Spr98]) and S_{der} remains a fundamental torus in G_{der}. By Proposition 4.2 we have $S_{\mathrm{der}}(\mathbf{R}) = S_{\mathrm{der}}(\mathbf{R})^0$.

Corollary 4.3. *The fundamental torus S of the reductive group G may be decomposed as*

$$S(\mathbf{R}) = Z_G(\mathbf{R})\, S(\mathbf{R})^0 = Z_G(\mathbf{R})\, S_{\mathrm{der}}(\mathbf{R})^0.$$

Proof. The group Z_G^0 is a torus (Proposition 7.3.1 [Spr98]) and G/Z_G^0 is a connected semisimple algebraic group (Proposition 5.5.10 and Corollary 8.1.6 [Spr98]) with maximal torus S/Z_G^0. Clearly, we may identify $R(G,S)$ with $R(G/Z_G^0, S/Z_G^0)$, and as the former has no real roots, neither does the latter. This is to say that $(S/Z_G^0)(\mathbf{R})$ remains fundamental. We may therefore apply Proposition 4.2 to conclude that $(S/Z_G^0)(\mathbf{R})$ is connected. Now the short exact sequence of tori

$$1 \to Z_G^0 \to S \to S/Z_G^0 \to 1$$

splits (Corollary 8.5 [Bor91]). Therefore we obtain an exact sequence of Γ-modules

$$1 \to X_*(Z_G^0) \to X_*(S) \to X_*(S/Z_G^0) \to 1.$$

In the long exact sequence of group cohomology the portion

$$\to H^2(\Gamma, X_*(Z_G^0)) \to H^2(\Gamma, X_*(S)) \to H^2(\Gamma, X_*(S/Z_G^0)) \to$$

may be rewritten as

$$\to \frac{X_*(Z_G^0)^\Gamma}{(1+\sigma)X_*(Z_G^0)} \to \frac{X_*(S)^\Gamma}{(1+\sigma)X_*(S)} \to \frac{X_*(S/Z_G^0)^\Gamma}{(1+\sigma)X_*(S/Z_G^0)} \to$$

(Theorem 6.2.2 [Wei94]). As we have seen in Lemma 4.2, these groups are isomorphic to the component groups of the three tori and the group on the right is trivial so that

$$\to Z_G^0(\mathbf{R})/Z_G(\mathbf{R})^0 \to S(\mathbf{R})/S(\mathbf{R})^0 \to 1.$$

This means that the canonical map $Z_G^0(\mathbf{R})/Z_G(\mathbf{R})^0 \to S(\mathbf{R})/S(\mathbf{R})^0$ is surjective and we may choose elements $z_1, \ldots, z_m \in Z_G^0(\mathbf{R})$ so that $S(\mathbf{R})$ is the disjoint union of the cosets $z_1 S(\mathbf{R})^0, \ldots, z_m S(\mathbf{R})^0$. In conclusion,

$$Z_G(\mathbf{R})S(\mathbf{R})^0 \subset S(\mathbf{R}) = \cup_{j=1}^m z_j S(\mathbf{R})^0 \subset Z_G(\mathbf{R})S(\mathbf{R})^0$$

implies that $S(\mathbf{R}) = Z_G(\mathbf{R})S(\mathbf{R})^0$. The second identity of the corollary follows from $S(\mathbf{R})^0 = Z_G(\mathbf{R})^0 S_{\mathrm{der}}(\mathbf{R})^0$ (Corollary 1.53 [Kna96]). □

Corollary 4.4. *The quotient group $G(\mathbf{R})/Z_G(\mathbf{R})G_{\mathrm{der}}(\mathbf{R})^0$ has representatives in $\Omega_{\mathbf{R}}(G,S)$. In particular the subgroup $Z_G(\mathbf{R})G_{\mathrm{der}}(\mathbf{R})^0$ has finite index in $G(\mathbf{R})$.*

Proof. For any $x \in G(\mathbf{R})$ the group $xS(\mathbf{R})x^{-1}$ is a fundamental torus. Therefore there exists $x_1 \in G(\mathbf{R})^0$ such that

$$x_1 x S(\mathbf{R})(x_1 x)^{-1} = S(\mathbf{R})$$

(Lemma 2.3.4 [Wal88]) which proves that every coset in $G(\mathbf{R})/S(\mathbf{R})G(\mathbf{R})^0$ has a representative in $\Omega_{\mathbf{R}}(G,S)$. By Corollary 4.3 and Corollary 1.53 [Kna96], we have

$$S(\mathbf{R})G(\mathbf{R})^0 = Z_G(\mathbf{R})\, S_{\mathrm{der}}(\mathbf{R})G_{\mathrm{der}}(\mathbf{R})^0 = Z_G(\mathbf{R})G_{\mathrm{der}}(\mathbf{R})^0$$

(*cf.* page 134 [Lan89]). □

Corollary 4.3 tells us that the component structure of $S(\mathbf{R})$ is governed by $Z_G(\mathbf{R})$. We shall eventually use this fact to obtain a simple expression of the representations in Π_φ in terms of those in $\Pi_{\varphi,M}$ (see Remark 2 III.5 [Duf82]).

Let us return to the representations in $\Pi_{\varphi,M}$. Recall that $\varpi_\Lambda \in \Pi_{\varphi,M}$ is induced from an irreducible representation of $Z_M(\mathbf{R})M_{\mathrm{der}}(\mathbf{R})^0$. More precisely, there exists a square-integrable (*i.e.* discrete series) representation ϖ_0 of $M_{\mathrm{der}}(\mathbf{R})^0$ such that

$$\varpi_\Lambda \cong \mathrm{ind}_{Z_M(\mathbf{R})M_{\mathrm{der}}(\mathbf{R})^0}^{M(\mathbf{R})} (\chi_\varphi \otimes \varpi_0), \tag{4.4}$$

where χ_φ is the central character of ϖ_Λ (or any other representation in $\Pi_{\varphi,M}$). Using isomorphism (4.1), one may identify the infinitesimal character of ϖ_Λ with μ. In addition, since $\langle \mu, \alpha^\vee\rangle \in \mathbf{R}$ for all $\alpha \in R(M,S)$ (proof of Lemma 3.3 [Lan89]) and χ_φ is unitary on $Z_M(\mathbf{R})$ it follows from Corollary 6.49 [Kna96] that $i\mu \in \mathfrak{s}^*$ (*cf.* (4.2)). In particular

$$\sigma(\mu) = -\mu. \tag{4.5}$$

This infinitesimal character must satisfy three criteria in order to place $\mathrm{ind}_{P(\mathbf{R})}^{G(\mathbf{R})}\varpi_\Lambda$ into the category of fundamental series representations. Two of the three criteria are covered by Assumption 5. The $\mathfrak{g}\otimes\mathbf{C}$-regularity of μ fulfils the criterion that $i\mu$ be *bien polarisable* (Lemma 7 II and III.1 [Duf82]). The $\mathfrak{g}\otimes\mathbf{C}$-regularity of $\mathfrak{s}_a^*\otimes\mathbf{C}$-component of μ fulfils the criterion of $i\mu$ being *standard* ((ii) III.1 [Duf82]).

The third criterion is that $\mu - \rho_M$ must lift to a quasicharacter of $S(\mathbf{R})^0$, where ρ_M is the half-sum of the positive roots in $R(\mathfrak{g}\otimes\mathbf{C}, \mathfrak{s}\otimes\mathbf{C})$ determined by the regular element μ (Remark 2 II.2 [Duf82]). This is equivalent to $i\mu$ being *admissible* in the parlance of Duflo. This criterion is satisfied, as we are identifying $\mu - \rho_M$ with $\mu - \iota_M$ and the latter defines the restriction of the quasicharacter Λ to $S(\mathbf{R})^0$ (§4.1 [She81]). This lifting property persists when ρ_M is replaced by ρ, the half-sum of positive roots in $R(\mathfrak{g}\otimes\mathbf{C}, \mathfrak{s}\otimes\mathbf{C})$ determined by the regular element $\mu_{|\mathfrak{s}_a}$.

Lemma 4.5. *The linear form $\mu - \rho \in \mathfrak{s}^*\otimes\mathbf{C}$ lifts to a quasicharacter of $S(\mathbf{R})^0$.*

Proof. Since $S_d(\mathbf{R})^0 S_a(\mathbf{R})$ is a closed connected subgroup of the same dimension as $S(\mathbf{R})^0$, we see that $S(\mathbf{R})^0 = S_d(\mathbf{R})^0 S_a(\mathbf{R})$. It is clear from the isomorphism $\mathfrak{s}_d \cong S_d(\mathbf{R})^0$ that $(\mu-\rho)_{|\mathfrak{s}_d}$ lifts to a quasicharacter of $S_d(\mathbf{R})^0$. To lift $(\mu-\rho)_{|\mathfrak{s}_a}$ we observe that $-\sigma(\mu_{|\mathfrak{s}_a}) = \mu_{|\mathfrak{s}_a}$ (*cf.* (4.5)) so that the positive system of $R(\mathfrak{g}\otimes\mathbf{C}, \mathfrak{s}\otimes\mathbf{C})$ determined by $\mu_{|\mathfrak{s}_a}$ coincides with that given in Corollary 4.3. We may therefore decompose ρ according to $-\sigma$-orbits of positive roots as follows

$$\rho = \left(\frac{1}{2}\sum_{\text{imaginary}} \alpha\right) + \left(\frac{1}{2}\sum_{\text{complex}} \alpha + (-\sigma(\alpha))\right).$$

The first summand is ρ_M (Lemma 15.3.2 [Spr98]) and we already know that $(\mu+\rho_M)_{|\mathfrak{s}_a}$ lifts to $S_a(\mathbf{R})$. The lemma will therefore be complete once we show that the second summand lifts to $S_a(\mathbf{R})$. For this, we compute that

$$\frac{1}{2}\sum_{\text{complex}}(\alpha-\sigma(\alpha))_{|\mathfrak{s}_a}=\frac{1}{2}\sum_{\text{complex}}\alpha_{|\mathfrak{s}_a}+\alpha_{|\mathfrak{s}_a}=\sum_{\text{complex}}\alpha_{|\mathfrak{s}_a}$$

and note that all integer combinations of roots lift to $S(\mathbf{R})$ ((4.15) [Kna86]). $\square$

We have now established the conditions on $S(\mathbf{R})$ and μ for us to describe the representations in Π_φ in Duflo's framework.

Lemma 4.6. *The representation* $\mathrm{ind}_{P(\mathbf{R})}^{G(\mathbf{R})}\varpi_\Lambda$ *is irreducible and equivalent to*

$$\mathrm{ind}_{Z_G(\mathbf{R})G(\mathbf{R})^0}^{G(\mathbf{R})}\left((\chi_\varphi)_{|Z_G(\mathbf{R})}\otimes\mathrm{ind}_{P(\mathbf{R})\cap G(\mathbf{R})^0}^{G(\mathbf{R})^0}\varpi_1\right)\tag{4.6}$$

where ϖ_1 *is defined as*

$$\varpi_1=\mathrm{ind}_{Z_M(\mathbf{R})^0M_{\mathrm{der}}(\mathbf{R})^0}^{M(\mathbf{R})\cap G(\mathbf{R})^0}((\chi_\varphi)_{|Z_M(\mathbf{R})^0}\otimes\varpi_0).$$

Proof. We shall first prove that (4.6) is an irreducible representation and then complete the lemma by proving that it is equivalent to $\mathrm{ind}_{P(\mathbf{R})}^{G(\mathbf{R})}\varpi_\Lambda$. The irreducibility of (4.6) follows from Lemma 8 (i) III.6 [Duf82] once we see that it is equal to the representation (8) on page 172 [Duf82]. To make this equality apparent, we shall indicate how our notation matches with the notation of III [Duf82]. Representation (8) of [Duf82] is written as

$$T_{g,\tau}=\mathrm{Ind}_{Z_G(\mathbf{R})G(\mathbf{R})^0}^{G(\mathbf{R})}(\tau\otimes T_g^{G(\mathbf{R})^0}).\tag{4.7}$$

Here, $g\in\mathfrak{g}^*$ is an element which is *admissible*, *bien polarisable* and *standard*. As was discussed earlier, we may take $g=i\mu$. The expression $T_g^{G(\mathbf{R})^0}$ on the right of (4.7) is defined as $\mathrm{ind}_{P(\mathbf{R})\cap G(\mathbf{R})^0}^{G(\mathbf{R})^0}\varpi_1$ (*p.* 164 [Duf82]). The term τ in (4.7) is an irreducible representation of a metaplectic group, but only its restriction to $Z_G(\mathbf{R})$ is relevant above. This restriction is determined by $g=i\mu$ on $Z_G(\mathbf{R})^0$ and is arbitrary otherwise. We may therefore take $\tau_{|Z_G(\mathbf{R})}=(\chi_\varphi)_{|Z_G(\mathbf{R})}$. With these substitutions, one sees that (4.6) is equal to (4.7). Before we move on, we note that equation (4.7) is not a definition in [Duf82]. It is an identity which holds for us thanks to Corollary 4.3 (Remark 2 III.5 [Duf82]).

We shall prove the equivalence of

$$\mathrm{ind}_{P(\mathbf{R})}^{G(\mathbf{R})}\varpi_\Lambda\cong\mathrm{ind}_{P(\mathbf{R})}^{G(\mathbf{R})}\,\mathrm{ind}_{Z_M(\mathbf{R})M_{\mathrm{der}}(\mathbf{R})^0}^{M(\mathbf{R})}\,(\chi_\varphi\otimes\varpi_0)\tag{4.8}$$

with (4.6) through an identity of their distribution characters. It suffices to prove equality between their characters on the strongly regular subset of $S(\mathbf{R})$. Indeed, by Harish-Chandra's Uniqueness Theorem (Theorem 12.6 [Kna86]), the character of $\chi_\varphi\otimes\varpi_0$ is determined by its values on this subset. Moreover, the characters of both of the representations in question are determined by the character of $\chi_\varphi\otimes\varpi_0$ via induction.

All fundamental maximal tori are conjugate over $\mathbf{R}$ to $S(\mathbf{R})$ (Proposition 6.61 [Kna96]). Therefore the cosets in the quotient groups which correspond to the finite inductions occurring in (4.8) and (4.6) have representatives which normalize $S(\mathbf{R})$ (*cf.* proof of Corollary 4.4).

We first prove the character identity under the assumption that $Z_G(\mathbf{R}) = Z_G(\mathbf{R})^0$. In this case, the character of (4.6) at a strongly regular element $s \in S(\mathbf{R})$ is the sum over

$$\begin{aligned} x_1 &\in N_{G(\mathbf{R})}(S(\mathbf{R}))/N_{G(\mathbf{R})^0}(S(\mathbf{R})), \\ x_2 &\in N_{G(\mathbf{R})^0}(S(\mathbf{R}))/N_{M(\mathbf{R})\cap G(\mathbf{R})^0}(S(\mathbf{R})), \\ x_3 &\in N_{M(\mathbf{R})\cap G(\mathbf{R})^0}(S(\mathbf{R}))/Z_M(\mathbf{R})^0 N_{M_{\mathrm{der}}(\mathbf{R})^0}(S(\mathbf{R})) \end{aligned}$$

of the product of the reciprocals of the cardinalities of the first and third quotients above with

$$D_G^{-1}(s)\, D_M \circ \Theta_{\chi_\varphi \otimes \varpi_0}(x_1 x_2 x_3 s (x_1 x_2 x_3)^{-1})$$

(*cf.* Lemma 7.1.3 (ii) [Bou87]). Here, D_G and D_M are absolute values of Weyl denominators (§4.5 [KS99]), and $\Theta_{\chi_\varphi \otimes \varpi_0}$ denotes the character of $\chi_\varphi \otimes \varpi_0$. This is equal to the product of

$$\sum_{x \in N_{G(\mathbf{R})}(S(\mathbf{R}))/Z_M(\mathbf{R})^0 N_{M_{\mathrm{der}}(\mathbf{R})^0}(S(\mathbf{R}))} D_G^{-1}(s)\, D_M \circ \Theta_{\chi_\varphi \otimes \varpi_0}(x s x^{-1}) \tag{4.9}$$

with above reciprocals.

To compute the character of (4.8) at s, we use

$$Z_M(\mathbf{R}) M_{\mathrm{der}}(\mathbf{R})^0 / Z_M(\mathbf{R})^0 M_{\mathrm{der}}(\mathbf{R})^0 \cong S(\mathbf{R}) M_{\mathrm{der}}(\mathbf{R})^0 / M_{\mathrm{der}}(\mathbf{R})^0 \cong S(\mathbf{R})/S(\mathbf{R})^0 \tag{4.10}$$

By the connectedness of $Z_G(\mathbf{R})$ and Corollary 4.3 this quotient is trivial. In consequence, the character value of (4.8) is the sum over

$$\begin{aligned} y_1 &\in N_{G(\mathbf{R})^0}(S(\mathbf{R}))/N_{M(\mathbf{R})\cap G(\mathbf{R})^0}(S(\mathbf{R})), \\ y_2 &\in N_{M(\mathbf{R})}(S(\mathbf{R}))/Z_M(\mathbf{R})^0 N_{M_{\mathrm{der}}(\mathbf{R})^0}(S(\mathbf{R})) \end{aligned}$$

of the product of the reciprocal of the cardinality of the second quotient above with

$$D_G^{-1}(s)\, D_M \circ \Theta_{\chi_\varphi \otimes \varpi_0}(y_1 y_2 s (y_1 y_2)^{-1})$$

(Lemma 7.1.3 (ii) [Bou87]). We must reconcile this expression with (4.9). We begin by using the fact that

$$G(\mathbf{R})/G(\mathbf{R})^0 \cong T(\mathbf{R})/T(\mathbf{R}) \cap G(\mathbf{R})^0 \cong M(\mathbf{R})/M(\mathbf{R}) \cap G(\mathbf{R})^0 \tag{4.11}$$

for any maximally $\mathbf{R}$-split torus T of M (Theorem 14.4 [BT65]). Writing $M(\mathbf{R})/Z_M(\mathbf{R})^0 M_{\mathrm{der}}(\mathbf{R})^0$ as the quotient of (4.11) with

$$(M(\mathbf{R}) \cap G(\mathbf{R})^0)/Z_M(\mathbf{R})^0 M_{\mathrm{der}}(\mathbf{R})^0 \tag{4.12}$$

allows one to choose representatives for y_2 of the form x_1 above. The remaining portion of y_2 corresponding to (4.12) may be absorbed into the y_1 parameter by replacing

$$N_{G(\mathbf{R})^0}(S(\mathbf{R}))/N_{M(\mathbf{R})\cap G(\mathbf{R})^0}(S(\mathbf{R}))$$

with

$$N_{G(\mathbf{R})^0}(S(\mathbf{R}))/Z_M(\mathbf{R})^0 N_{M_{\mathrm{der}}(\mathbf{R})^0}(S(\mathbf{R})).$$

These changes amount to the taking sum over

$$y_1' \in N_{G(\mathbf{R})}(S(\mathbf{R}))/N_{G(\mathbf{R})^0}(S(\mathbf{R})),$$
$$y_2' \in N_{G(\mathbf{R})^0}(S(\mathbf{R}))/Z_M(\mathbf{R})^0 N_{M_{\mathrm{der}}(\mathbf{R})^0}(S(\mathbf{R}))$$

of the product of the reciprocals of the cardinalities of the expected quotient groups with

$$D_G^{-1}(s)\, D_M \circ \Theta_{\chi_\varphi \otimes \varpi_0}(y_1' y_2' s (y_1' y_2')^{-1}),$$

and this expression is equal to (4.9). This concludes the proof in case that $Z_G(\mathbf{R})$ is a connected manifold. In the case that it is disconnected the reader may verify that the tensor product in (4.6) compensates for the discrepancy in (4.8) given by (4.10). □

Corollary 4.7. *Every representation of $G(\mathbf{R})$ parabolically induced from an irreducible representation in $\Pi_{\varphi,M}$ is irreducible.*

Proof. The representations of $\Pi_{\varphi,M}$ are obtained by replacing ϖ_Λ by $\varpi_{w^{-1}\Lambda}$, where $w \in \Omega(M,S)/\Omega_{\mathbf{R}}(M,S)$ (see §4.1 [Mez12]). The arguments of the proof are unaffected by replacing Λ with $\dot{w}^{-1}\Lambda$ and μ by $\dot{w}^{-1}\cdot\mu$ for any $\dot{w} \in \Omega(M,S)$. □

Corollary 4.7 implies that every irreducible representation in Π_φ is fully induced from a unique representation in $\Pi_{\varphi,M}$. Hence, parabolic induction furnishes a bijection between $\Pi_{\varphi,M}$ and Π_φ.

Lemma 4.6 expresses $\mathrm{ind}_{P(\mathbf{R})}^{G(\mathbf{R})}\varpi_\Lambda$ in terms of an irreducible representation of the connected Lie group $G(\mathbf{R})^0$. After some additional bookkeeping, one may express $\mathrm{ind}_{P(\mathbf{R})}^{G(\mathbf{R})}\varpi$ in terms of an irreducible representation of the connected *semisimple* group $G_{\mathrm{der}}(\mathbf{R})^0$. This will be required in section 4.3.

Corollary 4.8. *The representation* $\mathrm{ind}_{P(\mathbf{R})}^{G(\mathbf{R})}\varpi_\Lambda$ *is equivalent to*

$$\mathrm{ind}_{Z_G(\mathbf{R})G(\mathbf{R})^0}^{G(\mathbf{R})}\left((\chi_\varphi)_{|Z_G(\mathbf{R})} \otimes \mathrm{ind}_{P(\mathbf{R})\cap G_{\mathrm{der}}(\mathbf{R})^0}^{G_{\mathrm{der}}(\mathbf{R})^0}\varpi_1\right) \tag{4.13}$$

where ϖ_1 is defined as

$$\mathrm{ind}_{Z_M(\mathbf{R})^0 M_{\mathrm{der}}(\mathbf{R})^0 \cap G_{\mathrm{der}}(\mathbf{R})^0}^{M(\mathbf{R})\cap G_{\mathrm{der}}(\mathbf{R})^0}((\chi_\varphi)_{|Z_M(\mathbf{R})^0\cap G_{\mathrm{der}}(\mathbf{R})^0} \otimes \varpi_0).$$

Proof. The principal observation in showing that (4.6) is equivalent to (4.13) is that $G(\mathbf{R})^0$ is the internal direct product of $Z_G(\mathbf{R})^0$ and $G_{\mathrm{der}}(\mathbf{R})^0$. The corollary results from removing $Z_G(\mathbf{R})^0$ from the second induction in (4.6) and absorbing it into the tensor product with $(\chi_\varphi)_{|Z_G(\mathbf{R})}$. □

4.2 A parameterization of stable data

There are two sorts of stable data underlying the spectral transfer identity (4.3). The first sort is geometric and is related to the pair of elements $\delta \in G(\mathbf{R})$ and $\gamma_1 \in p_H^{-1}(\gamma) \subset H_1(\mathbf{R})$. Explicitly, the stable geometric data are the θ-conjugacy classes under $G(\mathbf{R})$ of elements in $G(\mathbf{R})$ whose norm is γ_1. In less obtuse terms, the stable geometric data are the collection of sets

$$\{x^{-1}\delta'\theta(x) : x \in G(\mathbf{R})\}$$

where $\delta' \in G(\mathbf{R})$ runs through the representatives which have norm γ_1. By Assumption 2, δ is a representative of such a conjugacy class. This collection of sets is basic to geometric transfer (§5.5 [KS99]). When S is elliptic in G and θ is trivial, this collection of stable data is parameterized by the collection of cosets $\Omega(G,S)/\Omega_{\mathbf{R}}(G,S)$ (§6.4 [Lab08]). Our first effort will be to describe how this parameterizing set is altered when one removes these assumptions on $S(\mathbf{R})$ and θ.

The second sort of stable data is spectral and is related to representations in the L-packet Π_φ. Again, when S is elliptic in G and θ is trivial these representations are parameterized by $\Omega(G,S)/\Omega_{\mathbf{R}}(G,S)$ (§7.1 [Lab08]). We shall describe how this spectral parameterizing set is altered and actually reduces to an object attached to M in the general case. Upon having described parameterizing sets of the stable geometric and spectral sorts, we connect them through a canonical surjection.

Let us begin geometric parameterization by looking back to some cosets presented in §6.1 [Mez12]. One may dissect $\Omega(G,S)/\Omega_{\mathbf{R}}(G,S)$ and extract the coset space $N_G(S)/N_{G(\mathbf{R})}(S)$. When S is elliptic in G the elements in $N_G(S)$ act as $\mathbf{R}$-automorphisms of S (Lemma 6.4.1 [Lab08]). This is not so in general, and the elements of $N_G(S)$ which act as $\mathbf{R}$-automorphisms form the subgroup $N_G(S^\sigma) = N_G(S(\mathbf{R}))$. A moment's reflection reveals that $N_{G(\mathbf{R})}(S)$ and S are subgroups of $N_G(S(\mathbf{R}))$ so that we may consider the collection of double cosets

$$S\backslash N_G(S^\sigma)/N_{G(\mathbf{R})}(S).$$

This collection may be identified with

$$\Omega(G,S)^\sigma/\Omega_{\mathbf{R}}(G,S). \tag{4.14}$$

This will be seen to be the parameterizing set of the stable geometric data when θ is trivial. However, as seen in §6.1 [Mez12], twisting by θ forces us to consider the collection of double cosets

$$S^{\delta\theta}\backslash N_G(S^\sigma)/N_{G(\mathbf{R})}(S).$$

In fact, the only double cosets $S^{\delta\theta}xN_{G(\mathbf{R})}(S)$ which are of interest are those which satisfy

$$x^{-1}\delta\theta x(\delta\theta)^{-1} \in G(\mathbf{R}). \tag{4.15}$$

This being so, we define

$$S^{\delta\theta}\backslash(N_G(S^\sigma)/N_{G(\mathbf{R})}(S))^{\delta\theta} \tag{4.16}$$

to be the collection of double cosets whose representatives $x \in N_G(S^\sigma)$ satisfy (4.15). The following two results justify the above claims.

