Riemann's insight

Where it came from and where it has led

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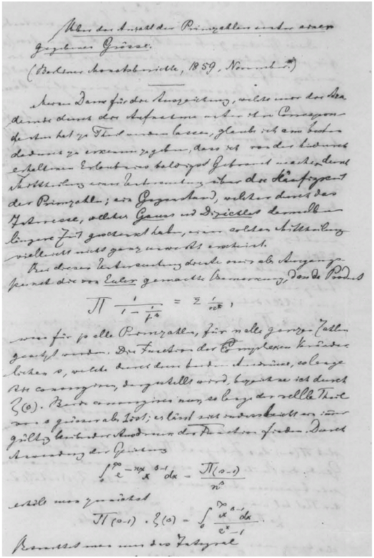
November 2009

BERNHARD RIEMANN 1826-1866

Ueber die Anzahl der Primzahlen unter einer gegeben Grösse (1859)



Ueber die Anzahl der Primzahlen unter einer gegebenen Grösse.

(Berliner Monatsberichte, 1859, November.)

$$\prod \frac{1}{1-\frac{1}{p^s}} = \sum \frac{1}{n^s},$$

$$\int_0^\infty e^{-nx} x^{s-1}\,dx = \frac{\Pi(s-1)}{n^s}$$

$$\Pi(s-1)\cdot\zeta(s) = \int_0^\infty \frac{x^{s-1}\,dx}{e^x-1}.$$

Riemann integral
Riemann sphere
Riemann-Schwarz principle
Riemann zeta function
Riemann-Hilbert correspondence
Riemann surface
Cauchy-Riemann equations
Riemann-Brill-Noether theorem
Riemannian geometry
Riemann hypothesis
Riemann sum
Riemann curvature tensor
Riemann mapping theorem
Riemann-Roch theorem

...

A short biography of Riemann

- **(1826)** Born in Breslenz (a small town in northern Germany).
- **(1840-1846)** Attended high school at Hanover and Lüneburg.
- **(1846)** Began study at University of Göttingen (where Gauss aged 69 was professor).
- **(1847-1849)** Continued studies in Berlin University taking courses from Dirichlet, Jacobi and Eisenstein.
- **(1849-1851)** Completed doctorate at Göttingen with Gauss.
- **(1851-1859)** Riemann's prolific period (9 of 11 papers published in his life time were published, another 7 were published after his death). Dedekind was a close colleague. Gauss died in 1855 and was replaced by Dirichlet as the professor of mathematics.
- **(1859)** Dirichlet died and Riemann was appointed as full professor.
- **(1862-1866)** Married Elize Koch in 1862 and daughter Ida born in 1863. Riemann's continuing ill health led to making extended visits to Italy. Died in Italy in 1866 (near Lago Maggiore) aged 39.

Influences on Riemann

EULER
1707-1783

FOURIER
1758-1830

CAUCHY
1789-1857

GAUSS
1777-1855

JACOBI
1804-1851

DIRICHLET
1805-1859

CHEBYSHEV
1821-1894

DEDEKIND
1831-1916

RIEMANN
1826-1866

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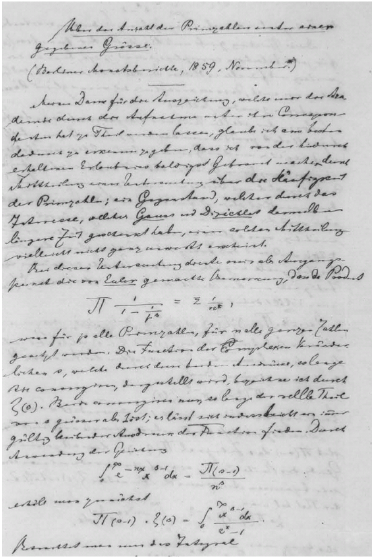

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Distribution of primes

$\pi(x)$ = no. of primes $p \leq x$.

- From extensive calculations Legendre estimated $\pi(x)$ is approximately $x/(\log(x) - 1.08366)$ and Gauss estimated that $\pi(x)$ is approximately $Li(x)$ where
$$Li(x) := \int_0^x \frac{dx}{\log x} = \frac{x}{\log x}\left\{1 + \frac{1}{\log x} + O\left(\frac{1}{(\log x)^2}\right)\right\}.$$
- (1849–1852) Chebyshev shows that Legendre's estimate is not asymptotically true, but provides theoretical evidence that $\pi(x) \sim Li(x)$. He also proves
$$\frac{Ax}{\log x} < \pi(x) < \frac{Bx}{\log x} \text{ for suitable constants } A, B > 0.$$
- The *Euler product* (1737)
$$\prod_{p \text{ prime}} \left(1 - \frac{1}{p^s}\right)^{-1} = \prod_{p \text{ prime}} \left(1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \ldots\right) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \zeta(s)$$

