Subtour Elimination Polytopes and Graphs of Inscribable Type

by

Kevin King Hin Cheung

A thesis
presented to the University of Waterloo
in fulfilment of the
thesis requirement for the degree of
Doctor of Philosophy
in
Combinatorics and Optimization

Waterloo, Ontario, Canada, 2003

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Abstract

The subtour-elimination polytope (SEP) of a graph G is the feasible region of the linear programming relaxation of an important integer linear programming formulation of the Symmetric Travelling Salesman Problem (STSP) on G.

The SEP is a well-studied combinatorial object at least in the case of the complete graph and is often used as a starting point for solving concrete STSPs. Remarkably, the SEP is related to the century-old geometry problem of determining if a 3-connected planar graph is of inscribable type, that is, realizable by a 3-dimensional polytope inscribed in the sphere. The problem was open for over a century until Rivin showed that G is of inscribable type if and only if there exists a point in the SEP of G that satisfies all the inequalities strictly.

In this thesis, we study the SEP of a general simple graph and graphs of inscribable type. In particular, questions on the existence of certain points in the SEP and the certificates of unsolvability of the system that defines the polytope are considered. Some graph operations that preserve the property that the system has a solution that satisfies all the inequalities strictly are also described. The SEP of a $(2k+1)$-edge-connected $(2k+1)$-regular graph is studied in detail. In particular, an efficient algorithm for computing the dimension of the SEP of such a graph is given. A detailed sketch of Rivin's elementary proof of his theorem is given in the context of the SEP. Descriptions of classes graphs of inscribable type are obtained in the thesis.

Acknowledgements

I am very thankful to my supervisor, Professor William Cunningham, for giving me guidance on my research.

I am also very thankful to my external examiner, Professor Komei Fukuda, and my internal examiners, Professor Timothy Chan, Professor James Geelen, and Professor Levent Tunçel for their insightful comments and suggestions.

Special thanks are due to the administrative staff at the Department of Combinatorics and Optimization for their help on myriads of administrative matters.

Sincere thanks are due as well to my colleagues at the Department of Combinatorics and Optimization, especially my fellow graduate students Ada Chan, Jason Chen, Youngbin Choe, Jackie Ho, Julia Huang, John Irving, Charles Lam, Marina Potaptchik, John Proos, Lawrence Tang, Kerri Webb, and Francisco Zaragoza for many stimulating and refreshing conversations.

I would like to thank some friends from my undergraduate years, in particular Gabriel Chow, Philip Ho, Viviffer Leong, Larry Leung, Silvia Tam, Maurice Wong, and Winnie Yeung, for sharing both good times and bad, then and now.

I would also like to thank my friends at 430, CMC, Opus Dei, and Unotone for their prayers and encouragement.

Finally, it was by the grace of God, the countless inspirations from my son Daniel, and the unconditional love and the undying support from my parents, my relatives, and my wife Jinah that I was able to come this far.

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Chapter 1

Introduction

A classical result that connects graph theory to geometry is the following:

Theorem 1.1. *(Steinitz' Theorem [46]) If G is a graph, then G is isomorphic to the graph of some 3-dimensional polytope P in $\mathbb{R}^3$ if and only if it is planar and 3-connected.*

In casual terms, the theorem asserts that a planar graph is 3-connected if and only if it can be realized by a 3-dimensional polytope in $\mathbb{R}^3$. Steinitz' Theorem has a number of extensions. For instance, Barnette and Grünbaum [1] showed that one can preassign the shape of a facet in the realization. There are also results on using only integral extreme points in the realization. However, one result that stands out among the numerous extensions of Steinitz' Theorem is the Koebe-Andreev-Thurston Circle Packing Theorem, which has garnered attention of graph theorists as well as geometers.

Other than extensions of Theorem 1.1, one can look for refinements of the result. For instance, one might ask what happens if the polytopes used in the realization are restricted to the ones that can be inscribed in a sphere. Graphs that can be so realized are said to be of inscribable type. The problem dates back to 1832 when Jakob Steiner in his book [45] asked the following question:

> In which cases does a convex polyhedron have a [combinatorial] equivalent which is inscribed in, or circumscribed about, a sphere?

Using a notion of duality, the problem of determining which convex polyhedra have a combinatorial equivalent which is circumscribed about a sphere can be reduced to the problem of determining which convex polyhedra have a combinatorial equivalent which is inscribed in a sphere. Hence, one can simply consider the following problem:

Which 3-connected planar graphs are of inscribable type?

The first examples of graphs that are not of inscribable type were found by Steinitz [47]. One of these examples is the truncated cube depicted in Figure 1.1.

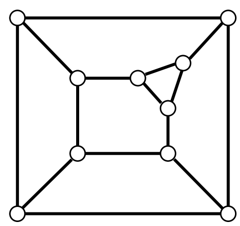

Figure 1.1: The truncated cube

The problem was open for well over a century until Rivin, in a series of papers published in the 1990's, gave a characterization of graphs of inscribable type in terms of the existence of an inner solution of a finite system of equations and linear inequalities. (An inner solution is one that satisfies all the inequalities of a system strictly.) Surprisingly, after scaling by a positive contant, the system is the same system that defines the subtour-elimination polytope of the graph. (The subtour-elimination polytope is the set of solutions of the linear programming relaxation of an important integer linear programming formulation of the Symmetric Travelling Salesman Problem.) Since one can find an inner solution of this system in polynomial time using linear programming methods, one can decide if a 3-connected planar graph is of inscribable type in polynomial time.

By studying the subtour-elimination polytope of 3-connected planar graphs, one can obtain a number of interesting necessary conditions as well as sufficient ones for graphs of inscribable type. For instance, Dillencourt and Smith [16] showed that 4-connected planar graphs and planar graphs obtained from 4-connected planar graphs by removing one vertex are of inscribable type. In [15], they gave necessary and sufficient conditions for a 3-connected 3-regular planar graph to be of inscribable type and a linear-time algorithm for recognizing such a graph. However, as Dillencourt and Smith [16] put it, "a general graph-theoretical characterization has remained elusive."

The remarkable, perhaps accidental, connection between the subtour-elimination polytope and the century-old geometry problem motivated the present study of the subtour-elimination polytope. In particular, the problem of determining when the system defining the subtour-elimination polytope has inner solutions is of great interest. The subtour-elimination polytope is a well-studied combinatorial object in the case of the complete graph because of its connection with the Symmetric Travelling Salesman Problem. However, the literature on the case when the graph is not necessarily complete is relatively sparse. One of the goals of the thesis is to obtain a deeper understanding of the subtour-elimination polytope of general graphs. Another goal is to further the work of Dillencourt and Smith on identifying more classes of graphs of inscribable type.

The thesis assumes the reader to be familiar with elementary Euclidean geometry and duality theory in the context of linear programming. Knowledge of linear programming algorithms is not required.

The rest of the thesis is organized as follows.

Chapter 2 contains basic notation and definitions on graphs and polytopes and states some existing results that are used in the thesis.

Chapter 3 begins with the definition of the system of linear inequalities defining the subtour-elimination polytope studied in this thesis. It then establishes some combinatorial properties of certain certificates of unsolvability of the system. Some simple sufficient conditions on the feasibility of the system and graphs that satisfy two notions of minimality are considered. The problem of determining if a graph with maximum degree three is feasible is then shown to be as difficult as the general problem. The chapter ends with a compact formulation of the subtour-elimination polytope in the case of planar graphs. This formulation is due to Rivin and will be used in Chapter 6 when Rivin's characterization is discussed.

Chapter 4 focuses on the study of the existence of inner solutions of the system. It begins with graph operations that preserve the existence of inner solutions. Many of these operations can be applied to graphs of inscribable type to obtain new ones. The most important of the operations discussed is the gluing operation, which allows one to construct a non-trivial class of maximal planar graphs of inscribable type that are not 4-connected. The chapter ends with a main result of the thesis, which gives a characterization for the system to have an inner solution. This characterization incorporates some combinatorial information which allows one to show that 4-connected planar graphs and 3-connected 3-regular planar bricks and braces are of inscribable type, thus providing, to a certain extent, a unified proof of two previous results obtained by Dillencourt and Smith.

Chapter 5 studies the subtour-elimination polytope of r-edge-connected r-regular graphs where r is odd and at least 3. In particular, the connection between the subtour-elimination polytope and the perfect matching polytope is exploited to obtain a dimension formula and an efficient combinatorial algorithm that computes the dimension. The algorithm can also be used to recognize 5-edge-connected 5-regular planar graphs that are of inscribable type. The results in this chapter are rather self-contained and can be read independently of the rest of the thesis.

Chapter 6 begins with a detailed sketch of Rivin's elementary proof of his characterization of graphs of inscribable type. It continues with a section on some graph-theoretical conditions. A refinement of a theorem of Wagner obtained by focusing on graphs of inscribable type ends the chapter.

Chapter 2

Preliminaries

2.1 Basic notation

In this thesis, $A \subseteq B$ means A is a subset of B and $A \subset B$ means A is a proper subset of B. The set of integers is denoted by $\mathbb{Z}$ and the set of real numbers is denoted by $\mathbb{R}$. $\mathbb{R}_+$ denotes the set $\{x \in \mathbb{R} : x \geq 0\}$ and $\mathbb{R}_{++}$ denotes the set $\{x \in \mathbb{R} : x > 0\}$. Vectors are written as columns unless otherwise stated. The vector of 1's is denoted by $\mathbf{e}$.

Let S be a finite set. Let $x, y \in \mathbb{R}^S$. If $A \subseteq S$, then $x(A)$ denotes the sum $\sum_{e \in A} x_e$. We write $x \geq y$ if $x_e \geq y_e$ for all $e \in S$. Similar notation can be defined for strict inequalities. If $T \subseteq S$, then the vector $x \in \mathbb{R}^S$ with $x_e = 1$ if $e \in T$ and $x_e = 0$ if $e \notin T$ is called the *incidence vector* of T.

2.2 Graphs

A *graph* G is an ordered pair (V, E) where V is a set whose elements are called *vertices* (or *nodes*) and E is a multi-set whose elements are two-element subsets of V called *edges*. Note that loops are not allowed under this definition. We can also denote the set of vertices by $V(G)$ and the set of edges by $E(G)$. We call $|V| + |E|$ the *size* of G.

An edge $\{u, v\}$ *joins* the vertices u and v and is denoted by uv for simplicity. The vertices u and v are said to be *adjacent* and are called *end-vertices* of the edge uv. Edges that have the same end-vertices are called *multiple edges*. If E happens to be a set, then G is said to be *simple*.

For a subset $S \subseteq V$, $G - S$ is the graph obtained from G by removing S from V and all the edges with an end-vertex in S from E. For a subset $F \subseteq E$, $G - F$ is the graph obtained from G by removing F from E. If u and v are non-adjacent, then $G + uv = (V, E \cup \{uv\})$.

Let G_1 and G_2 be two simple graphs. Then $G_1 \cup G_2$ denotes the graph $(V(G_1) \cup V(G_2), E(G_1) \cup E(G_2))$ and $G_1 \cap G_2$ denotes the graph $(V(G_1) \cap V(G_2), E(G_1) \cap E(G_2))$. G_1 and G_2 are said to be *isomorphic* (written as $G_1 \cong G_2$) if there exists a bijection $f : V(G_1) \to V(G_2)$ such that $f(u)f(v) \in E(G_2)$ if and only if $uv \in E(G_1)$.

Given a vertex v, any edge that has v as an end-vertex is said to be *incident with* v. Vertices adjacent to v are called the *neighbours* of v. The *degree* of v, denoted by $\deg(v)$, is the number of neighbours of v. A set $S \subseteq V$ is called an *independent set* if no two vertices in S are adjacent.

A graph $H = (V', E')$ is a *subgraph* of G if $V' \subseteq V$ and $E' \subseteq E$. If $V' = V$, H is a *spanning subgraph* of G. If $H \neq G$, then H is called a *proper subgraph* of G. If H contains all the edges of G that join two vertices in V', then H is said to be the subgraph *induced* by V' and is denoted by $G[V']$. If $H = G[V']$, then H is an *induced* subgraph of G.

A simple graph is *complete* if every vertex is adjacent to every other vertex. Since any two complete graphs having the same number of vertices are isomorphic, we simply call a complete graph on n vertices a K_n.

G is *connected* if for any distinct non-adjacent $u, v \in V(G)$, there exist $v_1, ..., v_k \in V(G)$ for some $k \geq 1$ such that $uv_1, v_1v_2, v_2v_3, ..., v_{k-1}v_k, v_kv \in E(G)$. G is *k-edge-connected* ($k \geq 2$) if it has at least two vertices and $G - F$ is connected for any $F \subseteq E$ with $|F| \leq k - 1$; G is *k-connected* ($k \geq 2$) if either G has a K_{k+1} as a spanning subgraph or it has at least $k+2$ vertices and $G - S$ is connected for every set $S \subseteq V$ with $|S| \leq k - 1$. A vertex v of a connected graph G is a *cut-vertex* if $G - v$ is not connected. A set $S \subseteq V$ is called a *k-separator* if $|S| = k$ and $G - S$ is not connected. An edge e of a connected graph G is a *bridge* if $G - e$ is not connected.

A maximal connected subgraph of G is called a *component* of G. If G is 2-connected, a subgraph B of G is a *block of* G if either B is a bridge (together with its end-vertices) or it is a maximal 2-connected subgraph of G.

A *cycle* is a connected simple graph such that every vertex has degree two. We say that a graph G has (or contains) a cycle if there exists a subgraph of G that is a cycle. It is not difficult to see that one can label the vertices of the cycle as $v_0, ..., v_{k-1}$ such that $v_i v_{i+1}$ is an edge for all $i = 0, ..., k-1$ where addition is performed modulo k. Such a labelling is called a *cyclic order* of the vertices. The edge-set of a cycle is called a *circuit*. A *tree* is a connected simple graph that has no cycles. It is known that the number of edges in a tree is one fewer than the number of vertices. A *leaf* of a tree is vertex of degree one. A *path* is a tree with exactly two leaves which the path *connects*. It is easy to see that every vertex on a path that is not a leaf has degree two and given any two distinct vertices in a tree, there is a unique path connecting them. In addition, it is not difficult to see that one can label the vertices of a path as $v_1, ..., v_k$ such that $v_i v_{i+1}$ is an edge for $i = 1, ..., k-1$. Such a labelling is called a *sequential order* of the vertices. A *rooted tree* is a tree with a distinguished vertex called the *root*. In a rooted tree T with root R, the *parent* of $v \in V(T)$ is the neighbour of v on the path connecting v and R. If vertices $u, v \in V(T)$ are such that v is on the path connecting u and R, then u is called a *descendant* of v and v is called an *ancestor* of u. A *child* of v is a descendant of v adjacent to v.

The number of edges in a path or a cycle is the *length*. The *distance* between two vertices is the length of the shortest path connecting them. An *odd cycle* is a cycle having odd length. An *odd circuit* is a circuit having an odd number of edges. An *even cycle* is a cycle having even length. An *even circuit* is a circuit having an even number of edges. A cycle of length three is called a *triangle*. A *path in* G is a subgraph of G that is a path. A *cycle in* G is a subgraph of G that is a cycle. A cycle C in G is a *chordless cycle* if there does not exists $u, v \in V(C)$ such that $uv \in E(G) \backslash E(C)$. A triangle T in G is *a separating triangle* if $G - V(T)$ is not connected.

G is *bipartite* if the vertices can be partitioned into two sets U and W such that no two vertices in U are adjacent and no two vertices in W are adjacent. In this case, (U, W) is called a *bipartition* and U and W are called *partitions*. It is easy to show that G is bipartite if and only if G has no odd cycle. A *complete bipartite graph* is a simple bipartite graph with bipartition (U, W) in which each vertex of U is joined to each vertex of W; if $|U| = m$ and $|W| = n$, such a graph is denoted by $K_{m,n}$.

G is *planar* if it can be drawn in the plane so that its edges intersect only at their ends. Such a drawing is called a *planar embedding* of G. A planar embedding $\hat{G}$ of G can be seen as a graph isomorphic to G with the understanding that $V(\hat{G})$ is the set of points representing vertices of G and $E(\hat{G})$ is the (multi-)set of lines representing edges of G and a vertex of $\hat{G}$ is incident with all the edges of $\hat{G}$ containing it. Therefore, we can refer to a planar embedding of a planar graph as a *plane graph*. A *maximal planar graph* is a simple planar graph having at least three vertices such that joining any two non-adjacent vertices will result in a non-planar graph. A planar embedding of a maximal planar graph is a *plane triangulation*.

A plane graph partitions the rest of the plane into a number of connected open regions. The closures of these regions are called the *faces*. Each plane graph has a unique unbounded face called the *exterior face*. The boundary of the exterior face is the *boundary of a plane graph*. A bounded face is called an *interior face*. A face is *incident* with the vertices and edges in its boundary. A triangle that bounds a face is called a *face triangle*. If e is a bridge in a plane graph, only one face is incident with e; otherwise two faces are incident with e. We say an edge *separates* the faces incident with it. A *Jordan curve* is a continuous non-self-intersecting curve whose origin and terminus coincide. The union of the elements of a circuit of a plane graph constitutes a Jordan curve. It is known that if v is a vertex of a planar graph H, then H can be embedded in the plane such that v is on the exterior face of the embedding.

If G is a 2-connected plane graph, the *planar dual* of G, denoted by G^*, is defined as follows: there is a vertex f^* of G^* corresponding to each face f of G and there is an edge e^* of G^* corresponding to each edge e of G; two vertices f^* and g^* are joined by the edge e^* in G^* if and only if their corresponding faces f and g are separated by the edge e in G.

A path in G that contains every vertex of G is called a *Hamiltonian path*. A cycle in G that contains every vertex of G is called a *Hamiltonian cycle*. The edge-set of a Hamiltonian cycle is called a *Hamiltonian circuit*. A graph is *Hamiltonian* if it has a Hamiltonian cycle. Tutte [49] showed that every 4-connected planar graph is Hamiltonian. The following strengthening of Tutte's result due to Sanders [42] will be used a few times in this thesis.

Theorem 2.1. *If G is a 4-connected planar graph, then there exists a Hamiltonian circuit through any two edges.*

For $S, T \subset V$, $\gamma(S, T)$ denotes the (multi-)set of edges joining an end-vertex in S and an end-vertex in T. For a subset S of V, let $N(S) = \{v \in V \backslash S : v$ is adjacent to a vertex in $S\}$. $N(\{v\})$ is abbreviated as $N(v)$. If $|S| > 1$, let $G \times S$ denote the graph obtained from G by *shrinking* S, that is, removing all the vertices in S and all the edges incident with a vertex in S from G and adding a new vertex v and edges uv for every edge $us \in E$ where $s \in S$. The new vertex v is called a *pseudo-vertex* of $G \times S$. By convention, $G \times \{v\} = G$ for all $v \in V$. Let $\delta_G(S)$ denote $\gamma(S, V \backslash S)$. If the graph G is clear in the context, we simply write $\delta(S)$. $\delta(S)$ is called a *cut* of G and if G is connected, $S, V \backslash S$ are the *shores* of the cut $\delta(S)$. A shore S is called a *proper shore* if $|S| \leq |V| - 2$. Cuts of the form $\delta(\{v\})$ (abbreviated as $\delta(v)$) where v is a vertex are called *trivial* cuts. All other cuts are called *non-trivial* cuts. We denote the set of non-trivial cuts of G by $C(G)$. Two cuts $\delta(S)$ and $\delta(T)$ are said to *cross* if the four sets $S \cap T$, $S \backslash T$, $T \backslash S$, and $V \backslash (S \cup T)$ are all non-empty. Two cuts that do not cross are said to be *non-crossing*.

The number of components of G is denoted by $\omega(G)$. G is *1-tough* if $|S| \geq \omega(G - S)$ for every subset S of V with $\omega(G - S) > 1$. G is *more-than-1-tough* if $|S| > \omega(G - S)$ for every subset S of V with $\omega(G - S) > 1$. Equivalently, G is more-than-1-tough if $G - v$ is 1-tough for every $v \in V$.

A family $\mathcal{F}$ of sets is called *nested* if for any non-disjoint distinct elements $S, T \in \mathcal{F}$, either $S \subset T$ or $T \subset S$. Given a nested family $\mathcal{F}$, one can construct a rooted tree $T(\mathcal{F})$ as follows: For each element in $\mathcal{F}$, create a vertex corresponding to the element. Create an extra vertex R to be the root. For each $S_1 \in \mathcal{F}$, let $S_2 \in \mathcal{F}$ be the minimal set properly containing S_1. If no such set exists, then put an edge between R and the vertex corresponding to S_1. Otherwise, put an edge between the vertex corresponding to S_1 and that corresponding to S_2.

For a graph $G = (V, E)$, a subset M of E is a *matching* of G if no two edges in M share a common endvertex. A *perfect matching* of G is a matching of G having cardinality $|V|/2$. The following characterization is due to Tutte.

Theorem 2.2. *G has a perfect matching if and only if for every $S \subset V$, $odd(G-S)$ is at most $|S|$. (Here, $odd(H)$ denotes the number of components of H having an odd number of vertices.)*

A cut $A \in C(G)$ is *tight* if every perfect matching of G uses exactly one edge in A. G is called *matching-covered* if for every edge $e \in E$, there exists a perfect matching that contains e. G is said to be *bicritical* if $G - \{u, v\}$ has a perfect matching for every pair $u, v \in V$. A 3-connected bicritical graph is called a *brick*. A bipartite graph G with bipartition (U, W) is called a *brace* if $G - \{u, w, u', w'\}$ has a perfect matching for any two vertices $u, u' \in U$ and two vertices $w, w' \in W$. It can be shown that a bipartite graph G is a brace if and only if $|U| = |W|$ and for any subset X of U (or of W) with $2 \leq |X| \leq |U| - 2$ (or $2 \leq |X| \leq |W| - 2$, $|N(X)| \geq |X| + 2$.

Let PM(G) denote the convex hull of incidence vectors of perfect matchings of G. An important result in matching theory is the following:

Theorem 2.3. *(Edmonds [17])* PM(G) *is the set of solutions to the system:*

$$\begin{aligned} x(\delta(v)) &= 1 \qquad \forall\, v \in V, \\ x(\delta(S)) &\geq 1 \qquad \forall\, S \subset V, 3 \leq |S| \leq \tfrac{|V|}{2}, |S| \text{ is odd} \\ x &\geq 0. \end{aligned}$$

An immediate consequence of the above theorem is the following:

Corollary 2.4. *If* PM(G) *is non-empty, then G has a perfect matching. Furthermore, if* PM(G) *contains a point $\hat{x} > 0$, then G is matching-covered.*

We will use the two results above a few times in the thesis.

From now on, simple graphs are called graphs. A graph that is not simple will be called a *multigraph*.

2.3 Polytopes

A set $C \subseteq \mathbb{R}^n$ is *convex* if for every pair $x, y \in C$ and every $\lambda \in [0,1]$, $\lambda x + (1-\lambda)y \in C$.

Given $S \subseteq \mathbb{R}^n$, the *convex hull* of S, denoted by $\operatorname{conv}(S)$, is the smallest convex set containing S. A *polytope* in $\mathbb{R}^n$ is the convex hull of a finite subset of $\mathbb{R}^n$. If $P \subseteq \mathbb{R}^n$ is a polytope, its *dimension* is one less than the maximum cardinality of an affinely-independent set $S \subseteq P$. If the dimension of P is n, then P is said to be *full-dimensional* or *of full dimension*.

A linear inequality $a^T x \leq b$ is *valid* for a polytope P if $P \subseteq \{x : a^T x \leq b\}$. If $a^T x \leq b$ is a valid inequality, the set $\{x \in P : a^T x = b\}$ is called the *face* of P induced by the inequality. A *proper face* of P is a face not equal to P. A maximal proper face of P is called a *facet*. An important result in polyhedral theory states that given a polytope, there exists a finite system of linear inequalities whose set of solutions is the polytope. If the polytope is full-dimensional, then it has a unique (up to positive scalar multiples) minimal defining system consisting of only the facet-inducing inequalities. The dimension of a facet is exactly one less than the dimension of the polytope.

We call $x \in P$ an *extreme point* of P if there do not exist $x', x'' \in P$, $x' \neq x''$ and $\lambda \in (0,1)$ such that $x = \lambda x' + (1-\lambda)x''$. Two extreme points of P are said to be *adjacent* in P if they are in the same 1-dimensional face of P.

A *d-simplex* in $\mathbb{R}^n$ is the convex hull of some $d+1$ affinely-independent points. A d-dimensional polytope is called *simplicial* if all its facets are $(d-1)$-simplices.

If P is a 3-dimensional polytope, we denote the graph of P by $G(P)$; i.e. $G(P)$ is the graph (V, E) where V is the set of extreme points of P and $uv \in E$ if and only if u and v are adjacent in P. It is not difficult to see that if P is simplicial, then $G(P)$ is a maximal planar graph. If $G \cong G(P)$ for some 3-dimensional polytope P inscribed in a sphere, then G is said to be of *inscribable type*.

Chapter 3

Feasibility

Let G be a graph having at least three vertices. Let $\mathrm{sys}(G)$ denote the system

$$\begin{aligned} x(\delta(v)) &= 2 \;\; \forall\, v \in V(G), \\ x(A) &\geq 2 \;\; \forall\, A \in C(G), \\ x &\geq 0 \end{aligned}$$

The *subtour-elimination polytope of* G is defined as

$$\mathrm{SEP}(G) := \{x \in \mathbb{R}^E : x \text{ is a solution of } \mathrm{sys}(G)\}.$$

The constraints

$$x(\delta(v)) = 2 \;\; \forall\, v \in V(G)$$

are called *degree constraints* and the constraints

$$x(A) \geq 2 \;\; \forall\, A \in C(G)$$

are called *subtour-elimination constraints.*

G is said to be *feasible* if $\mathrm{SEP}(G)$ is non-empty. Otherwise, G is said to be *infeasible.*

It is easy to see that the convex hull of integral points in $\mathrm{SEP}(G)$ is the same as the convex hull of incidence vectors of Hamiltonian circuits of G. As a result, the subtour-elimination polytope of a graph gives a natural relaxation of the Travelling Salesman Problem (TSP). Despite the exponential number of constraints, using the equivalence of separation and optimization (see Grötschel et al. [24]), one can optimize a linear function over SEP(G) in time polynomial in the size of the graph G. Over the years, SEP(G) has been studied by various researchers in the case when G is a complete graph. For instance, Boyd and Pulleyblank [4] studied the structure of optimal solutions when certain cost functions are optimized over the subtour-elimination polytope. They also showed that the extreme points are not "nice" in general.

In this chapter, we study properties of SEP(G) when G is not necessarily complete. In particular, we look at some extreme cases in our study. We first derive necessary and sufficient conditions for SEP(G) to be empty and study the certificates of unsolvability of sys(G). We then look at some simple results on feasible graphs. Two notions of minimality are discussed. The problem of determining if a graph with maximum degree three is feasible is shown to be as difficult as the general problem. We conclude the chapter by giving the compact formulation of the subtour-elimination polytope of 2-connected planar graphs due to Rivin.

3.1 Certificates of unsolvability

In this section, we consider the question of when a graph is feasible. The class of feasible graphs sits between the class of Hamiltonian graphs and the class of 1-tough graphs in the sense that every Hamiltonian graph is feasible and every feasible graph is 1-tough but not all 1-tough graphs are feasible and not all feasible graphs are Hamiltonian. Since testing if a graph is feasible is polynomial-time solvable whereas the problem of determining if a graph is Hamiltonian and the problem of determining if a graph is 1-tough are both $\mathcal{NP}$-hard, one might ask if there is a simple combinatorial characterization of feasible graphs.

As sys(G) is a system of linear equations and inequalities, the Farkas Lemma gives a necessary and sufficient condition for it to have no solution. As a step towards obtaining a simple combinatorial characterization of feasible graphs, the goal of this section is to explore the combinatorial structure of "certificates of unsolvability" and to obtain strengthenings of the Farkas Lemma for the system sys(G). In particular, we observe that non-1-tough graphs are characterized by "nice" certificates of unsolvability. We show that, however, for a class of infeasible graphs, there are no nice certificates of unsolvability.

Now, let us recall one form of the Farkas Lemma.

Theorem 3.1. *Let $A \in \mathbb{R}^{p\times n}, B \in \mathbb{R}^{q\times n}$, $b \in \mathbb{R}^p$, $d \in \mathbb{R}^q$. The system*

$$Ax = b,\ \ Bx \geq d,\ \ x \geq 0$$

has no solution if and only if there exist $y \in \mathbb{R}^q_+$ and $z \in \mathbb{R}^p$ such that

$$A^T z - B^T y \geq 0 \text{ and } d^T y > b^T z. \tag{3.1}$$

A pair y, z that satisfies (3.1) is often called a *certificate of unsolvability.* Observe that in the case when A, B, b and d are rational, if there is a certificate of unsolvability, there exists one that is rational. In fact, we can assume it to be integral because the linear inequalities that it need to satisfy are homogeneous (that is, having constant term 0).

Let sys$'(G)$ denote the system

$$\begin{aligned} z_u + z_v - \sum_{uv\in A\in C(G)} y_A &\geq 0 \qquad \text{for all } uv \in E(G) \\ y &\geq 0. \end{aligned}$$

We obtain the following refinement of the Farkas Lemma for the system sys(G).

Theorem 3.2. *If G is connected, then* SEP(G) *is empty if and only if there exist $\bar{y}, \bar{z}$ feasible for sys$'(G)$ such that $\sum_{A\in C(G)} \bar{y}_A > \sum_{v\in V(G)} \bar{z}_v$ and $\{A \in C(G) : \bar{y}_A > 0\}$ is a non-crossing family of cuts.*

The proof of Theorem 3.2 relies on the notion of uncrossing which we now describe. Let $\bar{y}, \bar{z}$ be integral and feasible for $\mathrm{sys}'(G)$. Let $\mathcal{A}(\bar{y})$ denote the set $\{A \in C(G) : \bar{y}_A > 0\}$. Let $\delta(S)$ and $\delta(T)$ be crossing cuts in $\mathcal{A}(\bar{y})$.

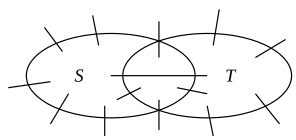

Figure 3.1: Crossing cuts

By *uncrossing* $\delta(S)$ and $\delta(T)$, we mean applying the following modifications: Let $\rho = \min\{\bar{y}_{\delta(S)}, \bar{y}_{\delta(T)}\}$. If $S \cap T = \{v\}$ for some $v \in V(G)$, then decrease $\bar{z}_v$ by ρ; otherwise, increase $\bar{y}_{\delta(S\cap T)}$ by ρ. If $V(G)\backslash(S \cup T) = \{v\}$ for some $v \in V(G)$, then decrease $\bar{z}_v$ by ρ; otherwise increase $\bar{y}_{\delta(S\cup T)}$ by ρ. Decrease $\bar{y}_{\delta(S)}$ and $\bar{y}_{\delta(T)}$ by ρ.

This technique of uncrossing is quite common in combinatorics. (See for instance Chapter 4 of [21].) The next result is a specialization of the technique for the purposes of the current thesis. The idea of the proof is very similar to the idea employed by Edmonds, Lovász, and Pulleyblank to prove Claim 1 of Theorem 4.7 in [18].

Lemma 3.3. *Given an integral pair $\bar{y}, \bar{z}$ feasible for $\mathrm{sys}'(G)$, one can obtain, by performing a finite number of uncrossings, an integral pair y', z' feasible for $\mathrm{sys}'(G)$ such that $\sum_{A \in C(G)} \bar{y}_A - \sum_{v \in V(G)} \bar{z}_v = \sum_{A \in C(G)} y'_A - \sum_{v\in V(G)} z'_v$ and $\{A \in C(G) : y'_A > 0\}$ is a non-crossing family of cuts.*

Proof. For $y \in \mathbb{Z}^{C(G)}$, let $M(y)$ denote $\sum_{A\in C(G)} \sum_{B \in C(G)} \pi_y(A,B)$ where

$$\pi_y(A,B) = \begin{cases} y_A y_B & \text{if } A, B \text{ cross;} \\ 0 & \text{otherwise.} \end{cases}$$

Let $\mathcal{A}(\bar{y})$ denote $\{A \in C(G) : \bar{y}_A > 0\}$. If $M(\bar{y}) = 0$, then $\mathcal{A}(\bar{y})$ is a non-crossing family of cuts and we are done. Suppose $M(\bar{y}) > 0$. Then there exist $S, T \subset V(G)$ such that $\delta(S), \delta(T) \in \mathcal{A}(\bar{y})$ cross. Pick any such pair S, T. Let $A = \delta(S)$ and $B = \delta(T)$. Uncross

A and B to obtain y', z'. It is not difficult to see that y', z' are still feasible for sys$'(G)$ and $\sum_{A \in C(G)} \bar{y}_A - \sum_{v \in V(G)} \bar{z}_v = \sum_{A \in C(G)} y'_A - \sum_{v \in V(G)} z'_v$. Now, given any cut $C \in C(G) \backslash \{A, B\}$, there are four possibilities:

(i) A and C cross but B and C do not cross.

(ii) B and C cross but A and C do not cross.

(iii) A and C do not cross and B and C do not cross.

(iv) None of the above.

It is not difficult to see that in any case, $\sum_{C' \in C(G)} \pi_{y'}(C, C') \leq \sum_{C' \in C(G)} \pi_{\bar{y}}(C, C')$. However, $\pi_{y'}(A, B) = 0 < \pi_{\bar{y}}(A, B)$. Hence, $M(y') < M(\bar{y})$. Since $M(y)$ is integral for all integral y, the result follows. □

Corollary 3.4. *If there exist $\bar{y}, \bar{z}$ feasible for sys$'(G)$ satisfying $\sum_{A \in C(G)} \bar{y}_A > \sum_{v \in V(G)} \bar{z}_v$, then there exist y', z' feasible for sys$'(G)$ such that $\sum_{A \in C(G)} y'_A > \sum_{v \in V(G)} z'_v$ and $\{A \in C(G) : y'_A > 0\}$ is a non-crossing family of cuts.*

Proof. We may assume that $\bar{y}$ and $\bar{z}$ are rational. Since sys$'(G)$ is homogeneous, we can clear fractions and assume that $\bar{y}$ and $\bar{z}$ are in fact integral. The result now follows from Lemma 3.3. □

Proof of Theorem 3.2. This follows directly from Theorem 3.1 and Corollary 3.4. □

In the case of non-1-tough graphs, we can say more about the certificates of unsolvability. Before we proceed to the discussion, we state a well-known technical result that will be used a few times later in the thesis.

Lemma 3.5. *Let $G = (V, E)$ be a connected graph. If $\mathcal{A}$ is a non-crossing family of cuts of G, then there exists a nested family $S(\mathcal{A}) \subset 2^V$ that contains precisely one proper shore of each cut in $\mathcal{A}$.*

Proof. For each cut $A \in \mathcal{A}$, pick a shore that has at most half the number of vertices in the graph and put it in $S(\mathcal{A})$. Clearly, $S(\mathcal{A})$ contains precisely one shore of each cut in $\mathcal{A}$.

We now show that $S(\mathcal{A})$ is a nested family. Suppose there exist $S, T \in S(\mathcal{A})$ such that $S \cap T \neq \emptyset$, $S \backslash T \neq \emptyset$, and $T \backslash S \neq \emptyset$. Since $\delta(S)$ and $\delta(T)$ do not cross, we must have $V \backslash (S \cup T) = \emptyset$. However, this is impossible since $|S|, |T| \leq \frac{|V|}{2}$. The result follows. □

Theorem 3.6. *If $G = (V, E)$ is connected, then G is not 1-tough if and only if there exist $\bar{y}, \bar{z}$ feasible for $sys'(G)$ such that $\bar{y} \in \{0,1\}^{C(G)}$, $\bar{z} \in \{-1, 0, 1\}^V$, $\sum_{A \in C(G)} \bar{y}_A > \sum_{v \in V} \bar{z}_v$, and $\{A \in C(G) : \bar{y}_A > 0\}$ is a non-crossing family of cuts.*

Proof. Let $\mathcal{C}$ denote the set of all the cuts of G.

Observe that such $\bar{y}, \bar{z}$ exist if and only if there exist $y' \in \{0,1\}^{\mathcal{C}}$ and $z' \in \{0,1\}^V$ satisfying

$$\begin{array}{lrcll} (D) & z'_u + z'_v - \sum_{uv \in A \in \mathcal{C}} y'_A & \geq & 0 & \text{for all } uv \in E \\ & y',\ z' & \geq & 0 & \end{array}$$

such that $\sum_{A \in \mathcal{C}} y'_A > \sum_{v \in V} z'_v$ and $\{A \in \mathcal{C} : y'_A > 0\}$ is non-crossing. Hence, it suffices to show that such y', z' exist if and only if G is not 1-tough.

Suppose G is not 1-tough. Then there exists $S \subset V$, $S \neq \emptyset$, such that $\omega(G - S) > |S|$. Let $S_1, ..., S_k$ denote the vertex-sets of the components of $G - S$. Construct $y' \in \{0,1\}^{\mathcal{C}}$ and $z' \in \{0,1\}^V$ as follows. For each $v \in S$, set $z'_v = 1$. For each $i \in \{1, ..., k\}$, set $y'_{\delta(S_i)} = 1$. Set all the remaining entries to zero. Clearly, $\sum_{A \in \mathcal{C}} y'_A = k > |S| = \sum_{v \in V} z'_v$. Moreover, it is easy to see that y', z' are feasible for (D) and $\{A \in \mathcal{C} : y'_A > 0\}$ is non-crossing.

Conversely, suppose such y', z' exist. Let $\mathcal{A} = \{A \in \mathcal{C} : y'_A = 1\}$. Choose y', z' so that $k = |\mathcal{A}|$ is as small as possible. By Lemma 3.5, there exists a nested family $S(\mathcal{A})$ that contains exactly one shore of each cut of G. Let $S_1, ..., S_k$ denote the elements in $S(\mathcal{A})$. Then for any distinct i, j, if $S_i \cap S_j \neq \emptyset$, either $S_i \subset S_j$ or $S_j \subset S_i$.

Observe that if i is such that $S_j \not\subset S_i$ for all $j \neq i$, then $z'_v = 0$ for all $v \in S_i$. Indeed, if $z'_v = 1$ for some $v \in S_i$, then setting $y'_{\delta(S_i)}$ and z'_v to zero gives a new pair y', z' that still satisfy all the conditions, but the number of ones in y' is $k - 1$. This contradicts our assumption of minimality.

We consider two cases.

Case 1. There does not exist j such that $S_i \subset S_j$ for some $i \in \{1, ..., k\} \backslash \{j\}$.

Let $U = \{v \in V : z'_v > 0\}$. Since $z'_v = 0$ for all $v \in S_i$, any edge $e \in \delta(S_i)$ must have one end in U. This implies that the number of components in $G - U$ is $\sum_A y'_A > \sum_v z'_v = |U|$. Hence, G is not 1-tough.

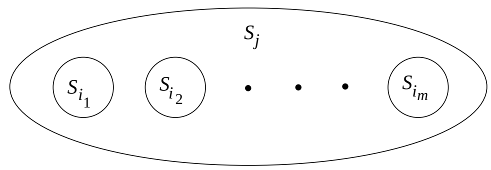

Figure 3.2: Illustration for Case 1

Case 2. There exist j and $\{i_1, ..., i_m\}$, $m \geq 1$, such that $S_{i_p} \subset S_j$ for $p = 1, ..., m$. (See Figure 3.2.)

Then there exists such a j so that for any $T \subset S_j$ with $T \notin \{S_{i_1}, ..., S_{i_m}\}$, $y'_{\delta(T)} = 0$. Note that $z'_u = 0$ for any $u \in S_{i_p}$, where $p = 1, ..., m$. Consider an edge $uv \in \delta(S_{i_p})$ with $u \in S_{i_p}$. If $v \notin S_j$, then $1 \geq z'_u + z'_v \geq \sum_{uv \in A} y'_A \geq y'_{\delta(S_j)} + y'_{\delta(S_{i_p})} = 2$, which is impossible. Hence, we must have $v \in S_j$. In addition, there cannot be an edge ww' with $w \in S_{i_p}$ and $w' \in S_{i_q}$ for any $p \neq q$. Let $W = \{v \in S_j : z'_v = 1\}$. Since $V \backslash S_j \neq \emptyset$, $\omega(G - W) \geq m + 1$. If $|W| \leq m$, then G is not 1-tough. Otherwise, set $z'_v = 0$ for all $v \in W$, $y'_{\delta(S_{i_p})} = 0$ for all $p = 1, ..., m$, and $y'_{\delta(S_j)} = 0$. The new y', z' still satisfy all the conditions but the number of ones in y' is strictly less than k, contradicting our assumption of minimality. □

We next give two results that show that non-1-tough (or infeasible) planar graphs can be "embedded" into maximal planar graphs that are non-1-tough (or infeasible). Both results depend on the following technical lemma which contains the bulk of the work.

Lemma 3.7. *Suppose G has a chordless cycle F of length at least four. Let $\bar{y}, \bar{z}$ be feasible for $sys'(G)$ such that $\mathcal{A} = \{A \in C(G) : \bar{y}_A > 0\}$ is a non-crossing family of cuts. Then* *there exist two non-adjacent vertices w, w' on F such that, where $H = G + ww'$ and*

$$y'_A = \begin{cases} \bar{y}_A & \text{if } ww' \notin A \\ \bar{y}_{A \setminus \{ww'\}} & \text{if } ww' \in A \end{cases}$$

for each $A \in C(H)$,

$$\bar{z}_w + \bar{z}_{w'} - \sum_{ww' \in A \in C(H)} y'_A \geq 0.$$

Furthermore, $\{A \in C(H) : y'_A > 0\}$ is a non-crossing family of cuts, the pair $y', \bar{z}$ is feasible for sys$'(H)$, and

$$\sum_{A \in C(H)} y'_A = \sum_{A \in C(G)} \bar{y}_A.$$

Theorem 3.8. *If G is a 3-connected planar graph that is not 1-tough, then G is a spanning subgraph of a maximal planar graph that is not 1-tough.*

Proof. Let G' be a non-1-tough planar graph having G as a spanning subgraph and having as many edges as possible. Note that G' is 3-connected. If G' is not a maximal planar graph, then there exists a chordless cycle F of length at least four such that in a planar embedding of G', F bounds a face. Since G' is not 1-tough, by Theorem 3.6, there exist $\bar{y} \in \{0,1\}^{C(G')}$ and $\bar{z} \in \{-1,0,1\}^{V(G')}$ feasible for sys$'(G')$ such that $\sum_{A \in C(G')} \bar{y}_A > \sum_{v \in V(G')} \bar{z}_v$ and $\{A \in C(G') : \bar{y}_A > 0\}$ is a non-crossing family of cuts. By Lemma 3.7, there exist non-adjacent vertices w, w' on F such that if $H = G' + ww'$ and $y'_A = \bar{y}_A$ for every $A \in C(H)$ with $ww' \notin A$ and $y'_A = \bar{y}_{A \setminus \{ww'\}}$ for every $A \in C(H)$ with $ww' \in A$, then $\{A \in C(H) : y'_A > 0\}$ is a non-crossing family of cuts and the pair $y', \bar{z}$ is feasible for sys$'(H)$ and satisfies $\sum_{A \in C(H)} y'_A > \sum_{v \in V(H)} \bar{z}_v$. Since $y' \in \{0,1\}^{C(H)}$ and $\bar{z} \in \{-1,0,1\}^{V(H)}$, by Theorem 3.6, H is not 1-tough. Now, H is a planar graph having G as a spanning subgraph and has one more edge than G'. We have a contradiction. □

A similar result holds for infeasible graphs in general.

Theorem 3.9. *If G is a 3-connected infeasible planar graph, then G is a spanning subgraph of an infeasible maximal planar graph.*

Proof. Using Theorem 3.2 and Lemma 3.7, one can prove this result using a similar argument used in the proof of the previous theorem. □

Proof of Lemma 3.7. The second part of the lemma follows immediately from the first part.

Let the vertices of F, in cyclic order, be $v_1, ..., v_k$. Let $U_i = \{A \in \mathcal{A} : v_{i-1}v_i \in A\}$. We use the convention that $v_0 = v_k$ and $v_{k+1} = v_1$. Let W_i denote the set $U_i \cap U_{i+1}$.

First, suppose there exists $i \in \{1, ..., k\}$ such that $\bar{z}_{v_i} \leq \sum_{A \in W_i} \bar{y}_A$.

Now

$$\begin{aligned} \bar{z}_{v_{i-1}} + 2\bar{z}_{v_i} + \bar{z}_{v_{i+1}} &= (\bar{z}_{v_{i-1}} + \bar{z}_{v_i}) + (\bar{z}_{v_i} + \bar{z}_{v_{i+1}}) \\ &\geq \sum_{v_{i-1}v_i \in A} \bar{y}_A + \sum_{v_i v_{i+1} \in A} \bar{y}_A \\ &= \sum_{A \in U_i} \bar{y}_A + \sum_{A \in U_{i+1}} \bar{y}_A \\ &= \sum_{A \in U_i \backslash W_i} \bar{y}_A + \sum_{A \in U_{i+1} \backslash W_i} \bar{y}_A + 2 \sum_{A \in W_i} \bar{y}_A \end{aligned}$$

Thus

$$\bar{z}_{v_{i-1}} + \bar{z}_{v_{i+1}} \geq \sum_{A \in U_i \backslash W_i} \bar{y}_A + \sum_{A \in U_{i+1} \backslash W_i} \bar{y}_A.$$

Add the edge $e = v_{i-1}v_{i+1}$ to G to form H. Let $C \in C(H)$ be such that $e \in C$. Then $C = \delta(S)$ for some $S \subset V$ such that S contains v_i and exactly one of v_{i-1} and v_{i+1}. From this we see that $C \backslash \{e\} \notin W_i$. Hence, either $C \backslash \{e\} \in U_i$ or $C \backslash \{e\} \in U_{i+1}$. It follows that $C \in (U_i \backslash W_i) \cup (U_{i+1} \backslash W_i)$. Forming y' with $w = v_{i-1}$ and $w' = v_{i+1}$, we obtain

$$\bar{z}_w + \bar{z}_{w'} \geq \sum_{e \in A} y'_A.$$

From now on, we may assume that $\bar{z}_{v_i} > \sum_{A \in W_i} \bar{y}_A$ for all $i \in \{1, ..., k\}$. We consider three cases.