Lemma 4.9. *Suppose $x \in G$ and $x^{-1}\delta\theta(x) \in G(\mathbf{R})$. Then $\mathrm{Int}(x^{-1})_{|S}$ is defined over $\mathbf{R}$. In particular, if x also belongs to $N_G(S)$ then $x \in N_G(S^\sigma)$.*

Proof. It suffices to show that $\mathrm{Int}(x\sigma(x^{-1}))_{|S}$ is the identity map. From (3.11) we know that γ_1 being a norm of δ entails that $\delta^* = g_{T'}m(\delta)\theta^*(g_{T'}^{-1})$ for some $g_{T'} \in G^*_{\mathrm{sc}}$. According to Lemma 4.4.A [KS99] the element $g_{T'}u_\sigma\sigma(g_{T'}^{-1})$ belongs to T'_{sc}. Likewise, $x^{-1}\delta\theta(x)$ has norm γ_1. Indeed, following the computations of §3.1 [KS99] we observe that

$$m(x^{-1}\delta\theta(x)) = \psi(x^{-1})\, m(\delta)\, \theta^*(\psi(x))$$

so that

$$\delta^* = g_{T'}\psi(x)\, m(x^{-1}\delta\theta(x))\, \theta^*(g_{T'}\psi(x))^{-1}.$$

We may thus apply Lemma 4.4.A [KS99] to the element $g_{T'}\psi(x)$ in place of $g_{T'}$, to find that $g_{T'}\psi(x)u_\sigma\sigma(g_{T'}\psi(x))^{-1}$ belongs to T'_{sc}. Therefore conjugation of T' by $g_{T'}\psi(x)u_\sigma\sigma(g_{T'}\psi(x))^{-1}$ is trivial. Under transport by (4.36), this implies that the restriction to S of

$$\psi^{-1}\mathrm{Int}(g_{T'})^{-1}\,\mathrm{Int}(g_{T'}\psi(x)u_\sigma\sigma(g_{T'}\psi(x))^{-1})\,\mathrm{Int}(g_{T'})\psi$$

is the identity map. For simplicity, we write $g = g_{T'}$ and compute

$$\begin{aligned}
&\psi^{-1}\mathrm{Int}(g)^{-1}\,\mathrm{Int}(g\psi(x)u_\sigma\sigma(g\psi(x))^{-1})\,\mathrm{Int}(g)\psi \\
&\quad= \mathrm{Int}(x)\,\psi^{-1}\mathrm{Int}(u_\sigma)\,\mathrm{Int}(\sigma(\psi(x^{-1})g^{-1}))\,\mathrm{Int}(g)\psi \\
&\quad= \mathrm{Int}(x)\,\sigma^{-1}\psi^{-1}\sigma\,\mathrm{Int}(\sigma(\psi(x^{-1})g^{-1}))\,\mathrm{Int}(g)\psi \\
&\quad= \mathrm{Int}(x\sigma(x^{-1}))\ (\sigma^{-1}(\mathrm{Int}(g)\psi)^{-1}\sigma\mathrm{Int}(g)\psi) \\
&\quad= \mathrm{Int}(x\sigma(x^{-1})),
\end{aligned}$$

where the last equality follows from (4.36) being defined over $\mathbf{R}$. □

The next lemma is a slightly amended version of Lemma 14 [Mez12]. Only the surjectivity argument is affected when S is not elliptic in G.

Proposition 4.10. *Suppose $x \in N_G(S^\sigma)$ satisfies (4.15). Then the map defined by*

$$x \mapsto x^{-1}\delta\theta(x)$$

passes to a bijection from (4.16) to the collection of θ-conjugacy classes under $G(\mathbf{R})$ of elements in $G(\mathbf{R})$ whose norm is γ_1.

Proof. Suppose $x \in N_G(S^\sigma)$ satisfies (4.15). Since δ belongs to $G(\mathbf{R})$, property (4.15) is equivalent to $x^{-1}\delta\theta(x) \in G(\mathbf{R})$. As γ_1 is a norm of δ it is also a norm of $x^{-1}\delta\theta(x)$. It is simple to verify that any element in the double coset $S^{\delta\theta}\backslash x/N_{G(\mathbf{R})}(S)$ maps to an element which is θ-conjugate to $x^{-1}\delta\theta(x)$ under $G(\mathbf{R})$. Thus, we have a well-defined map from (4.16) to the desired collection of θ-conjugacy classes.

To show that this map is surjective, suppose now that $x \in G$ is any element satisfying $x^{-1}\delta\theta(x) \in G(\mathbf{R})$, that is, an element in $G(\mathbf{R})$ whose norm is γ_1. The automorphism $\mathrm{Int}(x^{-1}\delta\theta(x))\theta$ is defined over $\mathbf{R}$. Therefore, the group $G^{x^{-1}\delta\theta(x)\theta}$ is defined over $\mathbf{R}$. The property that $x^{-1}\delta\theta(x) \in G(\mathbf{R})$ implies in turn that $x\sigma(x^{-1}) \in G^{\delta\theta}$ and

$$G^{x^{-1}\delta\theta(x)\theta}(\mathbf{R}) = (x^{-1}G^{\delta\theta}x)(\mathbf{R}) = x^{-1}G^{\delta\theta}(\mathbf{R})x.$$

The quotient $G^{x^{-1}\delta\theta(x)\theta}(\mathbf{R})/Z_G^\theta(\mathbf{R}) = x^{-1}(G^{\delta\theta}(\mathbf{R})/Z_G^\theta(\mathbf{R}))x$ is compact, for δ is θ-elliptic. Using Lemma 2.3.4 [Wal88] and the arguments of Lemma 4.1, one may show that there exists $g \in G(\mathbf{R})$ such that $g^{-1}G^{x^{-1}\delta\theta(x)\theta}(\mathbf{R})g$ lies in the torus $S(\mathbf{R})$. Hence,

$$S \supset g^{-1}G^{x^{-1}\delta\theta(x)\theta}g = (xg)^{-1}G^{\delta\theta}\, xg = (xg)^{-1}S^{\delta\theta}\, xg.$$

The group $S^{\delta\theta}$ contains strongly G-regular elements (*pp.* 227-228 [Art88]). The previous containment therefore implies that $xg \in N_G(S)$. Furthermore, the element $(xg)^{-1}\delta\theta(xg)$ belongs to $G(\mathbf{R})$ so that $xg \in N_G(S^\sigma)$ by Lemma 4.9. It is clear that $xg \in N_G(S^\sigma)$ maps to the same θ-conjugacy class as $x^{-1}\delta\theta(x)$ under $G(\mathbf{R})$, and surjectivity is proven.

To prove injectivity, suppose that $x_1, x_2 \in G$ are representatives for double cosets in (4.16) such that $x_1^{-1}\delta\theta(x_1)$ and $x_2^{-1}\delta\theta(x_2)$ belong to the same θ-conjugacy class under $G(\mathbf{R})$. Then there exists $g \in G(\mathbf{R})$ such that

$$x_1^{-1}\delta\theta(x_1) = (x_2g)^{-1}\delta\theta(x_2g)$$

and it follows that

$$x_2gx_1^{-1} \in G^{\delta\theta} = S^{\delta\theta}.$$

This implies that $g \in N_{G(\mathbf{R})}(S)$, and x_1 and x_2 represent the same double coset in (4.16). □

Let us point out that there is some redundancy in the notation of (4.16). If $x \in N_G(S)$ satisfies (4.15) then it satisfies $x^{-1}\delta\theta(x) \in G(\mathbf{R})$. Lemma 4.9 then tells us that $x \in N_G(S^\sigma)$. As a result, (4.16) could have been written more simply as

$$S^{\delta\theta}\backslash(N_G(S)/N_{G(\mathbf{R})}(S))^{\delta\theta}.$$

We prefer the notation of (4.16) as it highlights a distinction which is absent for elliptic tori, and reduces more readily to (4.14) when θ is trivial.

We now turn to the parameterization of the spectral data, namely the parameterization of Π_φ. Our assumptions on $\boldsymbol{\varphi}$ dictate that induction furnishes a bijection between $\Pi_{\varphi,M}$ and Π_φ (Lemma 4.6). The L-packet $\Pi_{\varphi,M}$ of essentially square-integrable representations of $M(\mathbf{R})$ is parameterized by the coset space $\Omega(M,S)/\Omega_{\mathbf{R}}(M,S)$ (see (21) [Mez12]). We wish to ascertain the cosets which parametrize the representations in Π_φ which are stable under twisting by (ω,θ). On the face of it, it is not clear that there exist any representations in Π_φ which are stable under twisting.

Lemma 4.11. *There exists $\pi \in \Pi_\varphi$ which is equivalent to $\omega \otimes \pi^\theta$.*

Proof. Assume first that G is quasisplit so that $G^* = G$, $T' = S$, etc. We are assuming that the admissible embedding (3.10) is defined via a θ^*-stable pair (B', T'). The group $B' \cap M$ is a Borel subgroup of M and $\Omega(M,T')$ acts simply transitively on the Weyl chambers determined by $(B' \cap M, T')$. The Weyl group $\Omega(M,T')$ also acts transitively on the L-packet $\Pi_{\varphi,M}$ through the quasicharacters of $T'(\mathbf{R})$ corresponding to each representation therein (§4.3 [Mez12], page 134 [Lan89]). These actions of the Weyl group allow one to choose an essentially square-integrable representation $\varpi_\Lambda \in \Pi_{\varphi,M}$ whose corresponding quasicharacter Λ has differential in the positive chamber determined by $B' \cap M$. This is equivalent to choosing $\mu \in \mathfrak{s}^* \otimes \mathbf{C}$ so that $\mu_{|\mathfrak{s}_a}$ belongs to the Weyl chamber determined by $B' \cap M$.

We know that parabolic induction furnishes a bijection between $\Pi_{\varphi,M}$ and Π_φ, and we are assuming that $\Pi_\varphi = \omega \otimes \Pi_\varphi \circ \theta^*$. Moreover, since $\delta^* \in T'(\mathbf{R})$ we have

$$\begin{aligned} \omega \otimes (\mathrm{ind}_{P(\mathbf{R})}^{G^*(\mathbf{R})} \varpi_\Lambda)^{\theta^*} &\cong \omega \otimes (\mathrm{ind}_{P(\mathbf{R})}^{G^*(\mathbf{R})} \varpi_\Lambda)^{\delta^*\theta^*} \\ &\cong \mathrm{ind}_{(\delta^*\theta^*)^{-1}\cdot P(\mathbf{R})}^{G^*(\mathbf{R})} \omega_{|M(\mathbf{R})} \otimes \varpi_\Lambda^{\delta^*\theta^*} \\ &\cong \mathrm{ind}_{P(\mathbf{R})}^{G^*(\mathbf{R})} \omega_{|M(\mathbf{R})} \otimes \varpi_\Lambda^{\delta^*\theta^*} \in \Pi_\varphi \end{aligned} \tag{4.17}$$

(see the proof of Proposition 4.1 [Mez07]). Therefore there exists $w \in \Omega(M,T')$ such that $\varpi_{w^{-1}\Lambda} = \omega_{|M(\mathbf{R})} \otimes \varpi_\Lambda^{\delta^*\theta^*}$. This identity passes to the level of quasicharacters, namely

$$w^{-1} \cdot \Lambda = \omega_{|T'(\mathbf{R})}(\Lambda \circ \theta^*_{|T'(\mathbf{R})}). \tag{4.18}$$

Since $\mathrm{Int}(\delta^*)\theta^*$ preserves $B' \cap M$ and ω is trivial on $T'_{\mathrm{der}}(\mathbf{R})$, the differential of the quasicharacter $w^{-1} \cdot \Lambda$ lies in the positive Weyl chamber, as does Λ. By Assumption 5, Λ is regular so the simply transitive action forces w to be trivial. This proves

$$\Lambda = \omega_{|T'(\mathbf{R})}(\Lambda \circ \theta^*_{|T'(\mathbf{R})}) \tag{4.19}$$

and

$$\varpi_\Lambda = \omega_{|M(\mathbf{R})} \otimes \varpi_\Lambda^{\delta^*\theta^*}. \tag{4.20}$$

The substitution of (4.20) into (4.17) completes the proof in the quasisplit case.

The proof in general follows by *transport de structure* via the map $\mathrm{Int}(g_{T'})\psi$, which determines an $\mathbf{R}$-isomorphism between the fundamental tori T' and S (see (4.36)). See Corollary 2 [Mez12] for the transport argument when S is elliptic in G. $\square$

Now that we know there are representations in Π_φ which are stable under twisting we may parametrize them using $\Omega(M,S)$. Define $(\Omega(M,S)/\Omega_{\mathbf{R}}(M,S))^{\delta\theta}$ to be the subset of those cosets in $\Omega(M,S)/\Omega_{\mathbf{R}}(M,S)$ which have a representative $w \in \Omega(M,S)$ satisfying

$$w^{-1}\,\delta\theta\,w(\delta\theta)^{-1} \in \Omega_{\mathbf{R}}(M,S). \tag{4.21}$$

Suppose ϖ_Λ is the representation $\Pi_{\varphi,M}$ of Lemma 4.11 which is stable under twisting. Suppose $w \in \Omega(M,S)$ is a representative of a coset in $\Omega(M,S)/\Omega_{\mathbf{R}}(M,S)$. Then according to Lemma 15 [Mez12]

$$\varpi_{w^{-1}\Lambda} \cong \omega_{|M(\mathbf{R})} \otimes \varpi^{\delta\theta}_{w^{-1}\Lambda}$$

if and only if w satisfies (4.21).

Proposition 4.12. *The subset of representations in Π_φ which are stable under twisting is*

$$\left\{\mathrm{ind}_{P(\mathbf{R})}^{G(\mathbf{R})}\,\varpi_{w^{-1}\Lambda} :\ w \in (\Omega(M,S)/\Omega_{\mathbf{R}}(M,S))^{\delta\theta}\right\}.$$

Proof. It is a consequence of (4.17) that the subset of representations in Π_φ which are stable under twisting contains the set in brackets. To prove the reverse inclusion, suppose $w \in \Omega(M,S)$ and $\mathrm{ind}_{P(\mathbf{R})}^{G(\mathbf{R})}\,\varpi_{w^{-1}\Lambda}$ is stable under $\delta\theta$. According to the Langlands Disjointness Theorem (*pp.* 149-151 [Lan89]), there exists $k \in N_{G(\mathbf{R})}(A_M)$ such that $\varpi_{w^{-1}\Lambda}$ is stable under $k\delta\theta$. Since $k \in G(\mathbf{R})$ the maximal torus kSk^{-1} is defined over $\mathbf{R}$ and also elliptic in M. As all elliptic tori of M are $M(\mathbf{R})$-conjugate, we may assume that k normalizes S while maintaining the stability of $\varpi_{w^{-1}\Lambda}$ under $k\delta\theta$. This stability implies

$$k\delta\theta w^{-1}\cdot\Lambda = w^{-1}\cdot\Lambda.$$

By assumption $\delta\theta\cdot\Lambda = \Lambda$ so that we may rewrite the above equation as

$$w_1^{-1}k\cdot\Lambda' = \Lambda'$$

where $\Lambda' = \delta\theta w^{-1}(\delta\theta)^{-1}\cdot\Lambda$ and $w_1 = w^{-1}\delta\theta w(\delta\theta)^{-1}$. The differential of the quasicharacter Λ' is G-regular so that $w_1^{-1}k$ is the identity in $\Omega(G,S)$ (Lemma B §10.3 [Hum94]). It follows that w_1 is represented by an element in $G(\mathbf{R})$. Looking back to (4.21), this means that $w \in (\Omega_{\mathbf{R}}(M,S)/\Omega_{\mathbf{R}}(M,S))^{\delta\theta}$. $\square$

This corollary tells us that $(\Omega(M,S)/\Omega_{\mathbf{R}}(M,S))^{\delta\theta}$ is a spectral parameterizing set. Despite appearances, it is not so different from the geometric parameterizing set $S^{\delta\theta}\backslash(N_G(S^\sigma)/N_{G(\mathbf{R})}(S))^{\delta\theta}$. The intermediary between the two sets is

$$S^{\delta\theta}\backslash(N_M(S)/N_{M(\mathbf{R})}(S))^{\delta\theta}$$

whose definition is given by substituting $M = G$ in (4.15).

Lemma 4.13. *Suppose $x \in G$ such that $x^{-1}\delta\theta(x) \in G(\mathbf{R})$. Then there exists $y \in G(\mathbf{R})$ such that $xy \in M$.*

Proof. Fix a maximally $\mathbf{R}$-split torus S' containing S_d and a positive system on $R(G, S')$. Choose $\beta^\vee \in X_*(S_d) \subset X_*(S')$ as regular as possible in the positive chamber and let $P(\beta^\vee)$ be its corresponding $\mathbf{R}$-parabolic subgroup (see Proposition 20.4 [Bor91]). By construction $P(\beta^\vee)$ has Levi decomposition MU. According to Lemma 4.9, the map $\mathrm{Int}(x^{-1})_{|S}$ is defined over $\mathbf{R}$ so that $x^{-1}S_d x$ is an $\mathbf{R}$-split torus. Consequently, $x^{-1}P(\beta^\vee)x = P(x \cdot \beta^\vee)$ is also an $\mathbf{R}$-parabolic subgroup. By Theorem 15.2.6 [Spr98] and Theorem 20.9 [Bor91], there exists $y \in G(\mathbf{R})$ such that $(xy)^{-1}S_d xy \subset S'$ and $(xy)^{-1}P(\beta^\vee)xy = P(\beta^\vee)$. The latter equation implies that $xy \in P(\beta^\vee)$ (Theorem 11.16 [Bor91]). Writing $xy = mu$ according to the Levi decomposition $P = MU$, the earlier containment implies that

$$u^{-1}m^{-1}smus^{-1} = u^{-1}sus^{-1} \in S' \cap U = \{1\},\ s \in S_d.$$

In other words, the element u belongs to $M = Z_G(S_d)$ and so $xy \in M$. $\square$

We remark that Lemma 4.9 and Lemma 4.13 do not rely on the θ-ellipticity of δ and so remain true without the assumption that S is fundamental in G. This fact will be used in section 7.

Proposition 4.14. *The canonical map from*

$$S^{\delta\theta}\backslash(N_M(S)/N_{M(\mathbf{R})}(S))^{\delta\theta} \text{ into } S^{\delta\theta}\backslash(N_G(S^\sigma)/N_{G(\mathbf{R})}(S))^{\delta\theta}$$

is a bijection.

Proof. The injectivity of this map follows from $N_{G(\mathbf{R})}(S) \cap M = N_{M(\mathbf{R})}(S)$. To prove surjectivity, suppose $x \in N_G(S^\sigma)$ is a representative of a double coset on the right. Then $x^{-1}\delta\theta(x) \in G(\mathbf{R})$ by Proposition 4.10. Choosing $y \in G(\mathbf{R})$ as in Lemma 4.13, we see that $xy \in M$. The map $\mathrm{Int}((xy)^{-1})_{|S}$ is defined over $\mathbf{R}$. Consequently, the torus $(xy)^{-1}S(xy)$ is also elliptic in M. After possibly multiplying y on the right by an element of $M(\mathbf{R})$ we may assume that $(xy)^{-1}S(xy) = S$ (Lemma 2.3.4 [Wal88]) so that $xy \in N_M(S)$ and $y \in N_{G(\mathbf{R})}(S)$. Finally, as M is preserved by $\mathrm{Int}(\delta)\theta$ we have $(xy)^{-1}\delta\theta xy(\delta\theta)^{-1} \in M$, and

$$(xy)^{-1}\delta\theta xy(\delta\theta)^{-1} = y^{-1}\,x^{-1}\delta\theta(x)\,\theta(y)\delta^{-1} \in G(\mathbf{R}) \cap M = M(\mathbf{R}).$$

This proves that $xy \in N_M(S)$ is a representative of a double coset on the left and that the canonical injection is surjective. $\square$

Proposition 4.15. *1. There is a canonical surjection*

$$S^{\delta\theta}\backslash(N_G(S^\sigma)/N_{G(\mathbf{R})}(S))^{\delta\theta} \to (\Omega(G,S)^\sigma/\Omega_{\mathbf{R}}(G,S))^{\delta\theta} \tag{4.22}$$

whose fibres are orbits of the kernel of the homomorphism

$$S/S(\mathbf{R})S^{\delta\theta} \xrightarrow{\delta\theta - 1} S/S(\mathbf{R}) \tag{4.23}$$

induced by $\mathrm{Int}(\delta)\theta - 1$. *Moreover, one may choose representatives* $z_1 \in S$ *for elements of this kernel such that* $(\delta\theta - 1)(z_1)$ *are elements in the component subgroup of* $S_d(\mathbf{R})$.

2. *The canonical injection*

$$\Omega(M,S)/\Omega_{\mathbf{R}}(M,S) \to \Omega(G,S)^{\sigma}/\Omega_{\mathbf{R}}(G,S) \tag{4.24}$$

passes to a bijection

$$(\Omega(M,S)/\Omega_{\mathbf{R}}(M,S))^{\delta\theta} \to (\Omega(G,S)^{\sigma}/\Omega_{\mathbf{R}}(G,S))^{\delta\theta}. \tag{4.25}$$

Proof. According to Proposition 2 [Mez12], it is proven that there is a canonical surjection

$$\begin{aligned} S^{\delta\theta}\backslash(N_M(S)/N_{M(\mathbf{R})}(S))^{\delta\theta} &\to S\backslash S(N_M(S)/N_{M(\mathbf{R})}(S))^{\delta\theta} \\ &\to (\Omega(M,S)/\Omega_{\mathbf{R}}(M,S))^{\delta\theta} \end{aligned} \tag{4.26}$$

whose fibres are orbits of the kernel of (4.23). The split component of the centre of $M = Z_G(S_d)$ is S_d (Proposition 20.6 [Bor91]). As $S_d(\mathbf{R})$ is isomorphic to a product of copies of $\mathbf{R}^{\times}$, there is an elementary 2-group $F \subset S_d(\mathbf{R})$ such that $S_d(\mathbf{R})$ is the internal direct product of F and $S_d(\mathbf{R})^0$. The component subgroup of $S_d(\mathbf{R})$ is isomorphic to the subgroup F. It is also proven in Proposition 2 [Mez12] that representatives $z_1 \in S$ for elements in the kernel of (4.23) may be chosen so that $(\delta\theta - 1)(z_1)$ belongs to F.

By following the arguments of Proposition 2 [Mez12], one sees analogously that there is a canonical surjection

$$\begin{aligned} S^{\delta\theta}\backslash(N_G(S^{\sigma})/N_{G(\mathbf{R})}(S))^{\delta\theta} &\to S\backslash S(N_G(S^{\sigma})/N_{G(\mathbf{R})}(S))^{\delta\theta} \\ &\to (\Omega(G,S)^{\sigma}/\Omega_{\mathbf{R}}(G,S))^{\delta\theta} \end{aligned}$$

whose fibres are orbits of the kernel of (4.23). The desired form of the representatives for elements in this kernel have been established above. Thus, the first assertion of the proposition is proven.

The second assertion follows from the bijection of Proposition 4.14 by identifying elements of the cosets in (4.25) with the orbits of the kernel of (4.23) of the double cosets. □

4.3 Twisted characters

The only representations in Π_{φ} which make any contribution to twisted endoscopy are those which are stable under twisting. These representations were identified in Lemma 4.11 and Proposition 4.12. Let us return to their proofs. We have a quasicharacter Λ of $S(\mathbf{R})$ which uniquely determines and essentially square-integrable representation ϖ_{Λ} in $\Pi_{\varphi,M}$. The analogue of (4.19) in the present context is

$$\Lambda = \omega_{|S(\mathbf{R})}(\delta\theta \cdot \Lambda), \tag{4.27}$$

for, unlike the quasisplit case where $\delta^* \in T'$, the element δ need not belong to S (§3.3 [KS99]). Taking the differential of (4.27) and restricting to $\mathfrak{s}_a \cap \mathfrak{m}_{\mathrm{der}}$, we find that

$$\mu_{|\mathfrak{s}_a \cap \mathfrak{m}_{\mathrm{der}}} = \delta\theta \cdot \mu_{|\mathfrak{s}_a \cap \mathfrak{m}_{\mathrm{der}}}. \tag{4.28}$$

The representation

$$\pi = \pi_\Lambda = \mathrm{ind}_{P(\mathbf{R})}^{G(\mathbf{R})} \varpi_\Lambda$$

is stable under twisting. More precisely, there exists a unitary linear operator $\mathsf{U} = \mathsf{U}_\pi$ on the space V_π of π such that

$$\mathsf{U} \circ \omega^{-1}(x)\pi(x) = \pi^\theta(x) \circ \mathsf{U},\ x \in G(\mathbf{R}). \tag{4.29}$$

We define the *twisted character* $\Theta_{\pi,\mathsf{U}}$ as the distribution on $G(\mathbf{R})$ defined by

$$f \mapsto \mathrm{tr} \int_{G(\mathbf{R})} f(x\theta)\, \pi(x)\, \mathsf{U}\, dx,\ f \in C_c^\infty(G(\mathbf{R})\theta) \tag{4.30}$$

(see (34) [Mez12]). This is the kind of distribution which appears on the right of (4.3). Our plan is to reduce this distribution to one on $G_{\mathrm{der}}(\mathbf{R})^0$. In this way, we may temporarily remove the quasicharacter ω from the picture and associate the resulting representations to those of a larger disconnected Lie group generated by $G_{\mathrm{der}}(\mathbf{R})^0$ and $\mathrm{Int}(\delta)\theta$. In this environment we obtain an explicit formula for the distribution of a twisted character on θ-regular and θ-elliptic elements of $G_{\mathrm{der}}(\mathbf{R})^0$. This is the formula which is amenable to comparison with the characters of the representations in $\Pi_{\varphi_{H_1}}$.

All of this was done in great detail in §§4-5 [Mez12] when $M = G$. Fortunately, the case at hand is encompassed by the same theory. Our exposition will therefore sketch the ideas present in [Mez12], bringing to the attention of the reader the few points in which the theory must be adjusted to allow for proper Levi subgroups M.

We wish to see how the stability of π under twisting behaves relative to decomposition (4.13). For the remainder of this section then then representation ϖ_1 is defined as in Corollary 4.8, not as in Lemma 4.6.

Lemma 4.16. *The representation* $\mathrm{ind}_{P(\mathbf{R}) \cap G_{\mathrm{der}}(\mathbf{R})^0}^{G_{\mathrm{der}}(\mathbf{R})^0} \varpi_1$ *is equivalent to* $(\mathrm{ind}_{P(\mathbf{R}) \cap G_{\mathrm{der}}(\mathbf{R})^0}^{G_{\mathrm{der}}(\mathbf{R})^0} \varpi_1)^{\delta\theta}$.