The rate of growth of the prime counting function $\pi(x)$

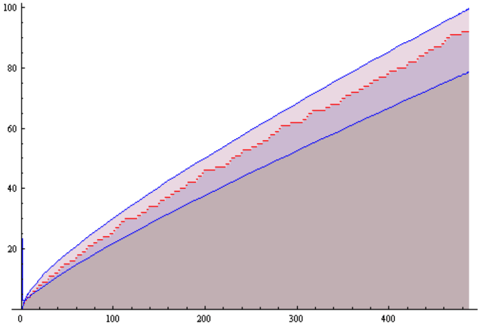


Figure: Graphs of $Li(x)$, $\pi(x)$ and $x/\log(x)$

where $Li(x) := \int_0^x \frac{dx}{\log(x)}$ (from Wolfram Mathworld)

The Riemann zeta function

- The *Riemann zeta function* $\zeta(s) := \sum_1^\infty n^{-s}$ for $\mathrm{Re}\, s > 1$.

Recall: for complex s we have $n^s := \exp(s \log n)$ and $|n^s| = n^\sigma$ when $s = \sigma + it$.

- $\zeta(s)$ can be extended by analytic continuation (uniquely) to a meromorphic function on $\mathbb{C}$ which is analytic everywhere except for a pole at $s = 1$
- $\zeta(s) - \frac{1}{s-1}$ is analytic in the whole plane.

Properties of the zeta function

- $\zeta(s) = \frac{s}{s-1} - s\int\limits_1^\infty (x - \lfloor x \rfloor) x^{-s-1} dx$ for $\sigma > 0$ (where $s = \sigma + it$).
- [left-right reflection formula] $\zeta(1-s) = \pi^{-s} 2^{s-2} \Gamma(s) \cos(\frac{1}{2}\pi s) \zeta(s)$ for all complex s.
- [top-bottom reflection] $\zeta(\bar{s}) = \overline{\zeta(s)}$ for all s.
- The *Euler product* (1737) for $\sigma > 1$:

$$\boxed{\prod_{p \text{ prime}} (1 - \tfrac{1}{p^s})^{-1} = \prod_{p \text{ prime}} (1 + \tfrac{1}{p^s} + \tfrac{1}{p^{2s}} + \ldots) = \textstyle\sum_{n=1}^\infty \frac{1}{n^s} = \zeta(s).}$$

Counting primes and the zeta function

- The step function $\pi(x^{1/k})$ has a jump of size 1 when $x = p^k$ so

$$\sum_{p \text{ prime}} p^{-ks} = s \int_0^\infty \pi(x^{1/k}) x^{-s-1} dx$$

-

$$\begin{aligned}
\log \zeta(s) &= -\sum_{p \text{ prime}} \log(1 - p^{-s}) \\
&= \sum_p \left\{ p^{-s} + \frac{1}{2} p^{-2s} + \frac{1}{3} p^{-3s} + \ldots \right\} \\
&= s \int_2^\infty \left\{ \pi(x) + \frac{1}{2}\pi(x^{1/2}) + \frac{1}{3}\pi(x^{1/3}) + \ldots \right\} x^{-s-1} dx.
\end{aligned}$$

- Putting
$P(x) := \pi(x) + \frac{1}{2}\pi(x^{1/2}) + \frac{1}{3}\pi(x^{1/3}) + \ldots = \pi(x) + O(\sqrt{x})$ gives

$$\boxed{\log \zeta(s) = s \int_0^\infty P(x) x^{-s-1} dx}$$

Riemann concludes that

$$\boxed{\log \zeta(s) = s \int_0^\infty P(x) x^{-s-1} dx}$$

where $P(x) = \pi(x) + \frac{1}{2}\pi(x^{1/2}) + \frac{1}{3}\pi(x^{1/3}) + \dots.$

- Inverting the transform Riemann obtains (at points of continuity of $P(x)$):

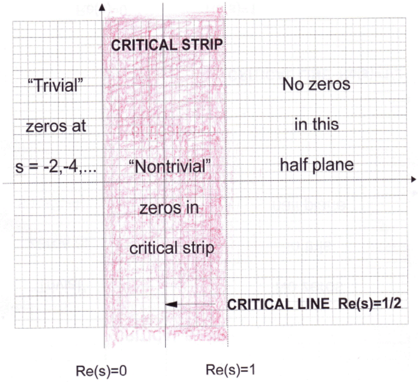
$$P(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\log \zeta(s)}{s} x^s ds \text{ for all real } c > 1.$$

- Assuming a factorization of $\xi(t)$ in terms of its roots he obtains

$$\boxed{P(x) = Li(x) - \sum_\rho Li(x^\rho) + \int_x^\infty \frac{1}{x^2-1} \frac{dx}{x \log x} - \log 2}$$

where ρ runs over the zeros of $\zeta(s)$ in the *critical strip* $0 \le \sigma \le 1$.