Case 1. For all $\delta(S) \in \mathcal{A}$, either $|S \cap V(F)| \leq 1$ or $|V(F)\backslash S| \leq 1$.

Thus, any cut in $\mathcal{A}$ that contains an edge in F must contain exactly two edges in F having a common end-vertex. It follows that $U_i = W_{i-1} \cup W_i$ for all $i \in \{1, ..., k\}$. Add the edge $e = v_1v_3$ to G to form H. Let $C \in C(H)$ be such that $e \in C$. Then $C = \delta(S)$ for some $S \subset V$ such that S contains v_2 and exactly one of v_1 and v_3. From this we see that either $C\backslash\{e\} \in U_2$ or $C\backslash\{e\} \in U_3$ and that $C\backslash\{e\} \notin W_2$. But $U_2 = W_1 \cup W_2$ and $U_3 = W_2 \cup W_3$. It follows that $C \in W_1 \cup W_3$. Forming y' with $w = v_1$ and $w' = v_3$, we obtain

$$\bar{z}_w + \bar{z}_{w'} > \sum_{A \in W_1} \bar{y}_A + \sum_{A \in W_3} \bar{y}_A = \sum_{e \in A \in C(H)} y'_A.$$

Case 2. There exists $S \in S(\mathcal{A})$ such that $S \cap V(F) = \{v_i, v_j\}$ or $V(F)\backslash S = \{v_i, v_j\}$ where v_i and v_j are non-adjacent. (Here, $S(\mathcal{A})$ is given by Lemma 3.5.)

Since $S(\mathcal{A})$ is a nested family, for any $T \subset S(\mathcal{A})$ such that T contains exactly one of v_i and v_j, we must have $T \subset S$ if $S \cap V(F) = \{v_i, v_j\}$ or $T \subset V\backslash S$ if $V(F)\backslash S = \{v_i, v_j\}$. Hence, $\delta(T) \in W_i \cup W_j$. Forming H and y' with $w = v_i$ and $w' = v_j$, we obtain

$$\bar{z}_w + \bar{z}_{w'} > \sum_{A \in W_i} \bar{y}_A + \sum_{A \in W_j} \bar{y}_A \geq \sum_{e \in A} y'_A.$$

Case 3. Neither Case 1 nor Case 2 holds.

Observe that there exists $S \in S(\mathcal{A})$ satisfying one of the following:

(i) $3 \leq |S \cap V(F)| \leq k - 3$;

(ii) $S \cap V(F) = \{v_j, v_{j+1}\}$ or $V(F)\backslash S = \{v_j, v_{j+1}\}$ for some $j \in \{1, ..., k\}$.

We claim that in either case, we can find distinct vertices $u_1, u_2, u_3, u_4 \in V(F)$ such that u_1u_2, $u_3u_4 \in \delta(S)$, $u_1, u_3 \in S$, and u_1u_4, u_2u_3 are not edges in G.

To prove our claim, assume without loss of generality that $v_1 \notin S$ and $v_2 \in S$. Let $j \geq 3$ be the smallest integer such that $v_j \in S$ and $v_{j+1} \notin S$. Observe that such j exists and $j \leq k - 1$. Take $u_1 = v_2$, $u_2 = v_1$, $u_3 = v_j$, $u_4 = v_{j+1}$. Then, u_1, u_2, u_3, u_4 are distinct. Since F is a chordless cycle, u_1 and u_4 are not adjacent and u_2 and u_3 are not adjacent. This proves our claim.

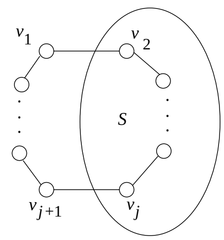

Figure 3.3: Illustration for Case 3

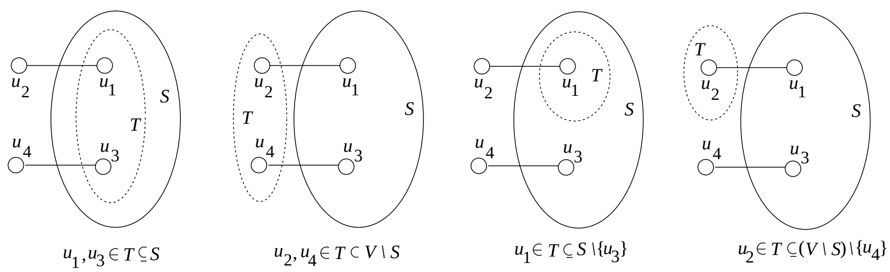

Figure 3.4: Illustrations of nested cuts

Since $S(\mathcal{A})$ is a nested family, we have

$$
\begin{aligned}
&(\bar{z}_{u_1} + \bar{z}_{u_2}) + (\bar{z}_{u_3} + \bar{z}_{u_4}) \\
\geq\ & \sum_{u_1 u_2 \in A} \bar{y}_A + \sum_{u_3 u_4 \in A} \bar{y}_A \\
=\ & \sum_{u_1, u_3 \in T \subsetneq S} \bar{y}_{\delta(T)} + \sum_{u_2, u_4 \in T \subset V \setminus S} \bar{y}_{\delta(T)} + \sum_{u_1 \in T \subsetneq S \setminus \{u_3\}} \bar{y}_{\delta(T)} + \sum_{u_2 \in T \subsetneq (V \setminus S) \setminus \{u_4\}} \bar{y}_{\delta(T)} \\
&+ \sum_{u_1, u_3 \in T \subsetneq S} \bar{y}_{\delta(T)} + \sum_{u_2, u_4 \in T \subset V \setminus S} \bar{y}_{\delta(T)} + \sum_{u_3 \in T \subsetneq S \setminus \{u_1\}} \bar{y}_{\delta(T)} + \sum_{u_4 \in T \subsetneq (V \setminus S) \setminus \{u_2\}} \bar{y}_{\delta(T)}.
\end{aligned}
$$

Hence, we have either

$$
\begin{aligned}
&\bar{z}_{u_1} + \bar{z}_{u_4} \\
\geq\ & \sum_{u_1, u_3 \in T \subsetneq S} \bar{y}_{\delta(T)} + \sum_{u_2, u_4 \in T \subset V \setminus S} \bar{y}_{\delta(T)} + \sum_{u_1 \in T \subsetneq S \setminus \{u_3\}} \bar{y}_{\delta(T)} + \sum_{u_4 \in T \subsetneq (V \setminus S) \setminus \{u_2\}} \bar{y}_{\delta(T)}
\end{aligned}
$$

or

$$
\begin{aligned}
& \bar{z}_{u_2} + \bar{z}_{u_3} \\
\geq & \sum_{u_1,u_3 \in T \subseteq S} \bar{y}_{\delta(T)} + \sum_{u_2,u_4 \in T \subset V \setminus S} \bar{y}_{\delta(T)} + \sum_{u_2 \in T \subseteq (V \setminus S) \setminus \{u_4\}} \bar{y}_{\delta(T)} + \sum_{u_3 \in T \subseteq S \setminus \{u_1\}} \bar{y}_{\delta(T)}.
\end{aligned}
$$

Without loss of generality, assume it is the former. Add the edge $e = u_1 u_4$ to G to form H. Let $C \in C(H)$ be such that $e \in C$ and $C \setminus \{e\} \in \mathcal{A}$. If $S' \in S(\mathcal{A})$ is a shore of C, then $S' \subseteq S$ or $S' \subseteq V(G) \setminus S$. Forming y' with $w = u_1$ and $w' = u_4$, we obtain

$$
\begin{aligned}
& \sum_{e \in A \in C(H)} y'_A \\
= & \sum_{u_1,u_3 \in T \subseteq S} y'_{\delta(T)} + \sum_{u_2,u_4 \in T \subset V \setminus S} y'_{\delta(T)} + \sum_{u_1 \in T \subseteq (V \setminus S) \setminus \{u_3\}} y'_{\delta(T)} + \sum_{u_4 \in T \subseteq S \setminus \{u_2\}} y'_{\delta(T)} \\
= & \sum_{u_1,u_3 \in T \subseteq S} \bar{y}_{\delta(T)} + \sum_{u_2,u_4 \in T \subset V \setminus S} \bar{y}_{\delta(T)} + \sum_{u_1 \in T \subseteq (V \setminus S) \setminus \{u_3\}} \bar{y}_{\delta(T)} + \sum_{u_4 \in T \subseteq S \setminus \{u_2\}} \bar{y}_{\delta(T)} \\
\leq & \bar{z}_{u_1} + \bar{z}_{u_4}.
\end{aligned}
$$

□

Since there always exists a 'nice' certificate of unsolvability of sys(G) if G is non-1-tough, one might ask if there are always 'nice' certificates for unsolvability for arbitrary infeasible graphs; in particular, whether or not the entries of the certificate can be required to be of small size. In the sequel, we show that the answer is "no."

Let G_1 be the graph shown in Figure 3.5. For $n \geq 2$, define G_n recursively (see Figure 3.6) as follows:

$$
\begin{aligned}
V(G_n) &= \cup_{i=1}^{3} \{s_{n,i}, t_{n,i}, u_{n,i}, v_{n,i}\} \cup V(G_{n-1}), \\
E(G_n) &= \cup_{i=1}^{2} \{s_{n,i}s_{n,i+1}, t_{n,i}t_{n,i+1}, u_{n,i}u_{n,i+1}, v_{n,i}v_{n,i+1}\} \\
&\quad \cup \{u_{n,1}v_{n,1}, u_{n,1}s_{n,1}, s_{n,1}u_{n-1,1}, v_{n,1}t_{n,1}, t_{n,1}v_{n-1,1}\} \\
&\quad \cup \{s_{n,3}t_{n,3}, u_{n,3}s_{n,3}, s_{n,3}u_{n-1,3}, v_{n,3}t_{n,3}, t_{n,3}v_{n-1,3}\} \cup E(G_{n-1}).
\end{aligned}
$$

For $n \geq 1$, let $G'_n = G_n - \{u_{n,1}, u_{n,2}, u_{n,3}, v_{n,1}, v_{n,1}, v_{n,3}\}$.

Lemma 3.10. *For $n \geq 1$, G_n and G'_n are Hamiltonian.*

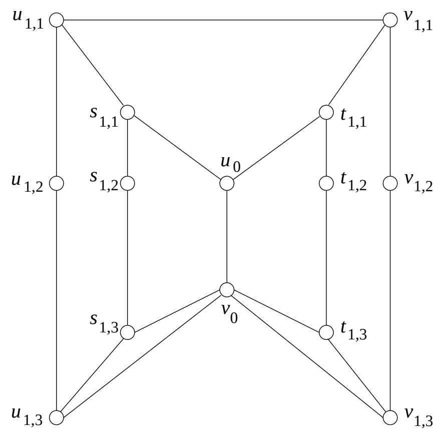

Figure 3.5: G_1

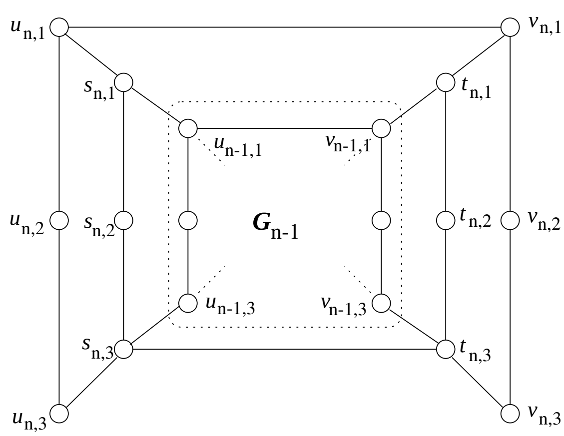

Figure 3.6: G_n

Proof. To prove that G_n is Hamiltonian, it suffices to show that there is a Hamiltonian path from $u_{n,1}$ to $v_{n,1}$ in G_n. We show this by induction on n. Observe that

$$u_{1,1}, u_{1,2}, u_{1,3}, s_{1,3}, s_{1,2}, s_{1,1}, u_0, t_{1,1}, t_{1,2}, t_{1,3}, v_0, v_{1,3}, v_{1,2}, v_{1,1}$$

are the vertices of a Hamiltonian path from $u_{1,1}$ to $v_{1,1}$ in G_1 in sequential order. Assume that G_n has a Hamiltonian path from $u_{n,1}$ to $v_{n,1}$ for some $n \geq 1$. Let P denote the sequence of vertices of this path in sequential order. Then $u_{n+1,1}, u_{n+1,2}, u_{n+1,3}, s_{n+1,3}, s_{n+1,2}, s_{n+1,1}, P, t_{n+1,1}, t_{n+1,2}, t_{n+1,3}, v_{n+1,3}, v_{n+1,2}, v_{n+1,1}, u_{n+1,1}$ are the vertices of a Hamiltonian path from $u_{n+1,1}$ to $v_{n_1,1}$ in G_{n+1} in sequential order. This completes the induction.

Now consider G'_n. Clearly, G'_1 is Hamiltonian. Consider the case when $n \geq 2$. From above, we see that there exists a Hamiltonian path from $u_{n-1,1}$ to $v_{n-1,1}$ in G_{n-1}. Let P denote the sequence of vertices of this path in sequential order. Then $s_{n,3}, s_{n,2}, s_{n,1}, P, t_{n,1}, t_{n,2}, t_{n,3}, s_{n,3}$ are the vertices of a Hamiltonian cycle in G'_n in cyclic order. □

For $n \geq 1$, let H_n be the graph with $V(H_n) = V(G_n) \cup \{w_0, l_1, l_2, r_1, r_2\}$, and $E(H_n) = E(G_n) \cup \{u_0w_0, w_0l_1, w_0r_1, l_1l_2, r_1r_2, l_2r_2, l_2u_{n,3}, r_2v_{n,3}\}$

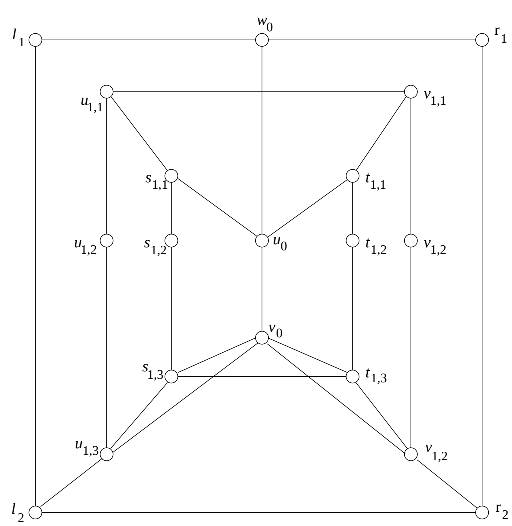

Figure 3.7: H_1

Lemma 3.11. *For $n \geq 1$, $H_n - V(G_m)$ and $H_n - V(G'_m)$ are Hamiltonian for all $m = 1, ..., n$.*

Proof. First, consider $H_n - V(G_m)$. The case $m = n$ is trivial.

We now prove by induction on k, $1 \leq k \leq n-1$ that $H_n - V(G_{n-k})$ has a Hamiltonian cycle using the edge $s_{n-k+1,3}, t_{n-k+1,3}$. When $k = 1$, it is easy to see that

$$w_0, l_1, l_2, u_{n,3}, u_{n,2}, u_{n,1}, s_{n,1}, s_{n,2}, s_{n,3}, t_{n,3}, t_{n,2}, t_{n,1}, v_{n,1}, v_{n,2}, v_{n,3}, r_2, r_1, w_0$$

are the vertices of a Hamiltonian cycle in $H_n - V(G_{n-k})$ in cyclic order.

Assume that $H_n - V(G_{n-k})$ has a Hamiltonian cycle using the edge $s_{n-k+1,3}, t_{n-k+1,3}$ for some $k \geq 1$. Therefore, there exists a Hamiltonian path from $t_{n-k+1,3}$ to $s_{n-k+1,3}$ in $H_n - V(G_{n-k})$. Let P denote the sequence of vertices of this path in sequential order. Then $u_{n-k,3}, u_{n-k,2}, u_{n-k,1}$, $s_{n-k,1}$, $s_{n-k,2}$, $s_{n-k,3}$, $t_{n-k,3}$, $t_{n-k,2}$, $t_{n-k,1}$, $v_{n-k,1}$, $v_{n-k,2}$, $v_{n-k,3}$, P, $u_{n-k,3}$ are the vertices of a Hamiltonian cycle using the edge $s_{n-k,3}, t_{n-k,3}$ in $H_n - V(G_{n-k-1})$ in cyclic order. This completes the induction.

Now, consider $H_n - V(G'_m)$. We prove by induction on k, $0 \leq k \leq n-1$ that $H_n - V(G'_{n-k})$ has a Hamiltonian cycle using the edge $u_{n-k,1}, v_{n-k,1}$.

When $k = 0$, the statement clearly holds.

Assume that $H_n - V(G'_{n-k})$ has a Hamiltonian cycle using the edge that joins $u_{n-k,1}$ and $v_{n-k,1}$ for some $k \geq 0$. Therefore, there exists a Hamiltonian path from $v_{n-k,1}$ to $u_{n-k,1}$ in $H_n - V(G'_{n-k})$. Let P denote the sequence of vertices of this path in sequential order. Then $s_{n-k,1}$, $s_{n-k,2}$, $s_{n-k,3}$, $u_{n-k-1,3}$, $u_{n-k-1,2}$, $u_{n-k-1,1}$, $v_{n-k-1,1}$, $v_{n-k-1,2}$, $v_{n-k-1,3}$, $t_{n-k,3}$, $t_{n-k,2}$, $t_{n-k,1}$, P, $s_{n-k,1}$ are the vertices of a Hamiltonian cycle using the edge that joins $s_{n-k,3}$ and $t_{n-k,3}$ in $H_n - V(G'_{n-k-1})$ in cyclic order. This completes the induction. □

Theorem 3.12. *There exists an integral pair $\bar{y}, \bar{z}$ feasible for $sys'(H_n)$ such that $\sum_{A \in C(H_n)} \bar{y}_A > \sum_{v \in V(H_n)} \bar{z}_v$. Furthermore, any such pair $\bar{y}, \bar{z}$ must have an entry having value at least n.*

Proof. Set

$\bar{z}_{u_0} = -1;$

$\bar{z}_{v_0} = 1;$

$\bar{z}_{w_0} = 4n + 2;$

$\bar{z}_{l_2} = \bar{z}_{r_2} = 2n + 1;$

$\bar{z}_{l_1} = \bar{z}_{r_1} = -(2n + 1);$

$\bar{z}_{s_{i,1}} = \bar{z}_{t_{i,1}} = \bar{z}_{u_{i,3}} = \bar{z}_{v_{i,3}} = 2$ for $i = 1, ..., n;$

$\bar{z}_{s_{i,3}} = \bar{z}_{t_{i,3}} = \bar{z}_{u_{i,1}} = \bar{z}_{v_{i,1}} = 1$ for $i = 1, ..., n;$

$\bar{z}_{u_{i,2}} = \bar{z}_{v_{i,2}} = \bar{z}_{s_{i,2}} = \bar{z}_{t_{i,2}} = -1$ for $i = 1, ..., n;$

$\bar{y}_{\delta(\{l_1,l_2\})} = \bar{y}_{\delta(\{r_1,r_2\})} = 2n + 1;$

$\bar{y}_{\delta(V(G_i))} = \bar{y}_{\delta(V(G_i'))} = 2$ for $i = 1, ..., n;$

$\bar{y}_{\delta(\{u_{i,1},u_{i,2}\})} = \bar{y}_{\delta(\{v_{i,1},v_{i,2}\})} = \bar{y}_{\delta(\{s_{i,2},s_{i,3}\})} = \bar{y}_{\delta(\{t_{i,2},t_{i,3}\})} = 1$ for $i = 1, ..., n;$

$\bar{y}_{\delta(\{u_0,v_0\})} = 1.$

It is not difficult to check that $\sum_{A \in C(H_n)} \bar{y}_A - \sum_{v \in V(H_n)} \bar{z}_v = 1$. We now show that the pair $\bar{y}, \bar{z}$ is feasible for $\text{sys}'(H_n)$. Clearly, $\bar{y} \geq 0$. Consider the following table

e	$\sum_{e \in A \in C(H_n)} \bar{y}_A$
u_0v_0, l_1l_2	0
u_0w_0	$\bar{y}_{\delta(\{u_0,v_0\})} + \sum_{i=1}^{n} \big(\bar{y}_{\delta(V(G_i))} + \bar{y}_{\delta(V(G_i'))}\big)$
$v_0u_{1,3}$	$\bar{y}_{\delta(\{u_0,v_0\})} + \bar{y}_{\delta(V(G_1'))}$
l_1w_0	$\bar{y}_{\delta(\{l_1,l_2\})}$
l_1r_2	$\bar{y}_{\delta(\{l_1,l_2\})} + \bar{y}_{\delta(\{r_1,r_2\})}$
$u_{i,1}u_{i,2}$	0
$u_{i,2}u_{i,3}$	$\bar{y}_{\delta(\{u_{i,1},u_{i,2}\})}$
$s_{i,1}s_{i,2}$	$\bar{y}_{\delta(\{s_{i,2},s_{i,3}\})}$
$s_{i,2}s_{i,3}$	0

e	$\sum_{e \in A \in C(H_n)} \bar{y}_A$
$u_{i,1}s_{i,1}$	$\bar{y}_{\delta(\{u_{i,1},u_{i,2}\})} + \bar{y}_{\delta(V(G'_i))}$
$u_{i,3}s_{i,3}$	$\bar{y}_{\delta(\{s_{i,2},s_{i,3}\})} + \bar{y}_{\delta(V(G'_i))}$
$u_{i,1}s_{i+1,1}$	$\bar{y}_{\delta(\{u_{i,1},u_{i,2}\})} + \bar{y}_{\delta(V(G_i))}$
$u_{i,3}s_{i+1,3}$	$\bar{y}_{\delta(\{s_{i+1,2},s_{i+1,3}\})} + \bar{y}_{\delta(V(G_i))}$
$u_{i,1}v_{i,1}$	$\bar{y}_{\delta(\{u_{i,1},u_{i,2}\})} + \bar{y}_{\delta(\{v_{i,1},v_{i,2}\})}$
$s_{i,3}t_{i,3}$	$\bar{y}_{\delta(\{s_{i,2},s_{i,3}\})} + \bar{y}_{\delta(\{t_{i,2},t_{i,3}\})}$

Using the information from the table, one can see that for any $uv \in E(H_n)$, we have

$$\bar{z}_u + \bar{z}_v \geq \sum_{uv \in A \in C(H_n)} \bar{y}_A.$$

Now consider any integral pair $\hat{y}, \hat{z}$ feasible for $\text{sys}'(H_n)$ such that $\sum_{A \in C(G)} \hat{y}_A > \sum_{v \in V(G)} \hat{z}_v$. By Lemma 3.10 and Lemma 3.11, we see that if we remove the constraint $x(\delta(S)) \geq 2$ from $\text{sys}(H_n)$ for any $S \in \cup_{i=1}^n \{V(G_i), V(G'_i)\}$, the resulting system has a solution. Hence, we must have $\hat{y}_{\delta(V(G_i))}, \hat{y}_{\delta(V(G'_i))} > 0$ for $i = 1, \ldots, n$. Since $u_0w_0 \in \delta(V(G_i)) \cap \delta(V(G'_i))$ for $i = 1, \ldots, n$, we have $\hat{z}_{u_0} + \hat{z}_{w_0} \geq \sum_{u_0w_0 \in A} \hat{y}_A \geq \sum_{i=1}^{n} (\hat{y}_{\delta(V(G_i))} + \hat{y}_{\delta(V(G'_i))}) \geq 2n$. The result follows. □

Observe that H_n is non-planar. One might ask if similar results can be obtained for planar graphs. We now show an infeasible planar graph which requires a certificate having entries larger than two in absolute value. However, it is not clear if a true analog of the previous theorem can be obtained when restricted to planar graphs.

Consider the graph in Figure 3.8. Call it H.

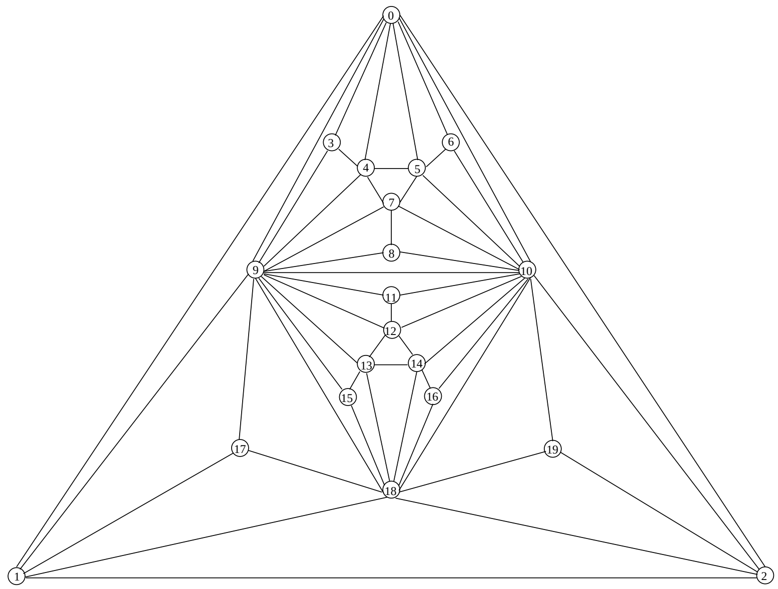

Figure 3.8: A more-than-1-tough infeasible maximal planar graph

Let $\bar{z} \in \mathbb{R}^{V(H)}$ be such that

$$\bar{z}_v = \begin{cases} 1 & \text{if } v \in \{4,5,7\}; \\ -1 & \text{if } v \in \{3,6,8\}; \\ 2 & \text{if } v \in \{0,1,2,12,13,14\}; \\ -2 & \text{if } v \in \{11,15,16,17,19\}; \\ 4 & \text{if } v \in \{9,10,18\}; \\ 0 & \text{otherwise.} \end{cases}$$

Let $\bar{y}_A \in \mathbb{R}^{C(H)}$ be such that

$$\bar{y}_A = \begin{cases} 1 & \text{if } A \in \{\delta(\{3,4\}), \delta(\{5,6\}), \delta(\{7,8\})\}; \\ 2 & \text{if } A \in \{\delta(\{11,12\}), \delta(\{13,15\}), \delta(\{14,16\}), \\ & \qquad \delta(\{1,17\}), \delta(\{2,19\}), \delta(\{0,3,4,5,6,7,8\})\}; \\ 0 & \text{otherwise.} \end{cases}$$

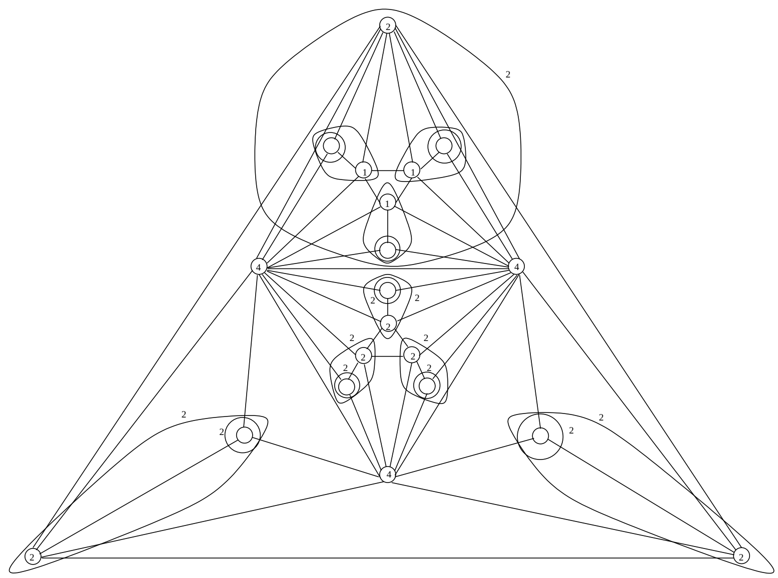

Figure 3.9: Certificate of infeasibility

One can check that the pair $\bar{y}, \bar{z}$ is feasible for $\mathrm{sys}'(H)$ and $\sum_A \bar{y}_A > \sum_v \bar{z}_v$. (See Figure 3.9.) By Theorem 3.2, $\mathrm{SEP}(H) = \emptyset$.

Now consider any integral $\bar{y}, \bar{z}$ feasible for $\mathrm{sys}'(H)$ and $\sum_A \bar{y}_A > \sum_v \bar{z}_v$. One can check that $\mathrm{sys}(H - 6)$ has a solution. (See the top figure in Figure 3.10.) Thus, we must have $\bar{z}_6 < 0$.

If we drop the contraint

$$x(\delta(\{5,6\})) \geq 2$$

from $\mathrm{sys}(H)$, the resulting system has a feasible solution. (See the middle figure in Figure 3.10.) Thus, we must have $\bar{y}_{\{5,6\}} > 0$.

If we drop the constraints

$$\begin{aligned} x(\delta(\{0,3,4,5,6\})) &\geq 2 \\ x(\delta(\{0,3,4,5,6,7\})) &\geq 2 \\ x(\delta(\{0,3,4,5,6,7,8\})) &\geq 2 \end{aligned}$$

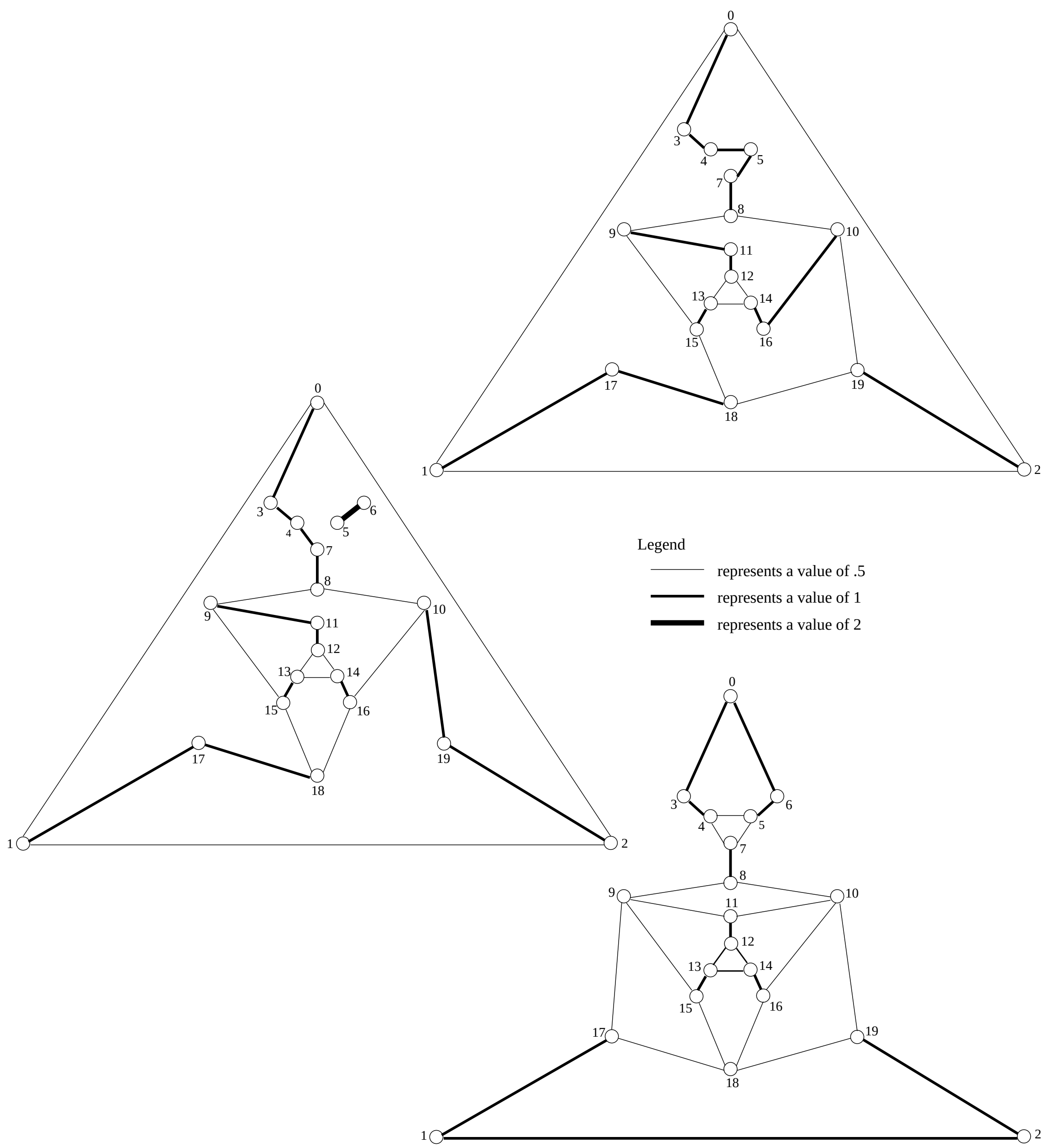

Figure 3.10: Various solutions

from sys(H), the resulting system has a feasible solution. (See the bottom figure in Figure 3.10.) Therefore, at least one of $\bar{y}_{\delta(\{0,3,4,5,6\})}$, $\bar{y}_{\delta(\{0,3,4,5,6,7\})}$, and $\bar{y}_{\delta(\{0,3,4,5,6,7,8\})}$ must be positive.

Consider the constraint $z_u + z_v - \sum_{uv \in A} y_A \geq 0$ in sys$'(H)$ where $u = 6$ and $v = 10$. Then

$$\bar{z}_{10} \geq -\bar{z}_6 + \bar{y}_{\{5,6\}} + \bar{y}_{\delta(\{0,3,4,5,6\})} + \bar{y}_{\delta(\{0,3,4,5,6,7\})} + \bar{y}_{\delta(\{0,3,4,5,6,7,8\}} \geq 3.$$

The graph H above is also somewhat interesting in a different context. Observe that by Theorem 3.6, H is 1-tough. In fact, by Proposition 3.13, it is more-than-1-tough. Hence, it is an infeasible, and thus non-Hamiltonian, more-than-1-tough maximal planar graph. There has been considerable interest in deciding if certain toughness conditions will guarantee Hamiltonicity. Nishizeki [35] mentioned that Chvátal raised the following question: Is 1-toughness a sufficient condition for a maximal planar graph to be Hamiltonian? Nishizeki constructed an infinite family of 1-tough non-Hamiltonian maximal planar graphs. However, the graphs are not more-than-1-tough. Harant and Owens [28] constructed an infinite family of $\frac{5}{4}$-tough maximal planar graphs that are non-Hamiltonian. Later on, Owens [36] constructed a sequence of maximal planar graphs whose toughness approaches $\frac{3}{2}$. However, it is not difficult, albeit tedious, to show that the graphs constructed in both papers are feasible.

Proposition 3.13. *The graph H (depicted in Figure 3.8) is more-than-1-tough.*

Proof. (Sketch.) Suppose H is not more-than-1-tough. Since H is 1-tough, there exists $S \subset V$, $|S| > 1$, such that $\omega(G - S) = |S|$.

First, one checks that $H - v$ is feasible, and therefore 1-tough, for all $v \in \{3, 6, 8, 17, 19\}$. Hence, $v \notin S$ for all $v \in \{3, 6, 8, 17, 19\}$.

Then, one checks that sys($H - v$) has a positive solution for all $v \in \{11, 15, 16\}$. By Corollary 4.13, sys($H - v$) has a point satisfying all the constraints strictly and hence, by Theorem 4.4, $H - v$ is more-than-1-tough. It follows that $11, 15, 16 \notin S$ and the neighbours of $11, 15, 16$ must all be in S.

Thus, $\{9, 10, 12, 13, 14, 18\} \subseteq S \subseteq \{0, 1, 2, 4, 5, 7, 9, 10, 12, 13, 14, 18\}$. By going through all the possibilities, one can see that S could not exist. Hence, H is more-than-1-tough. □

3.2 Points in the subtour-elimination polytope

Clearly, a graph G is Hamiltonian if and only if SEP(G) has an integral point. Thus, deciding if SEP(G) has an integral point is $\mathcal{NP}$-hard. Now, one could consider the problem of determining if SEP(G) contains a point with entries that have small size. Note that this problem is trivial if SEP(G) always contains such a point. In this section, we show that this is not the case. We show that for each $k \geq 1$, there exists a graph H_k such that SEP(H_k) contains exactly one point and that point has an entry with value $1/(2k+1)$.

Remark. Boyd and Pulleyblank [4] gave a class of extreme points of the subtour-elimination polytope of a complete graph that have entries with arbitrarily large size. However, there does not seem to be an obvious modification to the support graph of any of these extreme points to give a graph whose subtour-elimination polytope has no extreme points with entries having small size. (The support graph of an extreme point is the graph induced by the edges whose corresponding entries of the extreme point are positive.)

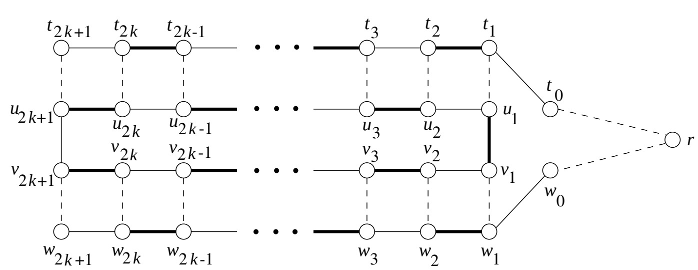

Figure 3.11: P_k

Let $k \geq 1$. We first construct a plane graph H'_k by joining together a number of gadgets and inserting edges. Let P_k denote the plane graph depicted in Figure 3.11.

P_k will be the gadget we use to construct H'_k. (For now, ignore the differences in the line-style of the edges. The differences will be useful for discussion later on.)

Form H'_k as follows. Join $2k+1$ copies of P_k by identifying the $2k+1$ vertices r. (In other words, all the vertices labelled r in the copies of P_k become one single vertex in H'_k but the rest of the vertices remain distinct.) Join the vertex labelled t_i of each copy to the vertex labelled w_i of the copy that follows immediately clockwise with a thick edge for all $i \in \{0, 2k+1\}$. (See Figure 3.12.) Figure 3.13 shows H_1 and H_2. The meaning of

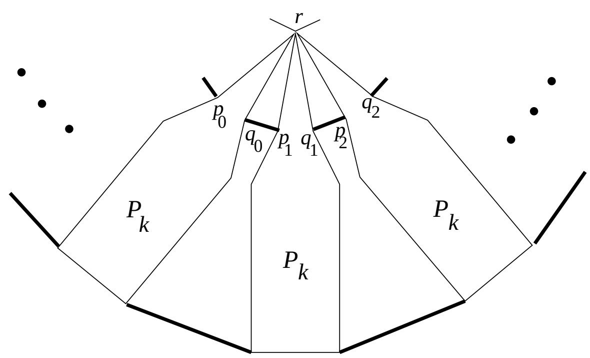

Figure 3.12: H'_k

the numbers will be explained later.

Now, let H_k denote the graph obtained by replacing each thick edge in H'_k by a path of length two. The next theorem is the main result of this section.

Theorem 3.14. *There is a unique point* $\bar{x}^k \in \mathrm{SEP}(H_k)$. *Furthermore, each entry of* $\bar{x}^k$ *has value* 1, $1/(2k+1)$, *or* $2k/(2k+1)$.

Proof. We first show that H_k is feasible. Define $x^k \in \mathbb{R}^{E(H'_k)}$ as follows. Set $x^k_e = 1/(2k+1)$ if e is a dashed edge in P_k. Set $x^k_e = 2k/(2k+1)$ if e is a thin solid edge in P_k. Set $x^k_e = 1$ for all the remaining edges e. By Proposition 3.15 below, $x^k \in \mathrm{SEP}^{E(H'_k)}$. Since $x^k_e = 1$ for all thick edges e, x^k can be converted to a point in $\mathrm{SEP}(H_k)$.

We now make an observation. If e is an edge incident with a degree-two vertex in H_k, then $\bar{x}_e = 1$ for all $\bar{x} \in \mathrm{SEP}(H'_k)$. Hence, it suffices to show that x^k is the only point in $\mathrm{SEP}(H'_k)$ that has the value 1 on all the thick edges.

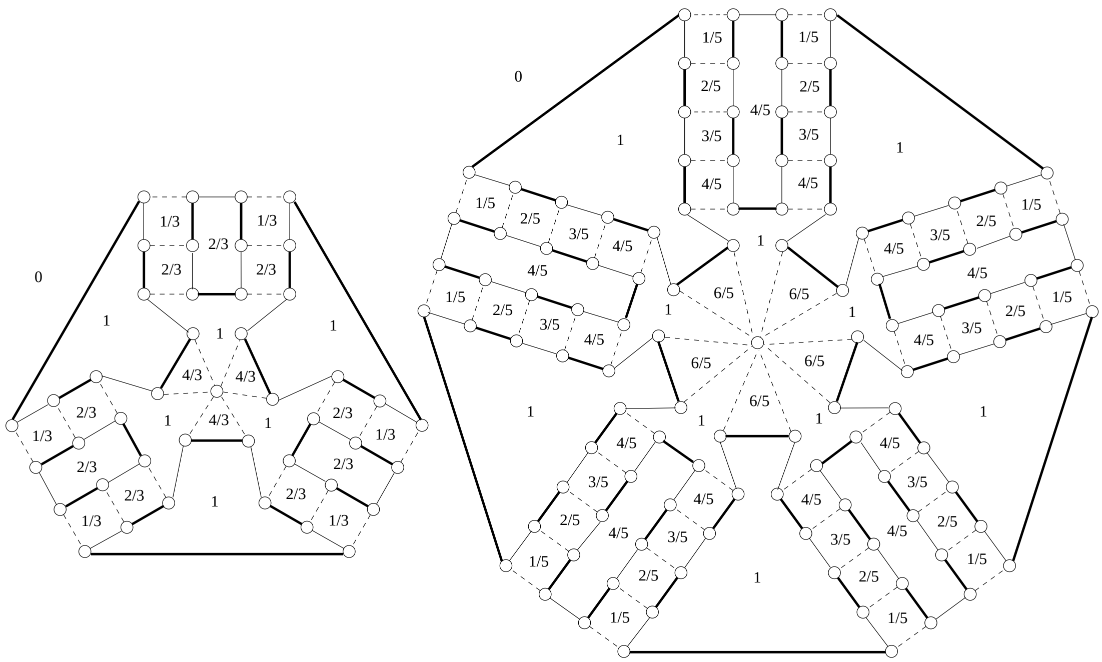

Figure 3.13: H_1 and H_2

Let $\bar{x} \in \mathrm{SEP}(H'_k)$ be such that it has the value 1 on all the thick edges. By the degree constraints, we see that the entries in $\bar{x}$ that correspond to all the dashed-edges in the same copy of P_k must have the same value. Note that this value must be at least $1/(2k+1)$ because $\bar{x}(\delta(S)) \geq 2$ for every set S whose elements are the vertices labelled $u_1, v_1, \ldots., u_{2k+1}, v_{2k+1}$ of the same copy of P_k. However, $\bar{x}(\delta(r)) = 2$ forces the value to be exactly $1/(2k+1)$. This implies that $\bar{x} = x^k$. The result follows. $\square$

Proposition 3.15. *Let $x^k \in \mathbb{R}^{E(H'_k)}$ be such that*

$$x^k_e = \begin{cases} 1/(2k+1) & \text{if } e \text{ is a dashed edge,} \\ 2k/(2k+1) & \text{if } e \text{ is a thin solid edge,} \\ 1 & \text{otherwise.} \end{cases}$$

Then $x^k \in \mathrm{SEP}(H'_k)$.

Proof. Clearly, $x^k > 0$. Also, it is not difficult to check that x^k satisfies the degree constraints. It remains to show that x^k satisfies the subtour-elimination constraints.

Let the neighbours of r, in anti-clockwise order, be $p_0, q_0, p_1, q_1 ..., p_{2k}, q_{2k}$ where p_i and q_i correspond to the vertices labelled t_0 and w_0, respectively, of the same copy of P_k. (See Figure 3.12.)

Let $S \subset V$ be the shore of a non-trivial cut of H'_k. Note that we may assume that both S and $V(H'_k)\backslash S$ induce connected sugraphs of H'_k. Without loss of generality, assume that $r \in S$. We consider two cases.

Case 1. S does not contain any vertices on the boundary of H'_k.

Suppose $p_i, q_i \in S$ for some $i \in \{0, ..., 2k\}$. Let E_0 denote the set of edges in the copy of P_k that contains p_i and q_i. Since S does not contain any boundary vertices, one can check that $x^k(\delta(S) \cap E_0) \geq 2(2k)/(2k+1)$. Observe that $|\delta(S)\backslash E_0|$ must be at least 2. Hence, $x^k(\delta(S)) \geq 4k/(2k+1) + 2/(2k+1) = 2$.

Suppose there does not exist $i \in \{0, ..., 2k\}$ such that $p_i, q_i \in S$. We may assume that if $q_i \in S$, then $p_{i+1} \in S$ because if $p_{i+1} \notin S$, then $x^k(\delta(S)) \geq x^k(\delta(S \cup \{p_{i+1}\}))$. (Indices are taken modulo $2k$.) Similarly, we may assume that if $p_i \in S$, then $q_{i-1} \in S$. Suppose there exists $i \in \{0, ..., 2k\}$ such that $q_i, p_{i+1} \in S$. Then $p_i, q_{i+1} \notin S$. It is not difficult to see that $x^k(\delta(S)) \geq 2$. Therefore, we may assume that there does not exists $i \in \{0, ..., 2k\}$ such that $q_i, p_{i+1} \in S$. But this means that $S = \{r\}$. Since $x^k(\delta(r)) = 2$, we see that $x^k(\delta(S)) \geq 2$ if S does not contain any vertices on the boundary of H'_k.

Case 2. S contains at least one vertex of the boundary of H'_k.

Let H^*_k denote the planar dual of H'_k. Let $\mathcal{O}$ denote the vertex of H^*_k that corresponds to the exterior face of H'_k. Observe that each edge of H^*_k crosses a unique edge of H'_k. If $e \in E(H'_k)$, we let e^* denote the edge in $E(H^*_k)$ that crosses e. Let $c \in \mathbb{R}^{E(H^*_k)}$ be such that $c_{e^*} = x^k_e$ for all $e \in E(H'_k)$.