Proof. Imitating the argument of (4.17), we obtain

$$(\mathrm{ind}_{P(\mathbf{R}) \cap G_{\mathrm{der}}(\mathbf{R})^0}^{G_{\mathrm{der}}(\mathbf{R})^0} \varpi_1)^{\delta\theta} \cong \mathrm{ind}_{\delta\theta \cdot P(\mathbf{R}) \cap G_{\mathrm{der}}(\mathbf{R})^0}^{G_{\mathrm{der}}(\mathbf{R})^0} \varpi_1^{\delta\theta} \cong \mathrm{ind}_{P(\mathbf{R}) \cap G_{\mathrm{der}}(\mathbf{R})^0}^{G_{\mathrm{der}}(\mathbf{R})^0} \varpi_1^{\delta\theta}.$$

The representation ϖ_1 is determined up to equivalence by the Harish-Chandra parameter $\mu_{|\mathfrak{s} \cap \mathfrak{m}_{\mathrm{der}}}$, and $\varpi_1^{\delta\theta}$ has Harish-Chandra parameter $\delta\theta \cdot \mu_{|\mathfrak{s}_a \cap \mathfrak{m}_{\mathrm{der}}}$. The lemma therefore follows from (4.28). ☐

Let $\delta = \delta_1,\ \delta_2, \ldots \delta_k \in G(\mathbf{R})$ be a complete set of representatives for the quotient group $G(\mathbf{R})/Z_G(\mathbf{R})G_{\mathrm{der}}(\mathbf{R})^0$ (Corollary 4.4).

Lemma 4.17. *Suppose $1 \leq j \leq k$. Then the representation $\mathrm{ind}_{P(\mathbf{R})\cap G_{\mathrm{der}}(\mathbf{R})^0}^{G_{\mathrm{der}}(\mathbf{R})^0} \varpi_1$ is equivalent to $(\mathrm{ind}_{P(\mathbf{R})\cap G_{\mathrm{der}}(\mathbf{R})^0}^{G_{\mathrm{der}}(\mathbf{R})^0} \varpi_1)^{\delta_j \theta}$ if and only if $j = 1$. In this case, the restriction of $\pi(\delta)\mathsf{U}_{|V_1}$ of $\pi(\delta)\mathsf{U}$ to the subspace $V_1 \subset V_\pi$ of $\mathrm{ind}_{P(\mathbf{R})\cap G_{\mathrm{der}}(\mathbf{R})^0}^{G_{\mathrm{der}}(\mathbf{R})^0} \varpi_1$ is an isomorphism which intertwines $\mathrm{ind}_{P(\mathbf{R})\cap G_{\mathrm{der}}(\mathbf{R})^0}^{G_{\mathrm{der}}(\mathbf{R})^0} \varpi_1$ with $(\mathrm{ind}_{P(\mathbf{R})\cap G_{\mathrm{der}}(\mathbf{R})^0}^{G_{\mathrm{der}}(\mathbf{R})^0} \varpi_1)^{\delta\theta}$.*

Proof. This is an exercise in decomposing a finitely induced representation using Clifford's theorem and Frobenius reciprocity. The proof when $M = G$ is given in Lemma 3 [Mez12]. The general proof follows by replacing ϖ_1 in Lemma 3 [Mez12] with $\mathrm{ind}_{P(\mathbf{R})\cap G_{\mathrm{der}}(\mathbf{R})^0}^{G_{\mathrm{der}}(\mathbf{R})^0} \varpi_1$. □

Lemma 4.17 provides us with an intertwining operator between representations on $G_{\mathrm{der}}(\mathbf{R})^0$. Let us denote it by $\mathsf{U}_1 = \pi(\delta)\mathsf{U}_{|V_1}$. In proving Lemma 4.17 one finds that the only operator $\pi(\delta_j)\mathsf{U}$ which preserves V_1 is $\pi(\delta_1)\mathsf{U} = \mathsf{U}_1$. This has the effect of reducing the twisted "trace" of the finitely induced representation π to that of $\mathrm{ind}_{P(\mathbf{R})\cap G_{\mathrm{der}}(\mathbf{R})^0}^{G_{\mathrm{der}}(\mathbf{R})^0} \varpi_1$. The precise statement of this fact is the next lemma, whose proof is that of Lemma 4 [Mez12] with V_1 replacing V_ϖ.

Lemma 4.18. *Suppose $f \in C_c^\infty(G(\mathbf{R})\theta)$ and define $f_1 \in C_c^\infty(G_{\mathrm{der}}(\mathbf{R})^0\delta\theta)$ by*

$$f_1(x\delta\theta) = \frac{1}{|G_{\mathrm{der}}(\mathbf{R})^0 \cap Z_G(\mathbf{R})|} \sum_{r=1}^{k} \omega(\delta_r) \int_{Z_G(\mathbf{R})} f(z\delta_r^{-1} x\delta\theta\delta_r)\, \chi_\varphi(z)\, dz,$$

for all $x \in G_{\mathrm{der}}(\mathbf{R})^0$. Then

$$\Theta_{\pi,\mathsf{U}}(f) = \int_{G_{\mathrm{der}}(\mathbf{R})^0} f_1(x\delta\theta)\, \mathrm{ind}_{P(\mathbf{R})\cap G_{\mathrm{der}}(\mathbf{R})^0}^{G_{\mathrm{der}}(\mathbf{R})^0} \varpi_1(x)\ \mathsf{U}_1\, dx.$$

We define the distribution $\Theta_{\varpi_1,\mathsf{U}_1}$ on $G_{\mathrm{der}}(\mathbf{R})^0\delta\theta$ by

$$\Theta_{\varpi_1,\mathsf{U}_1}(h) = \int_{G_{\mathrm{der}}(\mathbf{R})^0} h(x\delta\theta)\, \mathrm{ind}_{P(\mathbf{R})\cap G_{\mathrm{der}}(\mathbf{R})^0}^{G_{\mathrm{der}}(\mathbf{R})^0} \varpi_1(x)\, \mathsf{U}_1\, dx,\ h \in C_c^\infty(G_{\mathrm{der}}(\mathbf{R})^0\delta\theta).$$

So that the conclusion of Lemma 4.18 becomes

$$\Theta_{\pi,\mathsf{U}}(f) = \Theta_{\varpi_1,\mathsf{U}_1}(f_1). \tag{4.31}$$

We now present an alternative and more explicit description of $\Theta_{\varpi_1,\mathsf{U}_1}$ given in [Bou87]. We shall express $\Theta_{\varpi_1,\mathsf{U}_1}$ in terms of an ordinary representation character of the group generated by $G_{\mathrm{der}}(\mathbf{R})^0$ and $\delta\theta$. Let us begin with a better description of the group. We may regard $\mathrm{Int}(\delta)\theta$ as an automorphism of $G_{\mathrm{der}}(\mathbf{R})^0$ and define $L = G_{\mathrm{der}}(\mathbf{R})^0 \rtimes \langle\delta\theta\rangle$ to be the resulting semidirect product. According to (36) [Mez12], there exists $s_0 \in S_{\mathrm{der}}(\mathbf{R})$ and a positive integer ℓ such that

$$(\mathrm{Int}(\delta)\theta_{|G_{\mathrm{der}}(\mathbf{R})^0})^\ell = \mathrm{Int}(s_0). \tag{4.32}$$

In addtion, the abelian subgroup $Z_{G_{\mathrm{der}}(\mathbf{R})^0} \rtimes \langle s_0^{-1}(\delta\theta)^\ell\rangle$ is normal, centralizes $G_{\mathrm{der}}(\mathbf{R})^0$, and its product with $G_{\mathrm{der}}(\mathbf{R})^0$ is of finite index in L. These are the requirements for the group L to be compatible with (*) §1.2 [Bou87].

Another requirement pertaining to the structure of L is the existence of a maximal compact subgroup which is preserved by $\mathrm{Int}(\delta)\theta$.

Lemma 4.19. *There exists a maximal compact subgroup K of $G_{\mathrm{der}}(\mathbf{R})^0$ such that $\delta\theta \cdot K = K$.*

Proof. Let K' be a maximal compact subgroup of $G_{\mathrm{der}}(\mathbf{R})^0$. As $\mathrm{Int}(\delta)\theta$ is defined over $\mathbf{R}$, the group $\delta\theta \cdot K'$ is also a maximal compact subgroup of $G_{\mathrm{der}}(\mathbf{R})^0$. There exists $x \in G_{\mathrm{der}}(\mathbf{R})^0$ such that $x\delta\theta \cdot K' = K'$ (Corollary 5.3 [Spr79]). Since S_{der} is a fundamental torus its anisotropic component $S_{\mathrm{der},a} = S_{\mathrm{der}} \cap S_a$ defines a maximal torus in K'. We may assume that $x\delta\theta \cdot S_{\mathrm{der},a}(\mathbf{R}) = S_{\mathrm{der},a}(\mathbf{R})$ (Lemma 2.3.4 [Wal88]). The centralizer $Z_{G_{\mathrm{der}}}(S_{\mathrm{der},a})$ is a Levi subgroup whose roots are the real roots of $R(G_{\mathrm{der}}, S_{\mathrm{der}})$ (Lemma 15.3.2 (ii) [Spr98]). By assumption, there are no real roots in $R(G_{\mathrm{der}}, S_{\mathrm{der}})$, so that the Levi subgroup is minimal and $Z_{G_{\mathrm{der}}}(S_{\mathrm{der},a}) = S_{\mathrm{der}}$ (Proposition 8.1.1 (ii) [Spr98]). In consequence

$$x \cdot S_{\mathrm{der}} = x\delta\theta \cdot S_{\mathrm{der}} = x\delta\theta \cdot Z_{G_{\mathrm{der}}}(S_{\mathrm{der},a}) = Z_{G_{\mathrm{der}}}(x\delta\theta \cdot S_{\mathrm{der},a}) = S_{\mathrm{der}}.$$

This implies that $x \in N_{G_{\mathrm{der}}(\mathbf{R})}(S_{\mathrm{der}}(\mathbf{R}))$ and by Proposition 1.4.2.1 [War72] $x = x_1 x_2$, where $x_1 \in K$ and $x_2 \in S_{\mathrm{der}}(\mathbf{R}) \cap S_d(\mathbf{R})^0$. A substitution of this decomposition reveals

$$\delta\theta \cdot K' = x_2^{-1} x_1^{-1} \cdot K' = x_2^{-1} \cdot K'.$$

Finally, observe that the differential of $\mathrm{Int}(\delta)\theta$ has finite action on $\mathfrak{s}_{\mathrm{der},d}$ by (4.32). Therefore the differentials $\mathrm{Int}(\delta)\theta$ and $1 - \mathrm{Int}(\delta)\theta$ are semisimple. The latter fact yields a decomposition

$$\mathfrak{s}_{\mathrm{der},d} = \mathfrak{s}_{\mathrm{der},d}^{\delta\theta} \oplus (1 - \delta\theta)\mathfrak{s}_{\mathrm{der},d}.$$

The θ-ellipticity of δ forces $S_d^{\delta\theta}(\mathbf{R})$ to be a subgroup of $Z_G(\mathbf{R})$ and as a result $\mathfrak{s}_{\mathrm{der},d}^{\delta\theta} = \{0\}$. Together with the above decomposition of $\mathfrak{s}_{\mathrm{der},d}$ we conclude that there exists $x_3 \in S_d(\mathbf{R})^0$ such that $x_2 = (1 - \delta\theta)(x_3)$. Set $K = x_3^{-1} \cdot K'$. Then K is a maximal compact subgroup of $G_{\mathrm{der}}(\mathbf{R})^0$ and $\delta\theta \cdot K = K$. □

Having taken care of the requirements on L, we may give the description of $\Theta_{\varpi_1, \mathsf{U}_1}$ afforded by [Bou87]. This description is centred around the parameter $\Lambda_1 = \mu_{|\mathfrak{s}_{\mathrm{der}}}$. The parameter Λ_1 satisfies the following requirements of §5-6 [Bou87]. It is G-regular (Assumption 5), or *bien polarisable* in the parlance of §5.2 [Bou87]. It is *admissible* (Lemma 4.5). A superficial variant of equation (4.27) reads as $\delta\theta \cdot \Lambda_1 = \Lambda_1$, so that the centralizer of Λ_1 in L is $S_{\mathrm{der}}(\mathbf{R}) \rtimes \langle\delta\theta\rangle$ and $L = S_{\mathrm{der}}(\mathbf{R}) \rtimes \langle\delta\theta\rangle\, G_{\mathrm{der}}(\mathbf{R})^0$ (*cf.* §6 [Bou87]). In order for the results of §6 [Bou87] to apply, Λ_1 must also be *elliptic* in the sense that its restriction to $\mathfrak{s}_{\mathrm{der},d}$ is zero (§5.2 [Bou87]).

Lemma 4.20. *The parameter Λ_1 is elliptic.*

Proof. It suffices to prove that $\mu_{|\mathfrak{s}_d}$ is central (Corollary 1.53 [Kna96]), and to do this it suffices to prove that $\langle \mu_{|\mathfrak{s}_d}, \alpha\rangle = 0$ for all $\alpha \in R(G,S)^\vee$ (Proposition 8.1.8 (i) [Spr98]). Suppose $\alpha \in R(G,S)^\vee$ and recall the compatible isomorphisms of (3.6), (3.10), and (4.36). Using these isomorphisms and the definition of μ from μ_{H_1}, we may transport the pairing $\langle \mu_{|\mathfrak{s}_d}, \alpha\rangle$ to the context of $\alpha \in R(\hat{G}, \mathcal{T})$ and $\mu \in X_*((\mathcal{T}^{\hat{\theta}})^0) \otimes \mathbf{C}$ (see 4b in section 3.2). Thus, $\mu_{|\mathfrak{s}_d} \in X_*((\mathcal{T}^{\hat{\theta}})^0) \otimes \mathbf{C}$ and $\langle \mu_{|\mathfrak{s}_d}, \alpha\rangle = \langle \mu_{|\mathfrak{s}_d}, \alpha_{\mathrm{res}}\rangle$ where $\alpha_{\mathrm{res}} = \alpha_{|(\mathcal{T}^{\hat{\theta}})^0}$ (*cf.* Theorem 1.1.A [KS99]). If $\alpha_{\mathrm{res}} \neq 0$ then it may be identified with a root in $\alpha_1 \in R(H_1, T_{H_1})^\vee$ under the isomorphisms of (3.6) (see (1.3.4) [KS99] and (137) [Mez12]). In this manner, μ is identified with μ_{H_1}. The composition of these isomorphisms is defined over $\mathbf{R}$ so that the restriction $\mu_{|\mathfrak{s}_d}$ is identified with the restriction of μ_{H_1} to the split component of T_{H_1}. Since $T_{H_1} = H_1^{\gamma_1}$ is elliptic in H_1 (Lemma 12 [Mez12]), this split component is central and

$$\langle \mu_{|\mathfrak{s}_d}, \alpha\rangle = \langle (\mu_{H_1})_{|\mathfrak{t}_{H_1,d}}, \alpha_1\rangle = 0.$$

$\square$

With all requirements in place, we define a distribution character of a representation of L which is expressible as a locally integrable function on the regular elements of L. Proposition 6.1.2 [Bou87] gives an explicit formula of this function on L-regular elements in $S_{\mathrm{der}}(\mathbf{R}) \rtimes \langle \delta\theta\rangle$ (which normalize the maximal compact subgroup K). By Lemma 4.19 and the θ-ellipticity of δ, one may expect such a formula on L-regular elements in $S_{\mathrm{der}}^{\delta\theta}(\mathbf{R})\delta\theta \subset L$. A particular form of this formula is given in the next lemma whose proof is no different from the one given in §5.4 [Mez12].

Lemma 4.21. *Suppose $x \in S_{\mathrm{der}}^{\delta\theta}(\mathbf{R})$ such that $x\delta\theta$ is regular in L, $w \in \Omega(G_{\mathrm{der}}(\mathbf{R})^0, S_{\mathrm{der}}(\mathbf{R}))^{\delta\theta}$ and $\dot{w} \in G_{\mathrm{der}}(\mathbf{R})^0$ is any representative for w. Suppose further that $X \in \mathfrak{s}_{\mathrm{der}}$ satisfies $\exp(X) = x$. Then $\Theta_{\varpi_1, \mathsf{U}_1}(x\delta\theta)$ is equal to*

$$(-1)^{q^{-\Lambda_1}}\, \bar{\tau}_0(\delta\theta) \sum_{w \in \Omega(G_{\mathrm{der}}(\mathbf{R})^0, S_{\mathrm{der}}(\mathbf{R}))^{\delta\theta}} \frac{\det(w)\ \ e^{(wi\Lambda_1 - \rho)X}\ e^{i\Lambda_1 E(\dot{w})}}{\det(1 - \mathrm{Ad}(x\delta\theta))_{|\bar{\mathfrak{u}}}}, \tag{4.33}$$

where $\bar{\tau}_0$ is the quasicharacter of $S_{\mathrm{der}}(\mathbf{R})\langle\delta\theta\rangle$ which satisfies

$$\bar{\tau}_0(\exp(X)) = e^{(i\Lambda_1 - \rho)(X)},\ \ X \in \mathfrak{s}_{\mathrm{der}}$$

and

$$\bar{\tau}_0(\delta\theta) = \zeta e^{(i\Lambda_1 - \rho)(\log(s_0)/\ell)} \tag{4.34}$$

for some fixed ℓ^{th} root of unity $\zeta \in \mathbf{C}^\times$.

We must elucidate the expressions in Lemma 4.21. The regular element $\Lambda_1 \in \mathfrak{s}_{\mathrm{der}}^*$ determines a Borel subalgebra of $\mathfrak{g}_{\mathrm{der}} \otimes \mathbf{C}$ which is preserved by $\mathrm{Int}(\delta)\theta$. Let $\mathfrak{u}$ be the nil-radical of this complex Borel subalgebra and $\bar{\mathfrak{u}}$ be the nil-radical of the opposite Borel subalgebra. The term $q^{-\Lambda_1}$ is the number of negative eigenvalues of the matrix given by the Hermitian form

$$X \mapsto -i\Lambda_1([X,\bar{X}]),\ X \in \mathfrak{u}$$

(see §5.1 [Bou87]). The term $E(\dot{w}) \in \mathfrak{s}_{\mathrm{der}}$ is defined by

$$\exp(E(\dot{w})) = \dot{w}(\delta\theta)\dot{w}^{-1}(\delta\theta)^{-1} \in S_{\mathrm{der}}(\mathbf{R})$$

(see (45) [Mez12]).

The only remaining unspecified term is the ℓ^{th} root of unity ζ. Equation (4.32) implies that U_1^{ℓ} intertwines $\mathrm{ind}_{P(\mathbf{R})\cap G_{\mathrm{der}}(\mathbf{R})^0}^{G_{\mathrm{der}}(\mathbf{R})^0}\varpi_1$ with $(\mathrm{ind}_{P(\mathbf{R})\cap G_{\mathrm{der}}(\mathbf{R})^0}^{G_{\mathrm{der}}(\mathbf{R})^0}\varpi_1)^{s_0}$. Schur's Lemma then tells us that U_1^{ℓ} is a scalar multiple of $\mathrm{ind}_{P(\mathbf{R})\cap G_{\mathrm{der}}(\mathbf{R})^0}^{G_{\mathrm{der}}(\mathbf{R})^0}\varpi_1(s_0)$. Since the intertwining operator U of (4.29) is only defined up to a scalar multiple, we may normalize U so that $\mathsf{U}_1^{\ell} = \mathrm{ind}_{P(\mathbf{R})\cap G_{\mathrm{der}}(\mathbf{R})^0}^{G_{\mathrm{der}}(\mathbf{R})^0}\varpi_1(s_0)$. This normalization is unique only up to multiplication by an ℓ^{th} root of unity, and defines an extension of $\mathrm{ind}_{P(\mathbf{R})\cap G_{\mathrm{der}}(\mathbf{R})^0}^{G_{\mathrm{der}}(\mathbf{R})^0}\varpi_1$ to L (§5.4 [Mez12]).

On the other hand, using the theory of Duflo, the quasicharacter $\bar{\tau}_0$ of $S_{\mathrm{der}}(\mathbf{R}) \rtimes \langle\delta\theta\rangle$ fixes a unique representation of L. This representation fixes a unique operator which intertwines $\mathrm{ind}_{P(\mathbf{R})\cap G_{\mathrm{der}}(\mathbf{R})^0}^{G_{\mathrm{der}}(\mathbf{R})^0}\varpi_1$ with its $\delta\theta$-conjugate ((2) III.5 [Duf82], (37) [Mez12]). As we have already fixed the intertwining operator U_1 with this same property, Schur's Lemma dictates that the previous operator is a scalar multiple of U_1. As in the previous argument, this scalar must be an ℓ^{th} root of unity. We choose ζ in the definition of $\bar{\tau}_0(\delta\theta)$ so that the scalar multiple is one and the two operators coincide.

We shall encounter this ζ again in the definition of the spectral transfer factors, and it appears there so as to cancel with the ζ occurring in the twisted character $\Theta_{\varpi_1,\mathsf{U}_1}$. As a result, the right-hand side of (4.3) will be independent of this normalization of U_1.

Lemma 4.21 provides a fine expansion for the twisted character $\Theta_{\varpi_1,\mathsf{U}_1}$. We may alternatively write this twisted character as $\Theta_{\varpi_{\Lambda_1},\bar{\tau}_0}$, for ϖ_1 is determined by Λ_1 and U_1 is determined by $\bar{\tau}_0(\delta\theta)$. Writing the twisted character in this way suggests how one may obtain analogous expansions for the twisted characters in Proposition 4.12. For all $w \in (\Omega(M,S)/\Omega_{\mathbf{R}}(M,S))^{\delta\theta}$ we set $\pi_{w^{-1}\Lambda} = \mathrm{ind}_{P(\mathbf{R})}^{G(\mathbf{R})}\varpi_{w^{-1}\Lambda}$. By (4.13) it is apparent that

$$\pi_{w^{-1}\Lambda} = \mathrm{ind}_{Z_G(\mathbf{R})G(\mathbf{R})^0}^{G(\mathbf{R})}\left((\chi_\varphi)_{|Z_G(\mathbf{R})} \otimes \mathrm{ind}_{P(\mathbf{R})\cap G_{\mathrm{der}}(\mathbf{R})^0}^{G_{\mathrm{der}}(\mathbf{R})^0}\varpi_{w^{-1}\Lambda_1}\right). \tag{4.35}$$

Given any representative $\dot{w} \in N_M(S)$ for w, one may verify that the theory of Duflo and Bouaziz underlying Lemma 4.21 applies equally well with Λ_1 replaced by $w^{-1}\Lambda_1$, $\bar{\tau}_0$ replaced by $\bar{\tau}_0^{\dot{w}^{-1}}$ and $\delta\theta$ replaced by $\dot{w}^{-1}\delta\theta\dot{w}$. We therefore define $\Theta_{\pi_{w^{-1}\Lambda},\mathsf{U}_\pi}$ in terms of $\Theta_{\varpi_{w^{-1}\Lambda_1},\bar{\tau}_0^{\dot{w}^{-1}}}$ as in (4.31), after making the above replacements. For those $\pi \in \Pi_\varphi$ which are not stable under twisting, i.e. not given in Proposition 4.12, we set $\mathsf{U}_\pi = 0$ and $\Theta_{\pi,\mathsf{U}_\pi}$ equal to the zero distribution.

4.4 A bridge between S and endoscopic tori

Our goal here is to describe a bridge between the maximal torus $T_H = H^\gamma$ of H and the maximal torus S so that we may compare values of (twisted) characters on each of them. This bridge comes in three pieces. The first piece is isomorphism (3.12) which passes to an isomorphism

$$S^{\delta\theta}(\mathbf{R})^0 \overset{\mathrm{Int}(g_{T'})\psi}{\longrightarrow} (T')^{\theta^*}(\mathbf{R})^0.$$

In fact, this map extends to an isomorphism of the respective centralizers

$$S(\mathbf{R}) \overset{\mathrm{Int}(g_{T'})\psi}{\cong} T'(\mathbf{R}), \tag{4.36}$$

as the commutator of σ and $\mathrm{Int}(g_{T'})\psi$ lies in $\mathrm{Int}(T')$ ((3.3.6) [KS99]) and acts trivially on T'. The second piece is the homomorphism from $(T')^{\theta^*}(\mathbf{R})^0$ to $T'_{\theta^*}(\mathbf{R})^0$ defined by

$$t \mapsto t\,(1-\theta^*)T'(\mathbf{R}),\ t \in (T')^{\theta^*}(\mathbf{R})^0. \tag{4.37}$$

This homomorphism is surjective and has finite kernel (see the proof of Lemma 12 [Mez12]). The third piece is the restriction of the admissible embedding (3.10) to $T'_{\theta^*}(\mathbf{R})^0$ which yields an isomorphism

$$T'_{\theta^*}(\mathbf{R})^0 \cong T_H(\mathbf{R})^0.$$

We denote the composition of these three maps by

$$\eta : S^{\delta\theta}(\mathbf{R})^0 \to T_H(\mathbf{R})^0. \tag{4.38}$$

The map η is not canonical, depending as it does on the choices for B_H, B', $g_{T'}$, etc. However, our results are independent of these choices. Although η need not be an isomorphism, it is a *local* isomorphism. That is to say, there is an open subset of the identity $\mathcal{V} \subset S^{\delta\theta}(\mathbf{R})^0$ such that η maps $\mathcal{V}$ homeomorphically onto $\eta(\mathcal{V})$.

Let T_{H_1} denote the pre-image of T_H under the projection p_H in (3.3). On the set $\mathcal{V}$ we may extend η to a map $\eta_1 : \mathcal{V} \to T_{H_1}(\mathbf{R})^0$ by noting that (3.3) induces a split exact sequence of Lie algebras

$$0 \to \mathfrak{z}_1 \to \mathfrak{t}_{H_1} \to \mathfrak{t}_H \to 0 \tag{4.39}$$

and composing η with resulting local injection of $T_H(\mathbf{R})^0$ into $T_{H_1}(\mathbf{R})^0$.

According to Proposition 4.12 [Ren03], $\eta_1(x)\gamma_1$ is a norm of $x\delta$ for every $x \in S^{\delta\theta}(\mathbf{R})^0$ such that $x\delta$ is strongly θ-regular. As the latter elements form a dense subset of $S^{\delta\theta}(\mathbf{R})^0$, it follows that the set of norms of elements in $S^{\delta\theta}(\mathbf{R})^0\delta$ forms a dense subset of $T_H(\mathbf{R})^0$.

4.5 Transfer factors

The transfer factors in (4.3) have yet to be defined. For convenience, we begin by duplicating the initial presentation in §6.2 [Mez12] of the geometric transfer factors of Kottwitz and Shelstad (§§4-5 [KS99]). We set forth by fixing $\gamma_1^0 \in H_1(\mathbf{R})$ and strongly θ-regular $\delta^0 \in G(\mathbf{R})$ such that γ_1^0 is a norm of δ^0. One may choose $\Delta(\gamma_1^0, \delta^0)$ arbitrarily in $\mathbf{C}^\times$ and then set (see (5.1.1) [KS99])

$$\Delta(\bar{\gamma}_1, \bar{\delta}) = \Delta(\bar{\gamma}_1, \bar{\delta}; \gamma_1^0, \delta^0)\ \Delta(\gamma_1^0, \delta^0) \tag{4.40}$$

for any strongly θ-regular $\bar{\delta} \in G(\mathbf{R})$ with norm $\bar{\gamma}_1 \in H_1(\mathbf{R})$. The term $\Delta(\bar{\gamma}_1, \bar{\delta}; \gamma_1^0, \delta^0)$ is the product of four scalars $\Delta_I(\bar{\gamma}_1, \bar{\delta}; \gamma_1^0, \delta^0), \ldots, \Delta_{IV}(\bar{\gamma}_1, \bar{\delta}; \gamma_1^0, \delta^0)$. Other than $\Delta_{III}(\bar{\gamma}_1, \bar{\delta}; \gamma_1^0, \delta^0)$, these scalars are quotients of the form

$$\Delta_j(\bar{\gamma}_1, \bar{\delta}; \gamma_1^0, \delta^0) = \Delta_j(\bar{\gamma}_1, \bar{\delta}) / \Delta_j(\gamma_1^0, \delta^0),\ j = I,\, II,\, IV.$$

We choose

$$\Delta(\gamma_1^0, \delta^0) = \Delta_I(\gamma_1^0, \delta^0)\ \Delta_{II}(\gamma_1^0, \delta^0)\ \Delta_{IV}(\gamma_1^0, \delta^0)$$

so that (4.40) becomes

$$\Delta(\bar{\gamma}_1, \bar{\delta}) = \Delta_I(\bar{\gamma}_1, \bar{\delta})\ \Delta_{II}(\bar{\gamma}_1, \bar{\delta})\ \Delta_{III}(\bar{\gamma}_1, \bar{\delta}; \gamma_1^0, \delta^0)\ \Delta_{IV}(\bar{\gamma}_1, \bar{\delta}).$$

In the case that $\bar{\gamma}_1 = \eta_1(x)\gamma_1$ and $\bar{\delta} = x\delta$ the transfer factor $\Delta(\eta_1(x)\gamma_1, x\delta)$ is equal to

$$\Delta_I(\eta_1(x)\gamma_1, x\delta)\, \Delta_{II}(\eta_1(x)\gamma_1, x\delta)\, \Delta_{III}(\eta_1(x)\gamma_1, x\delta; \gamma_1^0, \delta^0)\, \Delta_{IV}(\eta_1(x)\gamma_1, x\delta) \tag{4.41}$$

This closes the duplicated text, but most of the difficult arguments in this section remain unchanged from those in §6.2 [Mez12]. As a matter of fact, of all four Δ-terms in (4.41) are the same as in the present case. The only term for which this claim requires any justification is Δ_{II}, as it is the only term which might depend on the $\mathbf{R}$-structure of S being fundamental rather than elliptic.

The term $\Delta_{II}(\eta_1(x)\gamma_1, x\delta)$ is a quotient (§4.3 [KS99]) which depends on a choice of a-data, and a choice of χ-data ((2.2) and (2.5) [LS87], §1.3 [KS99]). The numerator of $\Delta_{II}(\eta_1(x)\gamma_1, x\delta)$ is expressed in terms of products over representatives α_{res} for Galois orbits in $R_{\mathrm{res}}(G^*, T')$ (§1.3 [KS99])

$$\prod_{\alpha_{\mathrm{res}},\, \mathrm{type}\, R_1, R_2} \chi_{\alpha_{\mathrm{res}}}\left(\frac{N\alpha((x\delta)^*) - 1}{a_{\alpha_{\mathrm{res}}}}\right) \prod_{\alpha_{\mathrm{res}},\, \mathrm{type}\, R_3} \chi_{\alpha_{\mathrm{res}}}\left(\frac{N\alpha((x\delta)^*) + 1}{a_{\alpha_{\mathrm{res}}}}\right). \tag{4.42}$$

The the term $N\alpha$ and the three types of Galois orbits are detailed in §1.3 [KS99]. We wish to equate (4.42) with (75) [Mez12], which is a product over negative roots

$$(-i)^{\dim \mathfrak{u}_{(G^*)^{\theta^*}}} \frac{\left|\prod_{\alpha_{\mathrm{res}}<0,\, \mathrm{type}\, R_1, R_2} N\alpha((x\delta)^*) - 1 \prod_{\alpha_{\mathrm{res}}<0,\, \mathrm{type}\, R_3} N\alpha((x\delta)^*) + 1\right|}{\prod_{\alpha_{\mathrm{res}}<0,\, \mathrm{type}\, R_1, R_2} N\alpha((x\delta)^*) - 1 \prod_{\alpha_{\mathrm{res}}<0,\, \mathrm{type}\, R_3} N\alpha((x\delta)^*) + 1}. \tag{4.43}$$

In §6.2 [Mez12] this was achieved by first proving that each Galois orbit had a unique negative representative when the torus S was elliptic. The rest then follows by choosing a-data

$$a_{\alpha_{\rm res}} = \begin{cases} -i, & \alpha_{\rm res} > 0 \\ i, & \alpha_{\rm res} < 0 \end{cases},$$

and χ-data

$$\chi_{\alpha_{\rm res}}(z) = \begin{cases} |z|/z, & \alpha_{\rm res} > 0 \\ z/|z|, & \alpha_{\rm res} < 0 \end{cases} \quad z \in \mathbf{C}^\times.$$

The key step is the proof that that each Galois orbit had a unique negative representative, and for this we require a positive system on $R_{\rm res}(G^*, T')$. Recall from section 4.2 that we have fixed a Borel subgroup B' of G^* and that $\mathrm{Int}(g_{T'})\psi(\mu_{|\mathfrak{s}_a})$ is G^*-regular and lies in the positive chamber determined by B'. We choose the positive roots in $R(G^*, T')$ to be $R(B', T')$. Since $\mathrm{Int}(g_{T'})\psi$ is defined over $\mathbf{R}$ on $S(\mathbf{R})$ (see (4.36)), it follows from (4.5) that

$$\sigma(\mathrm{Int}(g_{T'}))\psi(\mu_{|\mathfrak{s}_a})) = \mathrm{Int}(g_{T'})\psi(\sigma(\mu_{|\mathfrak{s}_a})) = -\mathrm{Int}(g_{T'})\psi(\mu_{|\mathfrak{s}_a}),$$

and $\sigma(B')$ is opposite to B'. In particular, every Galois orbit in $R(G^*, T')$ has a unique negative representative. The same is true for $R_{\rm res}(G^*, T')$ by restriction. This justifies the equality of (4.42) with (4.43). The denominator of $\Delta_{II}(\eta_1(x)\gamma_1, x\delta)$ is identical to the one expressed in §6.2 [Mez12] and so $\Delta_{II}(\eta_1(x)\gamma_1, x\delta)$ is equal to

$$(4.44) \quad i^{\dim \mathfrak{u}_{(G^*)^{\theta^*}} - \dim \mathfrak{u}_H} \frac{\det(1 - \mathrm{Ad}(\eta(x)\gamma))_{|\bar{\mathfrak{u}}_H} \left|\det(1 - \mathrm{Ad}((x\delta)^*)\theta^*)_{|\bar{\mathfrak{u}}_{G^*}}\right|}{\det(1 - \mathrm{Ad}((x\delta)^*)\theta^*)_{|\bar{\mathfrak{u}}_{G^*}} \left|\det(1 - \mathrm{Ad}(\eta(x)\gamma))_{|\bar{\mathfrak{u}}_H}\right|}.$$

Taking for granted that the geometric transfer factors of [Mez12] are equally valid in the present context, we define the spectral transfer factors. By Proposition 4.12 the set of representation in Π_φ is

$$\left\{ \mathrm{ind}_{P(\mathbf{R})}^{G(\mathbf{R})} \varpi_{w^{-1}\Lambda} : \ w \in (\Omega(M,S)/\Omega_{\mathbf{R}}(M,S))^{\delta\theta} \right\}.$$

For simplicity, set $\pi_{w^{-1}\Lambda} = \mathrm{ind}_{P(\mathbf{R})}^{G(\mathbf{R})} \varpi_{w^{-1}\Lambda}$ for all $w \in (\Omega(M,S)/\Omega_{\mathbf{R}}(M,S))^{\delta\theta}$. Then we define the spectral transfer factor $\Delta(\boldsymbol{\varphi_{H_1}}, \pi_{w^{-1}\Lambda})$ to be the product of three scalars:

$$(4.45) \qquad \frac{n_\theta |\Omega(G_{\rm der}(\mathbf{R})^0, (S_{\rm der}^{\delta\theta}(\mathbf{R}))^0 \delta\theta)|}{|\Omega(G_{\rm der}(\mathbf{R})^0, S_{\rm der}(\mathbf{R}))^{\delta\theta}| \, |\det(1 - \mathrm{Ad}(\delta\theta))_{|\mathfrak{s}/\mathfrak{s}^{\delta\theta} \otimes \mathbf{C}}|},$$

$$(4.46) \qquad \frac{\mathrm{sgn}(H)}{(-1)^{q^{-\Lambda_1}} \zeta},$$

(4.47)

$$i^{\dim \mathfrak{u}_{G^\theta} - \dim \mathfrak{u}_H} \Delta_I(\gamma, \delta) \ \Delta_{III}(\gamma_1, \delta; \gamma_1^0, \delta^0) \ \langle (\delta^*, \gamma_1), a_{T'}^{-1} \rangle \langle \mathrm{inv}(\delta, \dot{w}^{-1}\delta\theta(\dot{w})), \kappa_\delta \rangle.$$

We refer the reader to [Mez12] for the exact definitions of the expressions occurring in each of these scalars. From a bird's-eye view, (4.45) is a constant which depends entirely on the structure of $S(\mathbf{R})$ under twisting by $\delta\theta$. The constant (4.46) is a vestige of the character expansions of Harish-Chandra or Bouaziz (*cf.* (4.33)). The scalar (4.47) is derived from the geometric transfer factor $\Delta(\gamma_1, \delta)$ after most of $\Delta_{II}(\gamma_1, \delta)$ and $\Delta_{IV}(\gamma_1, \delta)$ are removed. Scalars (4.45) and (4.47) appear to depend on the choice of δ, but it is proven[3] in §6.3.1 [Mez12] that any choice of strongly θ-regular and θ-elliptic element in $G(\mathbf{R})$ produces the same spectral transfer factors. As it stands, the spectral transfer factors appear to depend on a choice of a- and χ-data. We expect that there is in fact no such dependence for these spectral transfer factors ought to coincide with those of [She12] (*cf.* Lemma 7.5 [She]).

For those $\pi \in \Pi_\varphi$ which are not stable under twisting, i.e. not given in Proposition 4.12, we set $\Delta(\boldsymbol{\varphi_{H_1}}, \pi) = 0$.

4.6 Spectral comparisons

The terms in the spectral identity (4.3) are now all defined. We shall give a proof of (4.3) under some restrictions. First we prove that (4.3) holds for functions of small θ-elliptic support. To prove (4.3) for arbitrary functions in $C_c^\infty(G(\mathbf{R})\theta)$ we will later make the assumptions that the quasicharacter ω is trivial and θ is of finite order. The finiteness assumption shall be weakened in section 6.

We should perhaps qualify what is meant by "proof" in this section. As before, the lion's share of the proofs are already present in [Mez12]. Rather than enter the arguments therein we shall give an overview and indicate where adjustments must be made to accommodate fundamental series, as opposed to square-integrable, representations. In broad strokes, one must replace a square-integrable representation ϖ_1 with $\mathrm{ind}_{P(\mathbf{R})\cap G_{\mathrm{der}}(\mathbf{R})^0}^{G_{\mathrm{der}}(\mathbf{R})^0} \varpi_1$, and a parameterizing set $(\Omega(G,S)/\Omega_{\mathbf{R}}(G,S))^{\delta\theta}$ with $(\Omega(M,S)/\Omega_{\mathbf{R}}(M,S))^{\delta\theta}$. All other arguments remain unchanged.

We begin by specifying the small elliptic support of a function. There is an open subset of the identity $\mathcal{V} \subset S^{\delta\theta}(\mathbf{R})^0$ such that η maps $\mathcal{V}$ homeomorphically onto $\eta(\mathcal{V}) \subset T_H(\mathbf{R})^0$. Suppose first that $f \in C_c^\infty(G(\mathbf{R})\theta)$ has support in the $G(\mathbf{R})$-conjugates of $Z_G(\mathbf{R})^0\mathcal{V}\delta\theta$. By [She12] there exists a function f_{H_1} as in (3.13) which satisfies the geometric transfer identity (3.14).

By applying the Weyl integration formula and identifying $T_H(\mathbf{R})$ with the quotient $T_{H_1}(\mathbf{R})/Z_1(\mathbf{R})$, the left-hand side of (4.3) may be rewritten as

$$\frac{1}{|\Omega(H(\mathbf{R}), T_H(\mathbf{R}))|} \int_{T_H(\mathbf{R})} \sum_{\pi_{H_1} \in \Pi_{\varphi_{H_1}}} \Theta_{\pi_{H_1}}(t)\, \mathcal{O}_t(f_{H_1})\, D_H(t)^2\, dt. \tag{4.48}$$

Here D_H is the absolute value of the Weyl denominator for H (§4.5 [KS99]). We may replace the sum over $\Pi_{\varphi_{H_1}}$ with a sum over the set $\Omega(H, T_H)/\Omega_{\mathbf{R}}(H, T_H)$

[3] In the proof of Lemma 21 [Mez12] the connectedness, and not the compactness, of $S_{\mathrm{der}}(\mathbf{R})$ is the salient feature.

and make a change of variable $t \mapsto \dot{w}t\dot{w}^{-1}$ for representatives $\dot{w} \in N_H(T_H)$ of $w \in \Omega(H, T_H)/\Omega_{\mathbf{R}}(H, T_H)$. This introduces

$$\mathcal{SO}_t(f_{H_1}) = \sum_{w \in \Omega(H,T_H)/\Omega_{\mathbf{R}}(H,T_H)} \mathcal{O}_{\dot{w}t\dot{w}^{-1}}(f_{H_1})$$

into (4.48). The geometric transfer identity (3.14) applies, and (4.48) becomes

$$\begin{aligned} &\frac{1}{|\Omega(H,T_H)|} \int_{S^{\delta\theta}(\mathbf{R})^0} \sum_{w_1} \Theta_{\pi_{H_1}}(\dot{w}_1 \eta_1(x) \gamma_1 \dot{w}_1^{-1}) \\ &\quad \times \sum_{w} \Delta(\eta(x)\gamma, \dot{w}^{-1}x\delta\theta(\dot{w}))\, \mathcal{O}_{\dot{w}^{-1}x\delta\theta\dot{w}}(f)\, D_H(\eta(x)\gamma)^2\, dx, \end{aligned} \tag{4.49}$$

where the first and second sums are taken over $\Omega(H, T_H)/\Omega_{\mathbf{R}}(H, T_H)$ and $(S^{\delta\theta}\backslash(N_G(S)/N_{G(\mathbf{R})})^{\delta\theta}$ respectively.

The next manoeuvre is to express the character values $\Theta_{\pi_{H_1}}(\dot{w}_1\eta_1(x)\gamma_1\dot{w}_1^{-1})$ in terms of the quasicharacter data given by the admissible homomorphism φ_{H_1} ((102) [Mez12]) and thereupon combine that data with the four factors of $\Delta(\eta(x)\gamma, \dot{w}^{-1}x\delta\theta(\dot{w}))$ (see (4.41)). The combination with the Δ_{III}-factor ultimately produces a quasicharacter on $S(\mathbf{R})$ (Corollary 4 [Mez12]). The combination with the Δ_{II}- and Δ_{IV}-factors produces a quotient with the twisted Weyl denominator $D_{G\theta}(x\delta)$ and $\det(1-\mathrm{Ad}(x\delta)\theta)_{|\bar{\mathfrak{u}}}$ (Lemma 16 [Mez12], (4.44)). A further change of variable ((105) and Lemma 19 [Mez12]) recovers the twisted characters of Lemma 4.21 and places (4.49) into the form

$$\begin{aligned} &\sum_{w \in S^{\delta\theta}\backslash(N_G(S)/N_{G(\mathbf{R})}(S))^{\delta\theta}} \frac{\mathrm{sgn}(H) i^{\dim \mathfrak{u}_{G^\theta} - \dim \mathfrak{u}_H} \Delta_I(\gamma,\delta)\ \Delta_{III}(\gamma_1,\delta;\gamma_1^0,\delta^0)\ \langle(\delta^*,\gamma_1), a_{T'}^{-1}\rangle \langle \mathrm{inv}(\delta, \dot{w}^{-1}\delta\theta(\dot{w})), \kappa_\delta\rangle}{(-1)^{q-\Lambda_1}\zeta\ |\Omega(G_{\mathrm{der}}(\mathbf{R})^0, S_{\mathrm{der}}(\mathbf{R}))^{\delta\theta}|} \\ &\quad \times \int_{S_{\mathrm{der}}^{\delta\theta}(\mathbf{R})} \Theta_{\varpi_1,\mathsf{U}_1}(x\delta\theta) \int_{Z_G^{\delta\theta}(\mathbf{R})} \chi_\varphi(z)\ \mathcal{O}_{\dot{w}^{-1}zx\delta\theta\dot{w}}(f)\, dz\, D_{G\theta}(x\delta)^2\, dx. \end{aligned} \tag{4.50}$$

At this stage we are pleased to discern that the baroque expression preceding the integral is most of $\Delta(\boldsymbol{\varphi_{H_1}}, \pi_{w^{-1}\Lambda})$ (*cf.* (4.45)-(4.47)). In order to obtain all of $\Delta(\boldsymbol{\varphi_{H_1}}, \pi_{w^{-1}\Lambda})$ and arrive at the right-hand side of (4.3), we proceed by substituting $S^{\delta\theta}\backslash(N_M(S)/N_{M(\mathbf{R})}(S))^{\delta\theta}$ in place of $S^{\delta\theta}\backslash(N_G(S)/N_{G(\mathbf{R})}(S))^{\delta\theta}$ (Proposition 4.15). This leads to the correct parameterizing set for the representations in Π_φ ((4.24), Proposition 4.12). We then apply some vanishing results due to the support of f ((108) [Mez12]) and the twisted character (Lemma 20 [Mez12]). Finally, we utilize the twisted Weyl integration formula (Proposition 1 [Mez12]) as we did at the beginning of our comparison.

The above comparison achieves (4.3) for functions with small θ-elliptic support about any strongly θ-regular θ-elliptic element in $G(\mathbf{R})$ which has a norm in $H_1(\mathbf{R})$ (§6.3.1 [Mez12]). In order to prove (4.3) without restriction on the support, we appeal to a twisted version of the Harish-Chandra Uniqueness Theorem (Theorem 15.1 [Ren97]). In essence, this uniqueness theorem tells us that a certain kind of distribution on $G(\mathbf{R})\theta$ is determined everywhere by its values on the θ-elliptic set. In order to apply Theorem 15.1 [Ren97], we require that the distribution defined by

$$\Theta(f) = \sum_{\pi_{H_1} \in \Pi_{\varphi_{H_1}}} \Theta_{\pi_{H_1}}(f_{H_1}) - \sum_{\pi \in \Pi_\varphi} \Delta(\boldsymbol{\varphi_{H_1}}, \pi)\, \Theta_{\pi, \mathsf{U}_\pi}(f),\ f \in C_c^\infty(G(\mathbf{R})\theta)$$

be invariant under conjugation by $G(\mathbf{R})$ and an eigendistribution under the action of $\mathcal{Z}(\mathfrak{g} \otimes \mathbf{C})$. The property of $G(\mathbf{R})$-invariance holds only under the assumption that ω is trivial (see (121) [Mez12]), and so we make this assumption. The property of Θ being an eigendistribution is Lemma 24 [Mez12]. With these properties in place, Θ may be regarded as a locally integrable function on $G(\mathbf{R})\theta$ which satisfies a growth condition (Proposition 3.6.1 [Bou87]) required in Theorem 15.1 [Ren97]. Before applying the uniqueness theorem we must constrain ourselves to assuming that $G(\mathbf{R}) \rtimes \langle \theta \rangle$ has a finite number of connected components (§12 [Ren97]). This is equivalent to making the assumption that θ be of finite order. Under these assumptions, one may apply Theorem 15.1 [Ren97] to (4.3). Since Θ vanishes on the *elliptic Cartan subspace* $\mathfrak{s}^{\delta\theta}\delta\theta$ (Definitions 7.1 and 12.3 [Ren97]), it also vanishes everywhere (Theorem 1 [Mez12]). We conclude with a theorem that encapsulates these arguments.

Theorem 4.22. *Suppose ω is trivial and θ is of finite order. Then*

$$\int_{H_1(\mathbf{R})/Z_1(\mathbf{R})} f_{H_1}(h) \sum_{\pi_{H_1} \in \Pi_{\varphi_{H_1}}} \Theta_{\pi_{H_1}}(h)\, dh = \sum_{\pi \in \Pi_\varphi} \Delta(\boldsymbol{\varphi_{H_1}}, \pi)\, \Theta_{\pi, \mathsf{U}_\pi}(f) \tag{4.51}$$

for all $f \in C_c^\infty(G(\mathbf{R})\theta)$.

Under some additional restrictions on the structure of $G(\mathbf{R})$ one may extend Theorem 4.22 to include non-trivial ω (see Theorem 2 [Mez12]).

5 Spectral transfer for limits of fundamental series

In this section we work with the same framework as that of section 4, but we weaken Assumption 3 and remove Assumption 5. Let us concentrate on Assumption 5 for the moment. If we remove the $\hat{G}$-regularity of the parameters μ and $\mu_{|\mathfrak{s}_a}$ then the irreducible representations in Π_φ need no longer be fundamental series representations. As we shall see, these representations may be obtained using the method of *coherent continuation* or *Zuckerman tensoring.* When the Levi subgroup M of section 4 is equal to G then this method produces (essential) *limits of discrete series* (§7 XII [Kna86]). By analogy, when M is allowed to be a proper Levi subgroup we shall speak of (essential) *limits of fundamental series.* The goal then is to prove Theorem 4.22 for L-packets Π_φ consisting of limits of fundamental series. This was accomplished for limits of discrete series in §§7.2-7.3 [Mez12]. The proof for the limits of fundamental series is essentially the same once the requisite objects and assumptions are introduced.

To begin, we assume that ω is trivial and θ is of finite order as in Theorem 4.22. We assume that we have the same endoscopic data as in section 4 with the same Langlands parameter $\boldsymbol{\varphi_{H_1}}$. However, our assumptions on the Langlands parameter $\boldsymbol{\varphi^*}$ for G^* shall be weaker. The pair of elements $\delta \in G(\mathbf{R})$ and $\gamma_1 \in H_1(\mathbf{R})$ are as before, and these bring with them the same fundamental torus S and Levi subgroup M. We merely assume that $\boldsymbol{\varphi^*}$ has a representative φ^* which is an admissible homomorphism with respect to G (§8.2 [Bor79]). This amounts to the assumption that the image of φ^* is minimally contained in a parabolic subgroup of ${}^LG$ which is *relevant* (in the sense of §3.3 [Bor79]) with respect to G.

Lemma 5.1. *The image of φ^* is contained in a Levi subgroup ${}^LM$ dual to M (in the sense of (3) §3.3 [Bor79]).*

Proof. Without loss of generality, we assume that $\varphi_{H_1}(\mathbf{C}^\times) \subset \mathcal{T}_H$ (§4 [Mez12]). Let $\mathcal{M}$ be the centralizer in $\hat{G}$ of the subtorus equal to the identity component of the fixed point subgroup of $\mathcal{T}$ under conjugation by $\varphi^*(\sigma)$. Then $\mathcal{M}$ is a Levi subgroup of $\hat{G}$ (Propostion 20.4 [Bor91]). Let ${}^LM$ be the subgroup generated by $\mathcal{M}$ and $\varphi^*(\sigma)$. It is a Levi subgroup of ${}^LG$ by Lemma 3.5 [Bor79]. Furthermore, the image of φ^* is contained in the subgroup of ${}^LM$ generated by $\mathcal{T}$ and $\varphi^*(\sigma)$. The admissibility assumption on φ^* ((ii) §8.2 [Bor79]) implies that ${}^LM$ is dual to (a $G(\mathbf{R})$-conjugacy class of) an $\mathbf{R}$-Levi subgroup M' of G ((3) §3.3 [Bor79]). The action of $\varphi^*(\sigma)$ on $R(\mathcal{M}, \mathcal{T})$ is that of inversion (Lemma 15.3.2 [Spr98]). In the proof of Lemma 3.1 [Lan89] one sees that this implies that M' contains an elliptic maximal torus S' such that ${}^LS' \cong \langle \mathcal{T}, \varphi^*(W_{\mathbf{R}})\rangle$. By the conjugacy theorems, Corollary 4.35 [Kna96] and Corollary 5.31 [Spr79], we may assume that the anisotropic subtorus S'_a of S' is contained in S_a.

It follows from Assumption 1 on φ_{H_1} that $\varphi_{H_1}(\sigma)$ acts by inversion on the root lattice in $X^*(\mathcal{T}_H)$ ((17) [Mez12]). This implies that $\varphi^*(\sigma)$ acts by inversion on the corresponding root lattice in $X^*((\mathcal{T}^{\hat{\theta}})^0)$ (see (3.6)). Since $\hat{\theta}$ preserves the pair $(\mathcal{B}, \mathcal{T})$, there exists an element β in the root lattice of $R(\hat{G}, \mathcal{T})$ which lies in the Weyl chamber fixed by $\mathcal{B}$, and is invariant under the action of $\hat{\theta}$. In particular, β is $\hat{G}$-regular. The dual element $\beta^\vee$ (§2.2 [Spr79]) belongs to $X_*(\mathcal{T}^{\hat{\theta}})$. Since $\mathcal{T}^{\hat{\theta}}/(\mathcal{T}^{\hat{\theta}})^0$ is finite we may replace β by some integer mulitple and assume that $\beta^\vee \in X_*((\mathcal{T}^{\hat{\theta}})^0)$. From before we see that $\varphi^*(\sigma)$ acts by inversion on $\beta^\vee$. It therefore acts by inversion on β. This, together with the isomorphism $X^*(\mathcal{T}) \cong X_*(S')$, allows us to identify β with a regular element in $X_*(S'_a)$ (Proposition 13.2.4 [Spr98]). The regularity of β implies that $Z_{\hat{G}}(\mathrm{im}(\beta))$ is a maximal torus of G. Since $\mathrm{im}(\beta) \subset S'_a \subset S_a$, we find that this maximal torus is equal to both S' and S.