Notice that the set of edges $\delta(S)$ correspond to the edge-set of a cycle in H^*_k containing the vertex $\mathcal{O}$ having cost $x^k(\delta(S))$. Hence, it suffices to show that all cycles containing the vertex $\mathcal{O}$ in the graph H^*_k with edge-costs given by c has cost at least 2. We shall show this by computing lower bounds for shortest paths from $\mathcal{O}$.

We now give the remaining details for the cases when $k = 1$ or 2. It is not difficult to generalize the proof for higher values of k. The idea is to using the classical technique in network flow theory of assigning "potentials" to vertices of H_k^* that satisfy some inequalities. For each vertex $v \in V(H_k^*)$, we assign a value y_v such that $y_{\mathcal{O}} = 0$ and $y_u + c_{uw} \geq y_w$ for all $u, w \in V(H_k^*)$. The assignments for the cases when $k = 1$ and $k = 2$ are shown in Figure 3.13. The number on each face specifies the value assigned to the vertex that corresponds to the face. Let $u \in V(H_k^*)$. Let $v_1 = \mathcal{O}, v_2, ..., v_m = u$ be the vertex sequence of a path P from $\mathcal{O}$ to u in H_k^*. Then the cost of the path P is

$$\sum_{i=1}^{m-1} c_{v_i v_{i+1}} \geq \sum_{i=1}^{m-1} y_{v_{i+1}} - y_{v_i} = y_{v_m} = y_u.$$

Hence, the cost of a shortest path from $\mathcal{O}$ to u is bounded below by y_u. It follows that if $y_u \geq 1$, then any cycle in H_k^* that contains u and $\mathcal{O}$ has cost at least 2.

Consider a cycle containing $\mathcal{O}$ that does not contain any vertex u with $y_u \geq 1$. It is not difficult to see that the edges of such a cycle must cross only the edges of H_k' that belong to the same copy of P_k and then to check that such a cycle must have cost at least 2. □

Remark. Previously, it was not known if there exists an extreme point of the subtour-elimination polytope whose support graph has a vertex of degree higher than 6. Theorem 3.14 shows that the support graph of the extreme points can have a vertex of arbitrarily high degree.

3.3 Minimality

The graph H_k where $k \geq 1$ defined in the previous section has a unique positive point in the subtour-elimination polytope. Therefore, if we remove any edge from H_k, the resulting graph will no longer be feasible. This leads us to the next definition. G is said to be *minimally feasible* if G is feasible and for every $e \in E$, $G - e$ is not feasible. This definition gives us a notion of minimality concerning the subtour-elimination polytope. We can also consider minimality in terms of the dimension of the subtour-elimination polytope. If SEP(G) has exactly one point, then G is said to be *thin*. Note that H_k is both minimally feasible and thin for every $k \geq 1$.

Since every feasible graph contains a spanning subgraph that is minimally feasible, understanding the subtour-elimination polytope of minimally feasible graphs (in particular, the structure of the extreme points) will allow us to draw conclusions on the subtour-elimination polytopes of feasible graphs.

We call an edge e *useless* if there does not exist $x \in \text{SEP}(G)$ such that $x_e > 0$. Observe that a graph without useless edges is feasible. We begin this section by showing that the set of thin graphs with no useless edge is a proper subset of the set of minimally feasible graphs.

Proposition 3.16. *Let $G = (V, E)$ be a graph with no useless edge. If G is thin, then G is minimally feasible.*

Proof. Since G has no useless edge, $\text{SEP}(G) = \{\bar{x}\}$ for some $\bar{x} > 0$. If G is not minimally feasible, then there exists $e \in E$ and $\hat{x} \in \text{SEP}(G)$ such that $\hat{x}_e = 0$, which is a contradiction. □

The converse of Proposition 3.16 is not true. Consider the graph depicted in Figure 3.14. Call it G.

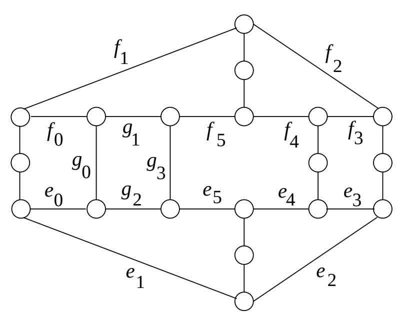

Figure 3.14: A minimally feasible graph that is not thin

Consider any $\bar{x} \in \text{SEP}(G)$. If e is an edge incident with a degree-two vertex, then $\bar{x}_e = 1$. In addition, the degree constraints force us to have $1-\bar{x}_{e_0} = \bar{x}_{e_1} = 1-\bar{x}_{e_2} = \bar{x}_{e_3} = 1-\bar{x}_{e_4} = \bar{x}_{e_5} = a$ for some $a \in [0,1]$ and $1-\bar{x}_{f_0} = \bar{x}_{f_1} = 1-\bar{x}_{f_2} = \bar{x}_{f_3} = 1-\bar{x}_{f_4} = \bar{x}_{f_5} = b$ for some $b \in [0,1]$.

From the subtour-elimination constraints, we see that

$$2a + 2b = \bar{x}_{e_1} + \bar{x}_{e_5} + \bar{x}_{f_1} + \bar{x}_{f_5} \geq 2 \text{ and } 2(1-a) + 2(1-b) = \bar{x}_{e_2} + \bar{x}_{e_4} + \bar{x}_{f_2} + \bar{x}_{f_4} \geq 2.$$

Hence, $a + b = 1$.

Again, from the subtour-elimination constraints, we see that

$$4a = \bar{x}_{e_1} + \bar{x}_{e_5} + \bar{x}_{f_2} + \bar{x}_{f_4} \geq 2 \text{ and } 4(1-a) = \bar{x}_{e_2} + \bar{x}_{e_4} + \bar{x}_{f_1} + \bar{x}_{f_5} \geq 2.$$

Therefore, $a = \frac{1}{2}$. Hence, $\bar{x}_e = \frac{1}{2}$ for all $e \in \{e_0, ..., e_5, f_0, ..., f_5\}$. But since $\bar{x} \leq \mathbf{e}$, we must have $\bar{x}_e > 0$ for all $e \in \{g_0, ..., g_3\}$. Hence, all the points in SEP(G) are positive. Thus, G is minimally feasible. However, G is not thin. Figure 3.15 shows two extreme points in SEP(G) with value 1 on the thick edges and value $\frac{1}{2}$ on the thin edges. Hence, the set of thin graphs with no useless edge is a proper subset of the set of minimally feasible graphs.

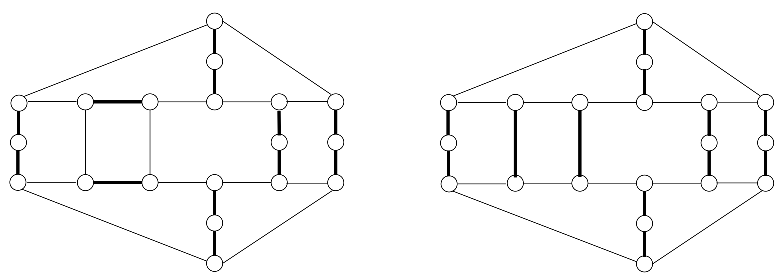

Figure 3.15: Two points in the subtour-elimination polytope

One might now ask how many extreme points the subtour-elimination polytope of a minimally feasible graph can have.

Consider the graph depicted in Figure 3.16. Call it G'. Let $\bar{x} \in$ SEP(G'). One can easily check that $\bar{x}$ must have value 1 on each thick edge and value $\frac{1}{2}$ on each thin solid edge. It follows that $\bar{x}_e > 0$ for all $e \in \{g_0, ..., g_3, h_0, ..., h_3\}$. Hence, G' is minimally feasible. Consider $\bar{x}^1, \bar{x}^2, \bar{x}^3, \bar{x}^4$ where $\bar{x}^i_e = \bar{x}_e$ for all $e \notin \{g_0, ..., g_3, h_0, ..., h_3\}$ and $\bar{x}^1_e = 1$ for all $e \in \{g_0, g_2, h_0, h_2\}$, $\bar{x}^1_e = \frac{1}{2}$ for all $e \in \{g_1, g_3, h_1, h_3\}$, $\bar{x}^2_e = 1$ for all $e \in \{g_0, g_2, h_1, h_3\}$, $\bar{x}^2_e = \frac{1}{2}$ for all $e \in \{g_1, g_3, h_0, h_2\}$, $\bar{x}^3_e = 1$ for all $e \in \{g_1, g_3, h_0, h_2\}$,

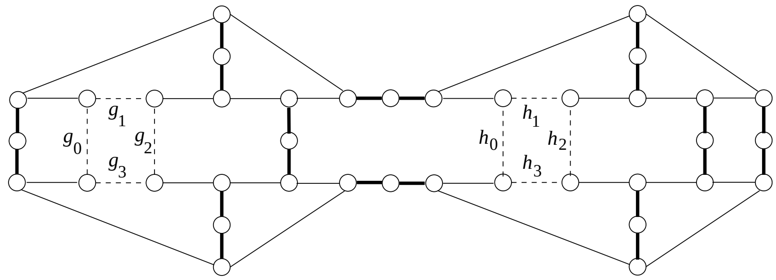

Figure 3.16: A minimally feasible graph with at least 4 extreme points

$\bar{x}^3_e = \frac{1}{2}$ for all $e \in \{g_0, g_2, h_1, h_3\}$, $\bar{x}^4_e = 1$ for all $e \in \{g_1, g_3, h_1, h_3\}$, $\bar{x}^4_e = \frac{1}{2}$ for all $e \in \{g_0, g_2, h_0, h_2\}$. Clearly, $\bar{x}^1, ..., \bar{x}^4$ are extreme points of $\mathrm{SEP}(G')$.

Now, it is not difficult to see that by "chaining" together k copies of G, one can generalize G' to obtain a minimally feasible graph whose subtour-elimination polytope has 2^k extreme points, k of which are affinely independent. Hence, we have the following:

Theorem 3.17. *For any integer $N > 0$, there exists a minimally feasible graph G_N with $\dim(\mathrm{SEP}(G_N)) \geq N$.*

Clearly, cycles are the only Hamiltonian minimally feasible graphs and the only Hamiltonian thin graphs with no useless edge. However, graph-theoretical characterizations of thin graphs and of minimally feasible graphs are not yet known. The remainder of this section is devoted to obtaining some classes of graphs that are not minimally feasible or not thin.

3.3.1 Some classes of graphs that are not minimally feasible

In this subsection, we look at some classes of graphs with no useless edge that are not minimally feasible. By Proposition 3.16, these graphs are not thin as well.

Theorem 3.18. *Let G be 3-connected with no useless edge. Suppose G_1 and G_2 are subgraphs of G such that $G = G_1 \cup G_2$ and $G_1 \cap G_2 = T$ where T is a triangle with $V(T) = \{u, v, w\}$. Assume that for any two edges $e, f \in E(T)$, there exists $\bar{x} \in \mathrm{SEP}(G_2)$ such that $\bar{x}_e = \bar{x}_f = 1$. Then G cannot be minimally feasible.*

Proof. Suppose G is minimally feasible. Let $x' \in \text{SEP}(G)$. Then $x' > 0$.

Let $U = V(G_2)\backslash\{u, v, w\}$. Let $a = x'(\gamma(\{u\}, U)), b = x'(\gamma(\{v\}, U)), c = x'(\gamma(\{w\}, U))$. Note that $2 \le x'(\delta(U \cup \{u\})) = b + c + 2 - a$. Hence, $b + c - a \ge 0$. Similarly, we have $a + b - c \ge 0$ and $a + c - b \ge 0$.

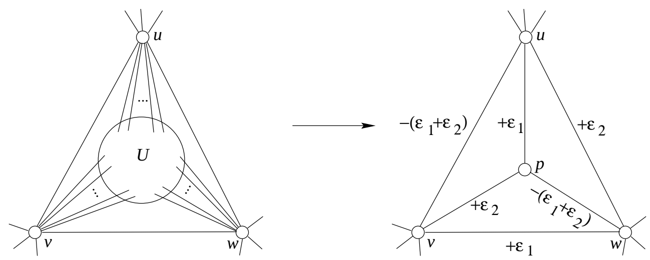

Figure 3.17: Illustration of modifications

Construct the graph G' by adding a new vertex p to G_1 and connecting p to u, v, and w. Let $D = x'_{uv} + x'_{vw} + x'_{uw} + (a + b + c)/2$. Since $x' > 0$ and $a + b + c \ge 2$, $D > 1$.

Form $\bar{x} \in \mathbb{R}^{E(G')}$ as follows:

$$\bar{x}_e = \begin{cases} (x'_e + (a+b-c)/2)(1 - 1/D) & \text{if } e = uv \\ (x'_e + (b+c-a)/2)(1 - 1/D) & \text{if } e = vw \\ (x'_e + (a+c-b)/2)(1 - 1/D) & \text{if } e = uw \\ 1 - \frac{1}{D}(x'_{vw} + \frac{b+c-a}{2}) & \text{if } e = pu \\ 1 - \frac{1}{D}(x'_{uw} + \frac{a+c-b}{2}) & \text{if } e = pv \\ 1 - \frac{1}{D}(x'_{uv} + \frac{a+b-c}{2}) & \text{if } e = pw \\ x'_e & \text{otherwise} \end{cases}$$

It is not difficult to check that $\bar{x} \in \text{SEP}(G')$. Modify $\bar{x}$ as follows. Since $\bar{x}_{uv} + \bar{x}_{pu} + \bar{x}_{pv} \le 2$, we see that there exist $\epsilon_1 > 0$ and $\epsilon_2 > 0$ such that $1 - \bar{x}_{pu} \ge \epsilon_1$, $1 - \bar{x}_{pv} \ge \epsilon_2$, and $\epsilon_1 + \epsilon_2 = \bar{x}_{uv}$. Note that $\bar{x}_{pu} \ge \bar{x}_{vw}$, $\bar{x}_{pv} \ge \bar{x}_{uw}$, $\bar{x}_{pw} \ge \bar{x}_{uv}$. Reduce $\bar{x}_{pw}$ and $\bar{x}_{uv}$ by $\epsilon_1 + \epsilon_2$. Increase $\bar{x}_{pu}$ and $\bar{x}_{vw}$ by ϵ_1, $\bar{x}_{pv}$ and $\bar{x}_{uw}$ by ϵ_2. It is easy to check that $\bar{x} \in \text{SEP}(G')$ with $\bar{x}_{uv} = 0$.

By assumption, there exist $\tilde{x}^1, \tilde{x}^2, \tilde{x}^3 \in \text{SEP}(G_2)$ such that $\tilde{x}^1_{uv} = \tilde{x}^1_{uw} = 1$, $\tilde{x}^2_{uv} = \tilde{x}^2_{vw} = 1$, and $\tilde{x}^3_{vw} = \tilde{x}^3_{uw}$. Let

$$x'' = (1 - \bar{x}_{pu})\tilde{x}^1 + (1 - \bar{x}_{pv})\tilde{x}^2 + (1 - \bar{x}_{pw})\tilde{x}^3.$$

Since x'' is a convex combination of points in SEP(G_2), $x'' \in$ SEP(G_2).

Construct $\bar{y} \in \mathbb{R}^{E(G)}$ as follows. Set $\bar{y}_e = \bar{x}_e$ if $e \in E(G_1)$. Set $\bar{y}_e = x''_e$ if $e \in E(G_2)\backslash\{uv, vw, uw\}$. (Note that $\bar{x}_{uv} = 0$.) Clearly $\bar{y} \geq 0$ and $\bar{y}(\delta(v')) = 2$ for all $v' \in V(G) - \{u, v, w\}$.

Now

$$\begin{aligned}\bar{y}(\delta(u)) &= x''(\delta_{G_2}(u)) - x''_{uv} - x''_{uw} + \bar{x}(\delta_{G_1}(u)) \\ &= 2 - x''_{uv} - x''_{uw} + \bar{x}(\delta_{G_1}(u)) \\ &= \bar{x}_{pu} + \bar{x}(\delta_{G_1}(u)) \\ &= \bar{x}(\delta_{G'}(u)) = 2\end{aligned}$$

Similarly, $\bar{y}(\delta(v)) = 2$ and $\bar{y}(\delta(w)) = 2$.

Consider $A \in C(G)$. If A does not contain any of uv, vw, uw, then clearly $\bar{y}(A) \geq 2$. Assume A contains at least one of uv, vw, uw. Then it must contain exactly two of them. Suppose $uw, vw \in A$. Let $S \subset V$ be a shore of A with $w \in S$. Then

$$\begin{aligned}\bar{y}(\delta(S)) &= x''(\delta_{G_2}(S \cap V(G_2)) - x''_{uw} - x''_{vw} + \bar{x}(\delta_{G_1}(S \cap V(G_1))) \\ &= \bar{x}_{pw} + \bar{x}(\delta_{G_1}(S \cap V(G_1))) \\ &= \bar{x}(\delta_{G'}(S \cap V(G_1)) \geq 2.\end{aligned}$$

Similarly, if $uv, vw \in A$ or if $uw, vw \in A$, $\bar{y}(A) \geq 2$. Hence $\bar{y}(A) \geq 2$ for any non-trivial cut A of G.

Thus, $\bar{y} \in$ SEP(G) with $\bar{y}_{uv} = \bar{x}_{uv} = 0$. This contradicts that G is minimally feasible.

□

Corollary 3.19. *A planar graph that is 4-connected or has a separating triangle cannot be minimally feasible.*

Proof. Let G be a planar graph. If G is 4-connected, then G is Hamiltonian by Theorem 2.1. Hence, G is not minimally feasible. So suppose G has a separating triangle with vertices u, v, w such that if we fix a planar embedding of G and let U be the set of vertices in the interior of the triangle induced by $\{u, v, w\}$, then $G_2 = G[U \cup \{u, v, w\}]$ is isomorphic to K_4 or is 4-connected. In either case, there is a Hamiltonian circuit in G_2 containing any two of uv, vw, and uw by Theorem 2.1. Let $G_1 = G - U$ and apply Theorem 3.18. □

We now move on to deriving some general conditions. We begin with two technical lemmas. The first is well known and the second follows from a stronger result due to Boyd and Pulleyblank (Theorem 4.7 in [4]). We will prove both lemmas for the sake of completeness.

Lemma 3.20. *Let $G = (V, E)$ be a connected graph having at least 3 vertices. If $\mathcal{C}$ is a non-crossing family of cuts of G, then $|\mathcal{C}| \leq 2|V| - 3$.*

Proof. By Lemma 3.5, there exists a nested family $\mathcal{S}$ containing precisely one proper shore of each cut in $\mathcal{C}$, and no other sets.

Without loss of generality, we may assume $\{v\} \in \mathcal{C}$ for each $v \in V$. Consider the rooted tree $T(\mathcal{C})$. Note that the root R has at least three children and that T has $|V|$ leaves with each non-leaf having at least two children. Let L denote the set of leaves of T. Then,

$$\begin{aligned} 2(|V(T)| - 1) = 2|E(T)| &\geq \deg(R) + \sum_{v \in L} \deg(v) + \sum_{v \notin L} \deg(v) \\ &\geq 3 + |V| + 3(|V(T)| - |V| - 1). \\ &= 3|V(T)| - 2|V| \end{aligned}$$

It follows that $|\mathcal{C}| < |V(T)| \leq 2|V| - 2$. □

Lemma 3.21. *Let $G = (V, E)$ be feasible with $|E| \geq 2|V| - 2$. If $\bar{x}$ is an extreme point of* SEP(G)*, then there exists $e \in E(G)$ such that $\bar{x}_e = 0$.*

Proof. Suppose $\bar{x} > 0$. By Theorem 4.9 in Cornuéjols et al. [10], there exists a non-crossing family of cuts $\mathcal{C}'$ such that

$$\{\bar{x}\} = \{x \in \mathbb{R}^E : x(C) = 2 \ \forall \ C \in \mathcal{C}'\}.$$

By Lemma 3.20, we know that $|\mathcal{C}'| \leq 2|V|-3$. Since $|E| > 2|V|-3 \geq |\mathcal{C}'|$, the system $x(C) = 2$ for any $C \in \mathcal{C}'$ cannot have a unique solution. This is a contradiction. □

Theorem 3.22. *If $G = (V, E)$ is minimally feasible and $|V| \geq 3$, then $|E| \leq 2|V| - 3$.*

Proof. This follows immediately from Lemma 3.21. □

Corollary 3.23. *Every 3-connected 4-regular graph is not minimally feasible.*

Proof. Let G be a 3-connected 4-regular graph. It is not difficult to see that G is 4-edge-connected. Thus, $\frac{1}{2}\mathbf{e} \in \mathrm{SEP}(G)$. It follows that G has no useless edge. Since $|E| = 2|V|$, the result follows from Theorem 3.22. □

Theorem 3.24. *Every 3-connected 3-regular bipartite graph is not minimally feasible.*

Proof. Let $G = (V, E)$ be a 3-connected 3-regular bipartite graph.

Let S be a minimal shore of a non-trivial 3-edge cut. If none exists, then let $S = V \backslash \{v\}$ for some $v \in V$. Let $u, w \in S$ be adjacent. Let w_1, w_2 be the neighbours of u distinct from w and u_1, u_2 be the neighbours of w distinct from u. Since G is bipartite, $u_1u_2, w_1w_2 \notin E$. Let H be the graph obtained from $G - uw$ by removing u and w and adding the edges u_1u_2 and w_1w_2.

It is easy to see that $G - uw$ is feasible if and only if there exists $\bar{x} \in \mathrm{SEP}(H)$ such that $\bar{x}_e = 1$ for all $e \in \{u_1u_2, w_1w_2\}$. We show that H is 3-edge-connected. If not, then there exists a non-trivial 3-edge cut $\delta(T)$ of G that contains uw. But $6 = |\delta(S)| + |\delta(T)| \geq |\delta(S \cap T)| + |\delta(S \cup T)| \geq 6$. Hence, $\delta(S \cap T)$ is a 3-edge cut. But this contradicts the choice of S since $S \cap T$ is a proper subset of S. Thus, H is 3-edge-connected and 3-regular.

Let $G' = G \times (V \backslash S)$. By Corollary 5.5, G' is bipartite and 3-regular, and has no non-trivial 3-edge cut. Since $\frac{1}{3}\mathbf{e} \in \mathrm{PM}(G')$, if A is a non-trivial cut, then $x(A) \geq \frac{4}{3}$ for all $x \in \mathrm{PM}(G')$. Hence, G' has no non-trivial tight cut. It is not difficult to see that G' is a brace as a result. Hence, $G' - \{u_1, u_2, w_1, w_2\}$ has a perfect matching M. It follows that $M' = M \cup \{u_1u_2, w_1w_2\} \backslash \{uw\}$ is a perfect matching of $H \times (V \backslash S)$. Since $G \times S$ is 3-connected and 3-regular, $\frac{1}{3}\mathbf{e} \in \mathrm{SEP}(G \times S)$. By Corollary 2.4, G is matching-covered and M' can be extended to a perfect matching N of H.

Let $\bar{x} = \frac{1}{2}(\chi^N + \mathbf{e})$ where χ^N denotes the incidence vector of M'. Since H is 3-edge-connected, $\bar{x} \in \text{SEP}(H)$. Furthermore, $\bar{x}_e = 1$ for all $e \in \{u_1u_2, w_1w_2\}$. The result follows. □

It is not known if there exist 3-connected minimally feasible graphs. In fact, the case when the graph is 3-regular and non-bipartite is still unresolved. However, evidence seems to point to the negative direction. The difficulty in constructing a 3-connected minimally feasible graph, if one exists, lies in identifying 3-connected graphs that are feasible but not Hamiltonian. We end this subsection with a question:

Problem 3.25. *Can a 3-connected graph be minimally feasible?*

3.3.2 Some classes of graphs that are not thin

In this subsection, we look at some classes of non-thin graphs with no useless edge that are not covered in the previous subsection.

We begin with 3-regular graphs. These graphs are easier to handle because, in the case when the graphs are 3-connected, we can construct points in the subtour-elimination polytope from perfect matchings.

Theorem 3.26. *Let $G = (V, E)$ be a 3-regular graph. If G is feasible, then G is not thin.*

Proof. Since G is feasible, G must be 2-connected. Suppose G is 3-connected. Then $\frac{1}{3}\mathbf{e} \in \text{PM}(G)$ and so G has a perfect matching M by Corollary 2.4. Hence, $\frac{2}{3}\mathbf{e}$ and $\frac{1}{2}(\chi^M + \mathbf{e})$ are two distinct points in $\text{SEP}(G)$.

Suppose G is not 3-connected. Since G is 2-connected and 3-regular, there must exist a 2-edge cut $\delta(S)$.

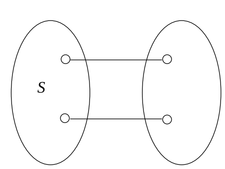

Figure 3.18: A 2-edge cut

Observe that if $x^1 \in \text{SEP}(G \times S)$ and $x^2 \in \text{SEP}(G \times (V \backslash S))$, then x^1 and x^2 can be spliced to give a point in $\text{SEP}(G)$. Namely, if $\bar{x} \in \mathbb{R}^E$ is such that $\bar{x}_e = 1$ for every $e \in \delta(S)$, $\bar{x}_e = x^1_e$ for every $e \in E(G[V \backslash S])$, and $\bar{x}_e = x^2_e$ for every $e \in E(G[S])$, then $\bar{x} \in \text{SEP}(G)$. Hence, it suffices to show that either $\text{SEP}(G \times S)$ or $\text{SEP}(G \times (V \backslash S))$ has more than one point.

Let v be the pseudo-vertex of $G \times S$. Note that v has exactly two neighbours, say u and w. Choose S so that $\{u, w\}$ is the only two-separator of $G \times S$. Observe that u and w cannot be adjacent in $G \times S$. Otherwise, $N(\{u, v, w\})$ will be another two-separator of $G \times S$. We show that $\text{SEP}(G \times S)$ has more than one point.

Form the 3-connected 3-regular graph H from $G \times S$ by removing v (and the incident edges) and adding in the edge $f = uw$. Observe that if $e \in \delta_{G \times S}(v)$, then $\hat{x}_e = 1$ for all $\hat{x} \in \text{SEP}(G \times S)$. Hence, any $x' \in \text{SEP}(H)$ with $x'_f = 1$ can be easily converted to a point in $\text{SEP}(G \times S)$. Therefore, it suffices to show that $\text{SEP}(H)$ has two points that have value one on the edge f.

As H is matching-covered, there is a perfect matching M' using f. Thus $x' = \frac{1}{2}(\chi^{M'} + \mathbf{e})$ is a point in $\text{SEP}(H)$ with $x'_f = 1$. If H has a Hamiltonian circuit using f, then the incidence vector of the Hamiltonian circuit gives us a second point.

Assume that H has no Hamiltonian circuit using f. We show that there is a perfect matching distinct from M' using f. Suppose there is a unique perfect matching M using f. Note that $\bar{x} = \frac{1}{3}\mathbf{e} \in \text{PM}(H)$. Since $\bar{x}$ is not an extreme point of $\text{PM}(H)$, $\bar{x}$ can be written as $\sum_{i=1}^{l} \lambda_i \chi^{M_i}$ where M_i, $i = 1, ..., l$, is a perfect matching of H, and $0 < \lambda_i < 1$ for each $i = 1, ..., l$ and $\sum_{i=1}^{l} \lambda_i = 1$. Since M is the only matching containing f, we see that $M = M_i$ for some i. Assume without loss of generality that $M_1 = M$. Then $\lambda_1 = \sum_{i=1}^{l} \lambda_i \chi^{M_i}_f = \bar{x}_f = \frac{1}{3}$. Now $(\bar{x} - \frac{1}{3}\chi^M)_{f'} = 0$ for all $f' \in M$. Since $\bar{x} - \frac{1}{3}\chi^M = \sum_{i=2}^{l} \lambda_i \chi^{M_i}$, it follows that M_i is a perfect matching of $H - M$ for any $i \in \{2, ..., l\}$. Hence, $M \triangle M_2$ is composed of even circuits and induces a spanning subgraph of H. Since H has no Hamiltonian circuit using f, there must be a circuit C in $M \triangle M_2$ not containing f. It follows that $M \triangle C$ is a perfect matching of H using f and distinct from M, a contradiction. □

Theorem 3.27. *Let G be a 3-connected graph that contains a degree-three vertex v whose neighbours induce a connected subgraph. If G has no useless edge, then G is not thin.*

Proof. Suppose $\mathrm{SEP}(G) = \{\bar{x}\}$. Then $\bar{x} > 0$.

Let r, s, t be the neighbours of v such that $rs, st \in E(G)$. Since each vertex has degree at least three and $\bar{x} > 0$, the degree constraints guarantees that either $\bar{x}_{rv}, \bar{x}_{st} < 1$ or $\bar{x}_{tv}, \bar{x}_{rs} < 1$. Without loss of generality, assume the former is true.

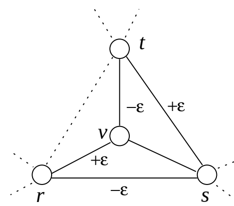

Figure 3.19: Illustration for Theorem 3.27

Let $\epsilon > 0$. Form $\hat{x}$ from $\bar{x}$ by decreasing $\bar{x}_{rs}$ and $\bar{x}_{tv}$ by ϵ and increasing $\bar{x}_{rv}$ and $\bar{x}_{st}$ by ϵ. We claim that if ϵ is sufficiently small, $\hat{x} \in \mathrm{SEP}(G)$.

Clearly, $\hat{x}(\delta(w)) = 2$ for all $w \in V(G)$ and for sufficiently small ϵ, we have $\hat{x} > 0$ and $\hat{x}(A) \geq 2$ for any $A \in C(G)$ that does not contain both rs and tv.

Let $A \in C(G)$ be such that $rs, tv \in A$. Let S be a shore of A containing v. If $s \in S$, then it is easy to see that $\hat{x}(A) = \bar{x}(A) \geq 2$. Suppose $r, v \in S$. Since $\bar{x}_{rv} < 1$, if ϵ is sufficiently small, we have $\hat{x}_{rv} < 1$, implying that $\hat{x}_{sv} + \hat{x}_{tv} > \hat{x}_{rv}$.

Thus,

$$2 \leq \hat{x}(\delta(S \backslash \{v\})) = \hat{x}(\delta(S)) - \hat{x}_{sv} - \hat{x}_{tv} + \hat{x}_{rv} < \hat{x}(\delta(S)).$$

Hence, $\hat{x} \in \mathrm{SEP}(G)$. The result follows. □

One can see from Theorem 3.22 and Theorem 3.27 that if G is a 3-connected thin graph with no useless edge, then G must have a vertex of degree 3 whose neighbours do not induce a connected subgraph. In addition, the subtour-elimination polytope of a 3-connected 3-regular graph always has at least two points. In light of these results, one might ask if there exist 3-connected thin graphs with no useless edge. Notice that no such graphs exist if the answer to Problem 3.25 is in the negative. Nevertheless, constructing thin graphs seem to be more difficult than constructing minimally feasible graphs. Therefore, we end the discussion on thin graphs with a conjecture.

Conjecture 3.28. *Every 3-connected graph with no useless edge is not thin.*

3.4 Reducing to graphs with low maximum degree

In this section, we show that deciding if a graph having maximum degree 3 is feasible is as difficult as deciding if a general graph is feasible. In particular, given a graph G, we construct, in time polynomial in the size of G, a graph G' with maximum degree 3 such that the size of G' is polynomial in the size of G and G' is feasible if and only if G is feasible. The idea is to "replace" each high-degree vertex by a graph with maximum degree 3. We first introduce an operation which preserves feasibility under certain conditions.

Let G_1 and G_2 be graphs such that $V(G_1) \cap V(G_2) = \emptyset$. Let $u_1 \in V(G_1)$ and $u_2 \in V(G_2)$ be vertices of degree $k \geq 1$. By *splicing* G_1 and G_2 with respect to u and v, we mean removing u_1 and u_2 (and all the incident edges) and adding k new edges, each joining a neighbour of u_1 and a neighbour of u_2, that form a matching.

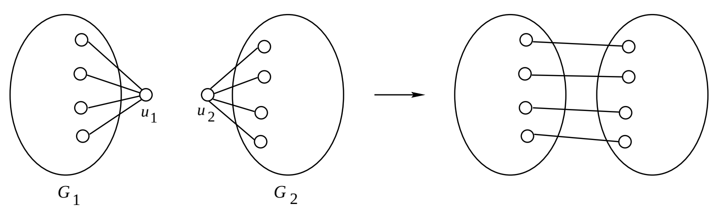

Figure 3.20: Splicing

The next result gives a condition under which splicing preserves feasibility.

Lemma 3.29. *Let G_1 and G_2 be feasible graphs. Suppose $u \in V(G_1)$ and $w \in V(G_2)$ are of degree k with $N(u) = \{u_1, ..., u_k\}$ and $N(w) = \{w_1, ..., w_k\}$. Let $G = (V(G_1 - u) \cup V(G_2 - w), E(G_1 - u) \cup E(G_2 - w) \cup \{u_1w_1, ..., u_kw_k\}$. If there exist $x^1 \in \mathrm{SEP}(G_1)$ and $x^2 \in \mathrm{SEP}(G_2)$ such that $x^1_{uu_i} = x^2_{ww_i}$ for all $i = 1, ..., k$, then G is feasible.*

Proof. Let $V = V(G)$ and $E = E(G)$. Let $G_1' = G_1 - u$ and $G_2' = G_2 - w$. For $i = 1, 2$, let $V_i = V(G_i')$ and $E_i = E(G_i')$.

Construct $\hat{x} \in \mathbb{R}^E$ as follows. For any $e \in E_1$, set $\hat{x}_e = x^1_e$. For any $e \in E_2$, set $\hat{x}_e = x^2_e$. For $i = 1, ..., k$, set $\hat{x}_{u_i w_i} = x^1_{uu_i}$. Clearly, $\hat{x} \geq 0$, $\hat{x}(\delta(v)) = 2$ for all $v \in V$ and $\hat{x}(\delta(V_1)) = 2$.

We now show that $\hat{x}(A) \geq 2$ for all $A \in C(G)\backslash\{\delta(V_1)\}$. Let $\delta(S) \in C(G)\backslash\{\delta(V_1)\}$. Clearly, if $S \subset V_1$ or $S \subset V_2$, or $V\backslash S \subset V_1$ or $V\backslash S \subset V_2$, then $\hat{x}(\delta(S)) \geq 2$. Suppose none of the four sets $S \cap V_1, S \cap V_2, (V\backslash S) \cap V_1$, and $(V\backslash S) \cap V_2$ is empty. Note that

$$\hat{x}(\delta(S)) = x^1(\delta(S) \cap E_1) + x^2(\delta(S) \cap E_2) + \hat{x}(\delta(S) \cap \delta(V_1)).$$

We claim that $x^1(\delta(S) \cap E_1) \geq 1$. Observe that

$$x^1(\delta(S) \cap E_1) = x^1(\delta(S \cap V_1)) - x^1(\gamma(\{u\}, S \cap V_1))$$

and that

$$x^1(\delta(S) \cap E_1) = x^1(\delta(S \cap V_1 \cup \{u\})) - x^1(\gamma(\{u\}, V_1\backslash S)).$$

Since $x^1(\delta(u)) = x^1(\gamma(\{u\}, S \cap V_1)) + x^1(\gamma(\{u\}, V_1\backslash S))$, we have

$$2x^1(\delta(S) \cap E_1) = x^1(\delta(S \cap V_1)) + x^1(\delta(S \cap V_1 \cup \{u\})) - x^1(\delta(u)) \geq 2 + 2 - 2 = 2.$$

It follows that $\hat{x}(\delta(S) \cap E_1) \geq 1$. Similarly, $x^2(\delta(S) \cap E_2) \geq 1$. Thus, $\hat{x}(\delta(S)) \geq 2$.

Therefore, $\hat{x} \in \text{SEP}(G)$ and so G is feasible. □

For each $k \geq 4$, we introduce a gadget H_k that is bipartite with the property that there is a vertex ξ of degree k and $\deg(u) \in \{3, 4\}$ for every vertex $u \in V(H_k)\backslash\{\xi\}$.

We shall not define H_4 formally. It is depicted on the left of Figure 3.21. Suppose $k \geq 5$. Let $G_{2,n}$ denote the graph with vertex-set $\{v_{i,j} : 0 \leq i \leq 2,\ 0 \leq j \leq n\}$ such that $v_{i,j}v_{i',j'}$ is an edge if and only if $|i - i'| + |j - j'| = 1$. Let H_k be the graph obtained from $G_{2,2(k-1)}$ by adding a vertex ξ and edges $v_{2,0}v_{2,3}$, $v_{2,2k-5}v_{2,2(k-1)}$, and $\xi v_{0,2j}$ for $j = 0, ..., k-1$. The graph H_5 is depicted on the right of Figure 3.21.

To facilitate discussion, we need a definition. Let $G = (V, E)$ be a graph and $v \in V$. Let $N(v) = \{v_1, ..., v_k\}$. Then G is said to be *spliceable at* v if for any $0 \leq \bar{y}_1, ..., \bar{y}_k \leq 1$

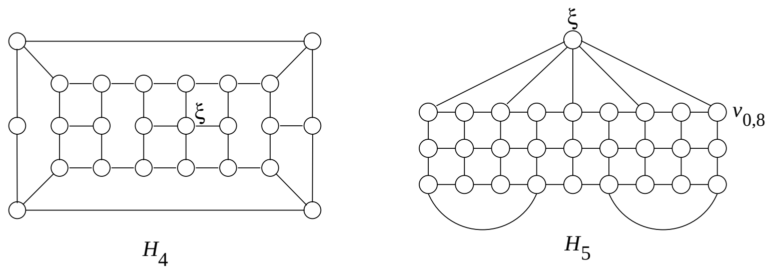

Figure 3.21: Gadgets H_4 and H_5

such that $\sum_{i=1}^{k} \bar{y}_i = 2$, there exists $\hat{x} \in \mathrm{SEP}(G)$ such that $\hat{x}_{vv_i} = \bar{y}_i$ for $i = 1, ..., k$.

Lemma 3.30. *H_k is spliceable at ξ for all $k \geq 4$.*

We postpone the proof to the end of this section. We now describe how to reduce the problem of determining if a graph is feasible to the class of graphs having maximum degree 3. Let G be a graph. Let $u \in V(G)$ be a vertex of degree k where $k \geq 4$. Let G' be the graph obtained from splicing G and H_k with respect to u and ξ. By Lemma 3.30, if G is feasible, then G' is also feasible. Suppose G' is feasible. Let $\bar{x} \in \mathrm{SEP}(G')$. Clearly, H_4 is bipartite and for every $k \geq 5$, H_k is bipartite with bipartition (U_k, W_k) where $U_k = \{v_{i,2j} : 0 \leq i \leq 2, 0 \leq j \leq k-1\} \cup \{v_{1,2j+1} : 0 \leq j \leq k-2\}$ and $W_k = V(H_k) \backslash U_k$. Since the two partitions in H_k have the same cardinality, we see that $\bar{x}(\delta(V(G-u))) = 2$. Hence, $G' \times V(H_k - v)$, and therefore, G is feasible.

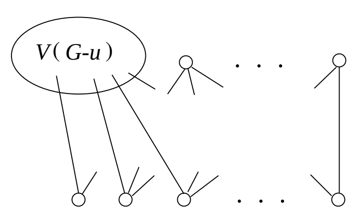

Figure 3.22: After splicing

It follows that given a graph G, one can obtain a graph G' by splicing using H_k for $k \geq 4$ such that G is feasible if and only if G' is and G' has maximum degree 3. Furthermore, if G is planar, we can require G' to be planar as well. Finally, it is not difficult to see that G' can be obtained in polynomial time and its size is polynomial in the size of the original graph.

While it is not clear if it is more convenient to study feasible graphs with low maximum degree, it is possible that a better understanding of these graphs might lead to answers to some questions on general graphs.

The remainder of this section is spent on completing the proof of Lemma 3.30. We begin with a sufficient condition for a graph to be spliceable at a vertex.

Lemma 3.31. *Let $G = (V, E)$ be a graph and $v \in V$. If for any distinct $e, f \in \delta(v)$, there exists $\bar{x} \in \mathrm{SEP}(G)$ such that $\bar{x}_e = \bar{x}_f = 1$, then G is spliceable at v.*

To prove this lemma, we need the following technical lemma.

Lemma 3.32. *Let $n \geq 2$. Let $x \in \mathbb{R}^n_+$ be such that $x_j \leq \frac{1}{2}\sum_{i=1}^n x_i$ for all j such that $x_j > 0$. For $i, j \in \{1, ..., n\}$, $i < j$, let $f_{ij} \in \mathbb{R}^n$ denote the vector having the value 1 in coordinates i and j and the value 0 everywhere else. Then there exist non-negative α_{ij}, $i, j \in \{1, ..., n\}$ and $i < j$, such that $x = \sum_{i<j} \alpha_{ij} f_{ij}$. Furthermore, $\sum_{i<j} \alpha_{ij} = \frac{1}{2}\sum_{i=1}^n x_i$.*

Proof of Lemma 3.31. Let $N(v) = \{v_1, ..., v_k\}$. For any distinct $v_i, v_j \in N(v)$, let $x^{\{i,j\}}$ be a point in $\mathrm{SEP}(G)$ such that $x^{\{i,j\}}_{vv_i} = x^{\{i,j\}}_{vv_j} = 1$.

Let $y'_1, ..., y'_k \in [0, 1]$ be such that $y'_1 + \cdots + y'_k = 2$. Clearly, $y'_i \leq \sum_{j=1}^k y'_i$ for all $i = 1, ..., k$. By Lemma 3.32, there exists non-negative α_{ij} such that $\sum_{i<j} \alpha_{ij} = \frac{1}{2}\sum_{i=1}^k y'_i = 1$ and $y'_i = \bar{x}_{vv_i}$ for all $i = 1, ..., k$ where $\bar{x} = \sum_{i<j} \alpha_{ij} x^{\{i,j\}}$. Therefore, $\bar{x}$ is a convex combination of points in $\mathrm{SEP}(G)$, implying that G is spliceable at v. □

Proof of Lemma 3.32. Clearly, if $x = \sum_{i<j} \alpha_{ij} f_{ij}$, then $\sum \alpha_{ij} = \frac{1}{2}\sum_{i=1}^n x_i$.

Observe that x has at least two non-zero entries. The proof is by induction on the number of non-zero entries in x. Suppose x has exactly 2 non-zero entries, x_i, x_j where $i < j$. Since $x_i, x_j \leq \frac{1}{2}(x_i + x_j)$, we see that $x_i = x_j$. Setting α_{ij} to x_i and all the other α's to zero, we get $x = \sum_{i<j} \alpha_{ij} f_{ij}$.

Assume that the result is true up to $k - 1$ non-zero entries for some $k \geq 3$. Suppose x has exactly k non-zero entries. Without loss of generality, assume that $x_1, ..., x_k > 0$

and $x_1 \geq \cdots \geq x_k$. Let $t_1 = x_k$ and $t_i = x_{k-i+1} - x_{k-i+2}$ for $i = 2, ..., k$. Let j be such that $\sum_{i=0}^{j-1}(i+1)t_{k-i} \leq x_k < \sum_{i=0}^{j}(i+1)t_{k-i}$. Let $s = x_k - \sum_{i=0}^{j-1}(i+1)t_{k-i}$. By our choice of j, we have $s < (j+1)t_{k-j}$.

For $i = 1, ..., j+1$, let $\alpha_{ik} = s/(j+1) + \sum_{m=0}^{j-i} t_{k-i+1-m}$. Let $x' = x - \sum_{i=1}^{j+1} \alpha_{ik} f_{ik}$. Then

$$\begin{aligned} x'_k = x_k - \sum_{i=1}^{j+1} \alpha_{ik} &= x_k - \sum_{i=1}^{j+1}(s/(j+1) + \sum_{m=0}^{j-i} t_{k-i+1-m}) \\ &= x_k - s - \sum_{i=0}^{j-1}(i+1)t_{k-i} = 0. \end{aligned}$$

Moreover, if $1 \leq l \leq j+1$,

$$\begin{aligned} x'_l = x_l - \alpha_{lk} &= \sum_{i=1}^{k-l+1} t_i - s/(j+1) - \sum_{i=k-j+1}^{k-l+1} t_i \\ &= \sum_{i=1}^{k-j} t_i - s/(j+1) \\ &= x_{j+1} - s/(j+1) > x_{j+1} - t_{k-j} = x_{j+2} \end{aligned}$$

Hence, $x'_1 = \cdots = x'_{j+1} > x'_{j+2} \geq \cdots \geq x'_k$. It follows that if $j \geq 1$, then $x'_m \leq \frac{1}{2}\sum_{i=1}^{n} x'_i$ for all $m = 1, ..., n$. If $j = 0$, it suffices to show that $x'_1 \leq \frac{1}{2}\sum_{i=1}^{n} x'_i$. Indeed, $x'_1 = x_1 - x_k \leq \frac{1}{2}\sum_{i=1}^{n} x_i - x_k = \frac{1}{2}\sum_{i=1}^{n} x'_i$. Now, applying the induction hypothesis to x' gives the desired result. □

We now show that H_k satisfies the condition in Lemma 3.31 by showing that H_k has a Hamiltonian circuit using vv_{2i} and vv_{2j} for every pair $i, j \in \{0, ..., k-1\}$. We first consider H_4.

Proposition 3.33. *For ever pair of edges $e, f \in \delta_{H_4}(v)$, there is a Hamiltonian circuit in H_4 using e and f.*

Proof. One can see from the Hamiltonian cycles shown in Figure 3.23 that the result is

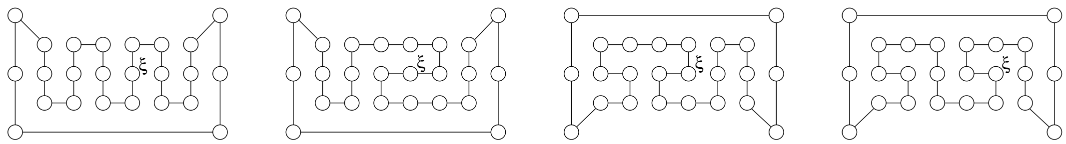

Figure 3.23: Some Hamiltonian circuits in H_4

true. □

Next, we consider H_k for $k \geq 5$. We first prove two technical results about H_k.