We deduce in turn that $S' = S$ is elliptic in M', $S_d \subset Z_{M'}$, and $M = Z_G(S_d) \supset M'$. On the other hand, the definition of $\mathcal{M}$ and the duality between $\mathcal{M}$ and M', and $\mathcal{T}$ and S, together imply that $Z_G(S_d) = M'$. We conclude that $M = M'$ and the lemma is complete. □

According to Lemma 5.1, the group $\varphi^*(W_{\mathbf{R}})$ is minimally contained in a Levi subgroup $M_1 \subset M$ of G. Moreover, M_1 is defined over $\mathbf{R}$ and corresponds to a Levi subgroup of ${}^LG$ (§§3.3-3.4 [Bor79]). This produces an admissible homomorphism $\varphi : W_{\mathbf{R}} \to {}^LG$ which we may view as a representative of a Langlands parameter for G or for M_1. We may view φ as a Langlands parameter of M as well.

Regardless of which perspective one takes, the admissible homomorphism φ is determined by a pair $\mu, \lambda \in X_*(\hat{S}) \otimes \mathbf{C}$. This pair is begotten from a defining pair $\mu_{H_1}, \lambda_{H_1} \in X_*(\hat{T}_{H_1}) \otimes \mathbf{C}$ for an admissible homomorphism $\varphi_{H_1} \in \boldsymbol{\varphi_{H_1}}$ (§3 [Lan89], §4.1 [Mez12]), and an application of the maps in (3.3), (3.6) and (4.36) (*cf.* 7 (b) [She10]). There are identifications of Borel subgroups implicit in the maps of (3.6). We may assume that μ_{H_1} is in the positive Weyl chamber determined by the Borel subgroup $B_H \supset T_H$ of H (Lemma 3.3 [Lan89], §4.1 [Mez12]). It follows from the identification of $\hat{B}_H$ with $\mathcal{B}_H$ and the containment $\xi(\mathcal{B}_H) \subset \mathcal{B} \cong \hat{B}'$ (section 3.3) that μ lies in the Weyl chamber determined by $\hat{B}' \cap \hat{M}_1$. To say precisely what this means, let us denote by B the image of B' under the inverse of $\mathrm{Int}(g_{T'})\psi$. Then the precise statement is that $\langle \mu, \alpha^\vee \rangle > 0$ for all $\alpha \in R(B \cap M_1, S)$ (Lemma 3.3 [Lan89]). This ensures the $\hat{M}_1$-regularity of μ, but not its $\hat{G}$-regularity.

5.1 Shifting to the context of section 4

We shall approach the representations in Π_φ indirectly by first shifting μ by a $\hat{G}$-regular element $\nu \in \mathrm{span}_{\mathbf{Z}} R(\hat{G}, \hat{S})^\vee \subset X_*(\hat{S})$. This will eventually give rise to a pair of matching admissible homomorphisms φ^ν and $\varphi^\nu_{H_1}$ which satisfy all of the assumptions of section 4. We may then apply coherent continuation to recover the representations in Π_φ. The analogue of Theorem 4.22 will follow from the compatibility of coherent continuation with spectral transfer.

Lemma 5.2. *There exists $\nu \in \mathrm{span}_{\mathbf{Z}} R(\hat{G}, \hat{S})^\vee$ which lies in the positive Weyl chamber determined by B and is fixed under the action of $\delta\theta$.*

Proof. Under transport by (4.36), this lemma is equivalent to proving that there is an element $\nu \in \mathrm{span}_{\mathbf{Z}} R(\hat{G}, \hat{T}')^\vee \cong \mathrm{span}_{\mathbf{Z}} R(G^*, T')$ which lies in the positive Weyl chamber determined by B' and is fixed under the action of θ^*. Let us prove this equivalent formulation. Recall from section 3.3 that the Borel subgroup $B' \supset T'$ is preserved by θ^*. Therefore this Borel subgroup fixes a Weyl chamber in $\mathrm{span}_{\mathbf{R}} R(\hat{G}, \hat{T}')^\vee$ which is preserved by the action of θ^*. Since $\mathrm{span}_{\mathbf{Z}} R(\hat{G}, \hat{T}')^\vee$ is a lattice of full rank, it has non-empty intersection with this Weyl chamber. Let ν' be an element in this intersection. As the automorphism θ^* has finite order on $X^*(\hat{T})$, we may define $\nu = \sum_{j=1}^{|\theta^*|} (\theta^*)^j(\nu')$ with the desired properties. $\square$

Let us fix ν as in Lemma 5.2. After possibly replacing it by some positive integer multiple we have $\mathrm{Re}\langle \mu + \nu, \alpha^\vee \rangle > 0$ for all $\alpha \in R(B, S)$, i.e. the element $\mu + \nu$ is $\hat{G}$-regular. This takes care of half of Assumption 5 in section 4. The other half requires an understanding of the action of σ on ν. We may represent the $\delta\theta$-invariance of ν by $\nu \in X_*(\hat{S}^{\delta\theta})$. Since $\hat{S}^{\delta\theta}/(\hat{S}^{\delta\theta})^0$ is finite, we may again replace ν by some positive integer multiple and assume without loss of generality that $\nu \in X_*((\hat{S}^{\delta\theta})^0)$. There is an isomorphism $X_*((\hat{S}^{\delta\theta})^0) \cong X^*(S_{\delta\theta})$, where $S_{\delta\theta} = S/(1-\delta\theta)S$, and a surjection $(S^{\delta\theta})^0 \to S_{\delta\theta}$ (see proof of Lemma 12 [Mez12]). Consequently, there is an injection $X^*(S_{\delta\theta}) \hookrightarrow X^*((S^{\delta\theta})^0)$. We may identify $\nu \in X_*((\hat{S}^{\delta\theta})^0)$ with its image under the map

$$X_*((\hat{S}^{\delta\theta})^0) \hookrightarrow X^*((S^{\delta\theta})^0) \tag{5.1}$$

of Γ-modules. By the θ-ellipticity of δ, the automorphism σ acts as inversion on $X^*((S^{\delta\theta})^0)$ modulo $X^*(Z_G^0)$ so that $\sigma(\nu) = -\nu$. The decomposition

$$X^*(S) \otimes \mathbf{R} \cong (X^*(S_a) \otimes \mathbf{R}) \oplus (X^*(S_d) \otimes \mathbf{R})$$

(8.15 [Bor91]) allows us to identify ν with its restriction $\nu_{|\mathfrak{s}_a}$ (*cf.* (4.2)). As before, we may assume that

$$\mathrm{Re}\langle (\mu+\nu)_{|\mathfrak{s}_a}, \alpha^\vee \rangle = \mathrm{Re}\langle \mu_{|\mathfrak{s}_a} + \nu, \alpha^\vee \rangle > 0$$

for all $\alpha \in R(B,S)$, so that the element $(\mu+\nu)_{|\mathfrak{s}_a}$ is $\hat{G}$-regular. At this point we have shown that Assumption 5 of section 4 holds for $\mu+\nu$.

We turn to the construction of matching admissible homomorphisms φ^ν and $\varphi^\nu_{H_1}$ which satisfy the remaining assumptions of section 4. First, since $\sigma(\nu) = -\nu$ is in the root lattice, it is easily verified that the pair $\mu+\nu, \lambda \in X_*(\hat{S}) \otimes \mathbf{C}$ corresponds to an admissible homomorphism $\varphi^\nu : W_{\mathbf{R}} \to {}^LG$ with $\varphi^\nu(\sigma) = \varphi(\sigma)$ (§4 [Mez12]). The pair also corresponds to a quasicharacter $\Lambda(\mu+\nu-\iota_M, \lambda)$ of $S(\mathbf{R})$ ((18) [Mez12]).

Lemma 5.3. *The image of $W_{\mathbf{R}}$ under φ^ν is not contained in a proper parabolic subgroup of ${}^LM$.*

Proof. In this proof we identify ${}^LS$ with the group $\langle \mathcal{T}, \varphi^\nu(W_{\mathbf{R}})\rangle \subset {}^LM$ so that $\nu \in \mathrm{span}_{\mathbf{Z}} R(\hat{G},\mathcal{T})^\vee$ and $\varphi^\nu(\sigma)\cdot\nu = -\nu$. Suppose ${}^LP$ is a parabolic subgroup of ${}^LM$ (as in §3.3 [Bor79]) containing $\varphi^\nu(W_{\mathbf{R}})$. Fix a Borel subgroup of ${}^LM$ containing $\mathcal{T}$. There exists $x \in \hat{M}$ such that $x\,{}^LPx^{-1}$ is standard with respect to this Borel subgroup (Theorem 15.4.6 [Spr98]) and normalizes $\mathcal{T}$ (Theorem 6.4.1). Without loss of generality, we may assume then that ${}^LP$ is standard. Let the connected component of ${}^LP$ equal $P(\beta)$ as in the proof of Proposition 8.4.5 [Spr98]. We may assume that $\beta \in \mathrm{span}_{\mathbf{Z}} R(\hat{M},\mathcal{T})^\vee$. Since $P(\beta)$ is standard, we have $\beta \geq 0$ relative to the positive system determined by the fixed Borel subgroup. Similarly, since

$$P(\beta) = \varphi^\nu(\sigma)P(\beta)\varphi^\nu(\sigma)^{-1} = P(\varphi^\nu(\sigma)\cdot\beta)$$

we have that $\varphi^\nu(\sigma)\cdot\beta \geq 0$. By the regularity of ν and

$$\langle \nu^\vee, \beta + \varphi^\nu(\sigma)\cdot\beta\rangle = \langle \varphi^\nu(\sigma)\cdot\nu^\vee, \beta+\varphi^\nu(\sigma)\cdot\beta\rangle = -\langle \nu^\vee, \beta+\varphi^\nu(\sigma)\cdot\beta\rangle$$

we deduce that $\beta = -\varphi^\nu(\sigma)\cdot\beta$. Since $\varphi^\nu(\sigma)\cdot\beta$ is both non-negative and non-positive, we conclude in turn that $\beta = 0$, $P(\beta) = \hat{M}$ and ${}^LP = {}^LM$. $\square$

By Lemma 5.3 we know that the L-packet $\Pi_{\varphi^\nu,M}$ consists of essentially square-integrable representations of $M(\mathbf{R})$ (§3 [Lan89], §4.1 [Mez12]). The central character of the representations in $\Pi_{\varphi^\nu,M}$ differs from the unitary central character of the representations in $\Pi_{\varphi,M}$ by the restriction of $\Lambda(\mu+\nu-\iota_M,\lambda)\Lambda(\mu-\iota_M,\lambda)^{-1}$ to $Z_M(\mathbf{R})$. This restriction depends only on the restriction of $\nu \in X_*(\hat{S})\otimes\mathbf{C} \cong X^*(S)\otimes\mathbf{C}$ to $Z_M \subset S$ (§9 [Bor79]). In order to show that Assumption 4 of section 4 holds for the central character of $\Pi_{\varphi^\nu,M}$ it suffices to show that the restriction of $\nu \in X^*(S)\otimes\mathbf{R}$ to the split componenet of Z_M is trivial. This is true, as the split component of Z_M is contained in S_d and the map $(1-\sigma)$ annihilates $X^*(S_d)$ (§8.15 [Bor91]), so that

$$\nu_{|S_d} = \frac{1-\sigma}{2}(\nu)_{|S_d} = \frac{1-\sigma}{2}(\nu_{|S_d}) = 0.$$

Thus far, see Assumptions 2, 4 and 5 of section 4 hold for φ^ν, and enough of Assumption 3 has been shown to hold to conclude that the L-packet Π_{φ^ν} is comprised of fundamental series representations (see 4.1). The θ-stability of the L-packet Π_{φ^ν} follows from the $\delta\theta$-invariance of ν ((136) [Mez12]). This means Assumption 6 of section 4 is satisfied, for ω is assumed to be trivial.

It remains to construct and admissible homomorhpism $\varphi^\nu_{H_1}$ such that Assumptions 1 and 3 hold. For this, we return to viewing ν as an element of $X^*((S_{\delta\theta})^0)$ as in (5.1). Isomorphism (4.36) passes to an isomorphism of $S_{\delta\theta}$ with $T'_{\theta *}$. Let ν_{H_1} be the image of ν under the composition of the isomorphisms $X^*(S_{\delta\theta}) \cong X^*(T'_{\theta *})$ and (3.6). By (3.4) we may regard ν_{H_1} as an element in $X_*(\hat{T}_{H_1})$. The positivity of ν with respect to the Borel subgroup B transfers to the postivity of ν_{H_1} with respect to the Borel subgroup B_H. Regarding ν as a $\hat{\theta}$-invariant element of $\mathrm{span}_{\mathbf{Z}} R(\hat{G},\mathcal{T})$ we may identify it with an element in $\mathrm{span}_{\mathbf{Z}} R_{\mathrm{res}}(\hat{G},\mathcal{T})$. This implies that ν_{H_1} is an element of $\mathrm{span}_{\mathbf{Z}} R(\hat{H}_1,\hat{T}_{H_1})$ ((1.3.4) [KS99], (137) [Mez12]). Let $\mu_{H_1},\lambda_{H_1} \in X_*(\hat{T}_{H_1})\otimes\mathbf{C}$ be a defining pair for φ_{H_1}. Then the pair $\mu_{H_1}+\nu_{H_1},\lambda_{H_1} \in X_*(\hat{T}_{H_1})\otimes\mathbf{C}$ defines an admissible homomorphism $\varphi^\nu_{H_1} : W_{\mathbf{R}} \to {}^L H_1$ for precisely the same reasons that the pair $\mu+\nu,\lambda$ defined the admissible homomophism φ^ν earlier on. Furthermore, Lemma 5.3 remains valid for $\varphi^\nu_{H_1}$, seeing as ν_{H_1} is a regular element in $\mathrm{span}_{\mathbf{Z}} R(\hat{H}_1,\hat{T}_{H_1})$ and T_{H_1} is elliptic (Corollary 3 [Mez12]). Thus, Assumption 1 of section 4 holds for $\varphi^\nu_{H_1}$. Finally, Assumption 3 of section 4 holds by virtue of the definition of $(\varphi^\nu)^*$ as $\xi\circ\xi_{H_1}^{-1}\circ\varphi^\nu_{H_1}$ (§6 [Mez12]) and the definition ν_{H_1}. Indeed, the image of $\mu_{H_1}+\nu_{H_1}$ under the maps induced by $\xi\circ\xi_{H_1}^{-1}$ corresponds to $\mu+\nu$ by the very construction of ν_H. This completes our task of constructing matching admissible homomorphisms φ^ν and $\varphi^\nu_{H_1}$ which satisfy all of the six assumptions of section 4.

5.2 Coherent continuation to limit of fundamental series representations

In what follows, we describe the relationship between the L-packets of φ and those of the shifted admissible homomorphism φ^ν. Our presentation follows §7

[Mez12]. The irreducible representations $\Pi_{\varphi^\nu,M}$ are parameterized by $\Omega(M,S)/\Omega_{\mathbf{R}}(M,S)$. Each irreducible representation in $\Pi_{\varphi,M}$ is an essential limit of discrete series obtained via Zuckerman tensoring a unique representation in $\Pi_{\varphi^\nu,M}$ ((1.10) [KZ84]). The converse is not always true, for the process of Zuckerman tensoring may result in a zero module. To explain these relationships better, let $w \in \Omega(M,S)/\Omega_{\mathbf{R}}(M,S)$ and $\varpi_{w^{-1}\Lambda} \in \Pi_{\varphi^\nu,M}$ be as in Proposition 4.12. Denote the distribution character of the representation obtained from $\varpi_{w^{-1}\Lambda}$ through Zuckerman tensoring by $\Theta(w^{-1}\mu, \lambda, w^{-1}\cdot \hat{B})$. Then the set of characters of the irreducible representations in $\Pi_{\varphi,M}$ is equal to the non-zero characters in

$$\{\Theta(w^{-1}\mu, \lambda, w^{-1}\cdot \hat{B}) : w \in \Omega(M,S)/\Omega_{\mathbf{R}}(M,S)\}.$$

Using Hecht-Schmid identities, one may parametrize this non-zero subset by taking only those w in $\Omega(M,S)/\Omega_{\mathbf{R}}(M,S)$ which lie in a subset of the form

$$\langle w_{\alpha_1^\vee}, \ldots, w_{\alpha_r^\vee}\rangle\, \Omega(M_1, S_{M_1})\Omega_{\mathbf{R}}(M,S)/\Omega_{\mathbf{R}}(M,S). \tag{5.2}$$

((142) [Mez12], page 408 [She82]). Here, S_{M_1} is an elliptic torus in M_1 obtained through Cayley transforms from S, and $\alpha_1^\vee, \ldots, \alpha_r^\vee$ are the positive real roots in $R(M, S_{M_1})$ (Lemma 31 [Mez12]). The Weyl group elements in (5.2) which belong to $\Omega(M, S_{M_1})$ are identified with their images in $\Omega(M,S)$ through the Cayley transforms.

The irreducible representations in Π_φ and Π_{φ^ν} are the irreducible subrepresentations of the representations induced from $\Pi_{\varphi,M}$ and $\Pi_{\varphi^\nu,M}$ respectively. In the present context parabolic induction and Zuckerman tensoring commute with one another (Corollary 5.9 [SV80]) and produce irreducible representations (when non-zero) (Theorem 5.15 [SV80]). Hence, parabolic induction furnishes a bijection between $\Pi_{\varphi,M}$ and Π_φ, just as it does between $\Pi_{\varphi^\nu,M}$ and Π_{φ^ν}. We may write the characters of the representations occurring in Π_φ as

$$\mathrm{ind}_{P(\mathbf{R})}^{G(\mathbf{R})}\, \Theta(w^{-1}\mu, \lambda, w^{-1}\cdot \hat{B}), \tag{5.3}$$

where w lies in (5.2). We call the representations corresponding to these characters *limit of fundamental series* representations. They are special cases of *limit of generalized principal series* representations given on *p.* 265 §5 [SV80].

This is an adequate description of Π_φ for our purposes. However, only the θ-stable representations in Π_φ contribute to twisted spectral transfer. Arguing as in (4.17), it is evident these representations are given by induction from the $\delta\theta$-stable representations in $\Pi_{\varphi,M}$. According to Proposition 4 [Mez12], characters of the latter are of the form

$$\Theta(w_1^{-1}w_2^{-1}\cdot \mu, \lambda, w_1^{-1}w_2^{-1}\cdot \hat{B}) \tag{5.4}$$

where w_1 runs through $(\langle w_{\alpha_1^\vee}, \ldots w_{\alpha_r^\vee}\rangle \Omega_{\mathbf{R}}(M,S)/\Omega_{\mathbf{R}}(M,S))^{\delta\theta}$ and w_2 runs through $(\Omega(M_1,S_{M_1})/\Omega_{\mathbf{R}}(M_1,S_{M_1}))^{\delta\theta}$.

To define *twisted* characters, we apply some results of [Duc02] to the algebraic group $G(\mathbf{R}) \rtimes \langle\theta\rangle$. Suppose that $\pi' \in \Pi_{\varphi^\nu}$ and U' is a choice of intertwining operator satisfying

$$\mathsf{U}' \circ \pi'(x) = (\pi')^{\theta}(x) \circ \mathsf{U}',\ x \in G(\mathbf{R})$$

(*cf.* (4.29)) with the normalization of section 4.3. Then π' lifts to a representation $\bar{\pi}'$ of $G(\mathbf{R}) \rtimes \langle\theta\rangle$ by setting $\bar{\pi}'(\theta) = \mathsf{U}'$. The irreducible representation $\bar{\pi}'$ is tempered and therefore has a distribution character $\Theta_{\bar{\pi}'}$ which one may identify with a locally integrable function on the regular elements of $G(\mathbf{R}) \rtimes \langle\theta\rangle$ (§3 [Bou87]). By construction, we have

$$\Theta_{\pi',\mathsf{U}'}(x\theta) = \Theta_{\bar{\pi}'}(x\theta) \tag{5.5}$$

for all regular $x\theta$ in $G(\mathbf{R}) \rtimes \langle\theta\rangle$. In other words, the twisted character is the restriction of the character of $\bar{\pi}$ to the component $G(\mathbf{R})\theta$. To keep the proliferation of subscripts at bay, we will abusively write $\Theta_{\bar{\pi}'}$ in place of $\Theta_{\pi',\mathsf{U}'}$

The representation $\bar{\pi}'$ falls under the classification of Theorem 9.6 [Duc02][4]. By Proposition 12.3 (c) [Duc02], one may extend the process of Zuckerman tensoring to $\bar{\pi}'$ and by means of that recover a representation $\bar{\pi}$ of $G(\mathbf{R}) \rtimes \langle\theta\rangle$, provided that the result is non-zero. The representation $\bar{\pi}$ is an extension of Zuckerman tensoring on $G(\mathbf{R})$ in the sense that the restriction of $\bar{\pi}$ to $G(\mathbf{R})$ is a limit of fundamental series representation π obtained from π' as above. It follows, that the representation π is θ-stable, with intertwining operator U_{π} defined to be $\bar{\pi}(\theta)$. We define the twisted character $\Theta_{\pi,\mathsf{U}_{\pi}}$ from $\Theta_{\bar{\pi}}$ as in (5.5).

We combine these results on $G(\mathbf{R}) \rtimes \langle\theta\rangle$ with the previous results on $G(\mathbf{R})$ as follows. Suppose $w = w_1 w_2$, where w_1 and w_2 are as in (5.4). Set $\pi_{w^{-1}\nu} = \mathrm{ind}_{P(\mathbf{R})}^{G(\mathbf{R})} \varpi_{w^{-1}\Lambda}$ for $\varpi_{\Lambda} \in \Pi_{\varphi^{\nu},M}$ and all $w = w_1 w_2$. Using the intertwining operators of section 4.3 we may extend $\pi_{w^{-1}\nu}$ to a representation $\bar{\pi}_{w^{-1}\nu}$ on $G(\mathbf{R}) \rtimes \langle\theta\rangle$. Let $\Psi_{w^{-1}\cdot\mu}^{w^{-1}\cdot(\mu+\nu)} \bar{\pi}_{w^{-1}\nu}$ denote the representation, obtained by Ducloux's extension of Zuckerman tensoring, whose restriction to $G(\mathbf{R})$ has character (5.3). This was denoted simply as $\bar{\pi}$ above. We write $\Psi_{w^{-1}\cdot\mu}^{w^{-1}\cdot(\mu+\nu)} \Theta_{\bar{\pi}_{w^{-1}\nu}}$ for the character of $\Psi_{w^{-1}\cdot\mu}^{w^{-1}\cdot(\mu+\nu)} \bar{\pi}_{w^{-1}\nu}$. This restricts to the twisted character $\Theta_{\pi,\mathsf{U}_{\pi}}$ above. When θ is the identity automorphism, we see that $\Psi_{w^{-1}\cdot\mu}^{w^{-1}\cdot(\mu+\nu)} \Theta_{\pi_{w^{-1}\nu}}$ is equal to (5.3). A precise expansion of $\Psi_{w^{-1}\cdot\mu}^{w^{-1}\cdot(\mu+\nu)} \Theta_{\bar{\pi}_{w^{-1}\nu}}$ on regular elements is given in Proposition 12.3 (c) [Duc02].

We are almost in the position to apply coherent continuation to the right-hand side of identity (4.3). Let $\boldsymbol{\varphi^{\nu}}$ and $\boldsymbol{\varphi^{\nu}_{H_1}}$ be the respective L-parameters of φ^{ν} and $\varphi^{\nu}_{H_1}$. We may now replace φ with φ^{ν} in the right-hand side of the spectral transfer identity (4.3). Let us define

$$\Delta(\boldsymbol{\varphi_{H_1}}, \Psi_{w^{-1}\cdot\mu}^{w^{-1}\cdot(\mu+\nu)} \pi_{w^{-1}\nu}) = \Delta(\boldsymbol{\varphi^{\nu}_{H_1}}, \pi_{w^{-1}\cdot\nu}) \tag{5.6}$$

for $w = w_1 w_2$ as in (5.4). This definition does not depend on the choice of ν (*cf.* (4.45)-(4.47)). If we apply coherent continuation to each of the twisted

[4]Its form is given in Definition 8.3 [Duc02] with $M_0' = G(\mathbf{R})$.

characters on the right, we obtain

$$
\begin{aligned}
& \sum_{w \in (\Omega(M,S)/\Omega_{\mathbf{R}}(M,S))^{\delta\theta}} \Delta(\boldsymbol{\varphi}^{\boldsymbol{\nu}}_{\boldsymbol{H_1}}, \pi_{w^{-1}\cdot\nu})\, \Psi^{w^{-1}\cdot(\mu+\nu)}_{w^{-1}\cdot\mu} \Theta_{\bar{\pi}_{w^{-1}\cdot\nu}} \\
= & \sum_{w_1, w_2} \Delta(\boldsymbol{\varphi}^{\boldsymbol{\nu}}_{\boldsymbol{H_1}}, \pi_{(w_1w_2)^{-1}\cdot\nu})\ \Psi^{(w_1w_2)^{-1}\cdot(\mu+\nu)}_{(w_1w_2)^{-1}\cdot\mu} \Theta_{\bar{\pi}_{(w_1w_2)^{-1}\cdot\nu}} \\
= & \sum_{\pi \in \Pi_\varphi} \Delta(\boldsymbol{\varphi}^{\boldsymbol{\nu}}_{\boldsymbol{H_1}}, \pi)\ \Theta_{\bar{\pi}} \\
= & \sum_{\pi \in \Pi_\varphi} \Delta(\boldsymbol{\varphi}^{\boldsymbol{\nu}}_{\boldsymbol{H_1}}, \pi)\ \Theta_{\pi, \mathsf{U}_\pi}
\end{aligned}
\tag{5.7}
$$

One may apply coherent continuation to the left-hand side of (4.3) in the same manner, so that

$$
\sum_{w \in \Omega(H_1, T_{H_1})/\Omega_{\mathbf{R}}(H_1, T_{H_1})} \Psi^{w^{-1}\cdot(\mu_{H_1}+\nu_{H_1})}_{w^{-1}\cdot\mu_{H_1}} \Theta_{\pi_{w^{-1}\cdot\nu_{H_1}}} = \sum_{\pi_{H_1} \in \Pi_{\varphi_{H_1}}} \Theta_{\pi_{H_1}}. \tag{5.8}
$$

We would like the right-hand sides of (5.7) and (5.8) to match under geometric transfer. This matching follows from a comparison of character values on $S^{\delta\theta}(\mathbf{R})\delta\theta$ and the twisted version of Harish Chandra's Uniqueness Theorem (Theorem 15.1 [Ren97]) as in the essentially square integrable case. The argument relies on character expansions of the limits of fundamental series (Lemma 5.5 [SV80], Proposition 12.3 (c) [Duc02]).

Theorem 5.4. *Suppose ω is trivial and θ is of finite order. Then*

$$
\int_{H_1(\mathbf{R})/Z_1(\mathbf{R})} f_{H_1}(h) \sum_{\pi_{H_1} \in \Pi_{\boldsymbol{\varphi}_{\boldsymbol{H_1}}}} \Theta_{\pi_{H_1}}(h)\, dh = \sum_{\pi \in \Pi_\varphi} \Delta(\boldsymbol{\varphi}_{\boldsymbol{H_1}}, \pi)\, \Theta_{\pi, \mathsf{U}_\pi}(f)
$$

for all $f \in C_c^\infty(G(\mathbf{R})\theta)$.

Proof. The proof is identical to that of Theorem 3 [Mez12]. □

6 A reduction to finite order automorphisms

In this section θ is an algebraic automorphism of G which is defined over $\mathbf{R}$ and has finite action on Z_G.

Proposition 6.1. *There exists $g_0 \in G_{\mathrm{der}}(\mathbf{R})$ such that $\mathrm{Int}(g_0)\theta$ has finite order and preserves a maximally $\mathbf{R}$-split maximal torus T of G.*

Proof. Since θ has finite order when restricted to Z_G, and $G = Z_G G_{\mathrm{der}}$ (Corollary 8.1.6 [Spr98]), we may assume without loss of generality that $G = G_{\mathrm{der}}$. Recall that in splitting (3.1), the maximal torus T is defined over $\mathbf{R}$ and contains a maximal $\mathbf{R}$-split torus of G. In other words, the split component of T

is a maximal $\mathbf{R}$-split torus in G. The split component of the maximal torus $\theta(T)$ is also a maximal $\mathbf{R}$-split torus of G, as θ is defined over $\mathbf{R}$. Therefore there exists $x_1 \in G(\mathbf{R})$ such that $\mathrm{Int}(x_1)\theta$ preserves T (Theorem 15.2.6 [Spr98]). We may therefore assume without loss of generality that θ itself preserves T. Splitting (3.1) affords a decomposition of $\mathrm{Aut}(G)$ as a split semidirect product of the group of inner automorphisms and the group of graph automorphisms of the Dynkin diagram (Corollary 2.14 [Spr79]). As the latter group is finite, there exists some positive integer ℓ_1 and an element $x_2 \in G$ such that $\theta^{\ell_1} = \mathrm{Int}(x_2)$. Since θ preserves T, so does $\mathrm{Int}(x_2)$ and this is the same as saying that x_2 is a representative of an element in $\Omega(G,T)$. The Weyl group $\Omega(G,T)$ is finite so that for some positive integer ℓ_2 we have $\theta^{\ell_1\ell_2} = \mathrm{Int}(x_3)$ where $x_3 = x_2^{\ell_2}$ belongs to T. The automorphism $\theta^{\ell_1\ell_2}$ commutes with σ and consequently $\mathrm{Int}(\sigma(x_3)x_3^{-1})$ is the identity automorphism. This implies that $\sigma(x_3)x_3^{-1}$ lies in the centre of the semisimple group G. The centre is finite, so there exists a positive integer ℓ_3 such that

$$\sigma(x_3^{\ell_3})x_3^{-\ell_3} = (\sigma(x_3)x_3^{-1})^{\ell_3} = 1.$$

This equation implies that $x_4 = x_3^{\ell_3} \in T(\mathbf{R})$. Similarly, $\mathrm{Int}(x_4) = \theta^{\ell_1\ell_2\ell_3}$ commutes with θ and this in turn implies that $\mathrm{Int}(\theta(x_4)x_4^{-1})$ is the identity automorphism and $\theta(x_4^{\ell_4}) = x_4^{\ell_4}$ for some positive integer ℓ_4. Set $x_5 = x_4^{\ell_4} \in T^\theta(\mathbf{R})$. Finally, being the real points of an algebraic group, the group $T^\theta(\mathbf{R})$ has finitely many connected components as a real manifold. Therefore, there is a positive integer ℓ_5 such that $y = x_5^{\ell_5}$ belongs to $T^\theta(\mathbf{R})^0$. Set $\ell = \ell_1 \cdots \ell_5$. Then $\theta^\ell = \mathrm{Int}(y)$ and there exists $Y \in \mathfrak{t}^\theta$ such that $\exp(Y) = y$. Let $g_0 = \exp(-\frac{1}{\ell}Y) \in T^\theta(\mathbf{R})^0$. Clearly,

$$(\mathrm{Int}(g_0)\theta)^\ell = \mathrm{Int}(g_0^\ell)\,\theta^\ell = \mathrm{Int}(y^{-1})\theta^\ell$$

is the identity automorphism. □

Corollary 6.2. *Let $g_0 \in G(\mathbf{R})$ be as in Proposition 6.1. Then the cyclic subgroup $\langle (g_0\theta)^\ell \rangle$ of $G(\mathbf{R}) \rtimes \langle\theta\rangle$ generated by $(g_0\theta)^\ell$ centralizes $G(\mathbf{R})$. Moreover, $\langle (g_0\theta)^\ell \rangle G(\mathbf{R})^0$ has finite index in $G(\mathbf{R}) \rtimes \langle\theta\rangle$.*

Proof. By hypothesis $(\mathrm{Int}(g_0)\theta)^\ell$ is the trivial automorphism of $G(\mathbf{R})$. An immediate consequence of this is that $\langle (g_0\theta)^\ell \rangle$ is centralizes $G(\mathbf{R})$. The second assertion follows from the fact that $G(\mathbf{R})/G(\mathbf{R})^0$ is finite ((c) (i) §C 24 V [Bor91]). □

Corollary 6.2 is significant in that it asserts that the Lie group $G(\mathbf{R}) \rtimes \langle\theta\rangle$ satisfies the requirements for Bouaziz' work on characters to apply ((*) §1.2 [Bou87]). In fact, one may prove twisted spectral transfer for limits of fundamental series by replacing θ everywhere in sections 4-5 by $\mathrm{Int}(g_0)\theta$. One might then hope that the following claim is true:

($\star$) Suppose $g_0 \in G(\mathbf{R})$ and spectral transfer holds for $\mathrm{Int}(g_0)\theta$. Then spectral transfer holds for θ.

If this claim is true then Proposition 6.1 implies that twisted spectral transfer holds for limits of fundamental series representations when θ is merely finite on the centre of G. We shall spend the rest of this section making this claim precise and giving justification for its truth. Many of the ideas that follow are also present in §11 [She].

Our starting point is a choice of endoscopic data $(H, \mathcal{H}, \mathsf{s}, \xi)$ for $(G, \theta, \mathbf{a})$ (section 3.2). We fix an L-parameter $\boldsymbol{\varphi_{H_1}}$ relative to a z-pair (H_1, ξ_{H1}) such that $\Pi_{\varphi_{H_1}}$ consists of essentially tempered representations whose central character is determined by λ_{Z_1} on $Z_1(\mathbf{R})$. We assume that the endoscopic data render a corresponding L-parameter $\boldsymbol{\varphi}$ for G, whose L-packet Π_φ also consists of essentially tempered representations. We assume that $\omega \otimes (\Pi_\varphi \circ \theta) = \Pi_\varphi$, and that there is a strongly θ-regular element $\delta \in G(\mathbf{R})$ which is has a norm $\gamma_1 \in H_1(\mathbf{R})$.

This backdrop places us in a context where spectral transfer is relevant with respect to twisting by θ. Now, fix $g_0 \in G(\mathbf{R})$. Our first undertaking is to furnish a backdrop for spectral transfer with respect to twisting by $\mathrm{Int}(g_0)\theta$. The key observation in this undertaking is that the passage from θ to the dual map $\hat{\theta}$ is insensitive to composition with inner automorphisms (section 3.1, *cf.* §1.3 [Bor79]). Indeed, $\hat{\theta}$ is obtained via an action on based root data and such actions are independent of composition by an inner automorphism. The upshot of this observation is that $\widehat{\mathrm{Int}(g_0)\theta} = \hat{\theta}$.

The definition of endoscopic data depends on $\hat{\theta}$, not θ per se. In consequence, the previous endoscopic data $(H, \mathcal{H}, \mathsf{s}, \xi)$ are also a choice of endoscopic data for $(G, \mathrm{Int}(g_0)\theta, \mathbf{a})$, and the L-parameter $\boldsymbol{\varphi_{H_1}}$ passes to $\boldsymbol{\varphi}$ as before. According to Lemma 2 [Mez12], twisted L-packets may be represented in terms of a representative cocycle $a \in \mathbf{a}$ and ${}^L\theta = \hat{\theta} \times 1_{W_\mathbf{R}}$, so that

$$\omega \otimes (\Pi_\varphi \circ \mathrm{Int}(g_0)\theta) = \Pi_{{}^L\theta \circ (a \cdot \varphi)} = \omega \otimes (\Pi_\varphi \circ \theta) = \Pi_\varphi.$$

It is simple to show that $\delta g_0^{-1} \in G(\mathbf{R})$ is a strongly $\mathrm{Int}(g_0)\theta$-regular element. We wish to prove that δg_0^{-1} has norm γ_1. To do this we must retrace the definitions of the maps in section 3.3. These maps are defined in terms of the endoscopic data and the automorphism θ^*. Replacing θ with $\mathrm{Int}(g_0)\theta$ does not have an effect on θ^* for the automorphism

$$\mathrm{Int}(g_\theta \psi(g_0)^{-1})\, \psi \mathrm{Int}(g_0)\theta\psi^{-1} = \theta^*$$

preserves the splitting $(B^*, T^*, \{X^*\})$. However, this replacement *does* have an effect on g_θ. The effect is to replace g_θ with $g_\theta \psi(g_0)^{-1}$, and this affects the definition of (3.8). By substituting $g_\theta \psi(g_0)^{-1}$ in place of g_θ and δg_0^{-1} in place of δ in (3.8), we find that

$$\psi(\delta g_0^{-1})(g_\theta \psi(g_0)^{-1})^{-1} = \psi(\delta) g_\theta^{-1},$$

and the expression on the right is equal to the image of δ under the original map (3.8). Conjugating this equation by $g_{T'}$ (*cf.* (3.11)), it is evident that δg_0^{-1}

corresponds to δ^* under twisting by $\mathrm{Int}(g_0)\theta$ in the same way that δ corresponds to δ^* under twisting by θ.

We should also observe that the change from g_θ to $g_\theta\psi(g_0)^{-1}$ does not affect the Γ-equivariance of m. To ensure this equivariance, we have assumed in section 3.3 that g_θ has been chosen so that (3.9) holds. Substituting $g_\theta\psi(g_0)^{-1}$ in place of g_θ into (3.9) yields

$$\begin{aligned}
&g_\theta\psi(g_0)^{-1}u_\sigma\sigma((g_\theta\psi(g_0)^{-1})^{-1})\theta^*(u_\sigma)^{-1}\\
&\quad = \quad g_\theta\psi(g_0)^{-1}\left(\mathrm{Int}(u_\sigma)\sigma\psi(g_0)\right)u_\sigma\sigma(g_\theta^{-1})\theta^*(u_\sigma)^{-1}\\
&\quad = \quad g_\theta\psi(g_0)^{-1}\,\psi(\sigma(g_0))\,u_\sigma\sigma(g_\theta^{-1})\\
&\quad = \quad g_\theta u_\sigma\sigma(g_\theta^{-1})\theta^*(u_\sigma)^{-1}\in(1-\theta^*)Z_{G^*_{\mathrm{sc}}}.
\end{aligned}$$

This ensures the Γ-equivariance of m relative to twisting by $\mathrm{Int}(g_0)\theta$. The rest being the same, we conclude that δg_0^{-1} has norm γ_1.

Given that the backdrop for spectral transfer is barely perturbed by replacing θ with $\mathrm{Int}(g_0)\theta$, we examine the effect of this replacement on twisted characters. Clearly, given any $\pi\in\Pi_\varphi$ satisfying (4.29), it also satisfies

$$\pi(g_0)\mathsf{U}\circ\omega^{-1}(x)\pi(x)=\pi^{g_0\theta}(x)\circ\pi(g_0)\mathsf{U},\ x\in G(\mathbf{R}). \tag{6.1}$$

This presents us with the intertwining operator $\mathsf{U}_\pi^{g_0}=\pi(g_0)\mathsf{U}$ and the corresponding twisted character $\Theta_{\pi,\mathsf{U}_\pi^{g_0}}$ defined by

$$f\mapsto \mathrm{tr}\int_{G(\mathbf{R})}f(x)\,\pi(x)\,\pi(g_0)\mathsf{U}\,dx,\ f\in C_c^\infty(G(\mathbf{R})), \tag{6.2}$$

(*cf.* (4.29))[5]. Let R. denote the right regular representation of $G(\mathbf{R})$ on $C_c^\infty(G(\mathbf{R}))$. Then

$$\Theta_{\pi,\mathsf{U}_\pi^{g_0}}(f)=\int_{G(\mathbf{R})}f(x)\,\pi(xg_0)\mathsf{U}\,dx=\Theta_{\pi,\mathsf{U}_\pi}(\mathrm{R}_{g_0^{-1}}f),\ f\in C_c^\infty(G(\mathbf{R})), \tag{6.3}$$

by the invariance of the Haar measure (*cf.* (5.1) [DM08]).

There is a dual identity to (6.3) for twisted orbital integrals. The orbital integral of $f\in C_c^\infty(G(\mathbf{R}))$ at δg_0^{-1} when twisted by $\mathrm{Int}(g_0)\theta$ is

$$\int_{G^{\delta\theta}(\mathbf{R})\backslash G(\mathbf{R})}\omega(g)\,f(g^{-1}\delta g_0^{-1}\,g_0\theta(g)g_0^{-1})\,dg=\mathcal{O}_{\delta\theta}(\mathrm{R}_{g_0^{-1}}f). \tag{6.4}$$

Let us denote the geometric transfer factors with respect to twisting by $\mathrm{Int}(g_0)\theta$ by Δ^{g_0}. It follows for (many small and) purely formal reasons that we may take $\Delta^{g_0}(\gamma_1,\delta' g_0^{-1})=\Delta(\gamma_1,\delta')$ for any pair (γ_1,δ') appearing in (3.14). Combining this identity with (6.4), we deduce that the geometric transfer of f with respect to twisting by $\mathrm{Int}(g_0)\theta$ may be taken to equal $(\mathrm{R}_{g_0^{-1}}f)_{H_1}\in C^\infty(H_1(\mathbf{R}),\lambda_{Z_1})$.

[5]In this discusion, we find it less confusing to take test functions in $C_c^\infty(G(\mathbf{R}))$ instead of $C_c^\infty(G(\mathbf{R})\theta)$.

Let us summarize where we stand with regard to the claim ($\star$). In replacing the automorphism θ with $\mathrm{Int}(g_0)\theta$ we retain the same endoscopic backdrop. A strongly θ-regular element δ with norm γ_1 is replaced by δg_0^{-1} and the norm remains the same. The geometric transfer factors are changed accordingly.

The final missing piece in the notion of spectral transfer is the definition of spectral transfer factors. Let us suppose that these have been defined relative to twisting by $\mathrm{Int}(g_0)\theta$ and denote them by $\Delta^{g_0}(\boldsymbol{\varphi_{H_1}}, \pi)$ for $\pi \in \Pi_\varphi$. We *define*[6] the spectral transfer factors relative to twisting by θ by

$$\Delta(\boldsymbol{\varphi_{H_1}}, \pi) = \Delta^{g_0}(\boldsymbol{\varphi_{H_1}}, \pi), \ \pi \in \Pi_\varphi. \tag{6.5}$$

Given these relationships, to say that spectral transfer holds for $\mathrm{Int}(g_0)\theta$ is to say that for all $f \in C_c^\infty(G(\mathbf{R}))$ we have

$$\int_{H_1(\mathbf{R})/Z_1(\mathbf{R})} (\mathrm{R}_{g_0^{-1}} f)_{H_1}(h) \sum_{\pi_{H_1} \in \Pi_{\varphi_{H_1}}} \Theta_{\pi_{H_1}}(h)\, dh = \sum_{\pi \in \Pi_\varphi} \Delta^{g_0}(\boldsymbol{\varphi_{H_1}}, \pi)\, \Theta_{\pi, \mathsf{U}_\pi^{g_0}}(f)$$

or equivalently

$$\int_{H_1(\mathbf{R})/Z_1(\mathbf{R})} f_{H_1}(h) \sum_{\pi_{H_1} \in \Pi_{\varphi_{H_1}}} \Theta_{\pi_{H_1}}(h)\, dh = \sum_{\pi \in \Pi_\varphi} \Delta^{g_0}(\boldsymbol{\varphi_{H_1}}, \pi)\, \Theta_{\pi, \mathsf{U}_\pi^{g_0}}(\mathrm{R}_{g_0} f).$$

The content of claim ($\star$) is that the right-hand side of this equation may be replaced with

$$\sum_{\pi \in \Pi_\varphi} \Delta(\boldsymbol{\varphi_{H_1}}, \pi)\, \Theta_{\pi, \mathsf{U}_\pi}(f),$$

and this is an immediate consequence of (6.3) and (6.5).

This completes our discussion of ($\star$). When taken together with Proposition 6.1, it results in a reduction of twisted spectral transfer to the case that θ is finite on Z_G.

Theorem 6.3. *Suppose ω is trivial and θ is of finite order on Z_G. Then, under the assumptions of section 5, we have*

$$\int_{H_1(\mathbf{R})/Z_1(\mathbf{R})} f_{H_1}(h) \sum_{\pi_{H_1} \in \Pi_{\varphi_{H_1}}} \Theta_{\pi_{H_1}}(h)\, dh = \sum_{\pi \in \Pi_\varphi} \Delta(\boldsymbol{\varphi_{H_1}}, \pi)\, \Theta_{\pi, \mathsf{U}_\pi}(f) \tag{6.6}$$

for all $f \in C_c^\infty(G(\mathbf{R})\theta)$.

7 A reduction of spectral transfer through parabolic induction

In this section we investigate the compatibility of twisted spectral transfer with parabolic induction. We assume that the quasicharacter ω is trivial and that θ is an algebraic automorphism of G defined over $\mathbf{R}$.

[6] One could also establish an independent definition and prove that the spectral transfer factors are equal.

The principal assumption of this section is that $\bar{P}$ is an $\mathbf{R}$-parabolic subgroup of G which is preserved by θ. Let $\bar{P} = \bar{M}\bar{N}$ be a Levi decomposition of $\bar{P}$. It follows that θ preserves $\bar{N}$. We also assume that θ preserves $\bar{M}$, that θ has finite action on $Z_{\bar{M}}$, and preserves a maximal compact subgroup K of $G(\mathbf{R})$ (*cf.* Lemma 4.19).

Suppose π is an irreducible tempered representation of $\bar{M}(\mathbf{R})$ such that

$$\mathsf{U} \circ \pi(x) = \pi^{\theta}(x) \circ \mathsf{U},\ x \in \bar{M}(\mathbf{R}) \tag{7.1}$$

for a non-zero intertwining operator U. The representation $(\mathrm{ind}_{\bar{P}(\mathbf{R})}^{G(\mathbf{R})}\pi)^{\theta}$ is equivalent to $\mathrm{ind}_{\bar{P}(\mathbf{R})}^{G(\mathbf{R})}\pi^{\theta} \cong \mathrm{ind}_{\bar{P}(\mathbf{R})}^{G(\mathbf{R})}\pi$. Indeed, we may define an operator T on the functions ϕ in the representation space of $\mathrm{ind}_{\bar{P}(\mathbf{R})}^{G(\mathbf{R})}\pi$ by

$$(\mathsf{T}\phi)(g) = \mathsf{U}\,\phi(\theta^{-1}(g)),\ g \in G(\mathbf{R}).$$

The reader may easily verify that $\mathrm{ind}_{\bar{P}(\mathbf{R})}^{G(\mathbf{R})}\pi$ is stable under θ and satisfies

$$\mathsf{T} \circ \mathrm{ind}_{\bar{P}(\mathbf{R})}^{G(\mathbf{R})}\pi(x) = (\mathrm{ind}_{\bar{P}(\mathbf{R})}^{G(\mathbf{R})}\pi)^{\theta}(x) \circ \mathsf{T},\ x \in G(\mathbf{R}) \tag{7.2}$$

(*cf.* proof of Proposition 3.1 [Mez07] and Lemma 5 (i)-(ii) [DM08]).

We wish to compute the twisted character[7] $\Theta_{\mathrm{ind}_{\bar{P}(\mathbf{R})}^{G(\mathbf{R})}\pi,\mathsf{T}}$ defined by

$$f \mapsto \mathrm{tr} \int_{G(\mathbf{R})} f(x)\, \mathrm{ind}_{\bar{P}(\mathbf{R})}^{G(\mathbf{R})}\pi(x)\, \mathsf{T}\, dx,\ f \in C_c^{\infty}(G(\mathbf{R}))$$

in terms of the twisted character of π. This amounts to a twisted version of a well-known descent formula ((10.21) [Kna86]), and the techniques are entirely the same. We begin with the twisted descent formula for functions in $C_c^{\infty}(G(\mathbf{R}))$. Suppose $f \in C_c^{\infty}(G(\mathbf{R}))$ and $\bar{\mathfrak{n}}$ is the real Lie algebra of the unipotent group $\bar{N}(\mathbf{R})$. Define

$$f^{(\bar{P})}(x) = |\det \mathrm{Ad}(x)_{|\bar{\mathfrak{n}}}|^{1/2} \int_K \int_{\bar{N}(\mathbf{R})} f(kxu\theta(k^{-1}))\, dn\, dk,\ x \in \bar{M}(\mathbf{R}).$$

This defines a smooth and compactly supported function on $\bar{M}(\mathbf{R})$ (*cf.* (10.22) [Kna86]). The expected identities for twisted geometric descent are given in the next lemma.

Lemma 7.1. *Suppose $f \in C_c^{\infty}(G(\mathbf{R}))$. Then*

$$f^{(\bar{P})}(x) = |\det \mathrm{Ad}(x)_{|\bar{\mathfrak{n}}}|^{1/2} |\det(\mathrm{Ad}(x\theta)^{-1} - 1)_{|\bar{\mathfrak{n}}}| \int_K \int_{\bar{N}(\mathbf{R})} f((nk)^{-1}x\theta(nk))\, dn\, dk \tag{7.3}$$

for all $x \in \bar{M}(\mathbf{R})$. Furthermore, for any strongly θ-regular element $\delta \in \bar{M}(\mathbf{R})$

$$\mathcal{O}_{\delta\theta}(f) = |\det(1 - \mathrm{Ad}(\delta\theta))_{|\mathfrak{g}/\mathfrak{m}}|^{-1/2}\ \mathcal{O}_{\delta\theta}(f^{(\bar{P})}). \tag{7.4}$$

[7] In this section it will again be less confusing to take test functions in $C_c^{\infty}(G(\mathbf{R}))$ due to the simplification of the geometric computations. The change is purely cosmetic.

Proof. Suppose $x \in \bar{M}(\mathbf{R})$. By making the change of variable

$$n \mapsto x^{-1}nx\theta(n^{-1}) = \mathrm{Int}(x^{-1})(n)\,\theta(n^{-1})$$

in the definition of $f^{(\bar{P})}$ one computes that $f^{(\bar{P})}(x)$ is equal to
(7.5)

$$|\det \mathrm{Ad}(x)_{|\bar{\mathfrak{n}}}|^{1/2} |\det(\mathrm{Ad}(x)^{-1} - \mathrm{Ad}(\theta))_{|\bar{\mathfrak{n}}}| \int_K \int_{\bar{N}(\mathbf{R})} f(knx\theta(n^{-1}k^{-1}))\,dn\,dk$$

(*cf.* Lemma 10.16 [Kna86]). By $\mathrm{Ad}(\theta)$ we mean the adjoint action in the Lie group $G(\mathbf{R}) \rtimes \langle\theta\rangle$. Since θ is assumed to be finite on $Z_{\bar{M}}$, it is also finite on $Z_G \subset Z_{\bar{M}}$. By Corollary 6.2 the Lie group $G(\mathbf{R}) \rtimes \langle\theta\rangle$ satisfies the hypotheses of [Bou87]. Thus, Lemma 1.6.1 [Bou87] applies to $G(\mathbf{R}) \rtimes \langle\theta\rangle$ and tells us that $\mathrm{Ad}(\theta)$, which preserves K, has eigenvalues which are all of modulus one. Equation (7.3) follows by multiplying $|\det(\mathrm{Ad}(x)^{-1} - \mathrm{Ad}(\theta))_{|\bar{\mathfrak{n}}}|$ by $|\det \mathrm{Ad}(\theta)_{|\bar{\mathfrak{n}}}|^{-1} = 1$ in (7.5).