Lemma 3.34. *Let $r \in \{0, ..., k-2\}$. There exists a path in H_k with vertex-set $\{v_{i,j} : 0 \leq i \leq 2,\ 0 \leq j \leq 2r\}$ connecting $v_{0,2r}$ and $v_{2,2r}$.*

Proof. The proof is by induction on i. If $r = 0$, we can take the path with edge-set $\{v_{0,0}v_{1,0}, v_{1,0}v_{2,0}\}$. Consider $r \geq 1$. Assume that there is a path P with vertex-set $\{v_{i,j} : 0 \leq i \leq 2,\ 0 \leq j \leq 2(r-1)\}$ connecting $v_{0,2(r-1)}$ and $v_{2,2(r-1)}$. Then the path with edge-set $E(P) \cup \{v_{0,2r}v_{1,2r}, v_{1,2r-1}v_{1,2r}, v_{0,2r-1}v_{0,2(r-1)}, v_{2,2(r-1)}v_{2,2r-1}, v_{2,2r-1}v_{2,2r}\}$ is a path with vertex-set $\{v_{i,j} : 0 \leq i \leq 2,\ 0 \leq j \leq 2r\}$ connecting $v_{0,2r}$ and $v_{2,2r}$. The result follows from induction. □

Lemma 3.35. *Let $r, t \in \{0, ..., k-1\}$ with $r < t$. There exists a path in H_k with vertex-set $\{v_{i,j} : 0 \leq i \leq 2,\ 2r+1 \leq j \leq 2(k-1)\}$ connecting $v_{0,2t}$ and $v_{2,2r+1}$.*

Proof. The path with the following edge-set is a desired path:

$$\begin{aligned} & \{v_{i,j-1}v_{i,j} : 0 \leq i \leq 1, 2(r+1) \leq j \leq 2t\} \cup \{v_{0,2r+1}v_{1,2r+1}\} \\ \cup\ & \{v_{1,2j}v_{1,2j+1} : t \leq j \leq k-2\} \cup \{v_{0,j}v_{1,j} : 2t+1 \leq j \leq 2(k-1)\} \\ \cup\ & \{v_{0,2j-1}v_{0,2j} : t+1 \leq j \leq k-1\} \cup \{v_{1,2(k-1)}v_{2,2(k-1)}\} \\ \cup\ & \{v_{2,j-1}v_{2,j} : 2(r+1) \leq j \leq 2(k-1)\} \end{aligned}$$

□

Proposition 3.36. *For every pair $r, t \in \{0, ..., k-1\}$ with $r < t$, H_k has a Hamiltonian circuit using vv_{2r} and vv_{2t}.*

Proof. It suffices to show that there is a Hamiltonian path in $H_k - v$ connecting v_{2r} and v_{2t}.

Let $V' = \{v_{i,j} : 0 \leq l \leq 2,\ 0 \leq j \leq 2r\}$. By Lemma 3.34, there is a path P connecting v_{2r} and $v_{2,2r}$ such that $V(P) = V'$. By Lemma 3.35, there is a path Q connecting $v_{0,2t}$ and $v_{2,2r+1}$ such that $V(Q) = V(H_k - v)\backslash V'$. Then the path with edge-set $E(P) \cup \{v_{2,2r}v_{2,2r+1}\} \cup E(Q)$ is a Hamiltonian path in $H_k - v$ connecting v_{2r} and v_{2t}. □

Proof of Lemma 3.30. This follows immediately from Lemma 3.31 and Propositions 3.33 and 3.36. □

3.5 Compact formulation in the planar case

Note that for any graph G, the number of constraints in $\mathrm{sys}(G)$ is exponential in $|V(G)|$. Using max-flow min-cut, Claus [6] obtained a compact formulation for the subtour-elimination polytope, that is, a formulation having a polynomial number of constraints and variables whose set of solutions when projected to the appropriate space is the subtour-elimination polytope.

In this section, we describe a compact formulation for the subtour-elimination polytope of a planar graph that is due to Rivin. It is based on certain geometric properties and is very different from the compact formulation due to Claus. The formulation will be useful in Chapter 6 when we give Rivin's elementary proof of the necessary and sufficient conditions for a 3-connected planar graph to be the graph of a polytope inscribed in a sphere.

We first begin with a technical lemma that allows us to drop certain redundant constraints from $\mathrm{sys}(G)$.

Lemma 3.37. *Let $G = (V, E)$ be 3-connected. Let $\mathcal{Q}$ denote the set of non-trivial cuts whose shores induce 2-connected subgraphs. Then the polytope*

$$\begin{aligned} Q(G) = \{x \in \mathbb{R}^E \ :\ & x(\delta(v)) = 2\ \forall\ v \in V,\ x(\delta(\{u,v\})) \geq 2\ \forall\ uv \in E, \\ & x(\delta(S)) \geq 2\ \forall\ S \in \mathcal{Q},\ x \geq 0\} \end{aligned}$$

is equal to SEP(G).

Proof. Clearly, SEP$(G) \subseteq Q(G)$. To show that $Q(G) \subseteq$ SEP(G), it suffices to show that if $\bar{x} \in Q(G)$, then $\bar{x}(A) \geq 2$ for all $A \in C(G)$.

Suppose there exists $\delta(S) \in C(G)$ such that $\bar{x}(\delta(S)) < 2$. Choose S such that the total number of blocks b in $G[S]$ and $G[V \backslash S]$ is as small as possible. Since we cannot have $S = \{u, v\}$ for any $uv \in E$, we must have $|S|, |V \backslash S| \geq 3$.

Without loss of generality, assume that the number of blocks in $G[S]$ is at most the number of blocks in $G[V \backslash S]$. If there is only one block in $G[V \backslash S]$, we are done. Thus, suppose there are at least two blocks in $G[V \backslash S]$. We consider two cases.

Case 1: $\omega(G[V \backslash S]) > 1$.

Let $S_1, ..., S_k$ be the vertex-sets of the components of $G[V \backslash S]$. Since G is 3-connected, the number of blocks in $G[S \cup S_1]$ is at most the number of blocks in $G[S]$. However, $\bar{x}(\delta(S \cup S_1)) \leq \bar{x}(\delta(S)) < 2$. This contradicts our assumption of minimality.

Case 2: $\omega(G[V \backslash S]) = 1$.

Let v be a cut vertex of $G[V \backslash S]$. Let $S_1, ..., S_k$ be the vertex-sets of the components of $G[V \backslash S] - v$. Since G is 3-connected, $G[S_i] \times S$ is 2-connected. Hence, $G[S_i]$ is a subgraph of a block in $G[S \cup S_i]$. But the edges in $\gamma(S_i, S)$ cannot be incident to the same vertex in S. Hence, for any subset $\{i_1, ..., i_m\}$ of $\{1, ..., k\}$, the number of blocks in $G[S \cup S_{i_1} \cup \cdots \cup S_{i_m}]$ is at most the number of blocks in $G[S]$. It follows that the total number of blocks in $G[S \cup S_1]$ and in $G[V \backslash (S \cup S_1)]$ is less than b. By minimality of b, we have $\bar{x}(\delta(S \cup S_1)) \geq 2$. We also have that the total number of blocks in $G[S \cup S_2 \cup \cdots \cup S_k]$ and in $G[V \backslash (S \cup S_2 \cup \cdots \cup S_k)]$ is less than b. By minimality of b, we have $\bar{x}(\delta(S \cup S_2 \cup \cdots \cup S_k)) \geq 2$.

But

$$\bar{x}(\delta(S)) = \bar{x}(\delta(S \cup S_1)) + \bar{x}(\delta(S \cup S_2 \cup \cdots \cup S_k)) - x(\delta(v)) \geq 2 + 2 - 2 = 2.$$

This is a contradiction, and the result follows. □

The next theorem allows us to obtain a compact formulation for the subtour-elimination polytope of planar graphs.

Theorem 3.38. *(Rivin [39], [41]) Let $H = (V, E)$ be a plane triangulation with at least four vertices and a distinguished vertex ∞ on the boundary. Define $\mathcal{F} := \{\{u, v, w\} \subset V \backslash \{\infty\} : u, v, w$ are vertices on some face triangle$\}$ and $\mathcal{I} := V \backslash (\{\infty\} \cup N(\infty))$. Let B be the set of edges on the boundary of $H - \infty$ and $E_0 \subset E$ be such that $E_0 \cap B = \emptyset$. Let $k > 0$. Let (CP_k) denote the following system:*

$$\sum_{w:\{u,v,w\}\in\mathcal{F}} \theta_{\{u,v,w\},w} + x_{uv} = k \quad \forall\ uv \in E(H - \infty), \tag{3.2}$$

$$\theta_{\{u,v,w\},u} + \theta_{\{u,v,w\},v} + \theta_{\{u,v,w\},w} = k \quad \forall\ \{u, v, w\} \in \mathcal{F}, \tag{3.3}$$

$$\sum_{v,w:\{u,v,w\}\in\mathcal{F}} \theta_{\{u,v,w\},u} = 2k \quad \forall\ u \in \mathcal{I}, \tag{3.4}$$

$$\sum_{v,w:\{u,v,w\}\in\mathcal{F}} \theta_{\{u,v,w\},u} + x_{u\infty} = k \quad \forall\ u \in N(\infty), \tag{3.5}$$

$$x_e = 0 \quad \forall\ e \in E_0, \tag{3.6}$$

$$x \geq 0, \tag{3.7}$$

$$\theta \geq 0. \tag{3.8}$$

If $(\bar{\theta}, \bar{x})$ is a solution to (CP_k), then $\bar{x}(\delta(v)) = 2k$ for all $v \in V$ and $\bar{x}(A) \geq 2k$ for all $A \in C(H)$. Furthermore, if $\bar{\theta} > 0$, then $\bar{x}(A) > 2k$ for all $A \in C(H)$.

In addition, for any $\bar{x} \in \mathbb{R}^E$ such that $\bar{x} \geq 0$, $\bar{x}(E_0) = 0$, $\bar{x}(\delta(v)) = 2k$ for all $v \in V$ and $\bar{x}(A) \geq 2k$ for all $A \in C(H)$, there exists $\bar{\theta}$ such that $(\bar{\theta}, \bar{x})$ is a solution to (CP_k). Furthermore, if $\bar{x}_e > 0$ for all $e \in B$ and $\bar{x}(A) > 2k$ for all $A \in C(H)$, then there exists $\bar{\theta} > 0$ such that $(\bar{\theta}, \bar{x})$ is a solution to (CP_k).

Remark. The purpose of the parameter k in the statement of the theorem is to faciliate discussion when Rivin's characterization of graphs of inscribable type is considered in Chapter 6.

We can think of a solution of (CP_k) as an assignment of values to edges and angles of face triangles not containing ∞. In other words, for an edge uv, x_{uv} corresponds to the value assigned to that edge. For the angle at vertex w in the face triangle containing vertices u, v, w, $\theta_{\{u,v,w\},w}$ corresponds to the value assigned to it. (See Figure 3.24.)

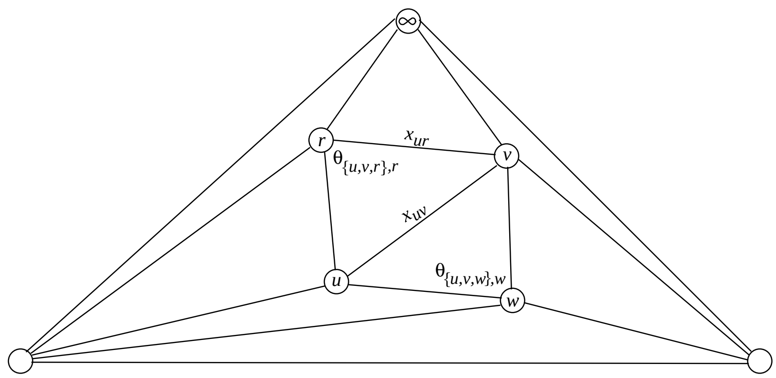

Figure 3.24: An illustration

Rivin did not state Theorem 3.38 the way it is stated here. The proof that follows is due to Rivin proof but written in the language of the current thesis. Some parts of his arguments have been elaborated.

Proof of Theorem 3.38. Let $(\bar{\theta}, \bar{x})$ be a solution of (CP_k).

Claim 1. If $v \in \mathcal{I}$, then $\displaystyle\sum_{u \in N(v)} \sum_{w:\{u,v,w\} \in \mathcal{F}} \bar{\theta}_{\{u,v,w\},w} = \sum_{u,w:\{u,v,w\} \in \mathcal{F}} (k - \bar{\theta}_{\{u,v,w\},v})$.

Proof. Note that $\displaystyle\sum_{u \in N(v)} \sum_{w:\{u,v,w\} \in \mathcal{F}} \bar{\theta}_{\{u,v,w\},w}$ is the sum of the values assigned to the marked angles in Figure 3.25 and that $\displaystyle\sum_{u,w:\{u,v,w\} \in \mathcal{F}} \bar{\theta}_{\{u,v,w\},v}$ is the sum of the values assigned to the unmarked angles in Figure 3.25. By (3.3), the claim follows. □

We first show that $\bar{x}$ satisfies the degree constraints.

If $v \in \mathcal{I}$, then

$$\begin{aligned}
\sum_{uv \in \delta(v)} (k - \bar{x}_{uv}) &= \sum_{u \in N(v)} \sum_{w:\{u,v,w\} \in \mathcal{F}} \bar{\theta}_{\{u,v,w\},w} && \text{(by (3.2))} \\
&= \sum_{u,w:\{u,v,w\} \in \mathcal{F}} (k - \bar{\theta}_{\{u,v,w\},v}) && \text{(by Claim 1)} \\
&= \deg(v)k - 2k.
\end{aligned}$$

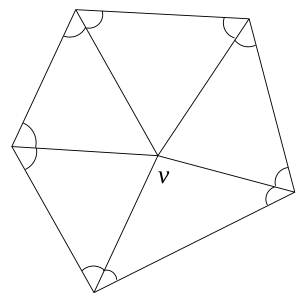

Figure 3.25: Illustration for Claim 1

Thus, $\sum_{uv\in\delta(v)} \bar{x}_{uv} = 2k$.

If $v \in N(\infty)$, then

$$\begin{aligned}\sum_{u\in N(v)} (k - \bar{x}_{uv}) &= (k - \bar{x}_{v\infty}) + \sum_{u\in N(v), u\neq\infty} \sum_{w:\{u,v,w\}\in\mathcal{F}} \bar{\theta}_{\{u,v,w\},w} \qquad \text{(by (3.2))}\\ &= \sum_{u,w:\{u,v,w\}\in\mathcal{F}} \bar{\theta}_{\{u,v,w\},v} + \sum_{u,w:\{u,v,w\}\in\mathcal{F}} (k - \bar{\theta}_{\{u,v,w\},v})\\ &= \sum_{u,w:\{u,v,w\}\in\mathcal{F}} k\\ &= (\deg(v) - 2)k.\end{aligned}$$

Thus, $\sum_{uv\in\delta(v)} \bar{x}_{uv} = 2k$.

Before we show that $\bar{x}(\delta(\infty)) = 2$, we need the following claim.

Claim 2. Let J be a Jordan curve in $H - \infty$. Consider the 2-connected plane graph $H' = H[V(J) \cup W]$ where W is the set of vertices not on the same side of J as ∞. Let $\mathcal{F}'$ denote the set $\{\{u,v,w\} \in \mathcal{F} : uv, vw, uw \in E(H')\}$. Then

$$\sum_{\{u,v,w\}\in\mathcal{F}':u\in V(J)} \bar{\theta}_{\{u,v,w\},u} = (|V(J)| - 2)k.$$

Proof. Applying Euler's formula to H', we obtain $|V(H')| - |E(H')| + |\mathcal{F}'| = 1$. Since H' has $|V(J)|$ boundary edges, we see that $|E(H')| = \frac{3|\mathcal{F}'|+|V(J)|}{2}$. It follows that $|\mathcal{F}'| = 2|V(H')| - 2 - |V(J)|$. Hence, $\sum_{\{u,v,w\}\in\mathcal{F}':u\in V(J)} \bar{\theta}_{\{u,v,w\},u} = |\mathcal{F}'|k - 2|V(H')\backslash V(J)|k =$

$(|V(J)| - 2)k$ as desired. □

Applying Claim 2 with the boundary of $H - \infty$ as the Jordan curve J, we get

$$\sum_{u \in N(\infty)} (k - \bar{x}_{u\infty}) = \sum_{\{u,v,w\} \in \mathcal{F}' : u \in N(\infty)} \theta_{\{u,v,w\},u} = (|N(\infty)| - 2)k.$$

It follows that $\sum_{u \in N(\infty)} \bar{x}_{u\infty} = 2k$.

Hence, $\bar{x}$ satisfies the degree constraints. We now show that $\bar{x}$ satisfies the subtour-elimination constraints,

Let $A \in C(H)$. We consider two cases.

Case 1: $A = \delta(S)$ for some $S = \{u, v\}$.

If $uv \in E(H - \infty)$, then $\bar{x}_{uv} = k - \sum_{w : \{u,v,w\} \in \mathcal{F}} \bar{\theta}_{\{u,v,w\},w} \leq k$. Hence, $\bar{x}(\delta(\{u, v\})) = 4k - 2\bar{x}_{uv} \geq 2k$. The inequality is strict if $\bar{\theta} > 0$.

If $uv \notin E$, then $\bar{x}(\delta(\{u, v\})) = 4k > 2k$.

If $u \in N(\infty)$, then $\bar{x}_{u\infty} = k - \sum_{vw : \{u,v,w\} \in \mathcal{F}} \bar{\theta}_{\{u,v,w\},u} \leq k$. Hence, $\bar{x}(\delta(\{u, v\})) \geq 2k$. Again, the inequality is strict if $\bar{\theta} > 0$.

Case 2: $A = \delta(S)$ such that both $H[S]$ and $H[V \backslash S]$ are 2-connected.

Assume without loss of generality that $\infty \notin S$. Denote the boundary vertices of $H[S]$ by U. Let $T = N(U) \backslash S$. Observe that each of U and T is the vertex-set of a Jordan curve in H. (See Figure 3.26.)

Suppose $S \subseteq \mathcal{I}$. Then T is the set of boundary vertices of $H[T \cup S]$, $U = N(T)$ in $H[T \cup S]$, and the number of face triangles in $H[T \cup S]$ incident with a vertex in T is $|\delta(S)|$. Let

$$A_1 = \sum_{u,v \in U, w \in T : \{u,v,w\} \in \mathcal{F}} \bar{\theta}_{\{u,v,w\},w},$$

$$A_2 = \sum_{u,v \in T, w \in U : \{u,v,w\} \in \mathcal{F}} \bar{\theta}_{\{u,v,w\},w},$$

and

$$A_3 = \sum_{uv \in \delta(S)} \sum_{w:\{u,v,w\} \in \mathcal{F}} \bar{\theta}_{\{u,v,w\},w}.$$

(Each term in A_i corresponds to an angle marked with i stripes in Figure 3.26.)

Observe that $A_1 + A_2 + A_3 = |\delta(S)|k$. Note that

$$A_3 = \sum_{e \in \delta(S)} (k - \bar{x}_e) = |\delta(S)|k - \bar{x}(\delta(S)).$$

Hence, $\bar{x}(\delta(S)) = A_1 + A_2$. Applying Claim 2 with the boundary of $H[T \cup S]$ as the Jordan curve J, we obtain

$$\sum_{u,v \in U \cup T, w \in T:\{u,v,w\} \in \mathcal{F}} \bar{\theta}_{\{u,v,w\},w} = (|T| - 2)k.$$

But

$$\begin{aligned}\sum_{u,v \in U \cup T, w \in T:\{u,v,w\} \in \mathcal{F}} \bar{\theta}_{\{u,v,w\},w} &= A_1 + \sum_{u,v \in T, w \in U:\{u,v,w\} \in \mathcal{F}} (k - \bar{\theta}_{\{u,v,w\},w}) \\ &= A_1 + |T|k - A_2.\end{aligned}$$

Thus, $(|T| - 2)k = A_1 + |T|k - A_2$. It follows that $A_2 - A_1 = 2k$. Hence, if $\bar{\theta} \geq 0$, $\bar{x}(\delta(S)) = A_1 + A_2 \geq 2k$ and if $\bar{\theta} > 0$, $\bar{x}(\delta(S)) = A_1 + A_2 > 2k$.

Now, suppose $S \not\subseteq \mathcal{I}$. Then, $S \cap N(\infty) \neq \emptyset$. Since $H[V \backslash S]$ is 2-connected, $\delta(S)$ has exactly two edges $u'v'$ and $u''v''$ in B. Otherwise, ∞ would be a cut-vertex of $H[V \backslash S]$. Assume without loss of generality that $u', u'' \in S$. (It is possible that $u' = u''$.) Let $N = N(\infty) \backslash \{u', u''\}$. Let

$$A_1 = \sum_{u,v \in U \backslash N, w \in T \backslash \{\infty\}:\{u,v,w\} \in \mathcal{F}} \bar{\theta}_{\{u,v,w\},w},$$

$$A_2 = \sum_{u,v \in T \backslash \{\infty\}, w \in U:\{u,v,w\} \in \mathcal{F}} \bar{\theta}_{\{u,v,w\},w},$$

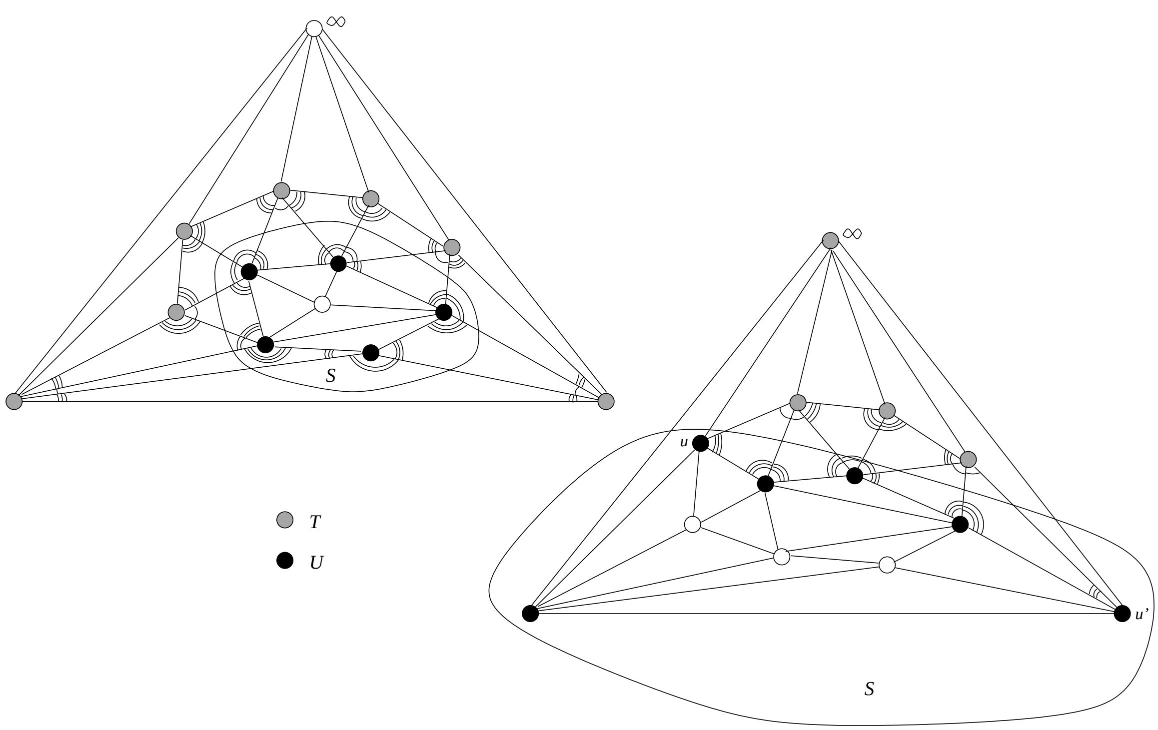

Figure 3.26: An illustration

and

$$A_3 = \sum_{uv \in \delta(S)\backslash\delta(\infty)} \sum_{w:\{u,v,w\}\in\mathcal{F}} \bar{\theta}_{\{u,v,w\},w}.$$

Observe that $A_1+A_2+A_3 = (|\delta(S)\backslash\delta(\infty)|-1)k$. Since $A_3 = \sum_{uv\in\delta(S)\backslash\delta(\infty)} \sum_{w:\{u,v,w\}\in\mathcal{F}} \bar{\theta}_{\{u,v,w\},w} = \sum_{e\in\delta(S)\backslash\delta(\infty)} (k - \bar{x}_e)$, we have $\bar{x}(\delta(S)\backslash\delta(\infty)) = A_1 + A_2 + k$.

Let $W = S \cup T\backslash\{\infty\}$. Observe that the set of boundary vertices of $H[W]$ is $(T\backslash\{\infty\}) \cup (U \cap N(\infty))$. Applying Claim 2 with the boundary of $H[W]$ as the Jordan curve J, we obtain

$$\begin{aligned} &\sum_{u,v\in U\cup T\backslash\{\infty\}, w\in T:\{u,v,w\}\in\mathcal{F}} \bar{\theta}_{\{u,v,w\},w} + \sum_{u,v\in U\cup T\backslash\{\infty\}, w\in U\cap N(\infty):\{u,v,w\}\in\mathcal{F}} \bar{\theta}_{\{u,v,w\},w} \\ = \;& (|T\backslash\{\infty\}| + |U\cap N(\infty)| - 2)k = (|T| + |U \cap N(\infty)| - 3)k. \end{aligned}$$

But

$$\sum_{u,v\in U\cup T\backslash\{\infty\}, w\in U\cap N(\infty):\{u,v,w\}\in\mathcal{F}} \bar{\theta}_{\{u,v,w\},w} = \sum_{u\in U\cap N(\infty)} (k - \bar{x}_{u\infty})$$

and

$$\begin{aligned} \sum_{u,v\in U\cup T\backslash\{\infty\}, w\in T\backslash\{\infty\}:\{u,v,w\}\in\mathcal{F}} \bar{\theta}_{\{u,v,w\},w} &= A_1 + \sum_{u,v\in T\backslash\{\infty\}, w\in U:\{u,v,w\}\in\mathcal{F}} (k - \bar{\theta}_{\{u,v,w\},w}) \\ &= A_1 + (|T|-2)k - A_2. \end{aligned}$$

Thus, $A_1 + (|T|-2)k - A_2 + \sum_{u\in U\cap N(\infty)} (k - \bar{x}_{u\infty}) = (|T| + |U\cap N(\infty)| - 3)k$. It follows that $\sum_{u\in U\cap N(\infty)} \bar{x}_{u\infty} = A_1 - A_2 + k$. Hence, $\bar{x}(\delta(S)) = \bar{x}(\delta(S)\backslash\delta(\infty)) + \sum_{u\in U\cap N(\infty)} \bar{x}_{u\infty} = 2A_1 + 2k \geq 2k$. The inequality is strict if $\bar{\theta} > 0$.

By Lemma 3.37, we see that $\bar{x}(A) \geq 2$ for any $A \in C(H)$ and the inequality is strict when $\bar{\theta} > 0$. This proves the first part of the theorem.

For the second part, let $\bar{x}$ be such that $\bar{x}_e = 0$ for all $e \in E_0$, $\bar{x}(\delta(v)) = 2k$ for all $v \in V$, $\bar{x}(A) \geq 2k$ for all $A \in C(H)$, and $\bar{x} \geq 0$. It is easy to see that if $S \subset V$,

$\bar{x}(\gamma(S)) \leq (|S|-1)k$. Let (P_k) denote the following linear programming problem:

$$\begin{array}{lrcll}
 & \max & 0 & & \\
\text{subject to} & \displaystyle\sum_{w:\{u,v,w\}\in\mathcal{F}} \theta_{\{u,v,w\},w} + \bar{x}_{uv} & = & k & \forall\ uv \in E(H-\infty), \\
 & \theta_{\{u,v,w\},u} + \theta_{\{u,v,w\},v} + \theta_{\{u,v,w\},w} & = & k & \forall\ \{u,v,w\} \in \mathcal{F}, \\
 & \displaystyle\sum_{v,w:\{u,v,w\}\in\mathcal{F}} \theta_{\{u,v,w\},u} & = & 2k & \forall\ u \in \mathcal{I}, \\
 & \displaystyle\sum_{v,w:\{u,v,w\}\in\mathcal{F}} \theta_{\{u,v,w\},u} + \bar{x}_{u\infty} & = & k & \forall\ u \in N(\infty), \\
 & \theta & \geq & 0. &
\end{array}$$

We first show that (P_k) has a feasible solution. Note that the last two sets of equations in (P_k) are superfluous since if $u \in \mathcal{I}$,

$$\begin{aligned}
\sum_{v,w:\{u,v,w\}\in\mathcal{F}} \theta_{\{u,v,w\},u} &= \sum_{v,w:\{u,v,w\}\in\mathcal{F}} (k - \theta_{\{u,v,w\},v} - \theta_{\{u,v,w\},w}) \\
&= \deg(u)k - \sum_{v\in N(u)} (k - \bar{x}_{uv}) = \sum_{e\in\delta(u)} \bar{x}_e = 2k,
\end{aligned}$$

and if $u \in N(\infty)$,

$$\begin{aligned}
\sum_{v,w:\{u,v,w\}\in\mathcal{F}} \theta_{\{u,v,w\},u} &= \sum_{v,w:\{u,v,w\}\in\mathcal{F}} (k - \theta_{\{u,v,w\},v} - \theta_{\{u,v,w\},w}) \\
&= (\deg(u)-2)k - \sum_{v\in N(u)\setminus\{\infty\}} (k - \bar{x}_{uv}) \\
&= (\deg(u)-2)k - \sum_{e\in\delta(u)} (k - \bar{x}_e) + k - \bar{x}_{u\infty} \\
&= (\deg(u)-2)k - \deg(u)k + 2k + k - \bar{x}_{u\infty} = k - \bar{x}_{u\infty}.
\end{aligned}$$

The dual of (P_k) without the last two sets of equations is

$$\begin{array}{llcl} & \min & & \sum_{e \in E \setminus \delta(\infty)} (k - \bar{x}_e) z_e + k \sum_{\{u,v,w\} \in \mathcal{F}} y_{\{u,v,w\}} \\ (D) & z_{uv} + y_{\{u,v,w\}} & \geq & 0 \ (\{u,v,w\}, w) \in \mathcal{F} \times V(H - \infty), \ uv \in E(H - \infty) \end{array}$$

Suppose (P_k) is infeasible. Since (D) is not infeasible (0 is a feasible solution), there exists a pair $\bar{z}, \bar{y}$ feasible to (D) having negative objective value. Choose one such solution that has the fewest distinct values in the entries of $\bar{y}$.

Suppose there are at least two distinct values in the entries of $\bar{y}$. Let $\rho = \min_{F \in \mathcal{F}} \bar{y}_F$. Let $\rho' = \min_{F \in \mathcal{F}, \bar{y}_F > \rho} \bar{y}_F$. Let $\mathcal{F}'' = \{\{u,v,w\} \in \mathcal{F} : \bar{y}_{\{u,v,w\}} = \rho\}$. Let $E'' = \{uv : \{u,v,w\} \in \mathcal{F}''$ for some $w\}$.

Let $E' \subseteq E''$ be such that $H[E']$ is a component of $H[E'']$. Let $S = V(H[E'])$ and $\mathcal{F}' = \{U \in \mathcal{F}'' : U \subseteq S\}$.

For each $e \in E'$, decrease $\bar{z}_e$ by $\rho' - \rho$. For each $\{u,v,w\} \in \mathcal{F}'$, increase $\bar{y}_{\{u,v,w\}}$ by $\rho' - \rho$. Clearly, the modified solution is still feasible. Let s denote the number of interior faces of the plane graph $H[E']$. Then by Euler's formula, we have $|E'| = |S| + s - 1$. Noting that $|\mathcal{F}'| \leq s$, we see that the change in the objective value is

$$\begin{aligned} (\rho' - \rho)(-\sum_{e \in E'} (k - \bar{x}_e) + k|\mathcal{F}'|) &\leq (\rho' - \rho)(-k(|S| + s - 1) + \bar{x}(E') + ks) \\ &\leq (\rho' - \rho)(\bar{x}(\gamma(S)) - (|S| - 1)k) \leq 0. \end{aligned}$$

By going through each component of $H[E'']$ and modifying the solution in the above manner, we obtain a feasible solution for which the number of distinct values of $\bar{y}$ is smaller and the objective value is not greater. This is a contradiction. Thus, we must have $\bar{y} = \lambda \mathbf{e}$ for some real number λ and the objective value is at least

$$\begin{aligned} & \lambda(-|E(H - \infty)|k + \bar{x}(\gamma(V(H - \infty))) + |\mathcal{F}|k) \\ = \ & \lambda(-(|V(H - \infty)| + |\mathcal{F}| - 1)k + (|V(H - \infty)| - 1)k + |\mathcal{F}|k) = 0. \end{aligned}$$

This contradicts that the objective value is negative. Hence, (P_k) must be feasible.

Now, assume further that $\bar{x}(A) > 2k$ for all $A \in C(H)$ and $\bar{x}_e > 0$ for all $e \in B$. Observe that $\bar{x}(\gamma(S)) < (|S|-1)k$ for all $S \subset V$, $2 \leq |S| \leq |V|-2$. In particular, $\bar{x}_e < k$ for all $e \in E$. This implies that all the coefficients in the objective function of (D) are positive.

From above, we know that (P_k) is feasible. Suppose there exists $(\{u,v,w\},w)$ such that $\theta'_{\{u,v,w\},w} = 0$ for any feasible solution θ' of (P_k). By strict complementarity, there exists an optimal solution $(\bar{z},\bar{y})$ of (D) that satisfies the inequality corresponding to $\theta'_{\{u,v,w\},w}$ strictly.

If $uv \in B$, we could reduce the value of $\bar{z}_{uv}$ and get a feasible solution having negative objective value. This is a contradiction. Hence, $uv \notin B$. Let w' be the vertex on the other face triangle that contains the edge uv. Clearly, $\bar{z}_{uv} + \bar{y}_{u,v,w'} = 0$. Let $\mathcal{F}'' = \{U \in \mathcal{F} : \bar{y}_U \leq \bar{y}_{\{u,v,w'\}}\}$. Let $S' = \bigcup_{U \in \mathcal{F}'} U$. Let $E'' = \{u'v' : \{u',v',w''\} \in \mathcal{F}''$ for some $w''\}$.

Let $E' \subseteq E''$ be such that $H[E']$ is a component of $H[E'']$. Let $S = V(H[E'])$ and $\mathcal{F}' = \{U \in \mathcal{F}'' : U \subseteq S\}$.

Let $\epsilon > 0$. For each $e \in E'$, decrease $\bar{z}_e$ by ϵ. For each $\{u,v,w\} \in \mathcal{F}'$, increase $\bar{y}_{\{u,v,w\}}$ by ϵ. Clearly, if ϵ is sufficiently small, the modified solution will still be feasible. Let s denote the number of interior faces of the plane graph $H[E']$. Then the change in the objective value is

$$\epsilon(-\sum_{e \in E'}(k-\bar{x}_e) + k|\mathcal{F}'|) \leq \epsilon(-(|S|+s-1)k + \bar{x}(E') + ks) \leq \epsilon(\bar{x}(\gamma(S)) - (|S|-1)k) \leq 0.$$

If $|S| \leq |V|-2$, then $\bar{x}(\gamma(S)) < (|S|-1)k$. Therefore, the left-hand side is negative. If $|S| = |V|-1$, then $S = V(H-\infty)$. If $B \backslash E' \neq \emptyset$, then $\bar{x}(E') < \bar{x}(\gamma(S))$ since $\bar{x}_e > 0$ for all $e \in B$. Hence, the left-hand side is negative. Suppose $B \subset E'$. If not all the interior faces of $H[E']$ are triangles, then $|\mathcal{F}'| < s$. Otherwise, $\{u,v,w\}$ is the set of vertices of a face in $H[E']$ but is not in $\mathcal{F}'$. Again, we have $|\mathcal{F}'| < s$. In either case, we see that the left-hand side is negative. Thus, the change in the objective value is negative. This contradicts the optimality of the original solution. Hence, (P_k) has a positive feasible solution. □

Observe that the number of constraints in (CP_k) is polynomial in $|V|$. From the above theorem, we see that projecting the set of solutions of (CP_1) onto the space of the x variables gives us SEP(H). To obtain a compact formulation for the subtour-elimination polytope of a 2-connected planar graph G, first embed G into a plane. Then pick any vertex on the boundary and call it ∞. Draw an edge from ∞ to any other boundary vertex that is not a neighbour of ∞. Triangulate all the interior faces and call the resulting graph H. Let E_0 be the set of all the added edges. Then the projection of the set of solutions to (CP_1) onto $\mathbb{R}^{E(H)\setminus E_0}$ is the same as SEP(G).

Chapter 4

Inner points

Let $G = (V, E)$ be a graph having at least three vertices. If $\hat{x} \in \text{SEP}(G)$ satisfies $\hat{x} > 0$ and $\hat{x}(A) > 2$ for all $A \in C(G)$, then $\hat{x}$ is called an *inner point* of $\text{sys}(G)$.

Remark. Note that our definition of an inner point depends on the system of linear inequalities defining $\text{SEP}(G)$ rather than the polytope itself. Hence, our definition is different from the definition of an inner point of a polyhedron given by Nemhauser and Wolsey [34]. They call a point $\hat{x} \in P := \{x : (a^i)^T x \leq b_i,\ i = 1, ..., m\}$ an inner point of P if $(a^i)^T \hat{x} < b_i$ for all $i \in M^{\leq}$ where $M^{\leq} := \{i \in \{1, ..., m\} : (a^i)^T x < b_i \text{ for some } x \in P\}$.

Observe that $\text{sys}(G)$ has an inner point if and only if the linear programming problem

$$\begin{aligned} \max \epsilon & \\ \text{subject to} & \\ x(\delta(v)) = 2 & \quad \forall\ v \in V, \\ x(A) - \epsilon \geq 2 & \quad \forall\ A \in\in C(G), \\ x_e \geq \epsilon & \quad \forall\ e \in E, \\ \epsilon \geq 0. & \end{aligned}$$

has a positive optimal value. Again, using the equivalence between separation and optimization (Grötschel et al. [24]), one can solve the above linear programming problem in polynomial time. Hence, one can determine if sys(G) has an inner point in polynomial time.

Determining if sys(G) has an inner point is of interest for two reasons. First, if sys(G) has an inner point, then dim(SEP(G)) can be determined easily. Second, we will see in Chapter 6 that if G is 3-connected and planar, G is isomorphic to the graph of a polytope inscribed in a sphere if and only if sys(G) has an inner point.

4.1 Main result

A cut A of G is said to be *constricted* if $x(A) = 2$ for all $x \in$ SEP(G). With this definition, we see that sys(G) has an inner point if and only if G has no useless edge and all constricted cuts of G are trivial. In this section, we obtain a necessary and sufficient condition for a graph G with no useless edge to be such that sys(G) has an inner point.

We first make a simple observation.

Lemma 4.1. *Let G be a feasible graph. If $\delta(S)$ is a non-trival constricted cut of G, then $G[S]$ and $G[V \backslash S]$ are connected.*

Proof. Suppose the statement is false. Without loss of generality, we may assume that $G[S]$ is not connected. Let T and U be non-empty proper subsets of S such that $S = T \cup U$, $T \cap U = \emptyset$, and there is no edge in $G[S]$ joining a vertex in T and a vertex in U. Then, for any $\bar{x} \in$ SEP(G),

$$\bar{x}(\delta(S)) = \bar{x}(\delta(T)) + \bar{x}(\delta(U)) \geq 2 + 2 = 4,$$

contradicting that $\delta(S)$ is constricted. The result follows. □

In this section, we will be working with the following primal-dual pair of linear programming problems.

$$
\begin{array}{llrcll}
(P) & & \max 0 & & & \\
& & \text{subject to} & & & \\
& & x(\delta(v)) & = & 2 & \forall\, v \in V(G) \\
& & -x(A) & \leq & -2 & \forall A \in C(G) \\
& & x & \geq & 0 & \\
(D) & & \min 2\sum_v z_v - 2\sum_A y_A & & & \\
& & \text{subject to} & & & \\
& & z_u + z_v - \sum_{uv\in A\in C(G)} y_A & \geq & 0 & \forall\, uv \in E(G) \\
& & y & \geq & 0. &
\end{array}
$$

We first give a simple sufficient condition for a cut to be constricted and an edge to be useless.

Lemma 4.2. *Let G be a feasible graph. If there exist $\bar{y} \in \mathbb{R}_+^{C(G)}$ and $\bar{z} \in \mathbb{R}^{V(G)}$ feasible for sys$'(G)$ be such that $\sum_{A\in C(G)} \bar{y}_A = \sum_{v\in V(G)} \bar{z}_v$, then all the cuts in $\{A : \bar{y}_A > 0\}$ are constricted and all the edges in $\{uv : \bar{z}_u + \bar{z}_v - \sum_{uv\in A} \bar{y}_A > 0\}$ are useless.*

Proof. Since G is feasible, (P) has a feasible solution and, therefore, an optimal solution. The result now follows immediately from complementary slackness. □

It is easy to see that for a graph G, if sys(G) has an inner point, then G satisfies

Property 4.3. *For every non-trivial cut A and $D \subset A$, there exists $\hat{x}$ in the subtour-elimination polytope such that not both $\hat{x}(D)$ and $\hat{x}(A\backslash D)$ are equal to 1.*

The next theorem is the main result of this chapter.

Theorem 4.4. *Let G be a 3-connected graph with no useless edge. Then sys(G) has an inner point if and only if G satisfies Property 4.3 and is more-than-1-tough or is a brace.*

The proof of Theorem 4.4 depends on the next two intermediate results.

Theorem 4.5. *Let G be 3-connected and more-than-1-tough. If G has no useless edge and satisfies Property 4.3, then sys(G) has an inner point.*

Theorem 4.6. *Let G be a brace. If G has no useless edge and satisfies Property 4.3, then $sys(G)$ has an inner point.*

The proofs of Theorem 4.5 and Theorem 4.6 will be given later.

Proof of Theorem 4.4. Since G has no useless edge, if G has no non-trivial constricted cut, then $\mathrm{sys}(G)$ has an inner point. So G satisfies Property 4.3.

Suppose G is non-bipartite. If G is not more-than-1-tough, then there exists $S \subset V$ such that $\omega(G - S) = |S| = k$ for some $k > 1$. Let $S_1, ..., S_k$ be the vertex-sets of the components of $G - S$. Construct $\bar{y}, \bar{z}$ as follows. For each $v \in S$, set $\bar{z}_v = 1$. For each $i \in \{1, ..., k\}$, if $S_i = \{v\}$ for some $v \in V$, set $\bar{z}_v = -1$. Otherwise, set $\bar{y}_{\delta(S_i)} = 1$. Note that the pair $\bar{y}, \bar{z}$ satisfies the conditions in Lemma 4.2. As G has no useless edge, S is an independent set. Since G is non-bipartite, there exists i such that S_i is not a singleton. Hence, by Lemma 4.2, $\delta(S_i)$ is a non-trivial constricted cut, which is a contradiction.

Suppose G is bipartite with bipartition (U, W). If G is not a brace, then, without loss of generality, there exists a subset X of U such that $0 < |X| < |U| - 1$ and $|N(X)| \leq |X| + 1$. Let $S = X \cup N(X)$. For any $\bar{x} \in \mathrm{SEP}(G)$, we have

$$2 \leq \bar{x}(\delta(S)) = \sum_{v \in N(X)} \bar{x}(\delta(v)) - \sum_{v \in X} \bar{x}(\delta(v)) = 2|N(X)| - 2|X| \leq 2.$$

Hence, equality holds throughout. This implies that $\delta(S)$ is a non-trivial constricted cut, which is a contradiction.

The converse follows from Theorem 4.5 and Theorem 4.6. □

Before we prove Theorem 4.5 and Theorem 4.6, we obtain necessary and sufficient conditions for $\mathrm{sys}(G)$ not to have an inner point in terms of solutions to $\mathrm{sys}'(G)$. We begin with a simple result.

Lemma 4.7. *Let G be a feasible graph. Then there exist $\bar{y} \in \mathbb{R}_+^{C(G)}$ and $\bar{z} \in \mathbb{R}^{V(G)}$ feasible for $sys'(G)$ such that $\sum_{A \in C(G)} \bar{y}_A = \sum_{v \in V(G)} \bar{z}_v$ and a cut A is constricted if and only if $\bar{y}_A > 0$ and an edge uv is useless if and only if $\bar{z}_u + \bar{z}_v - \sum_{uv \in A} \bar{y}_A > 0$.*

Proof. Since G is feasible, (P) has a feasible solution $\bar{x}$ such that $\bar{x}_e = 0$ if and only if e is useless and $\bar{x}(A) = 2$ if and only if A is constricted. The result now follows from strict complementarity. □

Next, we obtain a refinement of the previous lemma.

Lemma 4.8. *For any 3-connected feasible graph G, sys(G) has no inner points if and only if there exist $\bar{y} \in \mathbb{R}_+^{C(G)}$ and $\bar{z} \in \mathbb{R}^{V(G)}$ feasible for sys$'(G)$ such that $\sum_{A \in C(G)} \bar{y}_A = \sum_{v \in V(G)} \bar{z}_v$, $\{A \in C(G) : \bar{y}_A > 0\}$ is non-crossing, and that $\bar{y}_A > 0$ for some $A \in C(G)$ or $\bar{z}_u + \bar{z}_v - \sum_{uv \in A} \bar{y}_A > 0$ for some $uv \in E(G)$. (Here, "or" is not exclusive.)*

Proof. Sufficiency follows from Lemma 4.2.

Suppose sys(G) has no inner points. Then there exists either a constricted cut $C \in C(G)$ or a useless edge $e \in E(G)$. By Lemma 4.7, there exist an optimal solution $\bar{y}, \bar{z}$ for (D) such that $\bar{y}_A > 0$ for every non-trivial constricted cut A and $\bar{z}_u + \bar{z}_v - \sum_{uv \in A} \bar{y}_A > 0$ for every useless edge uv. Since the coefficients in (D) are integral, we may assume that $\bar{y}$ and $\bar{z}$ are rational. As the constraints of (D) are homogeneous and the objective value is zero, we may assume that $\bar{y}$ and $\bar{z}$ are integral. By Lemma 3.3 we may assume $\{A \in C(G) : \bar{y}_A > 0\}$ is a family of non-crossing cuts after uncrossing pairs of crossing cuts, if any.