To prove (7.4), suppose $\delta \in \bar{M}(\mathbf{R})$ is strongly θ-regular. Then δ is strongly θ-regular in $\bar{M}$ and $\bar{M}^{\delta\theta} = G^{\delta\theta}$. For compatible choices of Haar measures, we have

$$\begin{aligned}
\mathcal{O}_{\delta\theta}(f) &= \int_{G^{\delta\theta}(\mathbf{R})\backslash G(\mathbf{R})} f(g^{-1}\delta\theta(g))\,dg \\
&= \int_{\bar{M}^{\delta\theta}(\mathbf{R})\backslash \bar{M}(\mathbf{R})} \int_{\bar{N}(\mathbf{R})} \int_K f(k^{-1}n^{-1}m^{-1}\delta\,\theta(mnk))\,dk\,dn\,dm \\
&= \int_{\bar{M}^{\delta\theta}(\mathbf{R})\backslash \bar{M}(\mathbf{R})} |\det \mathrm{Ad}(x)_{|\bar{\mathfrak{n}}}|^{-1/2} |\det(\mathrm{Ad}(x\theta)^{-1} - 1)_{|\bar{\mathfrak{n}}}|^{-1} f^{(\bar{P})}(m^{-1}\delta\theta(m))\,dm
\end{aligned}$$

where $x = m^{-1}\delta\theta(m)$. Recall that $|\det \mathrm{Ad}(\theta)| = 1$ so that

$$\begin{aligned}
|\det \mathrm{Ad}(x)| &= |\det \mathrm{Ad}(m^{-1}\delta\theta(m))| \\
&= |\det \mathrm{Ad}(m^{-1}\delta\theta m\theta^{-1})| \\
&= |\det \mathrm{Ad}(m^{-1})||\det \mathrm{Ad}(\delta\theta)||\det \mathrm{Ad}(m)||\det \mathrm{Ad}(\theta^{-1})| \\
&= |\det \mathrm{Ad}(\delta\theta)|.
\end{aligned}$$

Similarly, $\mathrm{Ad}(x\theta) = \mathrm{Ad}(m^{-1})\mathrm{Ad}(\delta\theta)\mathrm{Ad}(m)$. As a result,

$$\begin{aligned}
&|\det \mathrm{Ad}(x)_{|\bar{\mathfrak{n}}}|^{-1/2} |\det(\mathrm{Ad}(x\theta)^{-1} - 1)_{|\bar{\mathfrak{n}}}|^{-1} \\
&= |\det \mathrm{Ad}(\delta\theta)_{|\bar{\mathfrak{n}}}|^{-1/2} |\det(\mathrm{Ad}(\delta\theta)^{-1} - 1)_{|\bar{\mathfrak{n}}}|^{-1} \\
&= |\det \mathrm{Ad}(\delta\theta)_{|\bar{\mathfrak{n}}}|^{1/2} |\det(1 - \mathrm{Ad}(\delta\theta))_{|\bar{\mathfrak{n}}}|^{-1} \\
&= |\det(1 - \mathrm{Ad}(\delta\theta))_{|\mathfrak{g}/\mathfrak{m}}|^{-1/2}.
\end{aligned}$$

Equation (7.4) now follows by substituting this expression into the earlier decomposition for $\mathcal{O}_{\delta\theta}(f)$. □

Let us return to spectral considerations. The analytic manipulations in §3 X [Kna86] give us the reduction

$$\Theta_{\mathrm{ind}_{\bar{P}(\mathbf{R})}^{G(\mathbf{R})}\pi,\mathsf{T}}(f) = \Theta_{\pi,\mathsf{U}}(f^{(\bar{P})}),\ f \in C_c^\infty(G(\mathbf{R})) \tag{7.6}$$

(*cf.* (10.21) [Kna86]). This reduction suggests a strategy for proving spectral transfer in the special case that π a limit of fundamental series representation as in section 5. The strategy is basically the same as Shelstad's in the ordinary case (§14 [She10]) and in the twisted case (§11 [She]).

Let us assume that $\boldsymbol{\varphi}$ is an L-parameter for G and that $\Pi_\varphi = \Pi_\varphi \circ \theta$. Let us further assume that each irreducible representation in Π_φ is a subrepresentation of $\mathrm{ind}_{\bar{P}(\mathbf{R})}^{G(\mathbf{R})}\pi$, where $\pi \in \Pi_{\varphi,\bar{M}}$ is a limit of fundamental series representation of $\bar{M}(\mathbf{R})$. Finally let us assume that the θ-stable representations in Π_φ are exactly the subrepresentations induced from the θ-stable representations in $\Pi_{\varphi,\bar{M}}$.

The restriction of θ to $\bar{M}$ might not be of finite order, but its restriction to $Z_{\bar{M}}$ is by assumption. The arguments of section 6 therefore apply. By Theorem 6.3 we have spectral transfer for the limits of fundamental series on $\bar{M}(\mathbf{R})$, as long as the assumptions of section 5 are met with G replaced by $\bar{M}$.

One may substitute $f^{(\bar{P})} \in C_c^\infty(\bar{M}(\mathbf{R}))$ into the right-hand side of (6.6) and reintroduce subscripts for U and T to obtain

$$(7.7)\quad \sum_{\pi \in \Pi_{\varphi,\bar{M}}} \Delta(\boldsymbol{\varphi}_{\bar{\boldsymbol{M}}_{\boldsymbol{H_1}}}, \pi)\, \Theta_{\pi,\mathsf{U}_\pi}(f^{(\bar{P})}) = \sum_{\pi \in \Pi_{\varphi,\bar{M}}} \Delta(\boldsymbol{\varphi}_{\bar{\boldsymbol{M}}_{\boldsymbol{H_1}}}, \pi)\, \Theta_{\mathrm{ind}_{\bar{P}(\mathbf{R})}^{G(\mathbf{R})}\pi,\mathsf{T}_\pi}(f).$$

In this equation we have written $\bar{M}_{H_1}$ in place of H_1 in order to emphasize that $\bar{M}_{H_1}$ must be an endoscopic group for $\bar{M}$, not G. If $\boldsymbol{\varphi}$ arises from an endoscopic group H_1 in the usual fashion (§6 [Mez12]) and $\bar{M}_{H_1}$ were suitably compatible with H_1 then one might *define* spectral transfer factors for $\pi' \in \Pi_\varphi$ by

$$(7.8)\qquad \Delta(\boldsymbol{\varphi}_{\boldsymbol{H_1}}, \pi') = \Delta(\boldsymbol{\varphi}_{\bar{\boldsymbol{M}}_{\boldsymbol{H_1}}}, \pi),$$

whenever π' is a subrepresentation of $\mathrm{ind}_{\bar{P}(\mathbf{R})}^{G(\mathbf{R})}\pi$ and $\pi \in \Pi_{\varphi,\bar{M}}$. If π is θ-stable in this definition then one could define the twisted character $\Theta_{\pi',\mathsf{T}_{\pi'}}$ by taking $\mathsf{T}_{\pi'}$ to be the resriction of T_π to the space of π' (Corollary 14.66 [Kna86], Theorem 2.3 (b) [KZ79]). With these definitions we recover

$$\begin{aligned}
\sum_{\pi' \in \Pi_\varphi} \Delta(\boldsymbol{\varphi}_{\boldsymbol{H_1}}, \pi')\, \Theta_{\pi',\mathsf{T}_{\pi'}}(f) &= \sum_{\pi \in \Pi_{\varphi,\bar{M}}} \Delta(\boldsymbol{\varphi}_{\bar{\boldsymbol{M}}_{\boldsymbol{H_1}}}, \pi)\, \Theta_{\mathrm{ind}_{\bar{P}(\mathbf{R})}^{G(\mathbf{R})}\pi,\mathsf{T}_\pi}(f)\\
&= \sum_{\pi \in \Pi_{\varphi,\bar{M}}} \Delta(\boldsymbol{\varphi}_{\bar{\boldsymbol{M}}_{\boldsymbol{H_1}}}, \pi)\, \Theta_{\pi,\mathsf{U}_\pi}(f^{(\bar{P})})\\
(7.9)\qquad &= \int_{\bar{M}_{H_1}(\mathbf{R})/Z_1(\mathbf{R})} (f^{(\bar{P})})_{\bar{M}_{H_1}}(h) \sum_{\pi_{\bar{M}_{H_1}} \in \Pi_{\varphi_{\bar{M}_{H_1}}}} \Theta_{\pi_{\bar{M}_{H_1}}}(h)\, dh
\end{aligned}$$

by applying Theorem 6.3. In order to continue with this strategy the "suitable compatibility" above would then need to also apply to geometric transfer. A natural expectation for this compatibility would be for $\bar{M}_{H_1}$ to be the Levi subgroup of a parabolic subgroup $\bar{P}_{H_1}$ of H_1 and

$$(7.10)\qquad (f^{(\bar{P})})_{\bar{M}_{H_1}} = (f_{H_1})^{(\bar{P}_{H_1})}.$$

Given such compatibility, one could apply the usual formula for induced characters and continue with

$$\begin{aligned} \sum_{\pi' \in \Pi_{\varphi}} \Delta(\boldsymbol{\varphi_{H_1}}, \pi') \, \Theta_{\pi', \mathsf{T}_{\pi'}}(f) &= \int_{\bar{M}_{H_1}(\mathbf{R})/Z_1(\mathbf{R})} (f_{H_1})^{(\bar{P}_{H_1})}(h) \sum_{\pi_{\bar{M}_{H_1}} \in \Pi_{\varphi_{\bar{M}_{H_1}}}} \Theta_{\pi_{\bar{M}_{H_1}}}(h) \, dh \\ (7.11) \qquad &= \int_{H_1(\mathbf{R})/Z_1(\mathbf{R})} f_{H_1}(h) \sum_{\pi_{H_1} \in \Pi_{\varphi_{H_1}}} \Theta_{\pi_{H_1}}(h) \, dh. \end{aligned}$$

This final equation would then constitute spectral transfer for $\boldsymbol{\varphi}$.

Let us formally state the required hypotheses in this strategy and give the explicit constructions for the earlier compatibilities.

1. θ preserves $\bar{P}$ and $\bar{M}$.
2. $\boldsymbol{\varphi_{H_1}}$ passes to an L-parameter of a Levi subgroup $\bar{M}_{H_1}$ of H_1 which is an endoscopic group of $\bar{M}$.
3. θ has finite order on $Z_{\bar{M}}$.
4. The assumptions of section 5 hold with G replaced by $\bar{M}$.
5. The θ-stable representations of Π_{φ} are precisely the subrepresentations of the representations induced from θ-stable representations of $\Pi_{\varphi, \bar{M}}$.

Let us assume these hypotheses to hold. Our assumption implicitly includes an admissible homomorphism $\varphi \in \boldsymbol{\varphi}$ whose image lies in a (standard) Levi subgroup $\mathcal{M} \rtimes W_{\mathbf{R}}$ of ${}^L G$ which is (identified with) ${}^L \bar{M}$. We assume that twisted characters $\Theta_{\pi, \mathsf{U}_{\pi}}$ have been specified as in section 5 for every $\pi \in \Pi_{\varphi, \bar{M}}$. According to (7.6), the twisted character $\Theta_{\mathrm{ind}_{\bar{P}(\mathbf{R})}^{G(\mathbf{R})} \pi, \mathsf{T}_{\pi}}$ is completely determined by the twisted character $\Theta_{\pi, \mathsf{U}_{\pi}}$, and the support of $\Theta_{\mathrm{ind}_{\bar{P}(\mathbf{R})}^{G(\mathbf{R})} \pi, \mathsf{T}_{\pi}}$ lies in the θ-conjugates in $G(\mathbf{R})$ of $\bar{M}(\mathbf{R})$ (see (7.3)). We know from section 4.6 that $\Theta_{\pi, \mathsf{U}_{\pi}}$ depends entirely on its values on elements of $\delta \in \bar{M}(\mathbf{R})$ which are strongly θ-regular and θ-elliptic in the sense that $\bar{M}^{\delta\theta}$ is abelian, and the centralizer of $Z_{\bar{M}}(\bar{M}^{\delta\theta})$ is a fundamental maximal torus S in $\bar{M}$ (Lemma 4.1). This dependence motivates our next assumption, which is the existence of an element $\delta \in \bar{M}(\mathbf{R})$ for which $G^{\delta\theta}$ is abelian, $S = Z_{\bar{M}}(\bar{M}^{\delta\theta})$ is a fundamental maximal torus in $\bar{M}$, and $\gamma_1 \in H_1(\mathbf{R})$ is a norm. The existence of a norm for δ allows us to engage the machinery of section 3.3.

We must establish endoscopic data for $\bar{M}$ and $\theta_{|\bar{M}}$ which are compatible with the underlying endoscopic data for G and θ. This is done in the appendix to [Mez12]. The resulting endoscopic data $(\bar{M}_H, \mathcal{H}_{\bar{M}}, \mathsf{s}_{\bar{M}}, \xi_{\bar{M}})$ are given through the standard Levi subgroup $\mathcal{M} \rtimes W_{\mathbf{R}}$ of ${}^L G$. To be more explicit $\mathsf{s}_{\bar{M}} = \mathsf{s}$, $\bar{M}_H$ is dual to the Levi subgroup $\hat{\bar{M}}_H = \xi^{-1}((\mathcal{M}^{\mathsf{s}\hat{\theta}})^0)$ of $\hat{H}$, $\mathcal{H}_{\bar{M}} = \hat{\bar{M}}_H \rtimes_c W_{\mathbf{R}}$ (see 2 in section 3.2), and $\xi_{\bar{M}} = \xi_{|\mathcal{H}_{\bar{M}}}$. In addition, we set $\bar{M}_{H_1} = p_H^{-1}(\bar{M}_H)$, which is a z-extension of $\bar{M}_H$ (see (3.3)).

The admissible homomorphism φ is obtained from an admissible homomorphism $\varphi_{H_1} \in \boldsymbol{\varphi_{H_1}}$ by $\varphi = \xi \circ \xi_{H_1}^{-1} \circ \varphi_{H_1}$. One could define

$$\varphi_{\bar{M}_{H_1}} = \xi_{H_1} \circ \xi_{\bar{M}}^{-1} \circ \varphi$$

to obtain an admissible homomorphism of $\bar{M}_{H_1}$ as in (7.7). If one substitutes $\varphi = \xi \circ \xi_{H_1}^{-1} \circ \varphi_{H_1}$ into this definition it is apparent that $\varphi_{\bar{M}_{H_1}}$ has the same values as φ_{H_1}. The only difference between $\varphi_{\bar{M}_{H_1}}$ and φ_{H_1} is their codomains. In the interest of reducing subscripts from the notation we shall identify the two from here on.

The spectral transfer factors in (7.7) are defined in (5.6), and are ultimately expressed as the product of (4.45)-(4.47). The only expression which depends on the geometric transfer factors is (4.47), which depends on the first and third part of the geometric transfer factors $\Delta_{\bar{M}}$ for $\bar{M}$. According to Lemma 11.4 [She], we may choose geometric transfer factors Δ for G and a compatible definition of norm between γ_1 and δ on the level of $\bar{M}$ ($z^{\dagger}$-norm) so that the first three parts of $\Delta_{\bar{M}}(\gamma_1, \delta)$ and $\Delta(\gamma_1, \delta)$ agree.

We may now rightly make definition (7.8) and proceed to equation (7.9).

Lemma 7.2. *Let $\bar{P}_{H_1}$ be a parabolic subgroup of H_1 with Levi subgroup $\bar{M}_{H_1}$. Then, under suitable normalization of geometric transfer factors and measures, we may assume that equation (7.10) holds.*

Proof. Suppose $f \in C_c^{\infty}(G(\mathbf{R}))$. Then $f^{(\bar{P})} \in C_c^{\infty}(\bar{M}(\mathbf{R}))$, and by (3.14) there exists a function $(f^{(\bar{P})})_{\bar{M}_{H_1}}$ such that

$$\sum_{\gamma_1'} \mathcal{O}_{\gamma_1'}((f^{(\bar{P})})_{\bar{M}_{H_1}}) = \sum_{\delta'} \Delta_{\bar{M}}(\gamma_1, \delta')\, \mathcal{O}_{\delta'\theta}(f^{(\bar{P})}). \tag{7.12}$$

The sum on the right is taken the θ-conjugacy classes under $\bar{M}(\mathbf{R})$ of elements in $\bar{M}(\mathbf{R})$ whose norm is γ_1. It follows from the remark following Lemma 4.13, that this collection of θ-conjugacy classes over $\bar{M}(\mathbf{R})$ is in bijection with the collection of θ-conjugacy classes under $G(\mathbf{R})$ of elements in $G(\mathbf{R})$ whose norm is γ_1. This bijection is necessary for us to convert the right-hand side of (7.12) into the analogous sum over $G(\mathbf{R})$. Towards this end, we also substitute (7.4) into right right-hand side, to obtain

$$\sum_{\gamma_1'} \mathcal{O}_{\gamma_1'}((f^{(\bar{P})})_{\bar{M}_{H_1}}) = \sum_{\delta'} \Delta_{\bar{M}}(\gamma_1, \delta') |\det(1 - \mathrm{Ad}(\delta'\theta))_{|\mathfrak{g}/\mathfrak{m}}|^{1/2}\, \mathcal{O}_{\delta'\theta}(f). \tag{7.13}$$

The normalization of geometric transfer factors as in Lemma 11.4 [She] yields

$$\Delta_{\bar{M}}(\gamma_1, \delta) = \frac{\Delta_{\bar{M},IV}(\gamma_1, \delta)}{\Delta_{IV}(\gamma_1, \delta)}\, \Delta(\gamma_1, \delta),$$

where

$$\Delta_{IV}(\gamma_1, \delta) = \frac{|\det(\mathrm{Ad}(\delta\theta) - 1)_{|\mathfrak{g}/\mathfrak{s}\otimes\mathbf{C}}|^{1/2}}{|\det(\mathrm{Ad}(\gamma) - 1)_{|\mathfrak{h}/\mathfrak{t}_H\otimes\mathbf{C}}|^{1/2}}$$

and

$$\Delta_{\bar{M},IV}(\gamma_1,\delta) = \frac{|\det(\mathrm{Ad}(\delta\theta)-1)_{|\mathfrak{m}/\mathfrak{s}\otimes\mathbf{C}}|^{1/2}}{|\det(\mathrm{Ad}(\gamma)-1)_{|\mathfrak{m}_H/\mathfrak{t}_H\otimes\mathbf{C}}|^{1/2}}$$

(§4.5 [KS99]). The quotient of the numerators of $\Delta_{\bar{M},IV}(\gamma_1,\delta)$ and $\Delta_{IV}(\gamma_1,\delta)$ equals $|\det(1-\mathrm{Ad}(\delta\theta))_{|\mathfrak{g}/\mathfrak{m}}|^{-1/2}$ and cancels with the corresponding term on the right-hand side of (7.12). The quotient of the denominators of $\Delta_{M,IV}(\gamma_1,\delta)$ and $\Delta_{IV}(\gamma_1,\delta)$ equals $|\det(1-\mathrm{Ad}(\gamma))_{|\mathfrak{h}/\mathfrak{m}_H}|^{1/2}$. Making the consonant substitutions in (7.13), applying (3.14) and (7.3), we find

$$\begin{aligned}\sum_{\gamma_1'}\mathcal{O}_{\gamma_1'}((f^{(\bar{P})})_{\bar{M}_{H_1}}) &= |\det(1-\mathrm{Ad}(\gamma))_{|\mathfrak{h}/\mathfrak{m}_H}|^{1/2}\sum_{\delta'}\Delta_G(\gamma_1,\delta')\,\mathcal{O}_{\delta'\theta}(f)\\ &= |\det(1-\mathrm{Ad}(\gamma))_{|\mathfrak{h}/\mathfrak{m}_H}|^{1/2}\sum_{\gamma_1'}\mathcal{O}_{\gamma_1'}(f_{H_1})\\ &= \sum_{\gamma_1'}\mathcal{O}_{\gamma_1'}((f_{H_1})^{(\bar{P}_{H_1})}).\end{aligned}$$

In the sums over γ_1' we have also used the bijection between stable conjugacy classes over H_1 and $\bar{M}_{H_1}$ (Lemma 4.13 with trivial θ, *cf.* §14 [She08a]). The above identity between orbital integrals justifies the asserted assumption of the lemma. □

Justification of equation (7.10) was the final step in the proof of the twisted spectral transfer identity (7.11) and this is now complete.

7.1 No norms from $\bar{M}(\mathbf{R})$

The purpose of this subsection is to persuade the reader that the assumption of the existence of θ-regular and θ-elliptic $\delta \in \bar{M}(\mathbf{R})$ with norm in $H_1(\mathbf{R})$ is a restriction to the only interesting case of spectral transfer.

Identity (7.10) allows us to reduce character values of functions in $C_c^\infty(H_1(\mathbf{R}),\lambda_{Z_1})$ to functions $(f^{(\bar{P})})_{\bar{M}_{H_1}}$ matching $f^{(\bar{P})}$. If there is no strongly θ-regular $\delta \in \bar{M}(\mathbf{R})$ with norm in $H_1(\mathbf{R})$ then $(f^{(\bar{P})})_{M_{H_1}}$ may be taken to equal zero. For this reason the character values on the left-hand side of the desired spectral identity (7.11) are zero. We may conclude that twisted spectral transfer holds by taking all spectral transfer factors equal to zero.

We assume then that there exists a strongly θ-regular element in $\bar{M}(\mathbf{R})$ which has a norm $H_1(\mathbf{R})$. If none of these elements is θ-elliptic then we set all spectral transfer factors equal to zero and prove twisted specral transfer as follows. By (7.11) it suffices to show that the distribution

$$f \mapsto \int_{\bar{M}_{H_1}(\mathbf{R})/Z_1(\mathbf{R})}(f_{H_1})^{(\bar{P}_{H_1})}(h)\sum_{\pi_{\bar{M}_{H_1}}\in\Pi_{\varphi_{\bar{M}_{H_1}}}}\Theta_{\pi_{\bar{M}_{H_1}}}(h)\,dh$$

vanishes. To this end, we show that this distribution is an invariant eigendistribution and apply the twisted version of Harish-Chandra's uniqueness theorem (*cf.* §6.5 and 7.3 [Mez12]). The invariance of this distribution follows from (121) [Mez12]. The eigendistribution property follows from Lemma 24 [Mez12] and Proposition 30 (38) §6 [Var77], which allow us to pull out the element z in the centre of the universal enveloping algebra from the parentheses in $(zf)^{(\bar{P}_{H_1})}$. The lack of a θ-elliptic element with norm implies that this distribution vanishes on any fundamental torus of $\bar{M}$. The uniqueness theorem Theorem 15.1 [Ren97] then tells us that the distribution vanishes everywhere when the L-packet $\Pi_{\varphi_{\bar{M}_{H_1}}}$ is essentially square-integrable.

Having disposed of the uninteresting spectral transfer identity "zero equals zero", one may assume that there exists a strongly θ-regular and θ-elliptic element $\delta \in \bar{M}(\mathbf{R})$ which has a norm $\gamma_1 \in H_1(\mathbf{R})$.

8 Spectral transfer for tempered representations

In this section we show how the framework of the previous section applies to tempered representations. Suppose, that we have an endoscopic quadruple $(H, \mathcal{H}, \mathsf{s}, \xi)$ and a compatible z-pair (H_1, ξ_{H1}) as in section 4 and that θ has finite order on Z_G. Suppose further that $\varphi_{H_1} : W_{\mathbf{R}} \to {}^L H_1$ is an admissible homomorphism such that $\Pi_{\varphi_{H_1}}$ consists of irreducible tempered representations. We shall further assume that the homomorphism $\varphi^* = \xi \circ \xi_{H_1}^{-1} \circ \varphi_{H_1}$ is *admissible* with respect to G (in the sense of §8.2 [Bor79]). As earlier, we denote φ^* by φ to distiguish the resulting admissible homomorphism of G. The condition that $\Pi_{\varphi_{H_1}}$ be tempered is equivalent to φ_{H_1} having bounded image (§(4) 10.3 [Bor79]). It follows from the continuity of ξ and $\xi_{H_1}^{-1}$ that φ has bounded image so that Π_φ is an L-packet consisting of irreducible tempered representations of $G(\mathbf{R})$. In order for this L-packet to have any bearing on twisted endsocopy, we assume that $\Pi_\varphi = \Pi_\varphi \circ \theta$.

Let ${}^L\bar{M}_{H_1}$ be the smallest Levi subgroup of ${}^L H_1$ containing the image of φ_{H_1}. By definition, the irreducible representations in $\Pi_{\varphi_{H_1}}$ are the irreducible subrepresentations of the representations induced from $\Pi_{\varphi_{H_1}, \bar{M}_{H_1}}$ (§11.3 [Bor79]).

We wish to produce an $\mathbf{R}$-Levi subgroup $\bar{M}$ of G and a correponding endoscopic quadruple $(\bar{M}_H, \mathcal{H}, \mathsf{s}_{\bar{M}}, \xi_{\bar{M}})$ such that $\bar{M}_{H_1}$ is a z-extension of $\bar{M}_H$. We will achieve this by running the arguments of Appendix A [Mez12] in reverse. In section 3.3 we have fixed a Borel subgroup $\mathcal{B}_H$ of $\hat{H}$ containing a maximal torus $\mathcal{T}_H$. We may assume that ${}^L\bar{M}_{H_1}$ is a standard Levi subgroup with respect to this fixed data (§3.3 [Bor79]). As such, ${}^L\bar{M}_{H_1}$ is determined by a Γ-stable set of simple roots $I \subset R(\hat{H}_1, \mathcal{T}_{H_1})$. Given that H_1 is a central extension of H, the root system $R(\hat{H}_1, \mathcal{T}_{H_1})$ may be identified with $R(\hat{H}, \mathcal{T}_H)$. Let $\bar{M}_H$ be the $\mathbf{R}$-Levi subgroup of H corresponding to $I \subset R(\hat{H}, \mathcal{T}_H)$. The root system $R(\hat{H}, \mathcal{T}_H)$ embeds into the system of indivisible roots in $R_{\mathrm{res}}(\hat{G}, \mathcal{T}) = R((\hat{G}^{\mathsf{s}\hat{\theta}})^0, (\mathcal{T}^{\hat{\theta}})^0)$ under ξ ((137) [Mez12]). This embedding preserves positivity, as $\xi(\mathcal{B}_H) \subset \mathcal{B}$ (section 3.3). Let us identify I with its image in $R_{\mathrm{res}}(\hat{G}, \mathcal{T})$ and define

$$\bar{I} = \{\alpha \in R(\hat{G}, \mathcal{T}) : \alpha_{|(\mathcal{T}^{\hat{\theta}})^0} \in I\}.$$

This is a set of positive roots (Theorem 1.1.A (2) [KS99]).