It is now sufficient to show that after the uncrossings, we do not end up with $\bar{y} = 0$ and $\bar{z}_u + \bar{z}_v - \sum_{uv \in A} \bar{y}_A = 0$ for all $uv \in E(G)$. The case when G has a useless edge uv is easy since uncrossings could not decrease the value of $\bar{z}_u + \bar{z}_v - \sum_{uv \in A} \bar{y}_A$, which initially was greater than zero. So, suppose G has no useless edge. Then G has at least one non-trivial constricted cut. We claim that uncrossing leaves at least one cut in $\{A \in C(G) : \bar{y}_A > 0\}$. Suppose at some point, we uncrossed $\delta(S)$ and $\delta(T)$ where $S \cap T = \{u\}$ and $V \backslash (S \cup T) = \{v\}$. Since G has no useless edge, there exists $\bar{x} \in \mathrm{SEP}(G)$ such that $\bar{x} > 0$. Then $4 = \bar{x}(\delta(S)) + \bar{x}(\delta(T)) = \bar{x}(\delta(S \cap T)) + \bar{x}(\delta(S \cup T)) + 2\bar{x}(\gamma(S \backslash T, T \backslash S)) \geq 4$. It follows that $\gamma(S \backslash T, T \backslash S) = \emptyset$. But this means $\{u, v\}$ is a 2-separator of G, which is a contradiction. Hence, each time we perform uncrossing, there is at least one non-trivial cut A such that $\bar{y}_A > 0$.

Therefore, the resulting $\bar{y}, \bar{z}$ is feasible for sys$'(G)$ such that $\sum_{A\in C(G)} \bar{y}_A = \sum_{v\in V(G)} \bar{z}_v$, $\{A \in C(G) : \bar{y}_A > 0\}$ is non-crossing, and that $\bar{y}_A > 0$ for some $A \in C(G)$ or $\bar{z}_u + \bar{z}_v - \sum_{uv\in A} \bar{y}_A > 0$ for some $uv \in E(G)$. □

Using Lemma 3.7 and Lemma 4.8, one can prove the next result using a similar argument used in the proof of Theorem 3.8.

Theorem 4.9. *If G is a 3-connected feasible planar graph such that sys(G) has no inner points, then G is a spanning subgraph of a maximal planar graph H such that sys(H) has no inner points.*

Proof of Theorem 4.5. Our proof uses ideas from the proof of Theorem 4.7 in [18] which states that bricks have no non-trivial tight cuts.

Since G has no useless edge, by Lemma 4.8, there exists a pair $\bar{y}, \bar{z}$ feasible for sys$'(G)$ such that $\sum_{A\in C(G)} \bar{y}_A = \sum_{v\in V(G)} \bar{z}_v$, $\mathcal{A}(\bar{y}) = \{A \in C(G) : \bar{y}_A > 0\}$ is non-crossing, and that $\bar{y}_A > 0$ for some $A \in C(G)$ and $\bar{z}_u + \bar{z}_v - \sum_{uv\in A} \bar{y}_A = 0$ for all $uv \in E(G)$.

Let $C \in \mathcal{A}(\bar{y})$ and $e = uv \in C$. Since $\bar{z}_u + \bar{z}_v - \sum_{uv\in A} \bar{y}_A = 0$, at least one of $\bar{z}_u$ and $\bar{z}_v$ is positive. Without loss of generality, assume that $\bar{z}_u > 0$. Let S be a shore of C containing u. Since C is a non-trivial cut, neither S nor $V\backslash S$ can be a singleton. Choose C, e, and u so that $|S|$ is as small as possible.

Let $S_1, .., S_k$ be the maximal proper subsets of S which are shores of cuts in $\mathcal{A}(\bar{y})$. Since $\mathcal{A}(\bar{y})$ is non-crossing, $S_1, ..., S_k$ are disjoint. Let $X = S\backslash(S_1 \cup \cdots \cup S_k)$ and let $Y = \{v \in X : \bar{z}_v > 0\}$. Observe that $u \in Y$ by the minimality of S.

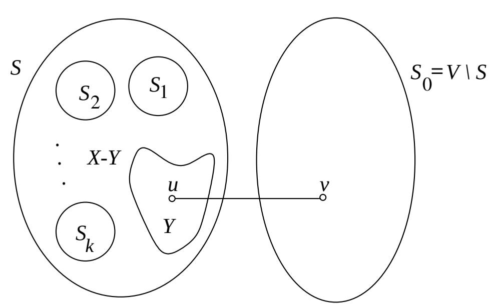

Figure 4.1: Illustration of S and $S_1, ..., S_k$

We first show that $\gamma(S_i, S_j) = \emptyset$ for all i, j. Assume that there exists an edge $f = ww'$ with $w \in S_i$ and $w' \in S_j$ for some i, j. Then, either $\bar{z}_w > 0$ or $\bar{z}_{w'} > 0$. Say $\bar{z}_w > 0$. Then $\delta(S_i)$, f, and w should have been chosen, contradicting the minimality of S. Similarly, one can show that $\gamma(S_i, X \backslash Y) = \emptyset$ for all i.

Next, observe that $\gamma(Y, Y) = \gamma(Y, X \backslash Y) = \emptyset$. Indeed, if ww' is an edge in either of the two sets, then $\bar{z}_w + \bar{z}_{w'} > 0$ but $\bar{y}_A = 0$ for all A containing ww'. This contradicts that $\bar{z}_w + \bar{z}_{w'} - \sum_{ww' \in A} \bar{y}_A = 0$.

By Lemma 4.1, $G[S]$ is connected. Hence, $X \backslash Y = \emptyset$.

Let $\bar{x} \in \mathrm{SEP}(G)$ with $\bar{x} > 0$. Since Y is an independent set, we obtain $\bar{x}(\delta(Y)) = \sum_{v \in Y} \bar{x}(\delta(v)) = 2|Y|$. Since $\gamma(S_i, S_j) = \emptyset$ for all i, j and $X \backslash Y = \emptyset$, we see that

$$\sum_{i=0}^{k} \bar{x}(\delta(S_i)) - 2\sum_{i=1}^{k} \bar{x}(\gamma(S_0, S_i)) = \bar{x}(\delta(Y)) = 2|Y|$$

where $S_0 = V \backslash S$. As $\bar{x}(\delta(S_i)) = 2$ for $i = 0, ..., k$ and $\sum_{i=1}^{k} \bar{x}(\gamma(S_0, S_i)) \leq \bar{x}(\delta(S_0)) - \bar{x}_e < 2$, it follows that $k \leq |Y| \leq k + 1$.

Suppose $|Y| = k$. Consider any $x' \in \mathrm{SEP}(G)$. Then,

$$2k = 2|Y| = x'(\delta(Y)) = \sum_{i=0}^{k} x'(\delta(S_i)) - 2\sum_{i=1}^{k} x'(\gamma(S_0, S_i)) = 2(k+1) - 2\sum_{i=1}^{k} x'(\gamma(S_0, S_i)).$$

Hence, $\sum_{i=1}^{k} x'(\gamma(S_0, S_i)) = x'(\gamma(S_0, Y)) = 1$. But this contradicts that G satisfies Property 4.3.

Hence, we must have $|Y| = k + 1$. But in this case, we must have $\gamma(S_0, S_i) = \emptyset$ for $i = 1, ..., k$. This implies $\omega(G - Y) = k + 1 = |Y|$. However, G is more-than-1-tough. We must have $Y = \{u\}$ and $X = S$. Since $X \backslash Y = \emptyset$, this contradicts that $|S| > 1$. The result follows. □

Proof of Theorem 4.6. Suppose G has a non-trivial constricted cut $\delta(S)$. Let $X = S \cap U$ and $Y = S \cap W$. Observe that neither X nor Y can be empty. Otherwise, S would be a singleton. Without loss of generality, assume that $|X| \geq |Y|$. Let $x \in \mathrm{SEP}(G)$. Now,

$$\sum_{v\in X} x(\delta(v)) - x(\gamma(X, W\backslash Y)) = x((X,Y)) = \sum_{v\in Y} x(\delta(v)) - x(\gamma(Y, U\backslash X)).$$

It follows that $2|X| - 2|Y| = x(\gamma(X, W\backslash Y)) - x(\gamma(Y, U\backslash X)) \leq 2$. If $|X| = |Y|$, we have $x(\gamma(X, W - Y)) = x(\gamma(Y, U\backslash X)) = 1$. Since G satisfies Property 4.3, this is impossible. Hence, $|X| = |Y| + 1$ and so $x(\gamma(Y, U\backslash X)) = 0$. As G has no useless edge, we must have $\gamma(Y, U\backslash X) = \emptyset$. Therefore, $X = N(Y)$ with $|X| = |Y| + 1$, which contradicts that G is a brace. □

Using Theorem 4.4, one obtains a different (but not necessarily easier) proof of the following result due to Dillencourt and Smith [16].

Theorem 4.10. *If G is a 4-connected planar graph, then sys(G) has an inner point.*

Remark. The proof by Dillencourt and Smith relies on the fact that $G - v$ is Hamiltonian for every $v \in V(G)$. They showed that an inner point of sys(G) can be constructed from the incidence vectors of elements in $\{C^v : v \in V(G)\}$, where C^v is a Hamiltonian circuit of $G - v$. Using a similar approach and the fact that every 4-connected planar graph minus any two vertices is Hamiltonian (Thomas and Yu [48]), they showed that if G is obtained from a 4-connected planar graph by removing a single vertex, then sys(G) has an inner point.

Proof of Theorem 4.10. By Theorem 2.1, G has no useless edge. Since each cut in G has cardinality at least four, G satisfies Property 4.3 by Theorem 2.1. Hence, it suffices to show that G is more-than-1-tough and the result will follow from Theorem 4.4.

Suppose G is not more-than-1-tough. Since G has no useless edge, there exists an independent subset S of $V(G)$ such that $\omega(G - S) = |S| = k$ for some $k > 1$. Let $S_1, ..., S_k$ be the vertex-sets of the components of $G - S$. Let $G' = G \times S_1 \times \cdots \times S_k$. Remove multiple edges from G', if any. Note that G' is bipartite having S as one of the partitions. Since G is 4-connected, each pseudo-vertex of G' is adjacent to at least four vertices in S. Since the partitions have the same cardinality, the average degree of a vertex in G' is at least 4. But this is impossible as G' is planar and bipartite. The result follows. □

As mentioned at the beginning of this section, checking if sys(G) has an inner point involves checking two conditions: Checking if G has a useless edge and checking if G has a non-trivial constricted cut. The next two results show that, in general, checking either condition is as hard as checking both of them.

Theorem 4.11. *Let G be a 3-connected $(2k+1)$-edge-connected $(2k+1)$-regular graph where k is a natural number. Then sys(G) has an inner point if and only if G is more-than-1-tough or is a brace.*

Proof. Let $\hat{x} = \frac{2}{2k+1}\mathbf{e}$. Then $\hat{x} > 0$ and $\hat{x} \in \mathrm{SEP}(G)$. Hence, G has no useless edge. For any cut A, if $D \subset A$, then neither $\hat{x}(D)$ nor $\hat{x}(A \backslash D)$ can be equal to 1. Hence, G satisfies Property 4.3. The result now follows from Theorem 4.4. □

Theorem 4.12. *Let G be a 3-connected plane graph such that each interior face is a triangle. Then sys(G) has an inner point if and only if G has no useless edge.*

Proof. Suppose there exists $\bar{x} \in \mathrm{SEP}(G)$ such that $\bar{x} > 0$. Assume that sys(G) does not have an inner point. Since G has no useless edge, G must have at least one non-trivial constricted cut. By Lemma 4.8, we see that there exist $\bar{y} \in \mathbb{R}_+^{C(G)}$ and $\bar{z} \in \mathbb{R}^{V(G)}$ feasible for sys$'(G)$ such that $\sum_{A \in C(G)} \bar{y}_A = \sum_{v \in V(G)} \bar{z}_v$, $\mathcal{A}(\bar{y}) = \{A \in C(G) : \bar{y}_A > 0\}$ is non-crossing, $\bar{y}_A > 0$ for some $A \in C(G)$, and $\bar{z}_u + \bar{z}_v - \sum_{uv \in A} \bar{y}_A = 0$ for all $uv \in E(G)$. Choose $\bar{y}, \bar{z}$ such that $|\mathcal{A}(\bar{y})|$ is as small as possible.

If $\bar{z}$ is not non-negative, pick w such that $\bar{z}_w < 0$ and let u, v be neighbours of w such that u and v are adjacent.

If $\bar{z} \geq 0$, choose a shore S of a cut in $\mathcal{A}(\bar{y})$ such that for all $T \subset S$, $\bar{y}_{\delta(T)} = 0$. Since G is feasible, $G[S]$ must be connected. If there exists $v \in S$ such that $\bar{z}_v > 0$, then there exists $w \in S$ such that vw is an edge and $\bar{z}_v + \bar{z}_w - \sum_{vw \in A} \bar{y}_A = \bar{z}_v + \bar{z}_w > 0$. This implies that $x_{vw} = 0$ for all $x \in \mathrm{SEP}(G)$, contradicting that G has no useless edge. Therefore, $\bar{z}_v = 0$ for all $v \in S$. It follows that if $u \in N(S)$, we must have $\bar{z}_u \geq \bar{y}_{\delta(S)}$. If $N(u) \subset S$, we can decrease $\bar{z}_u$ by $\bar{y}_{\delta(S)}$ and set $\bar{y}_{\delta(S)}$ to zero to obtain a new pair $\bar{y}, \bar{z}$ that is feasible for $\mathrm{sys}'(G)$. Observe that $\bar{y}$ and $\bar{z}$ cannot both be zero. Thus, if $\bar{y} = 0$, then $\bar{z} \neq 0$. As G has no useless edge, we have $\bar{z}_v + \bar{z}_w = 0$ for all $vw \in E(G)$. But this implies that G is bipartite, which is impossible. It follows that the pair $\bar{y}, \bar{z}$ certifies that $\mathrm{sys}(G)$ does not have an inner point with a smaller $\mathcal{A}(\bar{y})$, which is a contradiction. Thus, there exists $u \in N(S)$ that has a neighbour v' not in S. Let the neighbours of u be $\{v_0, v_1, ..., v_m\}$ where $v_i v_{i+1} \in E(G)$ for $i = 0, ..., m-1$. (Observe that if u is not on the boundary of G, then $v_0 v_m \in E(G)$.) Note that $v' = v_j$ for some $j \in \{0, ..., m\}$. Since $v' \notin S$ and u has a neighbour in S, we see that there exists i_1, i_2 such that $|i_1 - i_2| = 1$ and that $v_{i_1} \in S$ and $v_{i_2} \notin S$. Set $w = v_{i_1}$ and $v = v_{i_2}$. Then in G, w has u and v as neighbours with uv being an edge.

For an edge e, let U_e denote the set $\{A \in \mathcal{A}(\bar{y}) : e \in A\}$. Now $\bar{z}_u + \bar{z}_w - \sum_{A \in U_{uw}} \bar{y}_A = 0$ and $\bar{z}_v + \bar{z}_w - \sum_{A \in U_{vw}} \bar{y}_A = 0$. Hence, we have

$$\begin{aligned}
\bar{z}_u + \bar{z}_v &= \sum_{A \in U_{uw}} \bar{y}_A + \sum_{A \in U_{vw}} \bar{y}_A - 2\bar{z}_w \\
&= \sum_{A \in U_{uw} \setminus (U_{uw} \cap U_{vw})} \bar{y}_A + \sum_{A \in U_{vw} \setminus (U_{uw} \cap U_{vw})} \bar{y}_A + 2 \sum_{A \in U_{uw} \cap U_{vw}} \bar{y}_A - 2\bar{z}_w \\
&> \sum_{A \in U_{uw} \setminus (U_{uw} \cap U_{vw})} \bar{y}_A + \sum_{A \in U_{vw} \setminus (U_{uw} \cap U_{vw})} \bar{y}_A = \sum_{A \in U_{uv}} \bar{y}_A.
\end{aligned}$$

This contradicts that $\bar{z}_u + \bar{z}_v - \sum_{uv \in A} \bar{y}_A = 0$ for all $uv \in E(G)$. The result follows. □

Corollary 4.13. *Let G be a maximal planar graph. Then $sys(G)$ has an inner point if and only if G has no useless edge.*

4.2 Operations

In this section, we study graph operations that preserves the existence of inner points. The ultimate goal is to have a finite list of operations that permit one to construct inductively all the graphs G such that that $\mathrm{sys}(G)$ has an inner point starting from a (small) list of basis graphs. That goal is not attained in this thesis; however, some of the operations described in this section will be used in Chapter 6 to obtain large classes of graphs that are of inscribable type. As we will see in Chapter 6, a 3-connected planar graph is of inscribable type if and only if sys(G) has an inner point. Hence, operations that preserve the existence of inner points will preserve inscribability when restricted to planar graphs.

4.2.1 Adding and deleting edges

Dillencourt and Smith [14] gave conditions on the addition of edges to graphs of inscribable type to obtain graphs of inscribable. The next three propositions generalize their results.

Proposition 4.14. *Let G be a non-bipartite graph such that sys(G) has an inner point. Then sys($G + uv$) has an inner point for any non-adjacent $u,\ v \in V(G)$.*

Proof. Denote the set of contraints in sys(G) by $\{Ax = 2\mathbf{e},\ Bx \geq 2\mathbf{e},\ x \geq 0\}$ and the set of contraints in sys($G + uv$) by $\{A'y = 2\mathbf{e},\ B'y \geq 2\mathbf{e},\ y \geq 0\}$. Without loss of generality, assume that $A' = [A\ a]$ where $a \in \mathbb{R}^{V(G)}$ is such that

$$a_w = \begin{cases} 1 & \text{if } w \in \{u, v\} \\ 0 & \text{otherwise} \end{cases}.$$

Since sys(G) has an inner point, G must be connected. As G is non-bipartite, rank(A) $= |V(G)|$. It follows that a is in the column space of A and so there exists $d \in \mathbb{R}^{E(G+uv)}$ with $d_{uv} = 1$ and $A'd = 0$. If $\bar{x}$ is an inner point of sys(G) and $\epsilon > 0$ is sufficiently small, then $\begin{bmatrix} \bar{x} \\ 0 \end{bmatrix} + \epsilon d$ is an inner point of sys($G + uv$). □

Proposition 4.15. *Let G be a bipartite graph with bipartition (X, Y) such that sys(G) has an inner point. Then sys($G + uv$) has an inner point for any $u \in X$ and $v \in Y$ such that $uv \notin E(G)$.*

Proof. Since sys(G) has an inner point, G must be 2-connected. Hence, there exists a cycle C in $G + uv$ containing the edge uv. Since $G + uv$ is bipartite, $|E(C)|$ is even.

Denote the perfect matching of C containing uv by M. Let $d \in \mathbb{R}^{E(G+uv)}$ be such that

$$d_e = \begin{cases} 1 & \text{if } e \in M \\ -1 & \text{if } e \in E(C)\backslash M \\ 0 & \text{otherwise.} \end{cases}$$

Let $\hat{x} \in \mathbb{R}^{E(G+uv)}$ be such that $\hat{x}_{uv} = 0$ and $\hat{x}_e = \bar{x}_e$ if $e \in E(G)$ where $\bar{x}$ is an inner point of sys(G). Then, $\hat{x} + \epsilon d$ is an inner point of sys$(G + uv)$ if $\epsilon > 0$ is sufficiently small. $\square$

Remark. It is not difficult to see that if u, v are vertices in the same partition of a bipartite feasible graph G, then sys$(G + uv)$ does not have an inner point. In fact, $x_{uv} = 0$ for all $x \in \mathrm{SEP}(G + uv)$.

Proposition 4.16. *Let G be a bipartite graph with bipartition (X, Y) such that sys(G) has an inner point. Then sys$(G + uv + u'v')$ has an inner point for any distinct $u, v \in X$ and distinct $u', v' \in Y$.*

Proof. Since sys(G) has an inner point, G must be 2-connected. By Menger's Theorem, there are two edge-disjoint paths P and Q in G connecting u and u' and v and v', respectively. Since G is bipartite, both $|E(P)|$ and $|E(Q)|$ are odd. Let M be a maximum matching of P and N a maximum matching of Q. Let $d \in \mathbb{R}^{E(G+uv+u'v')}$ be such that

$$d_e = \begin{cases} 1 & \text{if } e \in \{uv, u'v'\} \cup (E(P)\backslash M) \cup (E(Q)\backslash N) \\ -1 & \text{if } e \in M \cup N \\ 0 & \text{otherwise.} \end{cases}$$

Let $\bar{x}$ be an inner point of sys(G). Let $\hat{x} \in \mathbb{R}^{E(G+uv+u'v')}$ be such that $\hat{x}_e = 0$ if $e \in \{uv, u'v'\}$ and $\hat{x}_e = \bar{x}_e$ if $e \in E(G)$. Then $\hat{x} + \epsilon d$ is an inner point of sys$(G + uv + u'v')$ if ϵ is sufficiently small. $\square$

The next result gives a condition when edge-deletion preserves the existence of inner points.

Proposition 4.17. *Let G be a graph having at least five vertices. Suppose G has a degree-three vertex u whose neighbours v_1, v_2, v_3 are pairwise adjacent. If $sys(G)$ has an inner point, then so does $sys(G - v_1v_2)$.*

Proof. Let $\bar{x}$ be an inner point of $\mathrm{sys}(G)$. Let $\alpha = \bar{x}_{v_1v_2}$. Let $\epsilon = \epsilon' + \max\{0, \alpha + \bar{x}_{uv_2} - 1)\}$ where $\epsilon' > 0$.

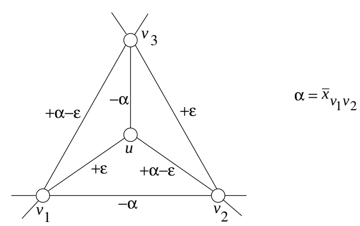

Figure 4.2: Deleting v_1v_2

Construct $x' \in \mathbb{R}^{E(G-v_1v_2)}$ as follows:

$$x'_e = \begin{cases} \bar{x}_e - \alpha & \text{if } e = uv_3 \\ \bar{x}_e + \alpha - \epsilon & \text{if } e = uv_2 \\ \bar{x}_e + \epsilon & \text{if } e = uv_1 \\ \bar{x}_e + \epsilon & \text{if } e = v_2v_3 \\ \bar{x}_e + \alpha - \epsilon & \text{if } e = v_1v_3 \\ \bar{x}_e & \text{otherwise.} \end{cases}$$

Clearly, $x'(\delta(v)) = 2$ for all $v \in V(G - v_1v_2)$.

Since $\bar{x}(\delta(u)) = 2$ and $\bar{x}_{uv_1} + \bar{x}_{uv_2} + \bar{x}_{v_1v_2} < 2$, we have $\bar{x}_{uv_3} > \bar{x}_{v_1v_2}$. Similary, one can show that $\bar{x}_{uv_2} > \bar{x}_{v_1v_3}$. Clearly, if ϵ' is sufficiently small, $x' > 0$ and $x'_e < 1$ for all $e \in E(G)$. It is now not difficult to check that if ϵ' is sufficiently small, $x'(A) > 2$ for all $A \in C(G - v_1v_2)$. Thus $\mathrm{sys}(G - v_1v_2)$ has an inner point. □

4.2.2 Gluing

In this subsection, we look at when we can glue two graphs together at a triangle to give a graph G such that $\mathrm{sys}(G)$ has an inner point. Gluing will be used in Chapter 6 to obtain a non-trivial class of maximal planar graphs of inscribable type that are not 4-connected.

We first show that if G is a graph with a 3-separator, then one can break it up into two pieces while preserving inner points.

Lemma 4.18. *Let G be 3-connected. Suppose there exist subgraphs G_1 and G_2 of G with $G = G_1 \cup G_2$ and $V(G_1) \cap V(G_2) = \{u, v, w\}$. For each $i \in \{1, 2\}$, let G_i' denote the graph obtained from G_i by joining non-adjacent pairs of vertices in $\{u, v, w\}$. If sys(G) has an inner point, then sys(G_i') has an inner point $\hat{x}$ such that $\hat{x}_{uv} + \hat{x}_{vw} + \hat{x}_{uw} > 1$ for all $i \in \{1, 2\}$.*

Proof. Let $x' \in \mathrm{SEP}(G)$ be an inner point. It suffices to consider G_1'. Let $U = V(G_2)\backslash\{u, v, w\}$. Let $a = x'((\{u\}, U))$, $b = x'((\{v\}, U))$, $c = x'((\{w\}, U))$. Note that $2 < x'(\delta_G(U \cup \{u\})) = b + c + 2 - a$. Hence, $b + c - a > 0$. Similarly, we have $a + b - c > 0$ and $a + c - b > 0$.

With the understanding that $x_e' = 0$ if $e \in \{uv, vw, uw\}\backslash E(G)$, construct $\hat{x} \in \mathrm{SEP}(G_1')$ as follows:

$$\hat{x}_e = \begin{cases} x_e' + (a + b - c)/2 & \text{if } e = uv \\ x_e' + (b + c - a)/2 & \text{if } e = vw \\ x_e' + (a + c - b)/2 & \text{if } e = uw \\ x_e' & \text{otherwise.} \end{cases}$$

Clearly, $\hat{x} > 0$. Since $x'(\delta(U \cup \{v, w\})) > 2$ and $x'(\delta(U \cup \{v, w\})) = a + (2 - b - x_{vw}') + (2 - c - x_{vw}')$, we have $1 + (a - b - c)/2 > x_{vw}'$. Hence, $\hat{x}_{vw} < 1$. Similarly, we have $\hat{x}_{uv} < 1$ and $\hat{x}_{uw} < 1$.

It is now easy to check that $\hat{x}(A) > 2$ for all A in $C(G_1')$. Finally, $\hat{x}_{uv} + \hat{x}_{vw} + \hat{x}_{uw} = x_{uv}' + x_{vw}' + x_{uw}' + (a + b + c)/2 > 1$ since $x' > 0$ and $a + b + c \geq 2$. □

With the above lemma, we give a condition on when one can glue two graphs together while preserving the existence of inner points.

Lemma 4.19. *Let G be 3-connected such that there exist subgraphs G_1 and G_2 of G with $G = G_1 \cup G_2$ and $G_1 \cap G_2 = T$ where T is a triangle with $V(T) = \{u, v, w\}$. Suppose sys(G_2) has an inner point and for any two edges $e, f \in E(T)$, there exists $\bar{x} \in \mathrm{SEP}(G_2)$*

such that $\bar{x}_e = \bar{x}_f = 1$. Then sys$(G)$ has an inner point if and only if sys(G_1) has an inner point $\hat{x}$ such that $\hat{x}_{uv} + \hat{x}_{vw} + \hat{x}_{uw} > 1$.

Proof. Necessity follows from Lemma 4.18.

Assume that sys(G_1) has an inner point $\hat{x}$ such that $\hat{x}_{uv} + \hat{x}_{vw} + \hat{x}_{uw} > 1$. Let $\hat{z}$ be an inner point of sys(G_2). Let $d = \hat{z}_{uv} + \hat{z}_{vw} + \hat{z}_{uw}$ and $D = \hat{x}_{uv} + \hat{x}_{vw} + \hat{x}_{uw}$.

Let

$$\theta_{uv} = \hat{x}_{uv}/D + \lambda(2-d)/3,\ \theta_{vw} = \hat{x}_{vw}/D + \lambda(2-d)/3,\ \theta_{uw} = \hat{x}_{uw}/D + \lambda(2-d)/3$$

for some $\lambda > 0$.

By assumption, there exist $\tilde{x}^1, \tilde{x}^2, \tilde{x}^3 \in$ SEP(G_2) such that $\tilde{x}^1_{uv} = \tilde{x}^1_{uw} = 1$, $\tilde{x}^2_{uv} = \tilde{x}^2_{vw} = 1$, $\tilde{x}^3_{uw} = \tilde{x}^3_{vw} = 1$.

Consider $x' = \alpha\tilde{x}^1 + \beta\tilde{x}^2 + \gamma\tilde{x}^3 + \lambda\hat{z}$ where

$$\begin{aligned}
\alpha &= \frac{1 - (\theta_{uv} + \theta_{uw} - \theta_{vw}) - \lambda(\hat{z}_{uv} + \hat{z}_{uw} - \hat{z}_{vw})}{2} \\
\beta &= \frac{1 - (\theta_{uv} + \theta_{vw} - \theta_{uw}) - \lambda(\hat{z}_{uv} + \hat{z}_{vw} - \hat{z}_{uw})}{2} \\
\gamma &= \frac{1 - (\theta_{uw} + \theta_{vw} - \theta_{uv}) - \lambda(\hat{z}_{uw} + \hat{z}_{vw} - \hat{z}_{uv})}{2}.
\end{aligned}$$

If λ is sufficiently small, we have $2\alpha = 1 - (\hat{x}_{uv} + \hat{x}_{uw} - \hat{x}_{vw})/D - \lambda(2-d)/3 - \lambda(\hat{z}_{uv} + \hat{z}_{uw} - \hat{z}_{vw}) > 0$. Hence, $\alpha > 0$. Similarly, $\beta, \gamma > 0$ if λ is sufficiently small. Further, $\alpha + \beta + \gamma = (3 - (\theta_{uv} + \theta_{vw} + \theta_{uw} - \lambda d)/2 = (3 - (1 + 2\lambda - \lambda d) - \lambda d))/2 = 1 - \lambda$. Hence, $\alpha + \beta + \gamma + \lambda = 1$ with $\alpha, \beta, \gamma, \lambda > 0$. It follows that x' is a convex combination of $\tilde{x}^1, \tilde{x}^2, \tilde{x}^3$, and $\hat{z}$, which are points in SEP(G_2). Since $\hat{z}$ is an inner point of sys(G_2), x' is an inner point of sys(G_2).

It is easy to check that

$$\begin{aligned}
x'_{uv} + x'_{uw} &= 2 - \theta_{uv} - \theta_{uw}, \\
x'_{uv} + x'_{vw} &= 2 - \theta_{uv} - \theta_{vw}, \\
x'_{uw} + x'_{vw} &= 2 - \theta_{uw} - \theta_{vw}.
\end{aligned}$$

Construct $\bar{y}$ as follows. For any $e \in E(G_1)\backslash E(T)$, set $\bar{y}_e = \hat{x}_e$. For any $e \in E(G_2)\backslash E(T)$, set $\bar{y}_e = x'_e$. Set $\bar{y}_{uv} = \hat{x}_{uv} - \theta_{uv}$, $\bar{y}_{vw} = \hat{x}_{vw} - \theta_{vw}$, $\bar{y}_{uw} = \hat{x}_{uw} - \theta_{uw}$. Observe that $\bar{y} > 0$ and $\bar{y}(\delta_G(v')) = 2$ for all $v' \in V(G)\backslash V(T)$.

Now

$$\begin{aligned}
\bar{y}(\delta_G(u)) - \bar{y}_{uv} - \bar{y}_{uw} &= x'(\delta_{G_2}(u)) - x'_{uv} - x'_{uw} + \hat{x}(\delta_{G_1}(u)) - \hat{x}_{uv} - \hat{x}_{uw} \\
&= 2 - (2 - \theta_{uv} - \theta_{uw}) + 2 - \hat{x}_{uv} - \hat{x}_{uw} \\
&= 2 - (\hat{x}_{uv} - \theta_{uv}) - (\hat{x}_{uw} - \theta_{uw}) = 2 - \bar{y}_{uv} - \bar{y}_{uw}.
\end{aligned}$$

Hence, $\bar{y}(\delta_G(u)) = 2$. Similarly, $\bar{y}(\delta_G(v)) = \bar{y}(\delta_G(w)) = 2$.

Consider $A \in C(G)$. If A does not contain any of uv, vw, uw, then $\bar{y}(A) > 2$. Assume A contains at least one of uv, vw, uw. Then it must contain exactly two of them. Without loss of generality, assume that $uv, uw \in A$. Let $S \subset V$ be such that $A = \delta_G(S)$ and $u \in S$. Then

$$\begin{aligned}
\bar{y}(\delta_G(S)) &= x'(\delta_{G_2}(S \cap V(G_2)) - x'_{uv} - x'_{vw} + \hat{x}(\delta_{G_1}(S \cap V(G_1))) - \theta_{uv} - \theta_{vw} \\
&> 2 - (2 - \theta_{uv} - \theta_{uw}) + 2 - \theta_{uv} - \theta_{vw} = 2.
\end{aligned}$$

Hence, $\bar{y}(A) > 2$ for any $A \in C(G)$. It follows that $\bar{y}$ is an inner point of $\text{sys}(G)$. □

Corollary 4.20. *Let G be 3-connected and G_1 and G_2 be such that $G \cap G_1 = G \cap G_2 = T$ where T is a triangle. Suppose sys(G_1) and sys(G_2) have inner points and for any two edges $e, f \in E(T)$, there exists $\bar{x} \in \text{SEP}(G_i)$ such that $\bar{x}_e = \bar{x}_f = 1$ for $i = 1, 2$. Then sys($G \cup G_1$) has an inner point if and only if sys($G \cup G_2$) does.*

Proof. This follows immediately from Lemma 4.19. □

Remark. If H is K_4 or a 4-connected planar graph containing a triangle, then $\text{sys}(H)$ has an inner point (see Theorem 4.10) and for every two distinct edges in $E(H)$, there exists a Hamiltonian circuit using them. Hence, H can be the graph G_1 or G_2 in the previous corollary.

Corollary 4.21. *Let G be a 3-connected graph such that there exist subgraphs G_1 and G_2 of G with $G = G_1 \cup G_2$ and $G_1 \cap G_2 = T$ where T is a triangle with $V(T) = \{u, v, w\}$. Suppose that at least one of u, v, w is a degree-three vertex in G_1, that sys(G_2) has an inner point, and that for any two edges $e, f \in E(T)$, there exists $\bar{x} \in \mathrm{SEP}(G_2)$ such that $\bar{x}_e = \bar{x}_f = 1$. Then sys$(G)$ has an inner point if and only if sys(G_1) has an inner point.*

Proof. Without loss of generality, assume that v is of degree three and let t be the remaining neighbour of v. Let $\bar{x}$ be an inner point of sys(G_1). Since $\bar{x}_{uv} + \bar{x}_{vw} + \bar{x}_{tv} = 2$ and $\bar{x} < 1$, we have $\bar{x}_{uv} + \bar{x}_{vw} > 1$. Thus, $\bar{x}_{uv} + \bar{x}_{vw} + \bar{x}_{uw} > 1$. The result now follows from Lemma 4.19. □

Proposition 4.22. *Let G be such that sys(G) has an inner point. Suppose G has a degree-three vertex u whose neighbours v_1, v_2, v_3 are such that $v_1v_2, v_1v_3 \in E(G)$. If G' is obtained from G by adding two new vertices u_0 and u_1 and edges u_0u, u_0v_1, u_0v_2, u_1u, u_1v_1, and u_1v_3, then sys(G') has an inner point. (See Figure 4.3.)*

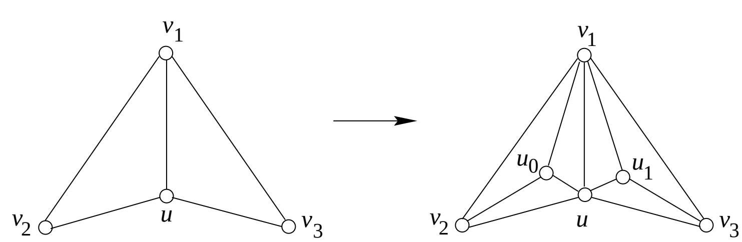

Figure 4.3: Double glue

Proof. Let $\bar{x}$ be an inner point of G. Let H denote the graph $G' - u_1$. Let $\epsilon > 0$ be such that $\epsilon < \bar{x}_{v_1v_2}$. Construct $x' \in \mathbb{R}^{E(H)}$ as follows:

$$x'_e = \begin{cases} 1 - \bar{x}_{uv_2} + \epsilon & \text{if } e = u_0v_1 \\ 1 - \epsilon & \text{if } e = uu_0 \\ \bar{x}_{uv_2} & \text{if } e = u_0v_2 \\ \bar{x}_e - \epsilon & \text{if } e = v_1v_2 \\ \bar{x}_e - (1 - \bar{x}_{uv_2}) & \text{if } e = uv_1 \\ \epsilon & \text{if } e = uv_2 \\ \bar{x}_e & \text{otherwise} \end{cases}$$

It is easy to check that x' is an inner point of sys(H).

Now, $x'_{uv_1} + x'_{uv_3} + x'_{v_1v_3} = \bar{x}_{uv_1} - (1 - \bar{x}_{uv_2}) + \bar{x}_{uv_3} + \bar{x}_{v_1v_3} = 1 + \bar{x}_{v_1v_3} > 1$. By Lemma 4.19, sys$(G')$ has an inner point. ☐

4.2.3 Splicing

We end the chapter by considering the splicing operation that was introduced in Section 3.4. The conditions considered in this thesis under which this operation preserves the existence of inner points are quite stringent. However, the cases for which the operation works are already strong enough to yield interesting results. We show the following.

Lemma 4.23. *Let G_1 and G_2 be non-bipartite graphs. Suppose $u \in V(G_1)$ and $w \in V(G_2)$ are of degree k with $N(u) = \{u_1, ..., u_k\}$ and $N(w) = \{w_1, ..., w_k\}$. Let $G = (V(G_1 - u) \cup V(G_2 - w), E(G_1 - u) \cup E(G_2 - w) \cup \{u_1w_1, ..., u_kw_k\})$. If sys$(G_1)$ has an inner point x^1 and sys(G_2) has an inner point x^2 such that $x^1_{uu_i} = x^2_{ww_i}$ for all $i = 1, ..., k$, then sys(G) has an inner point.*

The proof depends on the next lemma.

Lemma 4.24. *Let $G = (V, E)$ be a graph with no useless edge. Suppose there exists $\hat{x} \in \mathrm{SEP}(G)$ with $\hat{x} > 0$ such that $\delta(S)$ is the only non-trivial cut of G with $\hat{x}(\delta(S)) = 2$. Then sys(G) has an inner point if both $G \times S$ and $G \times (V \backslash S)$ are non-bipartite.*

Proof. It suffices to show that $\delta(S)$ is not a constricted cut.

Assume to the contrary that $\delta(S)$ is a constricted cut. Since $\hat{x} > 0$ and $\hat{x}(A) > 2$ for any non-trivial cut $A \neq \delta(S)$, by Lemma 4.7, there exist $\hat{y}, \hat{z}$ feasible for sys$'(G)$ such that $\hat{y}_{\delta(S)} > 0$, $\hat{y}_A = 0$ for all non-trivial cuts $A \neq \delta(S)$, $\hat{z}_v + \hat{z}_w = 0$ for all $vw \in E\backslash\delta(S)$, and $\hat{z}_v + \hat{z}_w = \hat{y}_{\delta(S)}$ for all $vw \in \delta(S)$.

Since $\hat{y}_{\delta(S)} > 0$, $\hat{z} \neq 0$. Note that $\hat{z}_u = -\hat{z}_v$ for every edge $uv \notin \delta(S)$. By Lemma 4.1, $G[S]$ and $G[V\backslash S]$ are connected. Hence, there exist $a, b \in \mathbb{R}_+$ such that $|\hat{z}_v| = a$ for all $v \in S$ and $|\hat{z}_v| = b$ for all $v \in V\backslash S$. Let $A = \{v \in S : \delta(v) \cap \delta(S) \neq \emptyset\}$ and $B = \{v \in V\backslash S : \delta(v) \cap \delta(S) \neq \emptyset\}$. If $a = 0$, then $\hat{z}_v = \hat{y}_{\delta(S)}$ for all $v \in B$. This implies that $G \times S$ is bipartite, which is a contradiction. Hence, $a > 0$. Similarly, $b > 0$.

If $\hat{z}_v$ has the same sign for all $v \in A$, then $G \times (V\backslash S)$ is bipartite, which is a contradiction. Hence, assume that $\hat{z}_v = -\hat{z}_u$ for some $u, v \in A$. Let $u', v' \in B$ be neighbours of u and v, respectively. Since $\hat{z}_u + \hat{z}_{u'} = \hat{y}_{\delta(S)}$ and $\hat{z}_v + \hat{z}_{v'} = \hat{y}_{\delta(S)}$, we have $\hat{z}_{u'} + \hat{z}_{v'} = 2\hat{y}_{\delta(S)}$. Since $\hat{y}_{\delta(S)} > 0$ and $|\hat{z}_{u'}| = |\hat{z}_{v'}|$, we must have $\hat{z}_{u'} = \hat{z}_{v'} = \hat{y}_{\delta(S)}$. But this implies that $\hat{z}_u = \hat{z}_v = 0$, contradicting our assumption. The result follows. □

Proof of Lemma 4.23. Let $V = V(G)$ and $E = E(G)$. Let $G_1' = G_1 - u$ and $G_2' = G_2 - w$. For $i = 1, 2$, let $V_i = V(G_i')$ and $E_i = E(G_i')$.

Construct $\hat{x} \in \mathbb{R}^E$ as follows. For any $e \in E_1$, set $\hat{x}_e = x^1_e$. For any $e \in E_2$, set $\hat{x}_e = x^2_e$. For $i = 1, ..., k$, set $\hat{x}_{u_i w_i} = x^1_{uu_i}$. Clearly, $\hat{x} > 0$, $\hat{x}(\delta(v)) = 2$ for all $v \in V$ and $\hat{x}(\delta(V_1)) = 2$.

We now show that $\hat{x}(A) > 2$ for all $A \in C(G)\backslash\{\delta(V_1)\}$. Let $\delta(S) \in C(G)\backslash\{\delta(V_1)\}$. Clearly, if $S \subset V_1$ or $S \subset V_2$, or $V\backslash S \subset V_1$ or $V\backslash S \subset V_2$, then $\hat{x}(\delta(S)) > 2$. Suppose none of the four sets $S \cap V_1, S \cap V_2, V\backslash S \cap V_1$, and $V\backslash S \cap V_2$ is empty. Note that

$$\hat{x}(\delta(S)) = x^1(\delta(S) \cap E_1) + x^2(\delta(S) \cap E_2) + \hat{x}(\delta(S) \cap \delta_G(V_1)).$$

We claim that $x^1(\delta(S) \cap E_1) > 1$. Observe that

$$x^1(\delta(S) \cap E_1) = x^1(\delta(S \cap V_1)) - x^1(\gamma(\{u\}, S \cap V_1))$$

and that

$$x^1(\delta(S) \cap E_1) = x^1(\delta(S \cap V_1 \cup \{u\})) - x^1(\gamma(\{u\}, V_1 \backslash S)).$$

Since $x^1(\delta(u)) = x^1(\gamma(\{u\}, S \cap V_1)) + x^1(\gamma(\{u\}, V_1 \backslash S))$, we have

$$\begin{aligned} 2x^1(\delta(S) \cap E_1) &= x^1(\delta(S \cap V_1)) + x^1(\delta(S \cap V_1 \cup \{u\})) - x^1(\delta(u)) \\ &> 2 + 2 - 2 = 2. \end{aligned}$$

It follows that $\hat{x}(\delta(S) \cap E_1) > 1$.

Similarly, $x^2(\delta(S) \cap E_2) > 1$. Thus, $\hat{x}(\delta(S)) > 2$.

Therefore, we have $\hat{x} \in \mathrm{SEP}(G)$ and $\hat{x}(A) > 2$ for all non-trivial cuts $A \neq \delta(V_1)$. By Lemma 4.24, sys(G) has an inner point. □

The next two results are immediate consequences of Lemma 4.23.

Corollary 4.25. *Let $G = (V, E)$ be a non-bipartite graph and $v \in V$. Let $G' = (V', E')$ be isomorphic to G and $v' \in V'$ is the image of v under some isomorphism f. Let $N(v) = \{v_1, ..., v_k\}$. Let $G'' = (V \cup V' \backslash \{v, v'\}, E(G - v) \cup E(G' - v') \cup \{v_1 f(v_1), ..., v_k f(v_k))\}$. If sys$(G)$ has an inner point, then so does sys(G'').*

The previous corollary basically states that G can be spliced with a copy of itself with respect to any vertex and its corresponding copy.

A non-bipartite graph G is said to be *strongly spliceable at v* if G is spliceable at v and if for any $\bar{y}_1, ..., \bar{y}_k \in (0, 1)$ satisfying $\sum_{i=1}^{k} \bar{y}_i = 2$, there exists an inner point $\hat{x}$ of sys(G) such that $\hat{x}_{vv_i} = \bar{y}_i$ for $i = 1, ..., k$.

Corollary 4.26. *Let G_1 and G_2 be non-bipartite graphs such that sys(G_1) and sys(G_2) have inner points. Suppose $u \in V(G_1)$ and $w \in V(G_2)$ are of degree k with $N(u) = \{u_1, ..., u_k\}$ and $N(w) = \{w_1, ..., w_k\}$. Let $G = (V(G_1 - u) \cup V(G_2 - w), E(G_1 - u) \cup E(G_2 - w) \cup \{u_1 w_1, ..., u_k w_k\})$. If G_2 is strongly spliceable at w, then sys(G) has an inner point.*

We now consider some classes of graphs that are strongly spliceable at selected vertices.

First, we prove the following.

Lemma 4.27. *Let $G = (V, E)$ be non-bipartite and $v \in V$. If sys(G) has an inner point and for any distinct $e, f \in \delta(v)$, there exists a point $\bar{x} \in \mathrm{SEP}(G)$ such that $\bar{x}_e = \bar{x}_f = 1$, then G is strongly spliceable at v.*

Proof. Let $N(v) = \{v_1, ..., v_k\}$. For any distinct $v_i, v_j \in N(v)$, let $x^{\{i,j\}}$ be a point in SEP(G) such that $x^{\{i,j\}}_{vv_i} = x^{\{i,j\}}_{vv_j} = 1$.

Let $\bar{y}_1, ..., \bar{y}_k \in (0, 1)$ be such that $\bar{y}_1 + \cdots + \bar{y}_k = 2$. Let $\hat{x}$ be an inner point of sys(G). For $i = 1, ..., k$, let $y'_i = \bar{y}_i - \epsilon \hat{x}_{vv_i}$ where $\epsilon > 0$. Since $\bar{y}_i < \sum_{j=1}^k \bar{y}_j$ for all $i = 1, ..., k$, if $\epsilon > 0$ is sufficiently small, then $y'_i \leq \sum_{j=1}^k y'_i$ for all $i = 1, ..., k$. By Lemma 3.32, there exists non-negative α_{ij} such that $\sum_{i<j} \alpha_{ij} = \frac{1}{2} \sum_{i=1}^k y'_i$ and that $y'_i = \tilde{x}_{vv_i}$ for all $i = 1, ..., k$ where $\tilde{x} = \sum_{i<j} \alpha_{ij} x^{\{i,j\}}$,

Let $\bar{x} = \epsilon \hat{x} + \tilde{x}$. Clearly, $\bar{x}_{vv_i} = \epsilon \hat{x}_{vv_i} + \tilde{x}_{vv_i} = \epsilon \hat{x}_{vv_i} + y'_i = \bar{y}_i$. Also,

$$\epsilon + \sum_{i<j} \alpha_{ij} = \epsilon + \frac{1}{2} \sum_{i=1}^k y'_i = \epsilon + \frac{1}{2} \sum_{i=1}^k \bar{y}_i - \frac{\epsilon}{2} \sum_{i=1}^k \hat{x}_{vv_i} = 1.$$

Hence, $\bar{x}$ is a convex combination of points in SEP(G). Since $\hat{x}$ is an inner point and the coefficient of $\hat{x}$ in the convex combination is positive, $\bar{x}$ is an inner point of sys(G). The result follows. □

Wheels are strongly spliceable at any of its vertices except at the hub. And clearly, K_4 is strongly spliceable at any of its vertices. The same can be said for 4-connected planar graphs, as the next result shows.

Corollary 4.28. *Let G be a 4-connected planar graph. Let v be any vertex of G. Then G is strongly spliceable at v.*

Proof. Note that G is non-bipartite. By Theorem 4.10, sys(G) has an inner point. By Theorem 2.1, there is a Hamiltonian cycle using any two distinct edges in $\delta(v)$. The result follows from Lemma 4.27. □

Chapter 5

$(2k+1)$-edge-connected $(2k+1)$-regular graphs

Theorem 4.11 of the previous chapter gives a graph-theoretical necessary and sufficient condition for sys(G) to have an inner point when G is a $(2k+1)$-edge-connected $(2k+1)$-regular graph. In this chapter, we study the subtour-elimination polytope of such graphs in more detail. In particular, we exploit the connection between the subtour-elimination polytope and the perfect matching polytope to obtain a dimension formula and an efficient combinatorial algorithm that computes the dimension. It is known that one can compute in polynomial time the dimension of the subtour-elimination polytope using the ellipsoid method (see Grötschel, Lovász, and Schrijver [24]). However, the ellipsoid method is not combinatorial and is not efficient in practice. Hence, one might ask if there exist efficient combinatorial algorithms for computing the dimension of the subtour-elimination polytope. The algorithm presented in this chapter is therefore a partial answer to the question.

We begin with some notation and definitions. If $\mathcal{S}$ is a set of disjoint subsets of V, then $G(\mathcal{S})$ denotes the graph obtained from G by shrinking each element in $\mathcal{S}$. In this chapter, we refer to a pseudo-vertex by the set that was shrunk to it. In other words, we make no distinction between the pseudo-vertex and the set from which it was shrunk.

Unless otherwise stated, $G = (V, E)$ denotes an r-edge-connected r-regular graph where r is odd and at least 3. Note that G is 3-connected and feasible and therefore 1-tough. Furthermore, as $\frac{1}{r} \in \mathrm{PM}(G)$, all tight cuts of G are r-edge cuts and by Corollary 2.4, G is matching-covered.

The next result relates the dimension of the subtour-elimination polytope and the dimension of the perfect matching polytope of G.

Theorem 5.1. *The dimension of* $\mathrm{PM}(G)$ *is the same as the dimension of* $\mathrm{SEP}(G)$. *Furthermore, a non-trivial cut* C *of* G *is tight if and only if it is constricted.*

Proof. Clearly, $\frac{1}{2}\mathrm{SEP}(G) \subseteq \mathrm{PM}(G)$. Hence, $\dim(\mathrm{SEP}(G)) \leq \dim(\mathrm{PM}(G))$.

We now show that $\dim(\mathrm{SEP}(G)) \geq \dim(\mathrm{PM}(G))$. Define the affine function $f : \mathbb{R}^E \to \mathbb{R}^E$ by $f(x) = (1/r)x + ((2r-1)/r^2)\mathbf{e}$. Let M be any perfect matching of G. Let $\hat{x} = f(\chi^M)$ where χ^M denotes the incidence vector of M. Then for any vertex $v \in V$,

$$\hat{x}(\delta(v)) = 1/r + \sum_{e\in\delta(v)} (2r-1)/r^2 = 1/r + r(2r-1)/r^2 = 2.$$

Consider $S \subset V$ such that $1 < |S| < |V|$ and $|S|$ is odd. Since $|\delta(S) \cap M| \geq 1$ and $|\delta(S)| \geq r$, we have

$$\hat{x}(\delta(S)) \geq 1/r + \sum_{e\in\delta(S)} (2r-1)/r^2 \geq 1/r + r(2r-1)/r^2 = 2.$$

Next, consider $S \subset V$ such that $1 < |S| < |V|$ and $|S|$ is even. Then $|\delta(S)| \geq r+1$. Hence,

$$\hat{x}(\delta(S)) = \sum_{e\in\delta(S)} (2r-1)/r^2 \geq (r+1)(2r-1)/r^2 > 2.$$

Hence, $\hat{x} \in \mathrm{SEP}(G)$. It follows that

$$f(\mathrm{PM}(G)) \subseteq \mathrm{SEP}(G).$$

As f is bijective, $\dim(f(\mathrm{PM}(G))) = \dim(\mathrm{PM}(G))$. Therefore, $\dim(\mathrm{PM}(G)) \leq \dim(\mathrm{SEP}(G))$. This proves the first part of the theorem.

We now prove the second part. Let C be a non-trivial cut. Suppose $\hat{x} \in \mathrm{PM}(G)$ is such that $\hat{x}(C) > 1$. Let $\hat{y} = f(\hat{x})$. Then $\hat{y}(C) > 2$ and $\hat{y} \in \mathrm{SEP}(G)$, implying that C is not a constricted cut. Suppose $\hat{x} \in \mathrm{SEP}(G)$ is such that $\hat{x}(C) > 2$. Then $\frac{1}{2}\hat{x}(C) > 1$. Since $\frac{1}{2}\hat{x} \in \mathrm{PM}(G)$, C is not a tight cut. The result now follows. □

The previous theorem tells us that we can determine the dimension of SEP(G) by determining the dimension of PM(G). Edmonds, Lovász, and Pulleyblank [18] showed that if the number of bricks in the brick decomposition of a matching-covered graph is β, then $\dim(\mathrm{PM}(G)) = |E| - |V| + 1 - \beta$. They also gave a combinatorial algorithm for obtaining the brick decomposition. Hence, there is a combinatorial algorithm to determine the dimension of SEP(G).

In a landmark paper on the matching lattice (the lattice of the incidence vectors of perfect matchings), Lovász [33] considered the tight cut decomposition of a matching-covered graph. One of the results he showed was that the number of bricks in the brick decomposition is the same as the number of bricks in the tight cut decomposition. The tight cut decomposition is important in matching theory and is the main method in Lovász' paper. The remainder of this chapter is devoted to presenting an $O(r^2|V|\log(|V|/r))$ algorithm using $O(r|V|)$ space that computes a representation of the tight cut decomposition. From such a representation, one can easily deduce the number of bricks and hence find the dimension of PM(G) and SEP(G). In the case of 3-regular planar graphs, one can improve the running time to $O(|V|)$.

We point out that having a tight cut decomposition of a matching-covered graph allows one to determine if the graph is bicritical. At the moment, the $O(|V||E|)$-algorithm by Lou and Zhong [32] has the best time-complexity bound for recognizing bicritical graphs and there is no known algorithm that finds a tight cut decomposition with a time-complexity bound better than $O(|V||E|)$. Hence, for any fixed r, the bound $O(r^2|V|\log(|V|/r))$ is currently asymptotically better than what one can achieve for general graphs.

The rest of the chapter is organized as follows. We first describe the tight cut decomposition procedure and establish a series of structural results that we need for the algorithm. We then describe the key data structure on which the algorithm depends. Essentially, the data structure is a tree-representation of all the r-edge cuts of G. Assuming that the data structure can be built efficiently, we then give a high-level description of the algorithm. We conclude the chapter by giving the technical details for building the data structure.

Remark. Dillencourt and Smith [15] gave a linear-time algorithm for determining if sys(G) has an inner point when G is 3-regular and planar. As we will see in Section 5.2, their algorithm is equivalent to determining if G is a brace or a brick. Our algorithm is based on generalizations of their ideas.

5.1 Tight cut decomposition

The tight cut decomposition procedure described by Lovász [33] hinges on the next result.

Lemma 5.2. *(Lovász [33]) A matching-covered graph has no non-trivial tight cuts if and only if it is either a brick or a brace.*

The proof of Lemma 5.2 will not be described here. It uses a non-trivial result by Edmonds, Lovász, and Pulleyblank [18] that states that bricks have no non-trivial tight cuts.

The tight cut decomposition procedure is as follows. Given a matching-covered graph H, find a non-trivial tight cut C. If all the tight cuts are trivial, stop. Otherwise, let S be a shore of C. Let $H_1 = H \times S$ and $H_2 = H \times (V(H) \backslash S)$. H_1 and H_2 are matching-covered as well. Apply the procedure to H_1 and H_2 and so forth. At the end, we have a list of bricks and braces.

Lovász [33] showed that the resulting list of bricks and braces is independent of the order in which the cuts are used.

It is not difficult to show that at any stage of the procedure, each of the graphs obtained can be obtained by shrinking shores of tight cuts of H. In particular, if H' is a graph obtained at some stage of the procedure and $\delta(S')$ is a cut of H', then $\delta(S')$ is a tight cut of H' if and only if $\delta(S)$ is a tight cut of H where $S \subset V(H)$ is such that $v \in S$

if and only if v is in S' or v is in a set that is shrunk to a pseudo-vertex in S'.

5.2 Structural results

In this section, we established a series of structural results that we will need for the algorithm. We first characterize when G is a brace.

Lemma 5.3. *If G is bipartite, then G is a brace if and only if G has no non-trivial r-edge cuts.*

Before we give the proof, we need a few technical results.

Lemma 5.4. *Suppose G is bipartite with bipartition (U, W). Let $S \subset V$ be a shore of an r-edge cut. Then either $N(S) \subset U$ or $N(S) \subset W$; moreover, $|S \cap U| - |S \cap W| \in \{-1, 1\}$.*

Proof. Since G is r-regular, $|U| = |W|$. Let $X = S \cap U$ and $Y = S \cap W$. Without loss of generality, assume that $|X| \geq |Y|$. Now,

$$\begin{aligned} r|X| - |\gamma(X, W\backslash Y)| &= \sum_{v \in X} |\delta(v)| - |\gamma(X, W\backslash Y)| \\ &= |\gamma(X, Y)| \\ &= \sum_{v \in Y} |\delta(v)| - |\gamma(Y, U\backslash X)| \\ &= r|Y| - |\gamma(Y, U\backslash X)|. \end{aligned}$$

It follows that $r|X| - r|Y| = |\gamma(X, W\backslash Y) - |\gamma(Y, U\backslash X)| \leq r$, implying that $|X| \leq |Y| + 1$. If $|X| = |Y|$, we have $|\gamma(X, W\backslash Y)| = |\gamma(Y, U\backslash X)|$. Since $|\gamma(X, W\backslash Y)| + |\gamma(Y, U\backslash X)| = |\delta(S)| = r$ and r is odd, this is impossible. Hence, $|X| = |Y| + 1$ and so $\gamma(Y, U\backslash X) = \emptyset$. It follows that $N(S) \subset W$. □

Corollary 5.5. *Let $S_1, ..., S_p \subset V$ be disjoint shores of non-trivial r-edge cuts of G. Then G is bipartite if and only if $G' = G \times S_1 \times \cdots \times S_p$ and $G \times (V\backslash S_i)$, $i = 1, ..., p$ are bipartite.*

Proof. It is immediate from Lemma 5.4 that if G is bipartite, then G' and $G \times (V\backslash S_i)$, $i = 1, ..., p$, are bipartite.

Suppose G' and $G \times (V \backslash S_i)$, $i = 1, ..., p$, are bipartite. If G is non-bipartite, then G has an odd circuit C. Since $G \times (V \backslash S_i)$ is bipartite, $G[S_i]$ must have an even number of edges from C. Therefore, the image of C under shrinking $S_1, ..., S_p$ must contain an odd circuit in G'. This contradicts that G' is bipartite. □

Proposition 5.6. *Suppose G is bipartite. If $S \subset V$ is a proper shore of a non-trivial r-edge cut, then $\delta(S)$ is a tight cut of G.*

Proof. Let the bipartition be (U, W). Denote $S \cap U$ and $S \cap W$ by X and Y respectively. Without loss of generality, assume $|X| \leq |Y|$. By Lemma 5.4, $|X| + 1 = |Y|$. Since G is r-regular, every perfect matching of G must have exactly one edge in $\delta(S)$. Hence, $\delta(S)$ is a tight cut. □

Proof of Lemma 5.3. By Proposition 5.6, we see that a non-trivial cut is tight if and only if it is an r-edge cut. The result now follows from Lemma 5.2. □

Next, we characterize when G is a brick.

Proposition 5.7. *If G is non-bipartite, then G is a brick if and only if G is more-than-1-tough.*

Proof. Since G is 3-connected, it suffices to show that G is bicritical if and only if G is more-than-1-tough.

Suppose G is not more-than-1-tough. Since G is 1-tough, then there exists $S \subset V$ such that $\omega(G - S) = |S| = k$ for some $k > 1$. Let $S_1, ..., S_k$ denote the vertex sets of the components of $G - S$. Since G is r-edge-connected, $|\delta(S_i)| \geq r$ for $i = 1, ..., k$. Since $\sum_{i=1}^{k} |\delta(S_i)| \leq \sum_{v \in S} deg(v) = rk$, we have that $|\delta(S_i)| = r$ for $i = 1, ..., k$. As r is odd, $|S_i|$ is odd for $i = 1, ..., k$. Hence, $odd(G - S) = |S|$, implying that G is not bicritical.

Conversely, suppose G is not bicritical. Then there exist two vertices $u, v \in V$ such that $H = G - \{u, v\}$ has no perfect matching. By Theorem 2.2, there exists $S \subset V(H)$ such that $odd(H - S) > |S|$. Observe that $odd(H - S)$ and $|S|$ must have the same parity. Hence, $odd(G - (S \cup \{u, v\})) \geq |S| + 2 = |S \cup \{u, v\}|$, implying that G is not more-than-1-tough. □

Putting together Lemma 5.2 and Proposition 5.7, we obtain the next result.

Theorem 5.8. *G has no non-trivial tight cut if and only if G is more-than-1-tough or G is a brace.*

Remark. By Theorem 5.1, Theorem 5.8 is equivalent to Theorem 4.11.

We now define an object that will allow us to give a more refined characterization for r-edge-connected r-regular bricks when $r \geq 3$ is odd. This object generalizes the notion of an s-localization of a 3-regular planar graph defined by Dillencourt and Smith [15]. We first make a simple observation that is well-known.

Proposition 5.9. *No two r-edge cuts of G cross.*

Proof. Suppose $\delta(S)$ and $\delta(T)$ are r-edge cuts of G that cross. Since G is r-edge-connected, we have

$$2r = |\delta(S)| + |\delta(T)| = |\delta(S \cap T)| + |\delta(S \cup T)| + 2|\gamma(S \backslash T, T \backslash S)| \geq 2r.$$

Hence, $\delta(S \cap T)$ and $\delta(S \cup T)$ are both r-edge cuts and $\gamma(S \backslash T, T \backslash S) = \emptyset$. It follows that

$$2r = |\delta(S)| + |\delta(T)| = |\delta(S \backslash T)| + |\delta(T \backslash S)| + 2|\gamma(S \cap T, V \backslash (S \cup T))| \geq 2r.$$

Hence, $\delta(S \backslash T)$ and $\delta(T \backslash S)$ are both r-edge cuts. Since r is odd, $|S|$, $|T|$, $|S \backslash T|$, $|T \backslash S|$, and $|S \cap T|$ must all be odd, which is impossible. The result follows. □

For a proper shore S of an r-edge cut of G, let $\mathcal{M}(G, S) := \{S_1, ..., S_p\}$ be the set of maximal shores of non-trivial r-edge cuts strictly contained in $V \backslash S$. As no two r-edge cuts cross, $S_1, ..., S_p$ are disjoint and uniquely determined. Let $H(G, S)$ denote the multigraph $G \times S \times S_1 \times \cdots \times S_p$. Observe that if $T \in \mathcal{M}(G, S)$, then $H(G, T) = H(G, S)$. In addition, $H(G, S)$ is r-edge-connected and r-regular. If $v \in V$, $H(G, \{v\})$ and $\mathcal{M}(G, \{v\})$ are abbreviated as $H(G, v)$ and $\mathcal{M}(G, v)$, respectively. Figure 5.1 shows a graph G_0 and $H(G_0, S)$ for various choices of S.

The next theorem is the main result of this section.

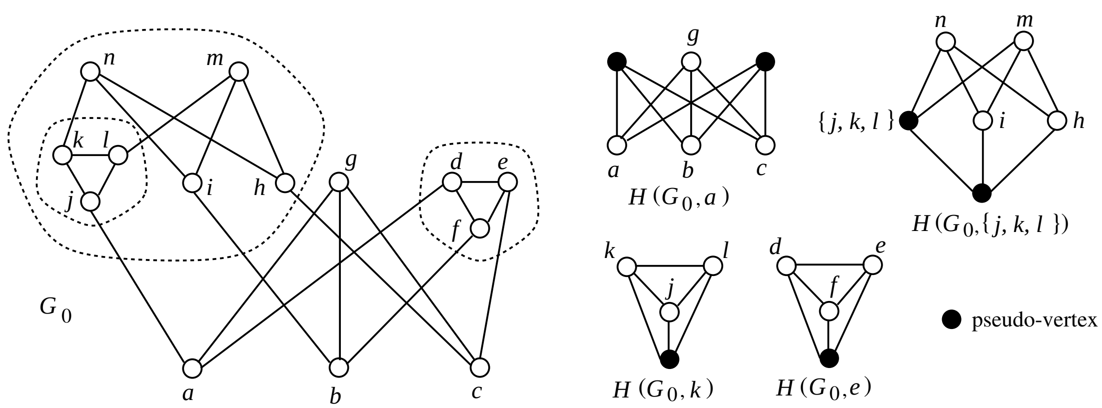

Figure 5.1: $H(G_0, v)$ for various v

Theorem 5.10. *Suppose G is non-bipartite. Then G is a brick if and only if there does not exist $v \in V$ such that $H(G, v)$ is bipartite with all the pseudo-vertices in the same partition.*

Before we give the proof, we need a few technical results.

Proposition 5.11. *If $v \in V$ is such that $H(G, v)$ is bipartite with bipartition (X, Y) and has all the pseudo-vertices in Y, then $odd(G - X) = |X|$.*

Proof. Consider $v \in V$ such that $H(G, v)$ is bipartite with bipartition (X, Y) and has all the pseudo-vertices in Y. Since $H(G, v)$ is r-edge-connected and r-regular, we must have $|X| = |Y|$. Since r is odd, if $S \in \mathcal{M}(G, v)$, then $|S|$ is also odd. Hence, $odd(G - X) = |Y| = |X|$. □

Corollary 5.12. *If $v \in V$ is such that $H(G, v)$ is bipartite and has all the pseudo-vertices in the same partition, then $\delta(S)$ is a tight cut of G for every $S \in \mathcal{M}(G, v)$.*

Proof. This follows immediately from Proposition 5.11. □

Lemma 5.13. *If G is non-bipartite, then G is more-than-1-tough if and only if there does not exist $v \in V$ such that $H(G, v)$ is bipartite with all the pseudo-vertices in the same partition.*

Proof. Suppose there exists $v \in V$ such that $H(G, v)$ is bipartite with bipartition (X, Y) and has all the pseudo-vertices in Y. By Proposition 5.11, we have $|X| = odd(G - X) \leq \omega(G - X)$. Hence, G is not more-than-1-tough.

Conversely, suppose G is not more-than-1-tough. Since G is 1-tough, there exists $S \subset V$ such that $\omega(G - S) = |S| = p$ for some $p > 1$. Choose a smallest such S. Since G is r-edge-connected and r-regular, S is an independent set. Let $S_1, ..., S_p$ denote the vertex sets of the components of $G - S$. Form the multigraph H from G by shrinking S_i for $i = 1, ..., p$. Then H is bipartite and all the pseudo-vertices are in the same partition. Suppose H has a non-trivial r-edge cut $\delta(T)$. By Lemma 5.4, either $\delta(T) = \gamma(S \backslash T, T)$ or $\delta(T) = \gamma(S \cap T, V \backslash T)$. Without loss of generality, assume the former. Obtain H' from H by shrinking T.

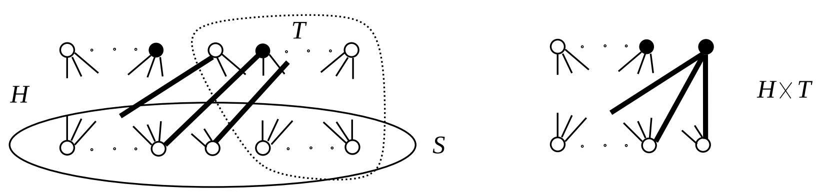

Figure 5.2: $\delta(T)$ and $H \times T$

By Corollary 5.5, H' is bipartite with all the non-pseudo-vertices in the same partition. Furthermore, the partitions have the same cardinality. This contradicts the minimality of $|S|$. Hence H has no non-trivial r-edge cuts. But this implies that $H = H(G, v)$ for any $v \in S$. □

Proof of Theorem 5.10. The result follows immediately from Proposition 5.7 and Lemma 5.13. □

5.3 Tree-representation

As no two r-edge cuts cross, by Lemma 3.5, there is a nested family $\mathcal{F}$ of proper shores of all the r-edge cuts with exactly one shore for each r-edge cut. Note that $\{v\} \in \mathcal{F}$ for all $v \in V$ since $|\delta(v)| = r$. We call $T = T(\mathcal{F})$ a *tree-representation* of the r-edge cuts of

G. (See Figure 5.3. Circles with a thick perimeter represent tight cuts. What this means exactly will become clear when we discuss the details of the algorithm.) From now on, we will abuse notation and treat each $S \in V(T)$ as the set of vertices of G that it represents. In other words, we think of $V(T)$ as $\mathcal{F} \cup \{V\}$.

Let $n = |V|$. Since T has n leaves and each non-leaf in $V(T)$ has at least two children, T has at most $2n-1$ vertices. Note that if $S \in V(T)$, then S is the union of the leaves of T that are descendants of S. For each $S \in V(T)$, let depth(S) denote the distance from R to S in T. Observe that given such a tree-representation, one can easily construct the family $\mathcal{F}$ associated with it. In Section 5.5, we will show how to construct T given G and discuss the running time.

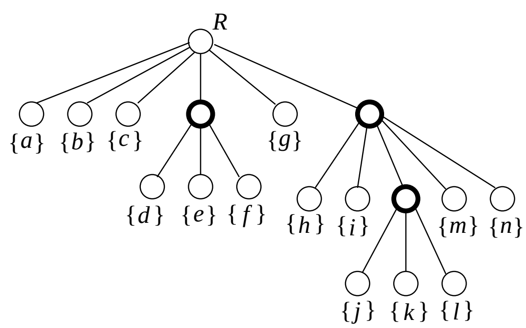

Figure 5.3: A tree-representation of 3-edge cuts of G_0

5.4 The algorithm

Let T be a tree-representation of all the r-edge-cuts of G as defined in the previous section. For our purposes, we need T to have the following attributes: type(R) (initially undefined) for the root R, traversed(S) (initially set to FALSE) for every $S \in V(T)\backslash\{R\}$, and the following for every non-leaf $S \in V(T)\backslash\{R\}$: bipartite$(S)$ (initially undefined), type(S) (initially undefined), and tight(S) (initially set to FALSE).

More attributes need to be added for the actual implementation of the algorithm. However, we will postpone the details to a later section.

We need an algorithm that has the following behaviour. Since T is a representation of all the potential tight cuts, we want the algorithm to set tight(S) to TRUE if and only if $\delta(S)$ is a tight cut of G. For each $S \in V(T)$, let $\mathcal{B}(S)$ denote the set $\{U \in V(T) : U$ is a maximal proper subset of S with tight(U) = TRUE$\}$. Notice that all the elements in $\mathcal{B}(S)$ are descendants of S in T. (For example, considering the tree in Figure 5.3, $\mathcal{B}(\{h,i,j,k,l,m,n\})$ contains only the set $\{j,k,l\}$ and $\mathcal{B}(R)$ contains the sets $\{d,e,f\}$ and $\{h,i,j,k,l,m,n\}$.) Once the tight cuts are known, it is not difficult to see that $G(\mathcal{B}(R))$ and $G(\mathcal{B}(S) \cup \{V \backslash S\})$ for all non-leaf $S \in V(T) \backslash \{R\}$ such that tight(S) = TRUE give the list of bricks and braces in the result of the tight cut decomposition procedure. Hence, we want the algorithm to set type(R) to BRICK (or BRACE) if and only if $G(\mathcal{B}(R))$ is a brick (or brace) and type(S) to BRICK (or BRACE) if and only if $G(\mathcal{B}(S) \cup \{V \backslash S\})$ is a brick (or brace) for each $S \in V(T)\backslash\{R\}$ such that tight(S) = TRUE.

We now establish some connections between T and the graphs $H(G,S)$ for any S that is a proper shore of an r-edge cut.

Proposition 5.14. *Let S be the shore of an r-edge cut.*

If $V \backslash S \in V(T)$, then $V(H(G,S)) = S \cup \{$the children of $V\backslash S\}$.

If $S \in V(T)\backslash\{R\}$, then

(i) if parent$(S) = R$, *then $V(H(G,S)) = S \cup \{$the siblings of $S\}$;*

(ii) if parent$(S) \neq R$, *then $V(H(G,S)) = S \cup \{$the siblings of $S\} \cup \{V\backslash$* parent$(S)\}$.

Proof. The result follows from the following observation: Let S be a proper shore of an r-edge cut. From the definition of T, if $V \backslash S \in V(T)$, then the children of $V\backslash S$ are all the maximal proper shores of r-edge cuts strictly contained $V\backslash S$. If $S \in V(T)$ and parent$(S) = R$, then the siblings of S are all the maximal proper shores of r-edge cuts strictly contained in $V\backslash S$. If $S \in V(T)$ and S is not a child of R, then the siblings of S together with $V\backslash$parent(S) are all the maximal proper shores of r-edge cuts strictly contained in $V\backslash S$. ☐

Proposition 5.14 shows that it is easy to identify the elements in $V(T)$ that are in $\mathcal{M}(G,S)$ for a given proper shore of an r-edge cut, S.

For the algorithm, we need a subroutine TRAVERSE_UP that takes any $S \in V(T)\backslash\{R\}$ as the input and sets traversed(U) to TRUE for every $U \in H(G,S)$ such that depth(U) = depth(S) and returns a 4-tuple (*is-bip*, Q, *same-pttn*, *all-bip*) such that

- *is-bip* = TRUE if and only if $H(G,S)$ is bipartite,
- $Q = \{U \in V(H(G,S)) : \text{tight}(U) = \text{FALSE and depth}(U) = \text{depth}(S)\}$,
- *same-pttn* = TRUE if and only if $Q = \emptyset$ or all the elements in Q are in the same partition,
- *all-bip* = TRUE if and only if bipartite(U) = TRUE for every $U \in Q$,

and a subroutine TRAVERSE_DOWN that takes a non-leaf $S \in V(T)$ as the input and returns a 4-tuple (*is-bip*, Q, *same-pttn*, *all-bip*) such that

- *is-bip* = TRUE if and only if $H(G, V\backslash S)$ is bipartite,
- $Q = \{U \in V(H(G,S)) : \text{tight}(U) = \text{FALSE, and depth}(U) = \text{depth}(S)+1\}$,
- *same-pttn* = TRUE if and only if $Q = \emptyset$ or all the elements in Q are in the same partition,
- *all-bip* = TRUE if and only if bipartite(U) = TRUE for every $U \in Q$.

Observe that we can implement TRAVERSE_UP and TRAVERSE_DOWN (using some sort of depth-first search algorithm) such that TRAVERSE_UP(S) and TRAVERSE_DOWN(S) take $O(r|V(H(G,S))|)$ time and $O(r|V(H(G,V\backslash S))|)$ time, respectively, provided that given any $U \in V(H(G,S))$ or any $U \in V(H(G,V\backslash S))$, we can find out in $O(r)$ time all the elements in $V(T)$ corresponding to the neighbours of U in $H(G,S)$ or in $H(G,V\backslash S)$. In Section 5.5, we will add more attributes to T to make it possible. Assuming that we can build T with the attributes we need to implement TRAVERSE_UP and TRAVERSE_DOWN, we claim that the algorithm below takes G as input and returns a tree-representation T of all the r-edge cuts with the attributes set properly as described at the beginning of this section.

DECOMPOSE(G)

1. Using G, build T with the necessary attributes
2. put the elements of $V(T)$ into a list L in non-increasing order of depth

```
while L is non-empty do
    remove first element S from L
    if traversed(S) = FALSE then
        (is-bip, Q, same-pttn, all-bip):= TRAVERSE_UP(S)
        if (is-bip = TRUE) and (same-pttn = TRUE) then
            if S ≠ R then
                tight(parent(S)) := TRUE
            endif
            type(parent(S)) := BRACE
            TRICKLE_DOWN(Q)
        else if (is-bip = TRUE) and (all-bip = TRUE) then
            bipartite(parent(S)) := TRUE
        else if S = R then
            type(R) := BRICK
        else
            bipartite(parent(S)) := FALSE
        endif
    endif
endwhile
return T
```

TRICKLE_DOWN(Q)

```
while Q ≠ ∅ do
    remove an element S from Q
    (is-bip, Q', same-pttn, all-bip) := TRAVERSE_DOWN(S)
    tight(S) := TRUE
    if (is-bip = TRUE) and (same-pttn = TRUE or all-bip = TRUE) then
        type(S) := BRACE
        TRICKLE_DOWN(Q')
    else
```

```
9.          type(S) := BRICK
10.     endif
11. endwhile
```

Figure 5.4 illustrates the execution of DECOMPOSE(G_0).

We now make some observations without proof. Let $\mathcal{G} = \{H(G,S) : S \in V(T)\backslash\{R\}\}$.

Observation 1. Every graph in $\mathcal{G}$ is traversed exactly once by TRAVERSE_UP. (This follows from Proposition 5.14 since TRAVERSE_UP(S) sets traversed(U) to TRUE for all the siblings U of S.)

Observation 2. Let $S \in V(T)\backslash\{R\}$ be not a leaf. Once tight(S) is set to TRUE, $H(G, V\backslash S)$ will not be traversed again by TRAVERSE_DOWN. (This is because tight(U) = FALSE for every U in Q passed to TRICKLE_DOWN.)

Observation 3. Each graph in $\mathcal{G}$ is traversed at most once by TRAVERSE_DOWN. (This follows from the previous observation since tight(S) is set to TRUE in line 4 of TRICKLE_DOWN after $H(G, V\backslash S)$ is traversed by TRAVERSE_DOWN.)

Theorem 5.15. *The running time of DECOMPOSE is O(time required to build T) + $O(rn)$.*

Proof. By Observation 1, we see that the total running time of TRAVERSE_UP over each graph in $\mathcal{G} = \sum_{G' \in \mathcal{G}} O(r|V(G')|) = O(r|V(T)|)$ because for each $S \in V(T)$, S appears in at most one $G' \in \mathcal{G}$ and so does $V\backslash S$. Similarly, by Observation 3, the total running time of TRAVERSE_DOWN over each graph in $\mathcal{G} = O(r|V(T)|)$. Now, what DECOMPOSE does in line 2 can be accomplished using a breadth-first search on T in $O(|V(T)|)$ time. Hence, the running time of DECOMPOSE is O(time required to build $T) + O(r|V(T)|) = O$(time required to build $T) + O(rn)$ since $|V(T)| \leq 2n-1$. □

We now show that DECOMPOSE is our desired algorithm.

Suppose DECOMPOSE has completed execution for a particular G. Let t_∞ denote the time when the algorithm terminated. We first establish the following:

Theorem 5.16. *Let $S \in V(T)\backslash\{R\}$ be not a leaf. Then* tight(S) = *TRUE at time t_∞ if and only if $\delta(S)$ is a non-trivial tight cut of G.*

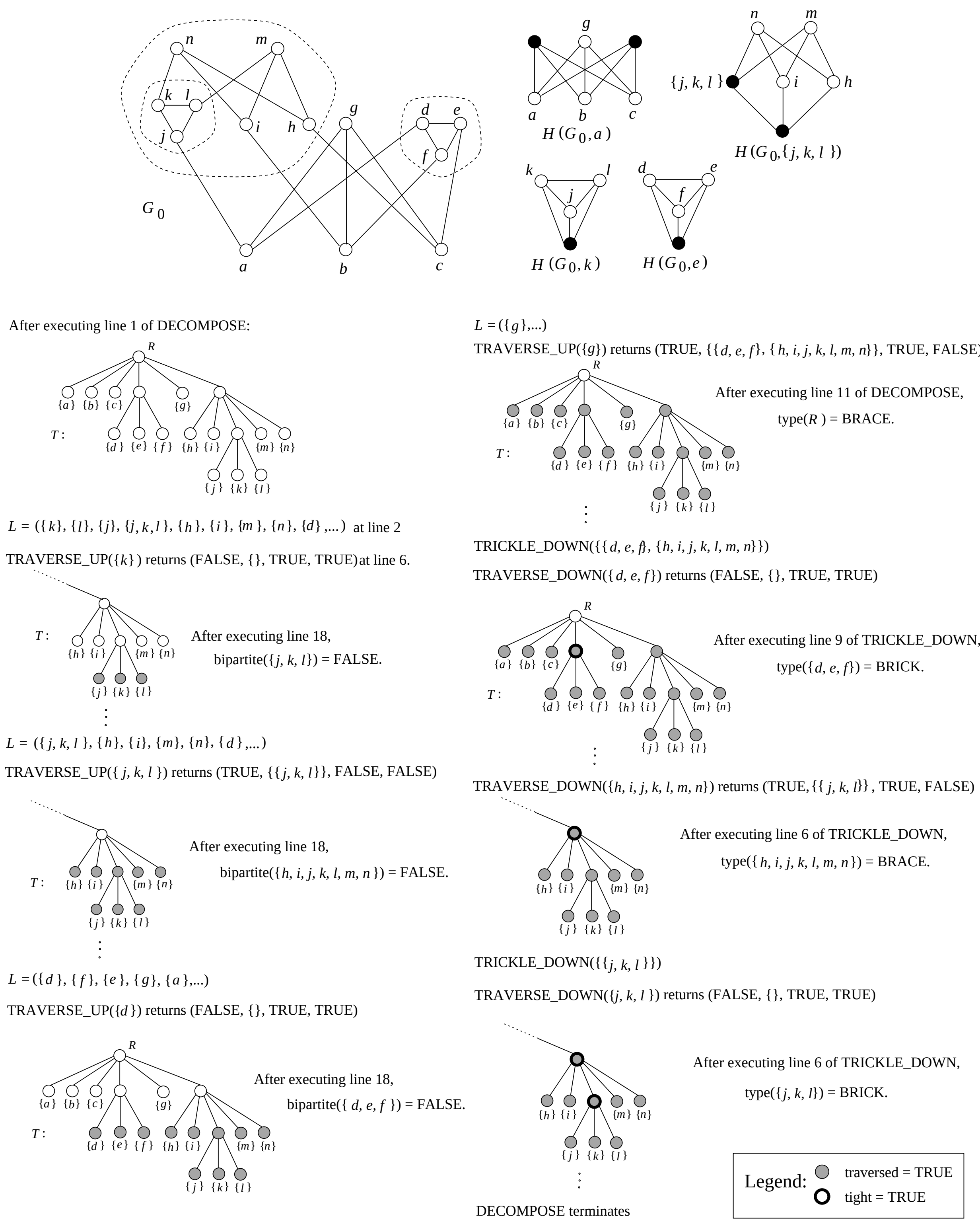

Figure 5.4: Execution of DECOMPOSE(G_0)

Before we present the proof, we need a few technical results. We first define some notation. For each $S \in V(T)\backslash\{R\}$ that is not a leaf, let $t_{tight}(S)$ denote the time when tight(S) was set to TRUE. Let $t_{up}(S)$ denote the time when DECOMPOSE began executing TRAVERSE_UP(U) for some child U of S. In other words, $t_{up}(S)$ denotes the time when TRAVERSE_UP began traversing $H(G, V\backslash S)$. Let $t_{up-done}(S)$ denote the earliest time DECOMPOSE reached line 19 after time $t_{up}(S)$. Let $\mathcal{B}_0(t,S) = \{U \in V(T) : U$ is a maximal proper subset of S such that $t_{tight}(U) \leq t\}$ and $\mathcal{B}(t,S) = \mathcal{B}_0(t,S) \cup \{V\backslash S\}$.

Proposition 5.17. *Let $U \in V(T)\backslash\{R\}$ be not a leaf and S be the parent of U. Then $t_{up-done}(U) < t_{up}(S)$.*

Proof. By Observation 1, each of $H(G, V\backslash S)$ and $H(G, V\backslash U)$ was traversed exactly once by TRAVERSE_UP. In addition, $H(G, V\backslash S)$ was traversed when DECOMPOSE executed TRAVERSE_UP(G, S') for some child S' of S and $H(G, V\backslash U)$ was traversed when DECOMPOSE executed TRAVERSE_UP(G, U') for some child U' of U. Since the elements of $V(T)$ are put into the list L in line 2 of DECOMPOSE in non-increasing order of depth and depth(U') = depth(U) + 1 = depth(S') + 1, U' preceded S' in L and the result follows. $\square$

Lemma 5.18. *If $S \in V(T)\backslash\{R\}$ is not a leaf, then $\mathcal{B}(t_{up}(S), S) = \mathcal{B}(t, S)$ for all t satisfying $t_{up}(S) \leq t < t_{tight}(S)$.*

Proof. If $t_{tight}(S) < t_{up-done}(S)$, then tight$(S)$ was set to TRUE in line 9 right after DECOMPOSE executed TRAVERSE_UP(U) for some child U of S. Clearly, $\mathcal{B}(t_{up}(S), S) = \mathcal{B}(t, S)$ for all t satisfying $t_{up}(S) \leq t < t_{tight}(S)$ in this case.

Otherwise, after DECOMPOSE executed TRAVERSE_UP(U) for some child U of S, the condition in line 7 was not satisfied. Thus, if U' is a descendant of S that is not a leaf, the value of tight(U') did not change between time $t_{up}(S)$ and time $t_{up-done}(S)$. In addition, after time $t_{up-done}(S)$, the value of tight(U') could not be changed from FALSE to TRUE until S was in some set that was later passed to TRICKLE_DOWN. However, this could not happen before time $t_{tight}(S)$. The result follows. $\square$

Lemma 5.19. *Let $S \in V(T)\backslash\{R\}$ be not a leaf. Let t be such that $t_{up-done}(S) \leq t < t_{tight}(S)$. Then at time t,* bipartite(S) *was not undefined. Furthermore,* bipartite$(S) =$ *TRUE at time t if and only if $G(\mathcal{B}(t, S))$ is bipartite.*

Proof. Since $t_{tight}(S) > t_{up-done}(S)$, we see that DECOMPOSE executed either line 14 or line 18 right before time $t_{up-done}(S)$. Hence, bipartite(S) was set to TRUE or FALSE before $t_{up-done}(S)$. Since the value of bipartite(S) could not change afterwards, this proves the first part of the lemma.

To prove the second part, suppose at some time t, $t_{up-done}(S) \leq t < t_{tight}(S)$, bipartite$(S) =$ TRUE if and only if $G(\mathcal{B}(t, S))$ is non-bipartite. Let $G' = G(\mathcal{B}(t_{up}(S), S))$. By Lemma 5.18, $G' = G(\mathcal{B}(t, S))$. Since bipartite$(S)$ was not undefined at time $t_{up-done}(S)$ and its value did not change afterwards, we have:

$$\text{bipartite}(S) = \text{TRUE at time } t_{up-done}(S) \text{ if and only if } G' \text{ is non-bipartite.} \tag{5.1}$$

Choose S so that $t_{up-done}(S)$ is as early as possible. By Proposition 5.17, if U is a non-leaf child of S, then $t_{up-done}(U) < t_{up}(S)$. Hence, if $t_{up}(S) \leq t_{tight}(U)$, by the first part of the lemma, bipartite(U) was not undefined at time $t_{up}(S)$ and by minimality of $t_{up-done}(S)$, we have

$$\text{bipartite}(U) = \text{TRUE at time } t_{up}(S) \text{ if and only if } G(\mathcal{B}(t_{up}(S), U)) \text{ is bipartite.} \tag{5.2}$$

Observe that DECOMPOSE set bipartite(S) to TRUE between time $t_{up}(S)$ and time $t_{up-done}(S)$ if and only if TRAVERSE_UP returned TRUE for both *all-bip* and *is-bip* if and only if $H(G, V\backslash S)$ is bipartite and $G(\mathcal{B}(t_{up}(S), U))$ is bipartite for every child U of S such that $t_{tight}(U) > t_{up}(S)$. But by Corollary 5.5, G' is bipartite if and only if $H(G', V\backslash S)$ is bipartite and $G(\mathcal{B}(t_{up}(S), U))$ is bipartite for every child U of S such that $t_{tight}(U) > t_{up}(S)$. Since $H(G', V\backslash S)$ is bipartite if and only if $H(G, V\backslash S)$ is and TRAVERSE_UP returned TRUE for *all-bip* if and only if bipartite$(U) =$ TRUE for every child U of S such that $t_{tight}(U) > t_{up}(S)$, it follows from (5.2) that G' is bipartite if and only if bipartite$(S) =$ TRUE by time $t_{up-done}(S)$, which contradicts (5.1). □

The next result proves one direction of Theorem 5.16.

Proposition 5.20. *Let $S \in V(T)\backslash\{R\}$ be not a leaf such that* tight(S) = *TRUE at time t_∞. Then $\delta(S)$ is a non-trivial tight cut of G.*

Proof. Suppose there exists a non-leaf $S \in V(T)\backslash\{R\}$ such that tight(S) = TRUE but $\delta(S)$ is not a non-trivial tight cut of G. Choose S so that $t_{tight}(S)$ is as early as possible.

Assume that tight(S) was set to TRUE in line 9 of DECOMPOSE or in line 4 of TRICKLE_DOWN called by DECOMPOSE. By Proposition 5.14, line 6 of DECOMPOSE must have at some point executed TRAVERSE_UP(S') for some S' that is either a child of S or a sibling of S and TRUE was returned for both *is-bip* and *same-pttn*. Let $G' = G(\mathcal{B}_0(t_{up}(S), S))$. Then, $H(G', S')$ is bipartite and has all its pseudo-vertices in the same partition. By Corollary 5.12, if $U \in \mathcal{M}(G', S')$, then $\delta(U)$ is a non-trivial tight cut of G'. By minimality of $t_{tight}(S)$, G' is obtained from G by shrinking shores of tight cuts of G. Hence, $\delta(U)$ is a non-trivial tight cut of G for all $U \in \mathcal{M}(G, S')$. Since $S \in \mathcal{M}(G, S')$, $\delta(S)$ is a tight cut of G, contradicting our assumption.

Suppose tight(S) was set to TRUE in line 4 of TRICKLE_DOWN after it was called by itself. Then at some point, line 3 of TRICKLE_DOWN executed TRAVERSE_DOWN$(\hat{S})$ where $\hat{S}$ = parent(S) and the value TRUE was returned for *is-bip* and for either *same-pttn* or *all-bip*. If TRUE was returned for *same-pttn*, then one can show as above that $\delta(S)$ is a non-trivial tight cut of G, contradicting our assumption. If TRUE was returned for *all-bip*, let $G' = G(\mathcal{B}(t_{tight}(\hat{S}), \hat{S}))$. By Lemma 5.19 and Corollary 5.5, G' is bipartite. By Proposition 5.6, $\delta(U)$ is a non-trivial tight cut of G' for all $U \in \mathcal{M}(G', V\backslash\hat{S})$. By minimality of $t_{tight}(S)$, we see that G' is obtained from G by shrinking shores of tight cuts of G. Hence, $\delta(U)$ is a non-trivial tight cut of G for all $U \in \mathcal{M}(G, V\backslash\hat{S})$. Since $S \in \mathcal{M}(G, V\backslash\hat{S})$, $\delta(S)$ is a tight cut of G, contradicting our assumption. □

To complete the proof of Theorem 5.16, we need the next two results.

Proposition 5.21. *Let $S \in V(T)\backslash\{R\}$ be not a leaf such that* tight(S) = *TRUE. Upon termination of DECOMPOSE, if* type(S) = *BRICK (BRACE) then $G(\mathcal{B}(t_\infty, S))$ is a brick (brace).*

Proof. Clearly, if tight(S) = TRUE at time t_∞, type(S) had been set to either BRICK or BRACE. In addition, once type(S) was set, its value could not change.

Let G' denote the graph $G(\mathcal{B}(t_\infty, S))$.

Suppose tight(S) was set to TRUE in line 9 of DECOMPOSE. Then type(S) was set to BRACE in line 11 and TRAVERSE_UP had returned TRUE for *is-bip* and TRUE for either *same-pttn* or *all-bip*. In any case, after executing TRICKLE_DOWN in line 11, tight(S') = TRUE for every child S' of S. Hence, G' is bipartite and has no non-trivial r-edge cuts. By Lemma 5.3, G' is a brace.

Now suppose tight(S) was set to TRUE in line 4 of TRICKLE_DOWN.

If type(S) was set to BRACE, then TRAVERSE_DOWN had returned TRUE for *is-bip* and TRUE for either *same-pttn* or *all-bip*. In any case, after line 7 of TRICKLE_DOWN, tight(S') = TRUE for every child S' of S. As before, G' is a brace.

Suppose type(S) was set to BRICK. By Theorem 5.10, to show that G' is a brick, it suffices to show that for any $U \in V(G')$, $H(G', U)$ is not a bipartite graph with all its pseudo-vertices in the same partition.