Lemma 8.1. *Every root in $\bar{I}$ is simple with respect to the positive system fixed by $\mathcal{B}$.*

Proof. Suppose $\beta \in \bar{I}$ so that the root $\beta_{|(\mathcal{T}^{\hat{\theta}})^0}$ is simple in $R_{\mathrm{res}}(\hat{G}, \mathcal{T})$. By way of contradiction, suppose that $\beta = \beta_1 + \beta_2$ for positive roots $\beta_1, \beta_2 \in R(\hat{G}, \mathcal{T})$. The restrictions of β_1 and β_2 to $(\mathcal{T}^{\hat{\theta}})^0$ are non-negative relative to $R((\mathcal{B}^{\mathsf{s}\hat{\theta}}), (\mathcal{T}^{\hat{\theta}})^0)$ (Theorem 1.1.A (2) [KS99]). By simplicity, one of the two must have trivial restriction. Suppose β_1 has trivial restriction. The automorphism $\hat{\theta}$ preserves the pair $(\mathcal{B}, \mathcal{T})$ (section 3.1) so that it has finite order l on $\mathcal{T}$ (Corollary 2.14 [Spr79]). Moreover, $\hat{\theta}^j \cdot \beta_1$ is another positive root for any $j \geq 1$. The trivial restriction of β_1 implies $0 = \sum_{j=1}^{l} \hat{\theta}^j \cdot \beta_1$, and this equation contradicts the positivity of the summands on the right. $\square$

By Lemma 8.1, we know that $\bar{I}$ is a set of simple roots. In addition, the set $\bar{I}$ is stable under the action of $\hat{\theta}$ ((1.3.1) [KS99]). As a result $\bar{I}$ corresponds to a unique standard parabolic subgroup of $\hat{G}$ which is preserved by $\hat{\theta}$. Since the action of Γ on $\hat{H}$ in $\mathcal{H}$ is transferred to the action of Γ on $(\hat{G}^{\mathsf{s}\hat{\theta}})^0$ in ${}^L G$ under ξ ((2.1.2), (2.1.4) [KS99]), it follows that $\bar{I}$ is preserved by Γ. Thus, the parabolic subgroup of $\hat{G}$ corresponding to $\bar{I}$ is the identity component of a parabolic subgroup ${}^L\bar{P}$ of ${}^L G$, and ${}^L\bar{P}$ is dual to an $\mathbf{R}$-parabolic subgroup of G^* ((2) §3.3 [Bor79]). The admissibility assumption on φ^* above tells us that ${}^L\bar{P} \supset \varphi^*(W_{\mathbf{R}})$ is *relevant* so that it is actually dual to (a $G(\mathbf{R})$-conjugacy class of) an $\mathbf{R}$-parabolic subgroup $\bar{P}$ of G. The subset $\bar{I}$ also corresponds to a unique $\hat{\theta}$-stable Levi subgroup $\mathcal{M}$ in $\hat{G}$ such that $\mathcal{M} \rtimes W_{\mathbf{R}}$ is a Levi subgroup of ${}^L\bar{P}$, and $\mathcal{M}$ is dual to the Levi subgroup $\bar{M}$ of $\bar{P}$. The definitions of $\bar{P}$ and $\bar{M}$ rely on a choice of a Borel subgroup and maximal torus in G. Let us choose this pair to be $T \subset B_T$ from (3.1). Using this pair of groups for the definitions of $\bar{P}$ and $\bar{M}$, the $\hat{\theta}$-stability of $\bar{I}$ translates into the $\mathrm{Int}(g_0)\theta$-stability of $\bar{P}$ and $\bar{M}$, for some $g_0 \in G(\mathbf{R})$ (*cf.* §1.3 [Bor79] and Theorem 20.9 (iii) [Bor91]). We define $\theta_1 = \mathrm{Int}(g_0)\theta$ so that the first hypothesis listed in section 7 is satisfied with θ_1 in place of θ. Recall from section 6 that $\hat{\theta}_1 = \hat{\theta}$. Our aim is to show that the remaining hypotheses of section 7 also hold for θ_1.

The group $(\mathcal{M}^{\mathsf{s}\hat{\theta}})^0$ is generated by the restriction to $(\mathcal{T}^{\hat{\theta}})^0$ of the roots generated by $\bar{I}$ (Theorem 1.1.A (2) [KS99], Proposition 8.1.1 (ii) [Spr98]). This implies that $(\mathcal{M}^{\mathsf{s}\hat{\theta}})^0$ is isomorphic to the dual of $\bar{M}_H$ by construction. Set $\mathcal{H}_{\bar{M}} = \xi^{-1}((\mathcal{M}^{\mathsf{s}\hat{\theta}})^0)$, $\xi_{\bar{M}} = \xi_{|\mathcal{H}_{\bar{M}}}$, $\mathsf{s}_{\bar{M}} = \mathsf{s}$ and $\theta_{\bar{M}} = (\theta_1)_{|\bar{M}}$. Then it is easily verified that $(\bar{M}_H, \mathcal{H}_{\bar{M}}, \mathsf{s}_{\bar{M}}, \xi_{\bar{M}})$ is an endoscopic datum for the pair $(\bar{M}, \theta_{\bar{M}})$, and that $\bar{M}_{H_1} = p^{-1}(\bar{M}_H)$ is a z-extension of $\bar{M}_H$ with corresponding homomorphism $(\xi_{H_1})_{|\mathcal{H}_{\bar{M}}}$ (see section 3.2).

Let ${}^L\bar{M} = \mathcal{M} \rtimes W_{\mathbf{R}}$. Then the image of φ is contained in ${}^L\bar{M}$ (*cf.* proof of Lemma 5.1). We may therefore regard φ as an admissible homomorphism of $\bar{M}$ obtained from an admissible homomorphism φ_{H_1} of $\bar{M}_{H_1}$. This is another way of saying that the second hypothesis listed in section 7 is satisfied.

Before showing that the third such hypothesis holds, it is convenient to jump to the fourth. We require that $\Pi_{\varphi,\bar{M}}$ consists of limits of fundamental series representations of $\bar{M}(\mathbf{R})$. This becomes evident from section 5 once we indicate how the assumptions of that section are satisfied with G replaced by $\bar{M}$, and H_1 replaced by $\bar{M}_{H_1}$. Looking back, we see that the assumptions of section 5 are a modified list of those given in section 4.

The first assumption in our context is that φ_{H_1} is not contained in a proper parabolic subgroup of ${}^L M_{H_1}$. This assumption is satisfied by the definition of M_{H_1}.

The second assumption is that there exists a θ_1-elliptic element in $\bar{M}(\mathbf{R})$ which has a norm in $\bar{M}_{H_1}(\mathbf{R})$. In section 7.1, we have argued that making the additional assumption that there exists θ_1-elliptic $\delta \in \bar{M}(\mathbf{R})$ with norm in $H_1(\mathbf{R})$ is innocuous. We make this assumption now.

The third assumption of section 5 is that φ^* is an admissible homomorphism, and this assumption has already been made here.

The fourth assumption is that the representations in $\Pi_{\varphi,M}$ have unitary central character. If the representations in $\Pi_{\varphi,M}$ do not have unitary central character, then $\Pi_{\varphi,M}$ does not consist of tempered representations. In that case φ is not bounded in M or G, so that Π_φ is not a tempered L-packet ((4) §10.3 [Bor79]). This would contradict Π_φ being a tempered L-packet. In consequence, the fourth assumption holds.

The fifth assumption of section 4 is eliminated from section 5, so that we are in the position to conclude that $\Pi_{\varphi,\bar{M}}$ consists of limits of fundamental series representations.

Let us show that θ_1 has finite order on $Z_{\bar{M}}$. We are assuming that $\mathrm{Int}(\delta)\theta_1$ determines a fundamental maximal torus S of $\bar{M}$ (Lemma 4.1). The automorphism $\mathrm{Int}(\delta)\theta_1$ preserves the torus S and has finite order on $Z_G \subset S$. In consequence, some power of this automorphism is trivial on S (*cf.* (36) [Mez12]). Since $Z_{\bar{M}}$ is contained in the maximal torus S and $\delta \in \bar{M}$, some power of θ_1 is trivial on $Z_{\bar{M}}$.

There remain two related results which we must verify in order for the mechanism of section 7 to apply. The first of the two is the final assumption of section 4. In the current context of twisted endoscopy for $(\bar{M}, \theta_{\bar{M}})$ this assumption takes the shape

$$\Pi_{\varphi,\bar{M}} = \Pi_{\varphi,\bar{M}} \circ \theta_1.$$

One would rightly suspect that this identity might be derived from the given assumption

$$\Pi_\varphi = \Pi_\varphi \circ \theta = \Pi_\varphi \circ \theta_1.$$

The other remaining result is the final hypothesis listed in section 7, namely the requirement that the $\mathrm{Int}(\delta)\theta_1$-stable representations in Π_φ be exactly the subrepresentations induced from the $\mathrm{Int}(\delta)\theta_1$-stable representations in $\Pi_{\varphi,\bar{M}}$. To prove both of these results, it helps to recall that any representation in $\Pi_{\varphi,\bar{M}}$ is of the form $\mathrm{ind}_{P(\mathbf{R})}^{\bar{M}(\mathbf{R})} \varpi$, where $P \subset \bar{M}$ is a parabolic subgroup with Levi subgroup $M = Z_{\bar{M}}(S_d)$, $\varpi \in \Pi_{\varphi,M}$ is an essential limit of discrete series representation of $M(\mathbf{R})$, *etc.* (see sections 4-5). The next lemma is a consequence of the Langlands Disjointness Theorem (*pp.* 149-151 [Lan89], Theorem 14.90 [Kna86]) given in the language of this description of $\Pi_{\varphi,\bar{M}}$ (*cf.* the proof of Proposition 4.12). The remaining two assumptions will follow as corollaries.

Lemma 8.2. *Suppose S is the fundamental torus of $\bar{M}$ generated by $\delta \in \bar{M}(\mathbf{R})$, $M = Z_{\bar{M}}(S_d)$, P is a parabolic subgroup of G with M as a Levi subgroup, and $\varpi, \varpi' \in \Pi_{\varphi,M}$ are essential limit of discrete series representations of $M(\mathbf{R})$ such that $\mathrm{ind}_{P(\mathbf{R})}^{G(\mathbf{R})}\varpi'$ and $(\mathrm{ind}_{P(\mathbf{R})}^{G(\mathbf{R})}\varpi)^{\delta\theta_1}$ share equivalent irreducible subrepresentations. Then ϖ' is equivalent to $\varpi^{\delta\theta_1}$.*

Proof. Suppose first that ϖ' and ϖ are essentially square-integrable representations. As in section 4.2, we may write $\varpi = \varpi_{w^{-1}\Lambda}$ and $\varpi' = \varpi_{(w')^{-1}\Lambda}$, where S is an elliptic torus of $M = \bar{M}$, Λ is an M-regular and $\delta\theta_1$-stable quasicharacter of $S(\mathbf{R})$, and $w, w' \in \Omega(M,S)$ (*cf.* Lemma 4.11). The Langlands Disjointness Theorem (*pp.* 149-151 [Lan89]) provides $k \in N_{G(\mathbf{R})}(M)$ such that ϖ' is equivalent to $(\varpi)^{k\delta\theta_1}$. As in Proposition 4.12, we may assume that $k \in N_{G(\mathbf{R})}(S)$ and identify it with an element of $\Omega_{\mathbf{R}}(G,S)$. We have the equation

$$k\delta\theta_1 w^{-1} \cdot \Lambda = (w')^{-1} \cdot \Lambda.$$

which may be rewritten in the form

$$w_1^{-1}k \cdot \Lambda' = \Lambda', \tag{8.1}$$

where $\Lambda' = \delta\theta_1 w^{-1}(\delta\theta_1)^{-1} \cdot \Lambda$ and $w_1 = (w')^{-1}\delta\theta_1 w(\delta\theta_1)^{-1}$.

The (differential of the) quasicharacter Λ' is M-regular. We choose a positive system on $R(G,S)$ so that its induced positive system on $R(M,S)$ corresponds to a Weyl chamber containing Λ'. Equation (8.1) implies that the element $w_1^{-1}k \in \Omega(G,S)$ is a product of reflections generated by simple roots in $R(G,S)$ which are orthogonal to Λ' (Lemma B §10.3 [Hum94]). Suppose α is such a simple root and let ρ_M be the half-sum of the positive roots of $R(M,S)$. The simple reflection s_α fixes Λ' and therefore stabilizes the system of positive roots for $R(M,S)$. This implies that s_α fixes ρ_M, or equivalently, that α is orthogonal to ρ_M.

Using the terminology of §3 [Vog82], this proves that α is a *quasisplit* root and that $w_1^{-1}k$ lies in the *quasisplit* Weyl group generated by the *quasisplit* roots. We also know that $w_1 \in \Omega(M,S)$ is defined over $\mathbf{R}$ (Lemma 6.4.1 [Lab08]) so that $w_1^{-1}k$ belongs to the subgroup of the quasisplit Weyl group whose elements are defined over $\mathbf{R}$. According to Vogan, this subgroup is a semidirect product of two groups (*p.* 961 [Vog82]) and each of these two is contained in $\Omega_{\mathbf{R}}(G,S)$ (Lemma 3.1 [Vog82]). In short, $w_1^{-1}k$ belongs to $\Omega_{\mathbf{R}}(G,S)$ so that $w_1 \in \Omega_{\mathbf{R}}(M,S)$ and

$$w'\Omega_{\mathbf{R}}(M,S) = (\delta\theta_1 \cdot w)\Omega_{\mathbf{R}}(M,S).$$

We deduce from §6.4 [Lab08] and a character comparison that

$$\varpi' = \varpi_{(w')^{-1}\Lambda} \cong \varpi_{\delta\theta_1 \cdot w^{-1}\Lambda} \cong (\varpi_{w^{-1}\Lambda})^{\delta\theta_1} = \varpi^{\delta\theta_1}.$$

Suppose now that $\varpi, \varpi' \in \Pi_{\varphi,M}$ are essential limit of discrete series representations. In the notation of section 5 we may write $\varpi = \Psi^{w^{-1}\cdot(\mu+\nu')}_{w^{-1}\cdot\mu}\varpi_{w^{-1}\Lambda}$ and $\varpi' = \Psi^{(w')^{-1}\cdot(\mu+\nu')}_{(w')^{-1}\cdot\mu}\varpi_{(w')^{-1}\Lambda}$, where $\varpi_{w^{-1}\Lambda}$ and $\varpi_{(w')^{-1}\Lambda}$ are essentially square-integrable representations as above. As in the previous case, the Langlands Disjointness Theorem supplies $k \in N_{G(\mathbf{R})}(S)$ such that ϖ' is equivalent to $\varpi^{k\delta\theta_1}$. By Theorem 1.1 (c) [KZ84], there exists $k_1 \in N_{\bar{M}(\mathbf{R})}(S)$ such that $k_1 k\delta\theta_1 w^{-1}\cdot\Lambda = (w')^{-1}\cdot\Lambda$. The previous argument for essentially square-integrable representations therefore applies after replacing k with k_1k. We conclude in turn that $\varpi_{(w')^{-1}\Lambda} \cong (\varpi_{w^{-1}\Lambda})^{\delta\theta_1}$ and

$$\varpi' = \Psi^{(w')^{-1}\cdot(\mu+\nu')}_{(w')^{-1}\cdot\mu}\varpi_{(w')^{-1}\Lambda} \cong \Psi^{\delta\theta_1\cdot w^{-1}\cdot(\mu+\nu')}_{\delta\theta_1\cdot w^{-1}\cdot\mu}(\varpi_{w^{-1}\Lambda})^{\delta\theta_1} \cong (\Psi^{w^{-1}\cdot(\mu+\nu')}_{w^{-1}\cdot\mu}\varpi_{w^{-1}\Lambda})^{\delta\theta_1} = \varpi^{\delta\theta_1}.$$

□

Corollary 8.3. *The L-packet $\Pi_{\varphi,\bar{M}}$ is equal to the L-packet $\Pi_{\varphi,\bar{M}}\circ\theta_1$.*

Proof. Suppose $\varpi \in \Pi_{\varphi,\bar{M}}$. Then the irreducible subrepresentations of $\mathrm{ind}_{\bar{P}(\mathbf{R})}^{G(\mathbf{R})}\varpi$ belong to Π_φ (§11.3 [Bor79]). We are assuming that $\Pi_\varphi = \Pi_\varphi\circ\theta_1$ so that there exists $\varpi' \in \Pi_{\varphi,\bar{M}}$ such that $\mathrm{ind}_{\bar{P}(\mathbf{R})}^{G(\mathbf{R})}\varpi'$ and $(\mathrm{ind}_{\bar{P}(\mathbf{R})}^{G(\mathbf{R})}\varpi)^{\delta\theta_1}$ have some equivalent irreducible subrepresentations. By Lemma 8.2 and the fact that the irreducible representations in $\Pi_{\varphi,\bar{M}}$ are induced from irreducible representations in $\Pi_{\varphi,M}$, ϖ' is equivalent to $\varpi^{\delta\theta_1} \cong \varpi^{\theta_1} \in \Pi_{\varphi,\bar{M}}\circ\theta_1$. Since L-packets with non-empty intersection are equal, the corollary is complete. □

Corollary 8.4. *Suppose $\pi \in \Pi_\varphi$ and $\varpi \in \Pi_{\varphi,\bar{M}}$ such that π is a subrepresentation of $\mathrm{ind}_{\bar{P}(\mathbf{R})}^{G(\mathbf{R})}\varpi$ (§11.3 [Bor79]). Then π is θ_1-stable if and only if ϖ is θ_1-stable.*

Proof. If π is $\delta\theta_1$-stable then ϖ is $\delta\theta_1$-stable by Lemma 8.2. Clearly, $\delta\theta_1$-stability is equivalent to θ_1 stability here so that one implication of the corollary is proven. Conversely, suppose ϖ is θ_1-stable. Then $\mathrm{ind}_{\bar{P}(\mathbf{R})}^{G(\mathbf{R})}\varpi$ is θ_1-stable (see the proof of Proposition 4.1 [Mez07]). Since every irreducible subrepresentation of $\mathrm{ind}_{\bar{P}(\mathbf{R})}^{G(\mathbf{R})}\varpi$ occurs with multiplicity one (Theorem 2.3 (b) [KZ79]) it follows that π is θ_1-stable. □

All of the hypotheses of section 7 have been shown to hold. We now conclude with the main theorem, whose objects have also been defined in section 7.

Theorem 8.5. *Suppose θ has finite order on Z_G, $(H, \mathcal{H}, \mathsf{s}, \xi)$ is an endoscopic datum for (G,θ) and (H_1, ξ_{H1}) is a compatible z-pair. Suppose further that $\varphi_{H_1} : W_{\mathbf{R}} \to {}^LH_1$ is a tempered admissible homomorphism which passes to an admissible homomorphism $\varphi : W_{\mathbf{R}} \to {}^LG$ such that $\Pi_\varphi = \Pi_\varphi\circ\theta$. Then*

$$\sum_{\pi\in\Pi_\varphi} \Delta(\boldsymbol{\varphi_{H_1}}, \pi)\,\Theta_{\pi,\mathsf{T}_\pi}(f) = \int_{H_1(\mathbf{R})/Z_1(\mathbf{R})} f_{H_1}(h) \sum_{\pi_{H_1}\in\Pi_{\varphi_{H_1}}} \Theta_{\pi_{H_1}}(h)\,dh$$

for all $f \in C_c^\infty(G(\mathbf{R})\theta)$.

Proof. The theorem holds for $\theta_1 = \mathrm{Int}(g_0)\theta$ by the arguments of section 7. It therefore follows for θ by $(\star)$ of section 6, as $g_0 \in G(\mathbf{R})$. $\square$

References

[Art] J. Arthur. The endoscopic classification of representations: orthogonal and symplectic groups. To appear in the proceedings of the International Colloquium on Automorphic Representations and L-functions held in January 2012 at the Tata Institute of Fundamental Research.

[Art88] J. Arthur. The local behaviour of weighted orbital integrals. *Duke Math. J.*, 56:223–293, 1988.

[Art08] J. Arthur. Problems for real groups. In *Representation theory of real reductive Lie groups*, number 472 in Contemp. Math., pages 39–62, 2008.

[Bor79] A. Borel. Automorphic l-functions. In *Automorphic forms, representations, and L-functions*, volume 33 of *Proc. Sympos. Pure Math.*, pages 27–61, 1979.

[Bor91] A. Borel. *Linear algebraic groups.* Springer, 1991.

[Bou87] A. Bouaziz. Sur les caractères des groupes de Lie réductifs non connexes. *J. Funct. Anal.*, 1987.

[Bou89] A. Bouaziz. Relèvement des caractères d'un groupe endoscopique pour le changement de base $\mathbf{C}/\mathbf{R}$. In *Orbites unipotentes et représentations*, number 171–172 in Astérisque, pages 163–194. Société Mathématique de France, 1989.

[Bou94] A. Bouaziz. Intégrales orbitales sur les groupes de Lie réductifs. *Ann. Sci. Ecole Norm. Sup.*, 27:573–609, 1994.

[BT65] A. Borel and J. Tits. Groupes réductifs. *Inst. Hautes Etudes Sci. Publ. Math.*, 27:55–150, 1965.

[CC09] G. Chenevier and L. Clozel. Corps de nombres peu ramifiés et formes automorphes autoduales. *J. Amer. Math. Soc.*, 22:467–519, 2009.

[Clo82] L. Clozel. Changement de base pour les représentations tempérées des groupes réductifs réels. *Ann. Sci. Ecole Norm. Sup.*, 15:45–115, 1982.

[Clo11] L. Clozel. Identités de caractères en la place archimédienne. In J.-P. Labesse L. Clozel, M. Harris and B.-C. Ngô, editors, *On the stabilization of the trace formula*, volume 1, pages 351–367. International Press, 2011.

[DM08] P. Delorme and P. Mezo. A twisted invariant Paley-Wiener theorem for real reductive groups. *Duke Math. J.*, 144:341–380, 2008.

[Duc02] J.-Y. Ducloux. Méthode des orbites et formules du caractère pour les représentations tempérées d'un groupe algébrique réel réductif non connexe. *J. Lie Theory*, 12:137–190, 2002.

[Duf82] M. Duflo. Construction de représentations unitaires d'un groupe de Lie. In *Harmonic analysis an group representations*, pages 129–221. Liguori, 1982.

[HC66] Harish-Chandra. Discrete series for semisimple Lie groups II. *Acta Math.*, 116:1–111, 1966.

[Hum75] J. E. Humphreys. *Linear algebraic groups.* Springer, 1975.

[Hum94] J. E. Humphreys. *Introduction to Lie algebras and representation theory.* Springer, 1994.

[Kna86] A. W. Knapp. *Representation theory of semisimple groups.* Princeton University Press, 1986.

[Kna96] A. W. Knapp. *Lie groups beyond and introduction.* Birkhäuser, 1996.

[KS99] R. E. Kottwitz and D. Shelstad. Foundations of twisted endoscopy. *Astérisque*, 255, 1999.

[KZ79] A. W. Knapp and G. Zuckerman. Normalizing factors, tempered representations, and l-groups. In *Automorphic forms, representations, and L-functions*, volume 33 of *Proc. Sympos. Pure Math.*, pages 93–106, 1979.

[KZ84] A. W. Knapp and G. Zuckerman. Classification of irreducible tempered representations of semisimple groups. *Ann. of Math.*, 116:389–501, 1984.

[Lab04] J.-P. Labesse. Stable twisted trace formula: elliptic terms. *J. Inst. Math. Jussieu*, 3:473–530, 2004.

[Lab08] J.-P. Labesse. Introduction to endoscopy. In *Representation theory of real reductive groups*, number 472 in Contemp. Math., pages 175–213, 2008.

[Lan79] R. P. Langlands. Stable conjugacy: definitions and lemmas. *Can. J. Math*, 31:700–725, 1979.

[Lan89] R. P. Langlands. On the classification of irreducible representations of real algebraic groups. In *Representation theory and harmonic analysis on semisimple Lie groups*, number 31 in Mathematical surveys and monographs, pages 101–170. American Mathematical Society, 1989.

[LS87] R. P. Langlands and D. Shelstad. On the definition of transfer factors. *Math. Ann.*, 278:219–271, 1987.

[Mez07] P. Mezo. Automorphism-invariant representations of real reductive groups. *Amer. J. Math.*, 129:1063–1085, 2007.

[Mez12] P. Mezo. Character identities in the twisted endoscopy of real reductive groups. *Mem. Amer. Math. Soc.*, 2012. Posted online. Too appear in print.

[Ren97] D. Renard. Intégrales orbitales tordues sur les groupes de Lie réductifs réels. *J. Funct. Anal.*, 145:374–454, 1997.

[Ren03] D. Renard. Twisted endoscopy for real groups. *J. Inst. Math. Jussieu*, 2:529–566, 2003.

[She] D. Shelstad. On spectral transfer factors in real twisted endoscopy. See http://andromeda.rutgers.edu/∼shelstad.

[She81] D. Shelstad. Embeddings of L-groups. *Canad. J. Math.*, 33:513–558, 1981.

[She82] D. Shelstad. L-indistinguishability for real groups. *Math. Ann.*, 259:385–430, 1982.

[She08a] D. Shelstad. Tempered endoscopy for real groups I. Geometric transfer with canonical factors. In *Representation theory of real reductive groups*, number 472 in Contemp. Math., pages 215–246, 2008.

[She08b] D. Shelstad. Tempered endoscopy for real groups. III. Inversion of transfer and L-packet structure. *Represent. Theory*, 12:369–402, 2008.

[She10] D. Shelstad. Tempered endoscopy for real groups II. Spectral transfer factors. In *Automorphic forms and the Langlands program*, volume 9 of *Advanced lectures in mathematics*, pages 236–276, 2010.

[She12] D. Shelstad. On geometric transfer in real twisted endoscopy. *Ann. of Math. (2)*, 176:1919–1985, 2012.

[Spr79] T. A. Springer. Reductive groups. In *Automorphic forms, representations, and L-functions*, volume 33 of *Proc. Sympos. Pure Math.*, pages 3–27, 1979.

[Spr98] T. A. Springer. *Linear algebraic groups.* Birkhäuser, 1998.

[Ste97] R. Steinberg. Endomorphisms of linear algebraic groups. In *Collected works.* American Mathematical Society, 1997.

[SV80] B. Speh and D. A. Vogan. Reducibility of generalized principal series representations. *Acta Math.*, 145:227299, 1980.

[Var77] V. S. Varadarajan. *Harmonic analysis on real reductive groups*, volume 576 of *Lecture notes in math.* Springer, 1977.

[Vog82] D. A. Vogan. Irreducible characters of semisimple Lie groups. IV. *Duke Math. J.*, 49:943–1073, 1982.

[Wal88] N. Wallach. *Real reductive groups I.* Academic Press, 1988.

[Wal13] J.-L. Waldspurger. Préparation à la stabilisation de la formule des traces tordue III: intégrales orbitales et endoscopie sur un corps local archimédien. Preprint, May 2013.

[War72] G. Warner. *Harmonic analysis on semi-simple Lie groups II.* Springer, 1972.

[Wei94] C. A. Weibel. *An introduction to homological algebra.* Cambridge University Press, 1994.