Let $\hat{G}$ denote the graph $G(\mathcal{B}(t_{down}, S))$ where t_{down} denotes the time when TRAVERSE_DOWN(S) finished. Note that TRAVERSE_DOWN(S) did not return TRUE for *is-bip*. It also did not return TRUE for at least one of *same-pttn* and *all-bip*. Hence, $H(\hat{G}, V\backslash S)$ is not bipartite with all its pseudo-vertices (that is, the vertices shrunk from vertices of $\hat{G}$) in the same partition. Since $t_{tight}(S)$ came right after time t_{down} and the condition in line 5 of TRICKLE_DOWN was not satisfied, by Observation 3, we see that after time t_{down}, $H(G, S')$ was not traversed by TRAVERSE_DOWN again for every descendant S' of S. Hence, $\hat{G} = G'$ and so $H(G', V\backslash S)$ is not bipartite with all its pseudo-vertices in the same partition.

Now, consider $U \in V(G')$ that is not a child of S. Let $\hat{S}$ = parent(U). Since $U \subset \hat{S}$ and $U \in V(G')$, tight($\hat{S}$) = FALSE at time t_∞. By Lemma 5.18, $G(\mathcal{B}(t_{up}(\hat{S}), \hat{S})) = G(\mathcal{B}(t_\infty, \hat{S}))$. Therefore, if $H(G', U)$ is bipartite with all its pseudo-vertices in the same partition, then $H(G(\mathcal{B}(t_{up}(\hat{S}), \hat{S})), U)$ is bipartite with all its pseudo-vertices in the same partition. This implies that DECOMPOSE would have set tight($\hat{S}$) to TRUE between time $t_{up}(\hat{S})$ and $t_{up-done}(\hat{S})$, which is a contradiction. □

Similarly, one can show the following:

Proposition 5.22. *Let $\mathcal{S} = \{S : S$ is a descendant of R and $t_{tight}(S) < t_\infty\}$. If type$(R)$ = BRICK (BRACE), then $G(\mathcal{S})$ is a brick (brace).*

Proof of Theorem 5.16. By Proposition 5.20, it suffices to show that if $S \in V(T)\backslash\{R\}$ is not a leaf and tight(S) = FALSE, then $\delta(S)$ is not a non-trivial tight cut of G.

Let $U \in V(T)$ be the ancestor of S such that tight(U) = TRUE and depth(U) is as large as possible. If $\delta(S)$ were a tight cut of G, then $G(B(t_\infty, U))$ would have a non-trivial tight cut. However, this contradicts that $G(B(t_\infty, U))$ is either a brick or a brace asserted by the previous two propositions. □

It is now clear from Theorem 5.16 and Propositions 5.21 and 5.22 that DECOMPOSE is our desired algorithm.

5.5 Building a desired tree-representation

In this section, we show how to build T with all the necessary attributes in $O(r^2 n \log(n/r))$ time. Then by Theorem 5.15, the running time of DECOMPOSE is $O(r^2 n \log(n/r))$.

In addition to the attributes mentioned in the previous section, we add to T the following attributes for each $S \neq R$ that allow TRAVERSE_UP and TRAVERSE_DOWN to traverse $H(G, S)$ for any $S \in V(T)\backslash\{R\}$ by navigating through T. For each leaf S, we add a neighbour-set $N(S)$. For each non-leaf $S \in V(T)\backslash\{R\}$, we make a clone of S to represent $V\backslash S$ and a neighbour-set for S and a neighbour-set for the clone of S denoted by $N(S)$ and $N(V\backslash S)$, respectively. The goal is to have $N(S)$ contain the pointers to nodes that correspond to neighbours of S in $H(G, S)$. Figure 5.5 gives a partial pictorial representation of the neighbour-sets for the example with the graph G_0.)

We now show how to build T. First, build a tree-representation of all the minimum cuts (that is, r-edge cuts) in $O(r^2 n \log(n/r))$ time consuming $O(rn)$ space using the algorithm by Gabow ([22], [23]). Then initialize depth(S), tight(S) (if S is not a leaf), and traversed(S) for each node S using a breadth-first search. This can be done in $O(rn)$

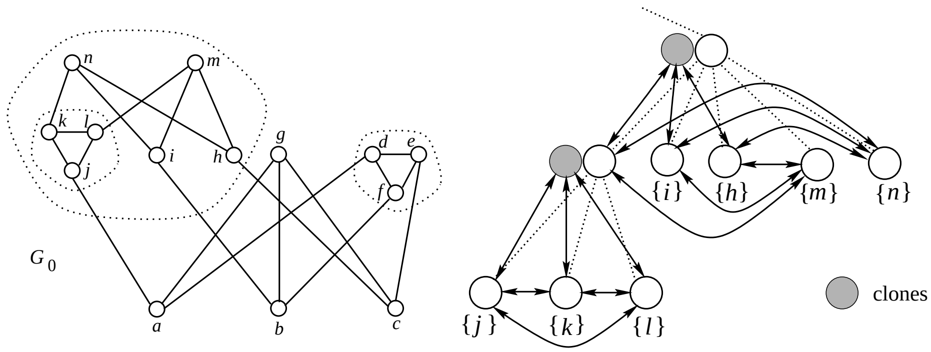

Figure 5.5: Illustration of neighbour-sets

time.

Finally, build the neighbour-sets using the following algorithm:

for each $uv \in E$ **do**
    if depth($\{u\}$) $>$ depth($\{v\}$) **then**
        $S_1 := \{u\}$, $S_2 := \{v\}$
    else
        $S_2 := \{u\}$, $S_1 := \{v\}$
    endif
    while depth(S_1) $>$ depth(S_2) **do**
        add to $N(S_1)$ a pointer to the clone of parent(S_1)
        add to $N(V \backslash S_1)$ a pointer to S_1
        $S_1 =$ parent(S_1)
    endwhile
    while parent(S_1) $\neq$ parent(S_2) **do**
        add to $N(S_2)$ a pointer to the clone of parent(S_2)
        add to $N(V \backslash S_2)$ a pointer to S_2
        $S_2 :=$ parent(S_2)
        add to $N(S_1)$ a pointer to the clone of parent(S_1)
        add to $N(V \backslash S_1)$ a pointer to S_1
        $S_1 :=$ parent(S_1)

19.   **endwhile**
20.   add to $N(S_2)$ a pointer to S_1
21.   add to $N(S_1)$ a pointer to S_2
22. **endfor**

It is not difficult to see that the above algorithm is correct with the following observation: Let $uv \in E$. Let $S \in V(T)$ be a proper shore of an r-edge cut. Without loss of generality, assume that $u \in S$. Let $\hat{S} = \text{parent}(S)$. If $\{v\}$ is not a descendant of $\hat{S}$, then $V \backslash \hat{S}$ is the maximal shore of an r-edge cut properly contained in $V \backslash S$ that contains v. If $\{v\}$ is a descendant of $\hat{S}$, then the child S' of $\hat{S}$ that has $\{v\}$ as a descendant is the maximal shore of an r-edge cut properly contained in $V \backslash S$ that contains v.

For the running time, note that each neighbour-set has exactly r elements and there are $O(n)$ such sets. Since in each of lines 8, 9, 13, 14, 16, 17, 20, 21, a new element is added to one of the sets, the above algorithm takes $O(rn)$ time.

Hence, the desired T can be built in $O(r^2 n \log(n/r))$ time.

5.6 3-connected 3-regular planar graphs

One can see from Theorem 5.15 that building a desired tree-representation of all the r-edge cuts is the bottle neck for the running time of DECOMPOSE. In this section, we give a sketch of how one can build a tree-representation in linear time when the graph is 3-connected, 3-regular, and planar. As a result, we have a linear-time algorithm for finding a representation of the result of the tight cut decomposition procedure of a 3-connected 3-regular planar graph.

The algorithm is an extension of the preprocessing step of the linear-time algorithm for recognizing 3-regular planar bricks given by Dillencourt and Smith [15]. Below is a description of the preprocessing step. First, construct G^*, the planar dual of a planar embedding of G. If x, y, and z are three vertices of a triangle (not necessarily a face triangle) of G^*, define $\mathbf{cwsucc}(x, y; z)$ to be the vertex $w \in V(G^*)$ such that x, y, and w are vertices of a triangle of G^* and w is the first vertex with this property that can be reached by starting at z and following one or more edges, moving clockwise about y through the neighbours of y. Dillencourt and Smith showed how to build in linear time a structure that makes it possible to compute **cwsucc**$(x, y; z)$ in $O(1)$ time given x, y, and z. Their algorithm involves enumerating all the triangles in G^*.

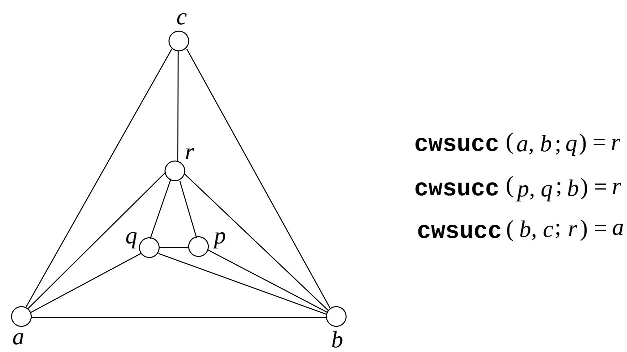

Figure 5.6: Illustration of the function **cwsucc**

Now, consider a triangle xyz in G^* where x, y, and z appear in anti-clockwise order. Let $w = \textbf{cwsucc}(x, y; z)$. Let $S_1 \subset V(G)$ be the set of vertices separated by xyz from the exterior face of G and let $S_2 \subset V(G)$ be the set of vertices separated by xyw from the exterior face of G. Then either S_2 is the minimal shore of a 3-edge cut strictly containing S_1 or S_2 is a maximal shore of a 3-edge cut strictly contained in $V(G)\backslash S_1$. It is the former if x, y, and w appear in anti-clockwise order and the latter otherwise.

Below is a high-level description of the algorithm. For each triangle xyz in G^*, create a node corresponding to it and denote the node by $n(xyz)$. For each non-boundary triangle xyz where x, y, and z apear in anti-clockwise order, compute $w_1 = \textbf{cwsucc}(x, y; z)$. $w_2 = \textbf{cwsucc}(y, z; x)$. $w_3 = \textbf{cwsucc}(z, x; y)$. Suppose x, y, and w_i do not appear in anti-clockwise order for all $i \in \{1, 2, 3\}$. If for some $j \in \{1, 2, 3\}$, $n(xyw_j)$ has already been made a child of some node, then make $n(xyz)$ a child of the parent of $n(xyw_j)$; otherwise, make a temporary connection between $n(xyz)$ and $n(xyw_j)$ for $j = 1, 2, 3$. Suppose x, y, and w_i appear in anti-clockwise order for some $i \in \{1, 2, 3\}$. (Note that there cannot be more than one such i.) Then make $n(xyw_i)$ the parent of $n(xyz)$ as well as all the nodes reachable from $n(xyz)$ via the temporary connections.

If we denote the node corresponding to the boundary triangle by R, then the above algorithm terminates with a tree-representation of the 3-edge cuts of G. It is not difficult to see that the running time is linear in the number of vertices of G.

Chapter 6

Inscribing a polytope in a sphere

In this chapter, we consider the problem of determining which 3-connected planar graphs are of inscribable type. We first begin with Rivin's characterization and give a detailed sketch of Rivin's elementary proof of the result. We then describe some classes of 3-connected planar graphs of inscribable type. We end the chapter by giving a refinement of a theorem of Wagner when restricted to graphs of inscribable type.

6.1 Rivin's characterization

The following characterization for graphs of inscribable type is due to Rivin.

Theorem 6.1. *Let G be a 3-connected planar graph. Then G is of inscribable type if and only if $sys(G)$ has an inner point.*

We first look at the history of this result. In 1992, Hodgson, Rivin, and Smith [26] announced Theorem 6.1 and an algorithm that decides if a 3-connected planar graph is of inscribable type having running time polynomial in the number of vertices. In a series of papers that characterized compact and ideal convex polyhedra in hyperbolic 3-space, Rivin obtained Theorem 6.1 by observing (see [37]) that Theorem 6.1 is a consequence of the fact that in the Klein model of hyperbolic space, an ideal convex hyperbolic polyhedron is represented by a convex Euclidean polyhedron inscribed in a sphere in $\mathbb{R}^3$.

A proof of the necessity part of Theorem 6.1 in the context of hyperbolic geometry first appeared in [37]. In [39], Rivin gave a more elementary proof (still in the context of hyperbolic geometry) and announced that a proof of the sufficiency part had also been found. An existence proof of sufficiency eventually appeared in [40].

In [41], Rivin described a different proof for Theorem 6.1, using linear programming techniques and results on Euclidean structures on simplicial surfaces ([38]). A polynomial-time algorithm for finding an explicit description of a polytope inscribed in a sphere realizing a graph of inscribable type based on solving a convex optimization problem was implicit in the results.

The results in [38] and [41] are more general than what Theorem 6.1 encompasses. In this section, we specialize Rivin's proofs of some of his results and present a detailed sketch of Rivin's elementary proof of Theorem 6.1 using the language of the current thesis without references to hyperbolic geometry or Euclidean structures on simplicial surfaces. In some cases, more details have been added to the original argument.

We first derive a necessary and sufficient condition for a 3-connected planar graph G to be of inscribable type.

Suppose G is of inscribable type. Pick a vertex of G and call it ∞. Let P be a polytope inscribed in a sphere S such that $G \cong G(P)$. Without loss of generality, assume that $G = G(P)$. Orient P in such a way that ∞ is at the north pole of S.

Perform a stereographic projection of S from ∞ onto the plane tangent to the south pole of S. For every adjacent pair $u, v \in V(G)$, connect their images on the plane with a straight line segment. (From now on, whenever the context is clear, we refer to the image of a vertex v of G on the tangent plane simply as v.) It is not difficult to see that the result is a straight-line drawing of $G - \infty$ satisfying the following properties:

1. $uv \notin \delta(\infty)$ is a boundary edge of the drawing of $G - \infty$ if and only if there is a face cycle of G containing u, v, and ∞. Furthermore, the boundary edges of the drawing of $G - \infty$ enclose a convex region. (A cycle of a 3-connected planar graph G is called a *face cycle* if it is the boundary of a face in a planar embedding of G. Face cycles are well-defined since one can show using a theorem of Whitney [51]

that they remain the same independent of the planar embedding.)

2. If u, v, w are in some face cycle of G containing ∞, then in the drawing of $G - \infty$, they are collinear. (In fact, they lie on the line that is the image of the circle circumscribing the facet of P whose extreme points are in the face cycle.)

3. The vertices on each face of the drawing of $G - \infty$ are concyclic (that is, they lie on the same circle.)

4. If C_1 and C_2 are the circles circumscribing two abutting faces F_1 and F_2 of the drawing of $G - \infty$, then no vertex on the boundary of F_1 lies in the interior of C_2 and no vertex on the boundary F_2 lies in the interior of C_1.

Remark. Triangulating the interior faces of the drawing of $G - \infty$ will result in a Delaunay triangulation.

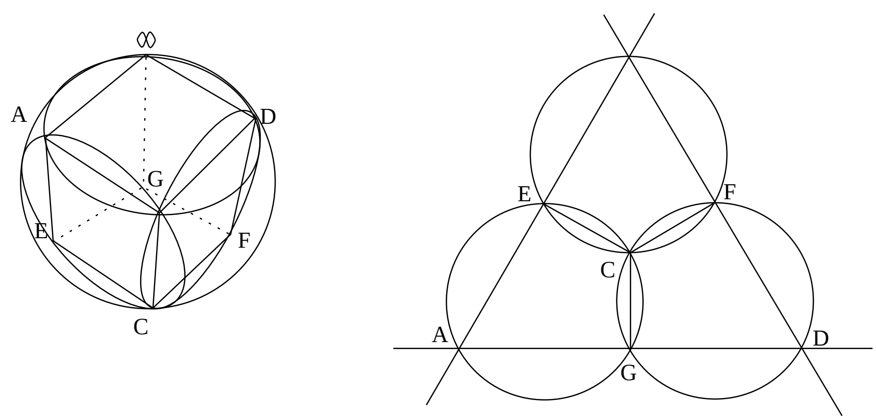

Figure 6.1: A cube inscribed in a sphere

The figure on the left of Figure 6.1 shows a cube along with three circles circumscribing three of its facets. The figure on the right shows the result after performing the stereographic projection and joining up pairs of end-vertices of edges. The straight-lines and circles are the images of the circumscribing circles of all the facets.

Conversely, it is not difficult to see that if we can find a straight-line drawing of $G - \infty$ satisfying the above properties, then G is of inscribable type.

We now capture all the conditions algebraically.

Let G be a 3-connected planar graph with a distinguished vertex ∞. Let H be a maximal planar graph having G as a spanning subgraph such that $u\infty \in E(H)$ for all $u \in V(F)$ where F is a face cycle of G containing ∞.

Define $\mathcal{F} := \{\{u,v,w\} \subseteq V(H)\backslash\{\infty\} : u, v, w$ are vertices on some face triangle of $H\}$ and $\mathcal{I} := V(H)\backslash(\{\infty\} \cup N(\infty))$.

Suppose G is of inscribable type. Then there is an planar embedding of H such that ∞ is on the boundary, $H - \infty$ is a straight-line drawing, and, where B denotes the boundary edges of $H - \infty$ and $\theta_{\{u,v,w\},w}$ denotes the value of the interior angle at vertex w on the triangle containing the vertices u, v, w for each ordered pair $(\{u,v,w\}, w) \in \mathcal{F} \times V(H)$, the following conditions hold:

- $\theta > 0$, $\theta_{\{u,v,w\},u} + \theta_{\{u,v,w\},v} + \theta_{\{u,v,w\},w} = \pi$ for all $\{u,v,w\} \in \mathcal{F}$, and

$$\sum_{v,w:\{u,v,w\}\in\mathcal{F}} \theta_{\{u,v,w\},u} = 2\pi$$

  for all $u \in \mathcal{I}$. These conditions come from the geometry of the drawing.

- if $u\infty$ is an edge not in G, then $\sum_{v,w:\{u,v,w\}\in\mathcal{F}} \theta_{\{u,v,w\},u} = \pi$ and if $u\infty$ is an edge in G, then $\sum_{v,w:\{u,v,w\}\in\mathcal{F}} \theta_{\{u,v,w\},u} < \pi$. These follow from the first two properties.

- If $uv \notin B$ is an edge not in G and w, w' are the two vertices opposite to uv, then $\theta_{\{u,v,w\},w} + \theta_{\{u,v,w'\},w'} = \pi$. This follows from the third property which requires that u, v, w, w' be concyclic.

- If $uv \notin B$ is an edge in G, then $\theta_{\{u,v,w\},w} + \theta_{\{u,v,w'\},w'} < \pi$. This follows from the fourth property.

Summarizing the above conditions as follows, we see that if G is of inscribable type and $E_0 = E(H)\backslash E(G)$, then (CP_π) (defined in Theorem 3.38) has a feasible solution $(\bar{\theta}, \bar{x})$ such that $\bar{\theta} > 0$ and $\bar{x}_e > 0$ for all $e \notin E_0$ where $E_0 = E(H)\backslash E(G)$. We call such $(\bar{\theta}, \bar{x})$ a *nice* solution.

Remarkably, the converse is also true and a proof will be given below. Hence, Theorem 6.1 follows immediately from Theorem 3.38.

Not every nice solution allows one to construct an equivalent straight-line drawing of $H - \infty$ with the angles of the faces prescribed by the values of $\bar{\theta}$. However, we shall show that if the system has a nice solution, then it has one that permits such a drawing to be made.

Given a triangle ABC, recall that the sine law gives $\frac{|AC|}{|AB|} = \frac{\sin(\angle B)}{\sin(\angle C)}$. Equivalently, $\log|AC| - \log|AB| = \log\sin(\angle B) - \log\sin(\angle C)$. Hence, it is necessary that

$$\sum_{i=0}^{l-1}(\log\sin(\theta_{\{v_i,v_{i+1},w\},v_i}) - \log\sin(\theta_{\{v_i,v_{i+1},w\},v_{i+1}})) = 0$$

where $w \in \mathcal{I}$ and $v_0, ..., v_{l-1} \in V(G - \infty)$ are such that $\{v_i, v_{i+1}, w\} \in \mathcal{F}$ and $v_{i+1}w$ is clockwise from $v_i w$. (Indices are taken modulo l.)

In fact, these conditions together with the condition that θ comes from a nice solution to (CP_π) are sufficient for one to construct such a drawing.

We first prove a technical result similar to the result by Di Battista and Vismara [13] for determining if there exists a straight-line drawing of a 3-connected planar graph with prescribed face angles.

Lemma 6.2. *Let G be a 2-connected plane graph obtained from a plane triangulation by deleting a boundary vertex. Let $\mathcal{I}$ denote the set of interior vertices of G. Suppose each pair $(\{u, v, w\}, u)$, where u, v, w are vertices on a face, is associated with a positive value $\theta_{\{u,v,w\},u}$. If θ satisfies*

$$\begin{array}{rcll} \theta_{\{u,v,w\},u} + \theta_{\{u,v,w\},v} + \theta_{\{u,v,w\},w} & = & \pi & \{u,v,w\} \text{ a vertex set of a face} \\ \sum_{v,w} \theta_{\{u,v,w\},u} & = & 2\pi & u \in \mathcal{I} \\ \sum_{v,w} \theta_{\{u,v,w\},u} & \leq & \pi & u \notin \mathcal{I} \end{array}$$

and $\sum_{i=0}^{l-1} \log\sin(\theta_{\{v_i,v_{i+1},w\},v_i}) - \log\sin(\theta_{\{v_i,v_{i+1},w\},v_{i+1}}) = 0$ *where $w \in \mathcal{I}$ and $v_0, ..., v_l = v_0 \in V(G)$ are such that $\{v_i, v_{i+1}, w\} \in \mathcal{F}$ and $v_{i+1}w$ is clockwise from $v_i w$ for $i = 0, ..., l-1$, then one can give a straight-line drawing of G with the angles of the faces* *having values prescribed by θ while maintaining the same orientation of the faces as in the given embedding of the plane graph G.*

Proof. The proof is by induction on the cardinality of $\mathcal{I}$.

If $|\mathcal{I}| = 0$, it is a simple induction on $|\mathcal{F}|$ that the conditions allow us to complete a straight-line drawing of G with angles prescribed by θ that also maintains the orientation of the faces.

Now, consider the case when $|\mathcal{I}| \geq 1$. Suppose the statement of the lemma holds for graphs with fewer interior vertices. Let $w \in \mathcal{I}$. First, complete a straight-line drawing $\mathcal{D}$ of the wheel W formed by w and its neighbours with angles prescribed by θ that maintains the orientation of the corresponding faces in the planar embedding of G. Clearly, this can always be done. Remove w (and the incident edges) from the drawing. We will be left with a polygon $\mathcal{P}$. Triangulate it using straight line segments.

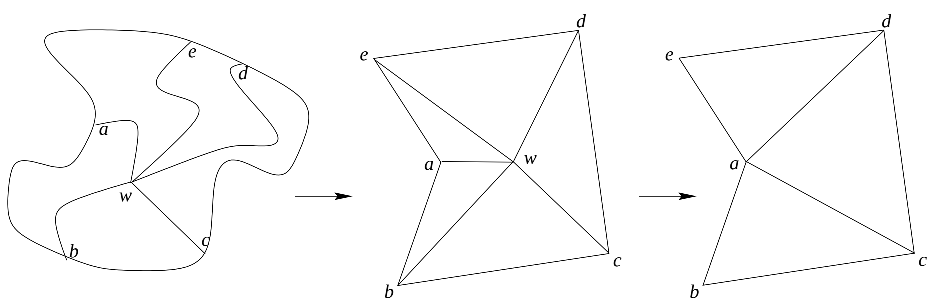

Figure 6.2: Drawing the wheel

Obtain G' from G by first removing w and then drawing in edges that correspond to the line segments added to $\mathcal{P}$, keeping the orientation of the faces as in the triangulation of $\mathcal{P}$.

Form θ' from θ as follows: Set $\theta'_{\{u,v,v'\},v'} = \theta_{\{u,v,v'\},v'}$ where $\{u, v, v'\}$ is not the vertex-set of a face in $\mathcal{D}$. Set $\theta'_{\{u,v,v'\},v'}$ to the value of the angle at vertex v' in the face of the triangulation $\mathcal{P}$ having vertex-set $\{u, v, v'\}$. One can check that θ' satisfies the conditions for G' in the statement of the lemma. By the induction hypothesis, we can obtain a straight-line drawing of G' with the angles prescribed by θ' that maintains the orientation of the faces.

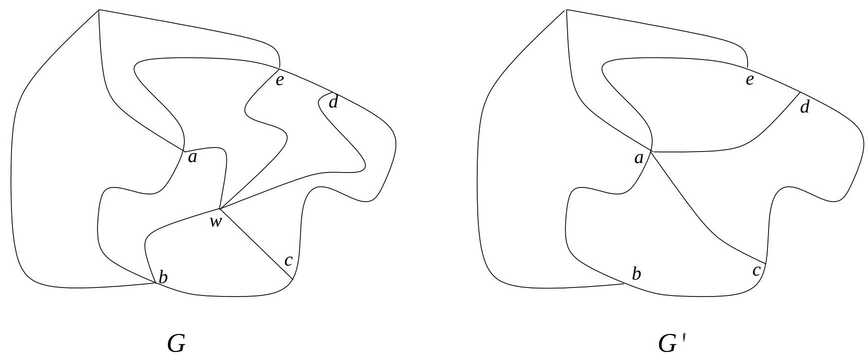

Figure 6.3: An example of G and G'

Consider the part of this drawing that corresponds to the triangulation of $\mathcal{P}$. Notice that we can now replace that part by a properly-scaled copy of $\mathcal{D}$ to obtain a straight-line drawing of G with the angles prescribed by θ that maintains the orientation of the faces. This completes the induction. □

Theorem 6.3. *If $(\bar{\theta}, \bar{x})$ is a nice solution to (CP_π) such that*

$$\sum_{i=0}^{l-1} \log \sin(\bar{\theta}_{\{v_i,v_{i+1},w\},v_i}) - \log \sin(\bar{\theta}_{\{v_i,v_{i+1},w\},v_{i+1}}) = 0$$

for all $w \in \mathcal{I}$ and $v_0, ..., v_{l-1} \in V(H - \infty)$ such that $\{v_i, v_{i+1}, w) \in \mathcal{F}$ and $v_{i+1}w$ is clockwise from v_iw for $i = 0, ..., l-1$, then there is a straight-line drawing of $H - \infty$ with the angles prescribed by the values of $\bar{\theta}$.

Proof. This follows immediately from the previous lemma. □

Remark. Rivin [38] proves an equivalent version of Theorem 6.3 using the notion of holonomy. His argument is much shorter than the proof of Lemma 6.2. There may be a shorter proof of Lemma 6.2 based on results in geometry.

We now show that if (CP_π) has a nice solution, then it has one that satisfies the conditions of Theorem 6.3.

Define $\mathcal{A} := \cup_{\{u,v,w\}\in\mathcal{F}}\{(\{u,v,w\},u), (\{u,v,w\},v), (\{u,v,w\},w)\}$.

Define $\mathcal{V}(\theta) := \sum_{A\in\mathcal{A}} \int_0^{\theta_A} \log(\sin s)ds$.

Let $P_k(H, x)$ denote the set of solutions to the system

$$\begin{aligned} \sum_{w:\{u,v,w\}\in\mathcal{F}} \theta_{\{u,v,w\},w} &= k - x_{uv} & uv \in E(H)\backslash\delta(\{\infty\}), \\ \theta_{\{u,v,w\},u} + \theta_{\{u,v,w\},v} + \theta_{\{u,v,w\},w} &= k & \{u,v,w\} \in \mathcal{F}, \\ \theta &\geq 0. \end{aligned}$$

The rest of the section is devoted to showing that one can obtain such a nice solution by minimizing $\mathcal{V}$ over $P_\pi(H, x)$.

Remark. Rivin's original proof considered maximizing Milnor's formula for the volume of an ideal simplex in hyperbolic 3-space $\mathcal{V}(\theta) = \sum_{A\in\mathcal{A}} L(\theta_A)$ where $L(x)$ is the Lobachevsky function

$$L(x) = -\int_0^x \log(2\sin s)ds.$$

For more details, see [38].

We first establish the following:

Lemma 6.4. *If $P_\pi(H, \bar{x})$ contains a positive point for some given $\bar{x}$ with $0 \leq \bar{x} \leq k\mathbf{e}$, then $\mathcal{V}$ achieves its minimum on $P_\pi(H, \bar{x})$ at a positive point.*

Before we give the proof, we need two technical results.

Lemma 6.5. *Given a fixed $\bar{x} \in \mathbb{R}^{E(H)}$, $\mathcal{V}$ is convex on $P_\pi(H, \bar{x})$.*

Proof. (Adapted from Theorem 2.1 in [38].)

It suffices to show that $f(\theta_A, \theta_B, \theta_C) = \int_0^{\theta_A} \log(\sin s)ds + \int_0^{\theta_B} \log(\sin s)ds + \int_0^{\theta_C} \log(\sin s)ds$ is convex on $\{(\theta_A, \theta_B, \theta_C) : \theta_A + \theta_B + \theta_C = \pi,\ \theta_A, \theta_B, \theta_C > 0\}$.

The Hessian of f at $(\bar{\theta}_A, \bar{\theta}_B, \bar{\theta}_C)$ is

$$\begin{pmatrix} \cot(\bar{\theta}_A) & 0 & 0 \\ 0 & \cot(\bar{\theta}_B) & 0 \\ 0 & 0 & -\cot(\bar{\theta}_A + \bar{\theta}_B) \end{pmatrix}.$$

One now needs to show that the matrix is positive definite on $L = \{(x,y,z) : x+y+z = 0\}$. Every point v in L can be written as $\lambda_1(1,0,-1)+\lambda_2(0,1,-1)$ for some λ_1 and λ_2. Hence,

$$\begin{aligned} v\nabla^2 f(\bar\theta_A,\bar\theta_B,\bar\theta_C)v^T &= \lambda_1^2(\cot(\bar\theta_A)-\cot(\bar\theta_A+\bar\theta_B)) + \lambda_2^2(\cot(\bar\theta_B)-\cot(\bar\theta_A+\bar\theta_B)) \\ &\quad -2\lambda_1\lambda_2(\cot(\bar\theta_A+\bar\theta_B)) \\ &= \frac{1}{\cot(\bar\theta_A)+\cot(\bar\theta_B)}((\lambda_1-\lambda_2)^2 + (\lambda_1\cot(\bar\theta_A)+\lambda_2\cot(\bar\theta_B))^2). \end{aligned}$$

The last quantity is strictly greater than zero provided that $\cot(\bar\theta_A)+\cot(\bar\theta_B) > 0$.

Recall that $0 < \bar\theta_A+\bar\theta_B < \pi$. If $\bar\theta_A, \bar\theta_B \le \frac{\pi}{2}$, we are done. Otherwise, assume that $\bar\theta_A > \frac{\pi}{2}$. Hence, $\frac{\pi}{2} > \pi-\bar\theta_A > \bar\theta_B$ and we obtain $\cot(\bar\theta_A)+\cot(\bar\theta_B) = -\cot(\pi-\bar\theta_A)+\cot(\bar\theta_B) > 0$ since $\cot(x)$ is a strictly decreasing function on $(0,\frac{\pi}{2})$. □

Lemma 6.6. *Let $\bar x \in \mathbb{R}^{E(H)}$ be such that $0 \le \bar x \le k\mathbf{e}$. Suppose $P_k(H,\bar x)$ has a positive point. If $\beta \in P(H,\bar x)$ is not positive, then there exists $\{u,v,w\} \in \mathcal{F}$ such that $(\beta_{\{u,v,w\},u}, \beta_{\{u,v,w\},v}, \beta_{\{u,v,w\},w}) = (0,s,k-s)$ for some $s > 0$.*

Proof. (Adapted from Lemma 6.11 in [38].)

Suppose no such $\{u,v,w\}$ exists. Since β is not positive, there must exist $A \in \mathcal{A}$ such that $\beta_A = 0$. It follows that there exists $\{u,v,w\} \in \mathcal{F}$ such that

$$(\beta_{\{u,v,w\},u}, \beta_{\{u,v,w\},v}, \beta_{\{u,v,w\},w}) = (0,0,k).$$

If $uv \in B$, we have $\bar x_{uv} = k$ and so $\alpha_{\{u,v,w\},u} = 0$ for all $\alpha \in P(H,\bar x)$, contradicting that $P(H,\bar x)$ has a positive point. Hence, $uv \notin B \cup \delta(\{\infty\})$.

Denote the face with vertex-set $\{u,v,w\}$ by T_0. Let w' be such that $\{u,v,w'\} \in \mathcal{F}$. Then $\beta_{\{u,v,w'\},w'} = 0$ and either $\beta_{\{u,v,w'\},v}$ or $\beta_{\{u,v,w'\},w'}$ equals k. Without loss of generality, assume that it is the former. Denote the face with vertex set $\{u,v,w'\}$ by T_1. As above, we must have $uw' \notin B$. Therefore there exists v' such that $\{u,v',w'\} \in \mathcal{F}$.

Since $\mathcal{F}$ is finite, it is not difficult to see that there exists, without loss of generality, a sequence $T_0,...,T_l$ with $T_0 = T_l$ such that T_i and T_{i+1} are face triangles that share a common edge, $\bar x_{e_i} = k$ where e_i denotes the edge incident with the faces T_i and T_{i+1}, and for each T_i, $\beta_{V(T_i),u} = k$ for some $u \in V(T_i)$. Without loss of generality, assume that $T_0, ..., T_l$ form the vertex sequence of a cycle in the planar dual of $H - \{\infty\}$.

Let ζ be a positive point in $P_k(H, \bar{x})$. Clearly, $\sum_{i=0}^{l-1} \sum_{u \in V(T_i)} \zeta_{V(T_i),u} = lk$. But $\sum_{i=0}^{l-1} \bar{x}_{e_i} = lk$. For each i, denote the vertices opposite e_i by u_i and v_i. Let $u' \in V(T_0) \cap V(T_1) \cap V(T_2)$. Then $0 = \sum_{i=0}^{l-1} \sum_{u \in V(T_i)} \zeta_{V(T_i),u} - \sum_{i=0}^{l-1} \bar{x}_{e_i} = \sum_{i=0}^{l-1} \sum_{u \in V(T_i)} \zeta_{V(T_i),u} - \sum_{i=0}^{l-1} (\zeta_{V(T_i),u_i} + \zeta_{V(T_{i+1}),v_i}) \geq \zeta_{V(T_1),u'} > 0$, which is a contradiction. □

Proof of Lemma 6.4. (Adapted from Theorem 6.10 in [38] with some more details and a minor correction.)

Since $\mathcal{V}$ is continuous and $P_\pi(H, \bar{x})$ is compact, $\mathcal{V}$ achieves its minimum on $P_\pi(H, \bar{x})$. Since $\mathcal{V}$ is convex on $P_\pi(H, \bar{x})$ by Lemma 6.5, it suffices to show that the minimum is not achieved at a point in $P_\pi(H, \bar{x})$ that is not positive.

Let β be a point that is not positive and α be a positive point in $P_\pi(H, \bar{x})$. Define $f(t) := \mathcal{V}(\beta + t(\alpha - \beta))$. We shall show that if $t > 0$ is sufficiently small, then $f(t) < f(0)$.

By the Mean-Value Theorem, if $t > 0$, then there exists $\zeta \in (0, t)$ such that

$$f(t) - f(0) = tf'(\zeta).$$

Let $\mathcal{B} = \{A \in \mathcal{A} : \beta_A = \pi\}$ and $\mathcal{Z} = \{Z \in \mathcal{A} : \beta_Z = 0\}$. Let $\mathcal{Z}' \subset \mathcal{Z}$ be such that for any $(\{u, v, w\}, w) \in \mathcal{Z}'$, neither $\beta_{\{u,v,w\},u}$ nor $\beta_{\{u,v,w\},v}$ equals zero. Since β is a not positive point, $\mathcal{Z}$ is non-empty. It follows from Lemma 6.6 that Z' is non-empty.

Consider $(\{u, v, w\}, w) \in \mathcal{B}$. Clearly, $\beta_{\{u,v,w\},u} = \beta_{\{u,v,w\},v} = 0$. Let A_1, A_2, and A_3

denote $(\{u,v,w\},u)$, $(\{u,v,w\},v)$, and $(\{u,v,w\},w)$, respectively. Then

$$\begin{aligned}
& \sum_{D\in\{A_1,A_2,A_3\}} (\alpha_D-\beta_D)\log(\sin(\beta_D+\zeta(\alpha_D-\beta_D))) \\
= \; & \alpha_{A_1}\log(\sin(\zeta\alpha_{A_1}))+\alpha_{A_2}\log(\sin(\zeta\alpha_{A_2})) \\
& +(\pi-\alpha_{A_1}-\alpha_{A_2}-\pi)\log(\sin(\pi+\zeta(\pi-\alpha_{A_1}-\alpha_{A_2}-\pi))) \\
= \; & \alpha_{A_1}\log(\sin(\zeta\alpha_{A_1}))+\alpha_B\log(\sin(\zeta\alpha_{A_2}))-(\alpha_{A_1}+\alpha_{A_2})\log(\sin(\zeta(\alpha_{A_1}+\alpha_{A_2}))) \\
= \; & \log\left(\frac{\sin(\zeta\alpha_{A_1})}{\sin(\zeta(\alpha_{A_1}+\alpha_{A_2}))}\right)^{\alpha_{A_1}}\left(\frac{\sin(\zeta\alpha_{A_2})}{\sin(\zeta(\alpha_{A_1}+\alpha_{A_2}))}\right)^{\alpha_{A_2}}.
\end{aligned}$$

Clearly, we can pick $\epsilon_{\{u,v,w\},w}>0$ small enough such that this last quantity is less than zero for all positive $\zeta<\epsilon_{\{u,v,w\},w}$. Hence, if $t<\min_{B\in\mathcal{B}}\epsilon_B$, then

$$f'(\zeta)\le\sum_{A\in Z'}\alpha_A\log(\sin(\zeta\alpha_A))+\sum_{A\in\mathcal{A}\setminus(\mathcal{B}\cup\mathcal{Z})}(\alpha_A-\beta_A)\log(\sin(\beta_A+\zeta(\alpha_A-\beta_A))).$$

If $\mathcal{A}\setminus(\mathcal{B}\cup\mathcal{Z})=\emptyset$, let $\epsilon'=1$. Otherwise, let $\epsilon'=\min_{A\in\mathcal{A}\setminus(\mathcal{B}\cup\mathcal{Z})}(\min(\beta_A,\pi-\beta_A))$. Observe that if $t<\frac{\epsilon'}{2\|\beta-\alpha\|}$, then $(\alpha_A-\beta_A)\log(\sin(\beta_A+\zeta(\alpha_A-\beta_A)))<\pi|\log(\sin(\epsilon'/2))|$ for all $A\in\mathcal{A}\setminus(\mathcal{B}\cup\mathcal{Z})$. Hence, if $t<\min(\frac{\epsilon'}{2\|\beta-\alpha\|},\min_{B\in\mathcal{B}}\epsilon_B)$,

$$f'(\zeta)\le\sum_{A\in Z'}\alpha_A\log(\sin(\zeta\alpha_A))+|\mathcal{A}\setminus(\mathcal{B}\cup\mathcal{Z})||\log(\sin(\epsilon'/2)|\pi.$$

Since $\log(\sin s)\to-\infty$ as $s\to 0$, we see that we can pick ϵ such that if $0<\zeta<t<\epsilon$, then $f'(\zeta)<0$. It follows that $f(t)=f(0)+tf'(\zeta)<f(0)$ as desired. $\square$

By Theorem 3.38, if $\bar{x}\in\pi\,\mathrm{SEP}(H)$ is such that $\bar{x}(A)>2\pi$ for all $A\in C(H)$, then $P_\pi(H,\bar{x})$ has a positive point. Let $\bar{\theta}$ be the unique minimizer of $\mathcal{V}$ promised by Lemma 6.4. Note that $\bar{\theta}>0$ and $(\bar{\theta},\bar{x})$ is a feasible solution to (CP_π).

Since the minimizer $\bar{\theta}$ of $\mathcal{V}$ is positive, we see that there is an open ball containing the minimizer on which $\mathcal{V}$ is continuously differentiable. By the principle of lagrange multipliers, we see that there exist $\bar{\mu}$ and $\bar{\lambda}$ such that $\log(\sin(\bar{\theta}_{\{u,v,w\},w}))=\bar{\mu}_{uv}+\bar{\lambda}_{\{u,v,w\}}$. So for each $\{u,v,w\}\in\mathcal{F}$, $\log(\sin(\bar{\theta}_{\{u,v,w\},u}))-\log(\sin(\bar{\theta}_{\{u,v,w\},v}))=\bar{\mu}_{vw}-\bar{\mu}_{uw}$.

It follows that if $w, v_0, ..., v_l = v_0$ are such that $\{v_i, v_{i+1}, w) \in \mathcal{F}$ and $v_{i+1}w$ is clockwise from $v_i w$, then

$$\sum_{i=0}^{l-1} \log(\sin(\bar{\theta}_{\{v_i,v_{i+1},w\},v_i})) - \log(\sin(\bar{\theta}_{\{v_i,v_{i+1},w\},v_{i+1}})) = \sum_{i=0}^{l-1} \bar{\mu}_{v_{i+1}w} - \bar{\mu}_{v_i w} = 0.$$

Hence, there exists a straight-line drawing of $H - \infty$ with angles of the faces prescribed by the values of $\bar{\theta}$.

6.2 Some classes of graphs of inscribable type

By Theorem 6.1, many results on the existence of inner points when restricted to the class of 3-connected planar graphs immediately give results on graphs of inscribable type. For instance, Theorem 4.10 says that every 4-connected planar graph is of inscribable type (Dillencourt and Smith [16]). In this section, we look at some more classes of graphs of inscribable type that have rather simple descriptions.

6.2.1 k-regular graphs

As an r-regular (simple) planar graph must have $r \leq 5$, Theorem 4.11 gives us the next result.

Theorem 6.7. *Let G be a $(2k+1)$-edge-connected $(2k+1)$-regular planar graph where $k \in \{1, 2\}$. Then G is of inscribable type if and only if it is more-than-1-tough or is a brace.*

Remark. The case when $k = 1$ was previously obtained by Dillencourt and Smith [15].

However, a 5-regular planar graph cannot be a brace because it cannot be bipartite. Hence, we have the following:

Theorem 6.8. *A 5-edge-connected 5-regular planar graph is of inscribable type if and only if it is more-than-1-tough.*

Note that the graph of the icosahedron (a platonic solid that can be inscribed in a sphere) is a 5-edge-connected, 5-regular graph that is of inscribable type.

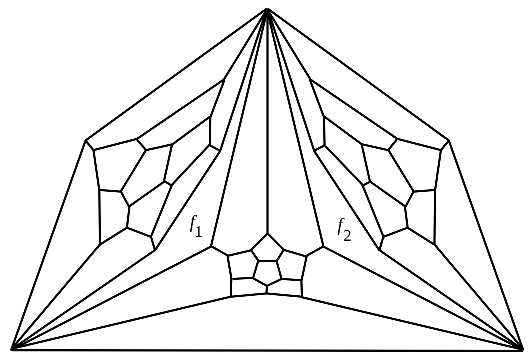

Figure 6.4: Planar dual of a 5-edge-connected 5-regular plane graph

Figure 6.4 shows the planar dual of a 5-edge-connected 5-regular plane graph G that is not of inscribable type. (Note that removing from G the vertices corresponding to f_1, f_2, and the exterior face will result in three components.)

With regards to 4-regular planar graphs, Eppstein [19] raised the following question: Is a more-than-1-tough 4-regular planar graph of inscribable type? The answer is in the negative as the next result shows.

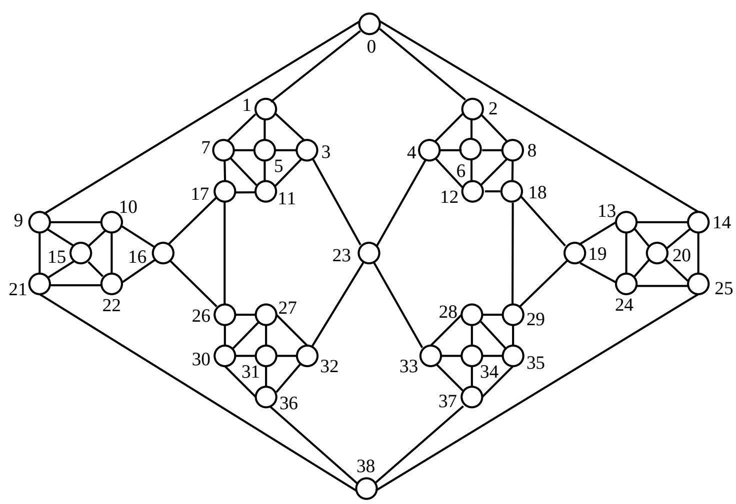

Figure 6.5: A more-than-1-tough 3-connected 4-regular planar graph

Proposition 6.9. *The graph G in Figure 6.5 is more-than-1-tough and is not of inscribable type.*

Proof. Since $\frac{1}{2}e \in \text{SEP}(G)$, G is feasible and has no useless edge.

Let $S_1 = \{1,3,5,7,11\}$, $T_1 = S_1 \cup \{17\}$, $S_2 = \{2,4,6,8,12\}$, $T_2 = S_2 \cup \{18\}$, $S_3 = \{9,10,15,21,22\}$, $T_3 = S_3 \cup \{16\}$, $S_4 = \{13,14,20,24,25\}$, $T_4 = S_4 \cup \{19\}$, $S_5 = \{27,30,31,32,36\}$, $T_5 = S_5 \cup \{26\}$, $S_6 = \{28,33,34,35,37\}$, $T_6 = S_6 \cup \{29\}$.

One can check that the only non-trivial cuts of cardinality four are $\delta(S_i)$ and $\delta(T_i)$, $i = 1, ..., 6$. They are the only possible non-trivial constricted cuts.

Let $\bar{z}$ and $\bar{y}$ be such that

$$\bar{z}_v = \begin{cases} 1 & \text{if } v \in \{16,17,18,19,26,29\} \\ 2 & \text{if } v \in \{0,23,38\} \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\bar{y}_A = \begin{cases} 1 & \text{if } A \in \{\delta(S_1), ..., \delta(S_6), \delta(T_1), ..., \delta(T_6)\} \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, $(\bar{y}, \bar{z})$ is a solution to sys$'(G)$ with $\sum_{v \in V(G)} \bar{z}_v = \sum_{A \in C(G)} \bar{y}_A$. By Lemma 4.2, $\delta(S_i)$ and $\delta(T_i)$, $i = 1, ..., 6$ are constricted. Hence, sys(G) does not have an inner point. By Theorem 6.1, G is not of inscribable type.

Suppose G is not more-than-1-tough. Since G is 1-tough and 4-regular, there exists an independent set $B \subset V(G)$ such that $\omega(G - B) = |B| = m$ for some integer $m \geq 2$. Let $R_1, ..., R_m$ be the vertex-sets of the components of $G - B$. Without loss of generality, assume that $\delta(R_1), ..., \delta(R_l)$ are non-trivial cuts and $\delta(R_{l+1}), ..., \delta(R_m)$ are trivial cuts. It is easy to see that $|R_i| = 1$ for $i = l+1, ..., m$. Since G is non-bipartite, $l \geq 1$. Let $\bar{z}$ and $\bar{y}$ be such that

$$\bar{z}_v = \begin{cases} 1 & \text{if } v \in B \\ -1 & \text{if } \{v\} \in \{R_{l+1}, ..., R_m\} \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\bar{y}_A = \begin{cases} 1 & \text{if } A \in \{\delta(R_1), ..., \delta(R_l)\} \\ 0 & \text{otherwise.} \end{cases}$$

Then, $(\bar{y}, \bar{z})$ is a solution to sys$'(G)$ with $\sum_{v \in V(G)} \bar{z}_v = \sum_{A \in C(G)} \bar{y}_A$. By Lemma 4.2, $\delta(R_i)$ is constricted for $i = 1, ..., l$. Hence, $\delta(R_i) \in \cup_{j=1}^{6} \{\delta(S_j), \delta(T_j)\}$ for $i = 1, ..., l$.

By symmetry, we may assume that at least one of $\bar{y}_{\delta(S_1)}$ and $\bar{y}_{\delta(T_1)}$ is positive.

Suppose $\bar{y}_{\delta(S_1)} > 0$. If $V(G) \backslash S_1 = R_i$ for some $i \in \{1, ..., l\}$, then we must have $\bar{z}_u = 1$ for all $u \in \{1, 3, 7, 11\}$. This contradicts that B is an independent set. Hence, $S_1 = R_i$ for some $i \in \{1, ..., l\}$. It follows that $\bar{z}_0 = \bar{z}_{17} = \bar{z}_{23} = 1$. But this means that $16 \in R_i$ and $26 \in R_j$ for some $i, j \in \{1, ..., m\}$. It is not difficult to see that we must have $i \neq j$. Thus, $0 > \bar{z}_{16} + \bar{z}_{26} - \sum_{\{16,26\} \in A} \bar{y}_A = 0$, which is a contradiction.

Now, suppose $\bar{y}_{\delta(T_1)} > 0$ and $\bar{y}_{\delta(S_1)} = 0$. Clearly, we must have $T_1 = R_i$ for some $i \in \{1, ..., l\}$. It follows that $\bar{z}_{16} = \bar{z}_{26} = 1$. Hence, $16, 26 \in B$. But this contradicts that B is an independent set.

It follows that such a set B could not exist. Hence, G is more-than-1-tough. □

Remark. The graph in Figure 6.5 is derived from the graph constructed by Dillencourt and Smith [15] as an example of a more-than-1-tough graph that is not of inscribable type. Their graph appears in Figure 6.9 below.

We do not yet have a characterization of 3-connected 4-regular planar graphs of inscribable type using graph-theoretical terms. However, we do have a sufficient condition for such a graph to be of inscribable type.

Theorem 6.10. *Let $G = (V, E)$ be a 3-connected 4-regular planar graph. If each nontrivial 4-edge cut is a matching of G, then G is of inscribable type.*

The proof makes use of the following simple lemma that appeared in Grünbaum [25]. The proof of the lemma is included here for the sake of completeness.

Lemma 6.11. *Let $G = (V, E)$ be a connected simple plane graph without vertices of degree two. Then there are at least eight degree-three vertices and degree-three faces in* *total.*

Proof. Let f denote the number of faces. Let n_i denote the number of vertices of degree i. Let f_i denote the number of faces having i edges on its boundary. Observe that $\sum_{i\geq 3} in_i = 2|E| = \sum_{i\geq 3} if_i$. By Euler's formula, $|V| - |E| + f = 2$. Hence, $8 = 4\sum_{i\geq 3} n_i - 4|E| + 4\sum_{i\geq f_i} \leq n_3 + f_3$, as desired. □

Proof of Theorem 6.10. Since $\frac{1}{2}e \in \mathrm{SEP}(G)$, G is feasible and has no useless edge. Suppose G is not of inscribable type. Then G must have a non-trivial constricted cut. By Lemma 4.8, there exist $\hat{y} \in \mathbb{R}_+^{C(G)}$ and $\hat{z} \in \mathbb{R}^V$ feasible for $D(G)$ such that $\sum\limits_{A\in C(G)} \hat{y}_A = \sum\limits_{v\in V} \hat{z}_v$, $\mathcal{A}(\hat{y}) := \{A \in C(G) : \hat{y}_A > 0\}$ is non-crossing, and $\hat{y}_A > 0$ for some $A \in C(G)$. Since $\mathcal{A}(\hat{y})$ is non-crossing, by Lemma 3.5, there exists a nested family $\mathcal{S}$ of subsets of V containing exactly one shore of each cut in $\mathcal{A}(\hat{y})$.

Since G is feasible, $G[S]$ is connected for all $S \in \mathcal{S}$. Choose $T \in \mathcal{S}$ such that there

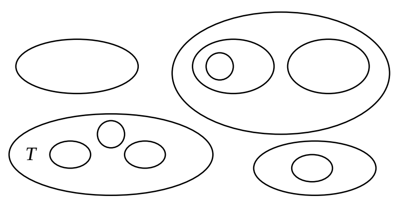

Figure 6.6: Sets in $\mathcal{S}$

exists a proper subset of T that is in $\mathcal{S}$ and for any proper subset R of T that is in $\mathcal{S}$, there is no proper subset of R that is in $\mathcal{S}$. If no such T exists, let $T = V$.

Let $\mathcal{S}' = \{S \in \mathcal{S} : S \subset T\}$. Observe that all the elements in $\mathcal{S}'$ are mutually disjoint. Consider the graph H obtained from $G[T]$ by shrinking each $S \in \mathcal{S}'$. It is connected, planar, and has exactly four vertices of degree three and no vertex of degree two. By Lemma 6.11, H has a triangle and therefore is non-bipartite.

Now consider any $S \in \mathcal{S}'$. Observe that $\hat{z}_u + \hat{z}_v = 0$ for all $uv \in E$ such that $u, v \in S$. By Lemma 6.11, $G[S]$ contains a triangle since it is a connected (simple) planar graph with exactly four vertices of degree 3 with no vertex of degree two. Hence, $\hat{z}_v = 0$ for all $v \in S$. But $\hat{y}_{\delta(S)} > 0$. Hence, the set of pseudo-vertices in H is independent. In addition, if v is a neighbour of a pseudo-vertex in H, then $\hat{z}_v > 0$. Since G has no useless edge, by Lemma 4.2, $\hat{z}_u + \hat{z}_v = 0$ for all $uv \in E$ such that $u, v \in T$ and $uv \notin \delta(S)$ for all $S \in \mathcal{S}'$. It follows that if $v \in T$ is not a pseudo-vertex in H, then $\hat{z}_v \neq 0$ since H is connected. Let $X = \{v \in T : \hat{z}_v > 0\}$. Then, X is an independent set in $G[T]$.

Now, let Y be the set containing all the vertices $v \in T$ such that $\hat{z}_v < 0$ and the pseudo-vertices. Clearly, Y is an independent set in H. Hence, H is a bipartite with bipartition (X, Y), which is impossible. The result follows. □

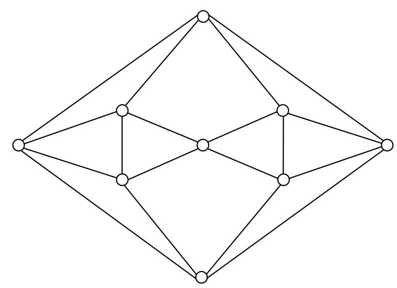

Figure 6.7: A 4-regular planar graph of inscribable type that is not 4-connected

Let $\mathcal{R}_4$ denote the class of graphs described in the previous theorem. Not all graphs in $\mathcal{R}_4$ are 4-connected. Figure 6.7 shows a 3-connected (but not 4-connected) 4-regular planar graph that has no non-trivial 4-edge cut. Observe that the class $\mathcal{R}_4$ is closed under splicing. In other words, if $G, H \in \mathcal{R}_4$ and $u \in V(G)$ and $v \in V(H)$ then splicing G and H with respect to u and v in such a way that planarity is preserved will result in a graph that is in $\mathcal{R}_4$. Hence, one can construct all the graphs in $\mathcal{R}_4$ starting from the ones that have no non-trivial 4-edge cuts.

Note that the analog of Theorem 6.10 does not hold if the graph is not required to be planar. More precisely, there exists a 3-connected 4-regular graph G whose non-trivial 4-edge cuts are matchings of G but sys(G) does not have an inner point. The graph depicted in Figure 6.8 is such a graph. Note that it is not more-than-1-tough and it has a non-trivial constricted cut. However, every non-trivial 4-edge cut of the graph is a matching. One might ask what happens if we restrict our attention to more-than-1-tough 4-regular graphs. We do not know the answer and so we have the following problem:

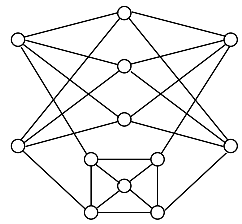

Figure 6.8: A 3-connected 4-regular graph whose subtour-elimination polytope has no inner points

Problem 6.12. *Let G be a more-than-1-tough 4-regular graph. If every non-trivial 4-edge cut of G is a matching, must $sys(G)$ have an inner point?*

6.2.2 Maximal planar graphs

In this subsection, we look at some conditions for a maximal planar graph to be of inscribable type.

Theorem 6.13. *A maximal planar graph is of inscribable type if and only if it has no useless edge.*

Proof. This is a restatement of Corollary 4.13. □

Theorem 6.14. *Let G be 3-connected and planar. If G is of inscribable type, then it is a spanning subgraph of a maximal planar graph that is of inscribable type. If G is not of inscribable type, then it is a spanning subgraph of a maximal planar graph that is not of inscribable type.*

Proof. Suppose G is of inscribable type. We may assume that G is non-bipartite. (Indeed, if G is bipartite, then G must have two even cycles of length at least four. By Proposition 4.16, we can add two chords to G, one to each cycle, in such a way that the resulting graph is non-bipartite and of inscribable type.) Since G is non-bipartite, we can keep applying Proposition 4.14 until we obtain a maximal planar graph of inscribable type.

Now, suppose G is not of inscribable type. By Theorem 6.1, it suffices to show that G is a spanning subgraph of a maximal planar graph H such that sys(H) has no inner points.

If G is infeasible, then the result follows from Theorem 3.9. If G is feasible, then the result follows from Theorem 4.9. □

Theorem 6.14 suggests that understanding maximal planar graphs is invaluable in the study of graphs of inscribable type.

Remark. Figure 6.9 shows two more-than-1-tough graphs given in [15] that are not of inscribable type. The graph depicted on the right is a spanning subgraph of the one depicted on the left.

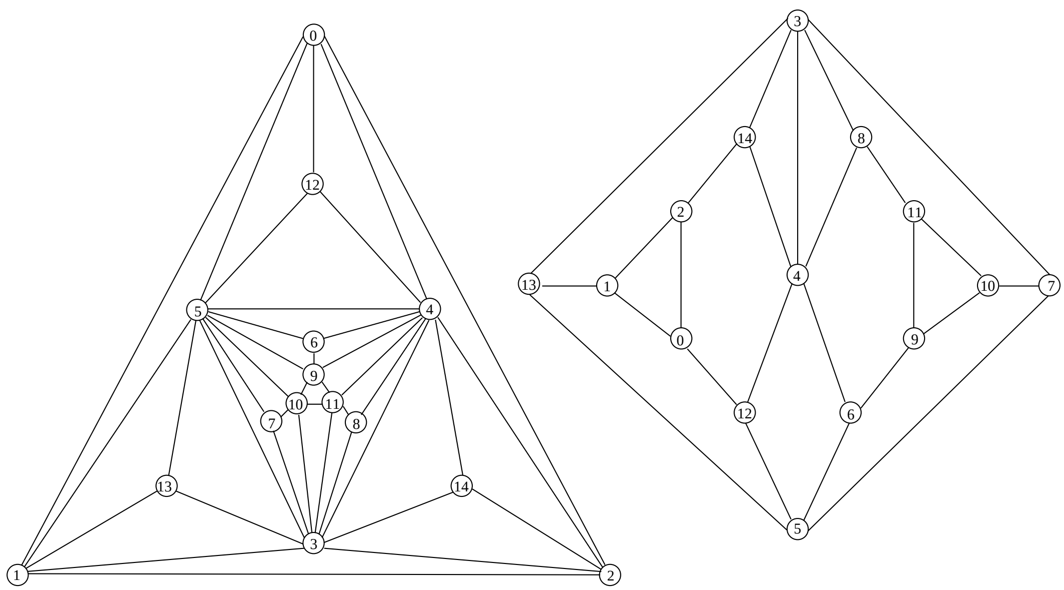

Figure 6.9: Two more-than-1-tough graphs that are not of inscribable type

Jucovič and Ševec [30] proved the following:

Theorem 6.15. *A plane triangulation is of inscribable type if it is obtained from the plane graph depicted in Figure 6.10 by successively applying the transformations* τ_1 *and*

τ_2 depicted in Figure 6.11.

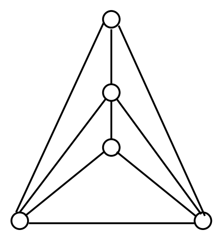

Figure 6.10: Bipyramid

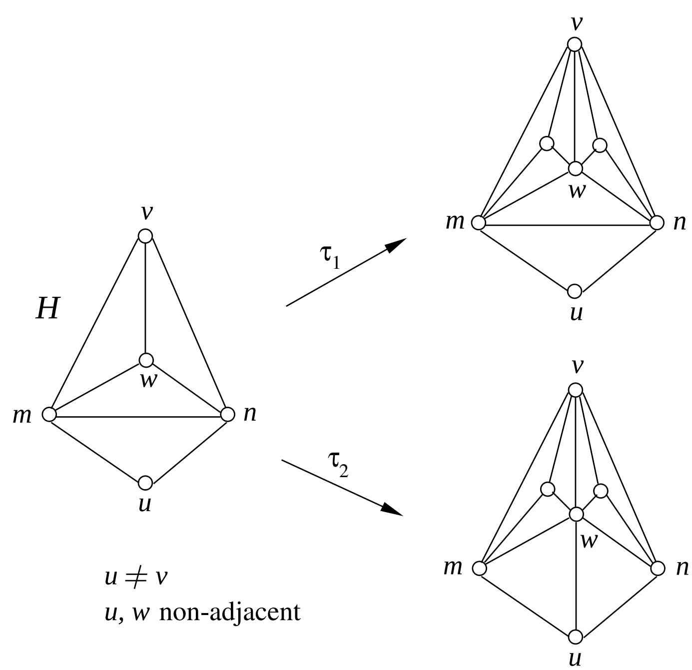

Figure 6.11: Transformations

We give an improvement of Theorem 6.15 using the results we have established.

Theorem 6.16. *Let G be of inscribable type. Suppose G has a degree-three vertex w whose neighbours m, n, and v are pairwise adjacent. Suppose $u \neq w$ is a common neighbour of m and n, G_1 and G_2 are such that G_i is either K_4 or a 4-connected planar graph for $i = 1, 2$, and $G \cap G_1 = T_1$ and $G \cap G_2 = T_2$ where T_1 is the triangle with vertices v, w, m*

and T_2 is the triangle with vertices v, w, n. Then $G \cup G_1 \cup G_2$ and $(G - mn + uw) \cup G_1 \cup G_2$ are of inscribable type. (See Figure 6.11.)

Proof. It suffices to show that $\operatorname{sys}(G \cup G_1 \cup G_2)$ and $\operatorname{sys}((G - mn + uw) \cup G_1 \cup G_2)$ have inner points.

Let $i \in \{1, 2\}$. If G_i is a K_4, then clearly $\operatorname{sys}(G_i)$ has an inner point and there exists $\bar{x} \in \operatorname{SEP}(G_i)$ such that $\bar{x}_e = \bar{x}_f = 1$ for any two edges $e, f \in E(G_i)$, If G_i is 4-connected, $\operatorname{sys}(G_i)$ has an inner point by Theorem 4.10. By Theorem 2.1 there exists $\bar{x} \in \operatorname{SEP}(G_i)$ such that $\bar{x}_e = \bar{x}_f = 1$ for any two edges $e, f \in E(G_i)$.

Let $G' = G - mn$. By Proposition 4.17, $\operatorname{sys}(G')$ has an inner point. Hence, by Proposition 4.22 and Corollary 4.20, $\operatorname{sys}(G' \cup G_1 \cup G_2)$ has an inner point.

Since $G' \cup G_1 \cup G_2$ is non-bipartite, the result follows from Proposition 4.14. □

Jackson and Yu [29] described a decomposition of a plane triangulation G as follows. For each separating triangle T, G can be separated into two graphs G_1 and G_2 such that $G = G_1 \cup G_2$, $G_1 \cap G_2 = T$. Note that for $i = 1, 2$, G_i has at least four vertices and G_i is a plane triangulation having T as a face. T is called a *marker triangle* in G_1 and G_2. The procedure is iterated for both G_1 and G_2 until one obtains a collection of plane triangulations S each of which has no separating triangles. The graphs in S are called *pieces* of G. Define a new graph D whose vertices are the pieces in S and in which two pieces are joined by an edge if they have a marker triangle in common. As Jackson and Yu remarked, it follows from the decomposition theory developed by Cunningham and Edmonds [11] that D is a tree and also that the set of pieces S and the tree D are uniquely defined by G. D is called *the decomposition tree* of G. They showed the following.

Theorem 6.17. *Let G be a 3-connected plane triangulation whose decomposition tree D has maximum degree at most three. Let H be a piece of G corresponding to a vertex of D of degree at most two and T be a face triangle of both H and G with $V(T) = \{u, v, w\}$. Then G has a Hamiltonian circuit through uv and uw.*

Using the above theorem, we obtain the next result.

Theorem 6.18. *Let G be a 3-connected plane triangulation. If the decomposition tree D of G has maximum degree at most three, then G is of inscribable type.*

Proof. The proof is by induction on $|V(D)|$. If D has only one vertex, then either $G = K_4$ or G is 4-connected. Clearly, K_4 is of inscribable type. If G is 4-connected, then G is of inscribable type by Theorem 4.10.

Assume that D has at least two vertices. Let H be a piece having degree one. Let T be the marker triangle in H. Let $V(T) = \{u, v, w\}$. Note that $H = K_4$ or H is 4-connected. Let $G' = G - (V(H)\backslash V(T))$. By the induction hypothesis, G' is of inscribable type. By Theorem 6.17, there is a Hamiltonian circuit of G' through uw and vw. Hence, $\text{sys}(G')$ has an inner point $\hat{x}$ such that $\hat{x}_{uv} + \hat{x}_{uw} + \hat{x}_{vw} > 1$. By Lemma 4.19, we see that G is of inscribable type. □

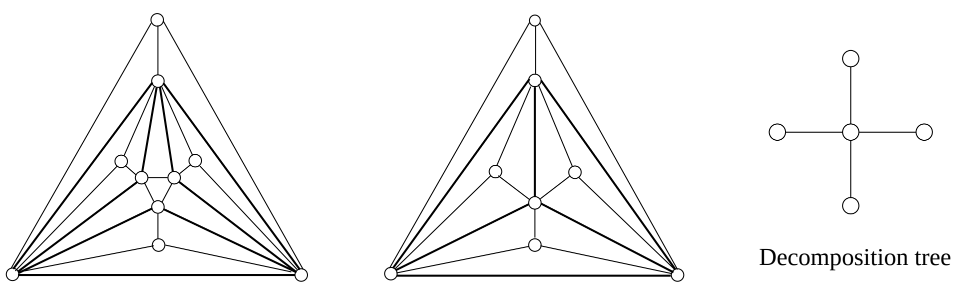

Figure 6.12: Two graphs having isomorphic decomposition trees

It is in general not possible to determine if a plane triangulation is of inscribable type just by considering the decomposition tree. The decomposition trees of the two graphs depicted in Figure 6.12 are isomorphic. However, the graph on the left is of inscribable type while the other is not.

6.3 A refinement of Wagner's Theorem

The main result in this section does not rely on the tools developed earlier. Nevertheless, it provides some insights on the structure of maximal planar graphs of inscribable type.

Given a maximal planar (or plane) graph G, if T_1 and T_2 are two triangles such that $V(T_1) = \{x, y, z\}$, $V(T_2) = \{x, y, w\}$, and $wz \notin E(G)$, we call the operation of removing the edge xy and adding the edge wz a *diagonal flip.*

Wagner [50] showed that given two maximal plane graphs with n vertices, one can be obtained from the other by performing a finite sequence of diagonal flips. Dewdney [12] subsequently extended the result to torus graphs. Similar results on other kinds of surfaces can be found in [7], [8], [9], and [31].

In this section, we show the following:

Theorem 6.19. *Let G_1 and G_2 be two maximal planar graphs of inscribable type having vertex set V. If $|V| > 4$, then there exists a sequence of maximal planar graphs $H_0 = G_1, H_1, ..., H_{k-1}, H_k = G_2$ such that H_i is of inscribable type for $i = 0, ..., k$ and H_i can be obtained from H_{i-1} via a diagonal flip for $i = 1, ..., k$.*

Given $\bar{x} \in \mathbb{R}^n$ and $r > 0$, let $B(\bar{x}, r)$ denote the set $\{x \in \mathbb{R}^n : \| x - \bar{x} \| < r\}$. We denote the boundary of a set $S \subseteq \mathbb{R}^n$ by $bd(S)$ and the unit sphere in $\mathbb{R}^n$ by S^{n-1}.

It is easy to see that the extreme points of a simplicial polytope can be "perturbed" without changing its combinatorial type. The next result is immediate.

Proposition 6.20. *Let G be a maximal planar graph of inscribable type having vertex set $\{v_1, ..., v_n\}$. Suppose $\{u^1, ..., u^n\} \in S^2$ are such that $G \cong G(\text{conv}(\{u^1, ..., u^n\})$ with u^i corresponding to v_i for $i = 1, ..., n$. Then there exists $\epsilon > 0$ such that if $w^i \in S^2 \cap B(u^i, \epsilon)$ for $i = 1, ..., n$, then $G \cong G(\text{conv}(\{w^1, ..., w^n\})$ with w^i corresponding to v_i for $i = 1, ..., n$.*

In this section, let P be a 3-dimensional simplicial polytope inscribed in S^2 and U be the set of extreme points of P.

Let $\mathcal{F}(P)$ denote the set of facets of P. For each $F \in \mathcal{F}(P)$, let $a_F x + b_F y + c_F z \leq d_F$ be an inequality that induces F. Let $S(P) = \{(x, y, z) \in S^2 : a_F x + b_F y + c_F z \neq d_F$ for all $F \in \mathcal{F}(P)\}$. Define the function $\phi_P : S(P) \to 2^{\mathcal{F}(P)}$ as follows: For any point $(\bar{x}, \bar{y}, \bar{z}) \in S(P)$, $\phi_P(\bar{x}, \bar{y}, \bar{z})$ is the maximal $\mathcal{A} \subseteq \mathcal{F}(P)$ such that $a_F\bar{x} + b_F\bar{y} + c_F\bar{z} < d_F$ for all $F \in \mathcal{A}$. Let $\Phi(P)$ denote the range of ϕ_P. For each $\mathcal{A} \in \Phi(P)$, let $S_{\mathcal{A}}$ denote the pre-image of $\mathcal{A}$. Then $S_{\mathcal{A}} = \{(x, y, z) \in S^2 : a_F x + b_F y + c_F z < d_F$ for all $F \in \mathcal{A}\}$ and the elements of the set $\{S_{\mathcal{A}} : \mathcal{A} \in \Phi(P)\}$ partition $S(P)$. We now establish a series of technical lemmas.

Proposition 6.21. *If $p \in S(P)$, then $Q = \text{conv}(U \cup \{p\})$ is simplicial.*

Proof. Suppose there exists a facet of Q that contains at least four extreme points of Q, say p^1, p^2, p^3, and p^4. Let $ax + by + cz \leq d$ be an inequality that induces this facet.

Without loss of generality, assume that $p^i \neq p$ for $i = 1, 2, 3$. Note that $ax+by+cz \leq d$ is a valid inequality for P. As p^1, p^2, and p^3 are non-collinear in P and lie on the plane defined by $ax + by + cz = d$, we see that $ax + by + cz \leq d$ induces a facet of P. Hence, $ax + by + cz \leq d$ is a scalar multiple of $a_F x + b_F y + c_F z \leq d_F$ for some $F \in \mathcal{F}(P)$ that contains p^1, p^2, and p^3. As P is simplicial, we must have $p^4 = p$. So p lies on the plane defined by $a_F x + b_F y + c_F z = d_F$, contradicting that $p \in S(P)$. □

The next result seems to be well-known but a reference could not be found. For the sake of completeness, a proof is given here.

Lemma 6.22. *Let Q be a full-dimensional simplicial polytope and $\{v^0, ..., v^n\}$ be the set of extreme points of Q. If v^0 and v^n are not adjacent in Q, then there exists $\lambda \in (0, 1)$ such that $\lambda v^0 + (1 - \lambda)v^n \in \text{conv}(\{v^1, ..., v^{n-1}\})$.*

Proof. Consider the following linear programming problem:

$$
\begin{array}{llrcll}
 & \text{minimize} & 0 & & & \\
(LP) & \text{subject to} & -(v^0)^T a + d & = & 0 & \\
 & & -(v^n)^T a + d & = & 0 & \\
 & & (v^i)^T a - d & \geq & 0 & \quad i = 1, ..., n-1
\end{array}
$$

The dual of (LP) is

$$
(DLP) \quad \text{subject to} \quad
\begin{aligned}
&\text{maxmize } 0\\
&\alpha_0 + \alpha_n - \sum_{i=1}^{n-1} \alpha_i = 0\\
&-\alpha_0 v^0 - \alpha_n v^n + \sum_{i=1}^{n-1} \alpha_i v^i = 0\\
&\alpha_i \geq 0 \qquad i = 1, ..., n-1
\end{aligned}
$$

As (DLP) has an optimal solution at $\alpha_i = 0$ for all $i = 0, ..., n$, every feasible solution to (LP) is optimal.

Since v^0 and v^n are not adjacent in Q, there does not exist a valid inequality $a^T v \geq b$ for Q such that $a^T v^0 = a^T v^n = b$ and $a^T v^i > b$ for all $i = 1, ..., n-1$. By strict complementarity, (DLP) has an optimal solution $\bar{\alpha}$ such that $\bar{\alpha}_j > 0$ for some $j \in \{1, ..., n-1\}$. Observe that at least one of $\bar{\alpha}_0$ and $\bar{\alpha}_n$ must be positive. Without loss of generality, assume that $\bar{\alpha}_0 > 0$. If $\bar{\alpha}_n \leq 0$, then $v^0 = \frac{\bar{\alpha}_1}{\bar{\alpha}_0} v^1 + \cdots \frac{\bar{\alpha}_{n-1}}{\bar{\alpha}_0} v^{n-1} + \frac{-\bar{\alpha}_n}{\bar{\alpha}_0} v^n$. Thus $v^0 \in \text{conv}(\{v^1, ..., v^n\})$, contradicting that v^0 is an extreme point of Q. Hence, $\bar{\alpha}_n > 0$. Let $\bar{\alpha} = \bar{\alpha}_0 + \bar{\alpha}_n = \sum_{i=1}^{n-1} \bar{\alpha}_i$. Then $\frac{\bar{\alpha}_0}{\bar{\alpha}} v^0 + \frac{\bar{\alpha}_n}{\bar{\alpha}} v^n = \frac{\bar{\alpha}_1}{\bar{\alpha}} v^1 + \cdots + \frac{\bar{\alpha}_{n-1}}{\bar{\alpha}} v^{n-1}$. Letting $\lambda = \frac{\bar{\alpha}_0}{\bar{\alpha}}$, we have $\lambda v^0 + (1-\lambda) v^n \in \text{conv}(\{v^1, ..., v^{n-1}\})$. □

Proposition 6.23. *Let $\bar{p} = (\bar{x}, \bar{y}, \bar{z}) \in S^2 \backslash U$ be such that $a_F \bar{x} + b_F \bar{y} + c_F \bar{z} = d_F$ for at most one $F \in \mathcal{F}(P)$. If $Q = \text{conv}(U \cup \{\bar{p}\})$, then for any $p' \in U$, $\bar{p}$ and p' are adjacent in Q if and only if there exists $F \in \mathcal{F}(P)$ such that $p' \in F$ and $a_F \bar{x} + b_F \bar{y} + c_F \bar{z} > d_F$. Moreover, if $u, v \in U$ are not both adjacent to $\bar{p}$ in Q, then u and v are adjacent in Q if and only if they are adjacent in P.*

Proof. Suppose that there exists $F \in \mathcal{F}(P)$ such that $p' \in F$ and $a_F \bar{x} + b_F \bar{y} + c_F \bar{z} > d_F$. Let $q(\lambda) = \lambda \bar{p} + (1-\lambda) p'$. Clearly, $q(\lambda) \notin P$ for all $\lambda \in (0,1)$. Hence, $q(\lambda) \notin \text{conv}(U \backslash \{p'\})$ for all $\lambda \in (0,1)$. By Lemma 6.22, $\bar{p}$ and p' are adjacent in Q.

Conversely, suppose that for any $F \in \mathcal{F}(P)$ such that $p' \in F$, we have $a_F \bar{x} + b_F \bar{y} + c_F \bar{z} \leq d_F$. Let $\mathcal{B} = \{F \in \mathcal{F}(P) : p' \in F\}$ and $p' = (x', y', z')$. Note that $a_F x' + b_F y' + c_F z' < d_F$ for all $F \notin \mathcal{B}$ and that $a_F \bar{x} + b_F \bar{y} + c_F \bar{z} = d_F$ for at most one $F \in \mathcal{B}$. If

$a_F\bar{x} + b_F\bar{y} + c_F\bar{z} < d_F$ for all $F \in \mathcal{B}$, then for sufficiently small $\epsilon > 0$, $p = p' + \epsilon(\bar{p} - p')$ satisfies $a_Fx + b_Fy + c_Fz < d_F$ for all $F \in \mathcal{F}(P)$. Hence, p is in the interior of $P \subset Q$ and so $\bar{p}$ and p' cannot be adjacent in Q. If F' is the only element in $\mathcal{B}$ such that $a_{F'}\bar{x} + b_{F'}\bar{y} + c_{F'}\bar{z} = d_{F'}$, then for sufficiently small $\epsilon > 0$, $p = p' + \epsilon(\bar{p} - p')$ satisfies $a_{F'}x + b_{F'}y + c_{F'}z = d_{F'}$ and $a_Fx + b_Fy + c_Fz < d_F$ for all $F \neq F'$. Hence, p is in the relative interior of F'. Since $F' \subset Q$, $\bar{p}$ and p' cannot be adjacent in Q. This proves the first part of the proposition.

For the second part, let $u, v \in U$, not both adjacent to $\bar{p}$ in Q.

Clearly, if u and v are adjacent in Q, then they are adjacent in P.

Conversely, if u and v are adjacent in P, then there exist $F_1, F_2 \in \mathcal{F}(P)$ such that u and v satisfy $a_{F_i}x + b_{F_i}y + c_{F_i}z \leq d_F$, $i = 1, 2$ with equality. If $\bar{p}$ violates one of these inequalities, then by the first part of the proposition, $\bar{p}$ will be adjacent to both u and v, contradicting our assumption. Thus, $\bar{p}$ satisfies both inequalities and therefore they induce distinct facets of Q. As u, v are in both facets, they must be adjacent in Q. □

Corollary 6.24. *Let $\mathcal{A} \in \Phi(P)$, $\bar{p} \in S_{\mathcal{A}}$, $W = \{p \in U : p \in F$ for some $F \in \mathcal{F}(P)\backslash\mathcal{A}\}$. If $Q = \text{conv}(U \cup \{\bar{p}\})$, then, for any $p' \in U$, $\bar{p}$ is adjacent to p' in Q if and only if $p' \in W$. Moreover, if $u, v \in U$ are not both adjacent to $\bar{p}$ in Q, then u and v are adjacent in Q if and only if they are adjacent in P.*

Proof. Follows immediately from Proposition 6.23. □

Observe that $S^2\backslash S(P)$ provides an embedding of some planar multi-graph, H, on S^2. The set of points in each face of the embedding is $S_{\mathcal{A}}$ for some $\mathcal{A} \in \Phi(P)$. For $\mathcal{A}, \mathcal{B} \in \Phi(P)$, $\mathcal{A} \neq \mathcal{B}$, we say $S_{\mathcal{A}}$ and $S_{\mathcal{B}}$ are adjacent if and only if their corresponding faces are adjacent.

Theorem 6.25. *Let $p_{\mathcal{A}} \in S_{\mathcal{A}}$ and $p_{\mathcal{B}} \in S_{\mathcal{B}}$ where $S_{\mathcal{A}}$ and $S_{\mathcal{B}}$ are adjacent. Let $P_{\mathcal{A}} = \text{conv}(U \cup \{p_{\mathcal{A}}\})$ and $P_{\mathcal{B}} = \text{conv}(U \cup \{p_{\mathcal{B}}\})$. For $i \in \{\mathcal{A}, \mathcal{B}\}$, let G_i denote the graph with vertex set $V = \{v_0\} \cup U$ such that for all $u, v \in U$, $uv \in E(G_i)$ if and only if u, v are adjacent in P_i and for all $u \in U$, $v_0u \in E(G_i)$ if and only if p_i, u are adjacent in P_i. Then $G_{\mathcal{A}}$ can be obtained from $G_{\mathcal{B}}$ via a diagonal flip.*

Before we proceed to the proof, we remark that $G_{\mathcal{A}}$ remains the same for any $p_{\mathcal{A}} \in S_{\mathcal{A}}$. (Similarly for $G_{\mathcal{B}}$.)

Proof. By Proposition 6.21, both $P_{\mathcal{A}}$ and $P_{\mathcal{B}}$ are simplicial. Let $\Psi = bd(S_{\mathcal{A}}) \cap bd(S_{\mathcal{B}})$ and $\bar{F} \in \mathcal{F}(P)$ be such that Ψ is in the plane defined by $a_{\bar{F}}x + b_{\bar{F}}y + c_{\bar{F}}z = d_{\bar{F}}$. Let p be a point in the relative interior of Ψ and $p', q, q' \in U$ denote the extreme points of P in $\bar{F}$.

Without loss of generality, we may assume that $a_{\bar{F}}x+b_{\bar{F}}y+c_{\bar{F}}z < d_{\bar{F}}$ for all $(x, y, z) \in S_{\mathcal{A}}$ and $a_{\bar{F}}x + b_{\bar{F}}y + c_{\bar{F}}z > d_{\bar{F}}$ for all $(x, y, z) \in S_{\mathcal{B}}$. By Corollary 6.24, p_B is adjacent to p', q', and q.

Let $Q = \text{conv}(U \cup \{p\})$. Observe that the inequality $a_{\bar{F}}x + b_{\bar{F}}y + c_{\bar{F}}z \leq d_{\bar{F}}$ induces a facet of Q. By Radon's Theorem, $\{p, p', q, q'\}$ can be partitioned into two sets A and A' such that $\text{conv}(A) \cap \text{conv}(A') \neq \emptyset$. Since p, p', q, and q' are extreme points of Q, we must have $|A| = |A'| = 2$. Without loss of generality, we may assume that $A = \{p, q\}$ and $A' = \{p', q'\}$. Hence, p and q are adjacent to both p' and q' in Q with p, q non-adjacent and p', q' non-adjacent in Q.

Claim: Let $i \in \{\mathcal{A}, \mathcal{B}\}$. If $F \neq \bar{F}$, then for any $\bar{p} \in S_i$, p satisfies $a_F x+b_F y+c_F z \leq d_F$ strictly if and only if $\bar{p}$ does.

From the way p is chosen, it is clear that there exists $\epsilon > 0$ such that for any $w \in B(p, \epsilon)$, p satisfies $a_F x + b_F y + c_F z < d_F$ if and only if w does. Now, $p \in bd(S_i)$ implies that there exists $w' \in B(p, \epsilon) \cap S_i$. Hence, p satisfies $a_F x + b_F y + c_F z < d_F$ if and only if w' does. The claim now follows from the definition of S_i.

We now establish a series of facts.

(i) If $v \in U$, then $p_{\mathcal{A}}$ is adjacent to v in $P_{\mathcal{A}}$ if and only if p is adjacent to v in Q.

(ii) If $v \in U \backslash \{q\}$, then $p_{\mathcal{B}}$ is adjacent to v in $P_{\mathcal{B}}$ if and only if p is adjacent to v in Q.

(iii) If $u, v \in U$ are not both adjacent to $p_{\mathcal{A}}$ in $P_{\mathcal{A}}$, then they are adjacent in $P_{\mathcal{A}}$ if and only if they are adjacent in Q.

(iv) If $u, v \in U$, are not both adjacent to $p_{\mathcal{B}}$ in $P_{\mathcal{B}}$, then they are adjacent in $P_{\mathcal{B}}$ if and only if they are adjacent in Q.

(v) p' and q' are adjacent in $P_{\mathcal{A}}$ and are non-adjacent in $P_{\mathcal{B}}$.

(vi) Let $i \in \{\mathcal{A}, \mathcal{B}\}$. Consider $u, v \in U$, both adjacent to p_i and $\{u, v\} \neq \{p', q'\}$. If they are adjacent in P_i, then they are adjacent in Q.

Proof of (i): By the claim, for any $F \in \mathcal{F}(P)$, p satisfies $a_F x + b_F y + c_F z > d_F$ if and only if $p_{\mathcal{A}}$ does. The result follows from Proposition 6.23.

Proof of (ii): Suppose $v \notin \bar{F}$. By Proposition 6.23, p is adjacent to v if and only if there exists $F \in \mathcal{F}(P)$ such that $v \in F$ and p satisfies $a_F x + b_F y + c_F z > d_F$. By Corollary 6.24, $p_{\mathcal{B}}$ is adjacent to v if and only if there exists $F \in \mathcal{F}(P)$ such that $v \in F$ and $p_{\mathcal{B}}$ satisfies $a_F x + b_F y + c_F z > d_F$. By the claim, for any $F \in \mathcal{F}(P)$ with $F \neq \bar{F}$, p satisfies $a_F x + b_F y + c_F z > d_F$ if and only if $p_{\mathcal{B}}$ does. Since $v \notin \bar{F}$, it follows that v is adjacent to p in Q if and only if v is adjacent to $p_{\mathcal{B}}$ in $P_{\mathcal{B}}$.

Now suppose $v \in \bar{F}$. Then $v \in \{p', q'\}$. Since p is adjacent to p' and q' in Q and $p_{\mathcal{B}}$ is adjacent to p' and q' in $P_{\mathcal{B}}$, the result follows.

Proof of (iii): By (i), u and v are not both adjacent to p in Q. By Proposition 6.23, u and v are adjacent in Q if and only if they are adjacent in P. Since, by Corollary 6.24, u and v are adjacent in $P_{\mathcal{A}}$ if and only if they are adjacent in P, the result follows.

Proof of (iv): First, observe that u and v cannot be both adjacent to p in Q. Indeed, if $q \in \{u, v\}$, then p is not adjacent to both u and v in Q since p is not adjacent to q. If $q \notin \{u, v\}$, then by (ii), u and v are not both adjacent to p. The result now follows from Proposition 6.23 and Corollary 6.24.

Proof of (v): Clearly, p' and q' are adjacent in $P_{\mathcal{A}}$ since $\bar{F}$ is also a facet of $P_{\mathcal{A}}$ containing p' and q'. Let F' be the other facet of P that contains p' and q'. If p' and q' are adjacent in $P_{\mathcal{B}}$, then they must be in a facet of $P_{\mathcal{B}}$ that is also a facet of P. Since $\bar{F}$ cannot be a facet of $P_{\mathcal{B}}$, such a facet must be F'. Hence, $p_{\mathcal{B}}$ satisfies $a_{F'} x + b_{F'} y + c_{F'} z \leq d_{F'}$ strictly. By the claim, p also satisfies $a_{F'} x + b_{F'} y + c_{F'} z \leq d_{F'}$ strictly. Hence, F' is also a facet of Q. Since F' has only three extreme points of Q, we see that p' and q' are adjacent in Q, which is a contradiction. Thus, p' and q' must be non-adjacent in $P_{\mathcal{B}}$.

Proof of (vi): Let $\mathcal{E}_1$ and $\mathcal{E}_2$ denote the inequalities that induce the two facets of P_i that contain both u and v. As each facet of P_i has only three extreme points, p_i must satisfy one of these two inequalities strictly, say $\mathcal{E}_1$. Since the facet induced by $\mathcal{E}_1$ contains only extreme points from U, we see that $\mathcal{E}_1$ induces a facet of P, say F'. Hence, p_i satisfies $a_{F'}x + b_{F'}y + c_{F'} \leq d_{F'}$ strictly, As $\{u, v\} \neq \{p', q'\}$, $F' \neq \bar{F}$ and so by the claim, p also satisfies the inequality strictly. It follows that u and v are two of the three extreme points of Q in the facet induced by $a_{F'}x + b_{F'}y + c_{F'}z \leq d_{F'}$. Hence, u and v are adjacent in Q.

Now, let G be the graph having vertex set V such that for all $u, v \in U$, $uv \in E(G)$ if and only if u, v are adjacent in Q and for all $u \in U$, $v_0u \in E(G)$ if and only if p, u are adjacent in Q. Then, in any planar embedding of G, v_0, p', q', q lie on the same face. Hence, $G + p'q'$ and $G + v_0q$ are planar. Using facts (i)–(vi), we see that $G_{\mathcal{A}}$ is a spanning subgraph of $G + p'q'$ and $G_{\mathcal{B}}$ is a spanning subgraph of $G + v_0q$. Since $G_{\mathcal{A}}$ and $G_{\mathcal{B}}$ are maximal planar graphs, we must have $G = G_{\mathcal{A}} - p'q' = G_{\mathcal{B}} - v_0q$. The result now follows. □

Proof of Theorem 6.19. Let the elements of V be $v_1, ..., v_n$. Let $U_1 = \{p^i \in S^2 : i \in \{1, ..., n\}\}$ be such that no four points in U_1 are coplanar and $G(\text{conv}(U_1)) \cong G_1$ with p^i corresponding to v_i. (This is possible by Proposition 6.20.) Let $U_2 = \{q^i \in S^2 : i \in \{1, ..., n\}\}$ be such that no four points in U_2 are coplanar, $U_2 \cap U_1 = \emptyset$, and $G(\text{conv}(U_2)) \cong G_2$ with p^i corresponding to v_i.

Let $\epsilon > 0$. For $i \in \{1, ...n\}$, denote the set $S^2 \cap B(q^i, \epsilon)$ by B_i. Assume that ϵ is sufficiently small so that $\cup_{i=1}^n B_i \cap U_1 = \emptyset$ and $B_i \cap B_j = \emptyset$ for any $i, j \in \{1, ..., n\}$, $i \neq j$.

We perform the following algorithm:

- Let $Q_1^0 = \text{conv}(U_1)$ and $G_{Q_1^0} = G_1$.
- Set $L = (G_{Q_1^0})$.
- for $i = 1, ..., n$, set $f(i) = p^i$.
- for $i = 1, ..., n$ do
  - Let $U = \{f(1), ..., f(n)\} \backslash f(i)$.

- Let $P = \text{conv}(U)$. (Note that P is 3-dimensional and simplicial.)
- Let $\mathcal{A} \in \Phi(P)$ be such that $f(i) \in S_{\mathcal{A}}$.
- Pick $p' \in B_i$ such that $p' \in S_{\mathcal{B}}$ for some $\mathcal{B} \in \Phi(P)$ and no four points in $U \cup \{p'\}$ are coplanar.
- Identify a sequence $\mathcal{A}_0 = \mathcal{A}, \mathcal{A}_1, ..., \mathcal{A}_{k_i} = \mathcal{B}$ such that $\mathcal{A}_j \in \Phi(P)$ for $j = 1, ..., k_i - 1$ and $\mathcal{A}_j, \mathcal{A}_{j+1}$ are adjacent for $j = 0, ..., k_i - 1$.
- for $j = 1, ..., k_i - 1$, pick a point $r^j \in S_{\mathcal{A}_j}$
- Let $r^{k_i} = p'$.
- for $j = 1, ..., k_i$ do
  * Set $f(i) = r^j$.
  * Let $Q_i^j = \text{conv}(\{f(1), ..., f(n)\})$
  * Let $G_{Q_i^j}$ be the graph having vertex set V such that $G_{Q_i^j} \cong G(Q_i^j)$ with $f(l)$ corresponding to v_l for $l = 1, ..., n$.
  * Append $G_{Q_i^j}$ to L.
- Let $Q_{i+1}^0 = Q_i^{k_i}$ (Note that no four extreme points of $Q_i^{k_i}$ are coplanar.)

By Theorem 6.25, $G_{Q_i^{j+1}}$ can be obtained from $G_{Q_i^j}$ via a diagonal flip for $j = 0, ..., k_i - 1$, $i = 1, ..., n$. Hence, for any two successive graphs in L, one can be obtained from the other via a diagonal flip.

At the end of the algorithm, we have $f(i) \in B_i$ for $i = 1, ..., n$. By Proposition 6.20, if ϵ is sufficiently small, the last graph in L (i.e. $G_{Q_n^{k_n}}$) is G_2. Since the first graph in L is G_1 and all the graphs in L are of inscribable type, the result follows.

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