Zeros of the Riemann zeta function $\zeta(s)$



$\mathfrak{Riemann\ Hypothesis}$: **all nontrivial zeros lie on critical line**

Prime number Thereom (PNT)

$$\pi(x) \sim \frac{x}{\log(x)}$$

- (1894) Based on Riemann's paper, von Mangoldt shows that the PNT is true provided there are no zeros of $\zeta(s)$ on the line $\sigma = 1$.
- (1896) de la Vallée Poussin and Hadamard independently use van Mangoldt's result to prove PNT.
- (1901) von Koch shows $\pi(x) = Li(x) + O(\sqrt{x}\log x)$ if and only if RH is true.
- (1949) Selberg and Erdös independently give an "elementary" proof of PNT (not using complex analysis).

Counting nontrivial zeros of the Riemann zeta function

- For a region R such that $\zeta(s)$ is analytic on R and its boundary ∂R and nonzero on ∂R the number of zeros in R is equal to

$$\frac{1}{2\pi i}\int_{\partial R} \frac{\zeta'(s)}{\zeta(s)} ds = \text{ no. times } \zeta(\partial R) \text{ winds around } 0.$$

- The number of zeros $\sigma + it$ in the *critical strip* with $0 \leq t \leq T$ is:

$$N(T) = \frac{T}{2\pi} \log \frac{T}{2\pi} - \frac{T}{2\pi} + O(\log T).$$

- There exists a realvalued function $\xi(t)$ such that $\xi(t) = 0 \iff \zeta(\frac{1}{2} + it) = 0$. Number of zeros of $\zeta(s)$ on the *critical line* with $0 \leq t \leq T$ is at least the number of sign changes in the xi function $\xi(t)$ over $[0, T]$. RH is true up to T if this latter number is $\geq N(T)$.
- At least 40% of the zeros lie on the critical line (Conrey 1989).
- The first 10^{13} (ten trillion) zeros all lie on the critical line (Gourdon 2004).

Equivalents of RH

- $1/\zeta(s) = \prod_p (1 - p^{-s}) = \sum_n \mu(n) n^{-s}$ where $\mu(1) = 1$, $\mu(n) = (-1)^k$ if n is a product of k distinct primes, and $\mu(n) = 0$ otherwise. RH is true $\iff \sum_{n \leq x} \mu(n) = O(x^{1/2+\varepsilon})$ for all $\varepsilon > 0$.
- Let $\sigma(n)$ be the sum of the positive integers dividing n. Then RH is true $\iff \sigma(n) < e^{\gamma} n \log\log n$ for all $n > 5040$.
- RH is true $\iff$

$$\sum_{k=1}^{\infty} \frac{(-1)^k x^k}{k!\zeta(2k+1)} = O(x^{-1/4}) \text{ as } x \to \infty.$$

- There are many others ...

Other "Riemann Hypotheses"

- [Extended Riemann Hypothesis] Let K be any algebraic number field, D be its ring of integers, $\mathcal{I}$ be the set of nonzero ideals in D, $\mathcal{P}$ be the prime ideals. Set $N(I) := |D/I|$ for $I \in \mathcal{I}$. Then
$$\zeta_K(s) := \sum_{I \in \mathcal{I}} N(I)^{-s} = \prod_{P \in \mathcal{P}} (1 - N(P)^{-s})$$
has properties similar to $\zeta(s)$. ERH: All the nontrivial zeros of $\zeta_K(s)$ lie on the critical line.
- [Generalized Riemann Hypothesis] All Dirichlet L-functions have their nontrivial zeros on the critical line.
- [Riemann Hypothesis for varieties over finite fields] An analogous zeta function $\zeta_L(s)$ can be defined over any field L which is a finite extension of the field $F(X)$ of rational functions over a finite field F. It turns out that $\zeta_L(s)$ is a rational function of $|F|^s$. Weil (1948) proved that all zeros of $\zeta_L(s)$ have real part $\frac{1}{2}$. Deligne (1974) finally proved Weil's conjecture that a similar theorem holds for general varieties.

Approaches to proving RH

- Riemann himself wrote "I have meanwhile temporarily put aside the search for [a proof] after some fleeting futile attempts as it appears unnecessary for the next objective of my investigation."
- Stieltjes (1885) claimed to prove $M(x) := \sum_{n \le x} \mu(n) = O(\sqrt{x})$. He never published his proof, and it is doubtful that the bound holds. Odlyzko and te Riele (1985) showed that sometimes $|M(x)| > \sqrt{x}$.
- Radamacher (1945) believed he had proved that RH was false. Paper withdrawn from Trans. AMS before journal went to press.
- Turán (1948) showed that RH is true if, for all sufficiently large n, the nth partial sum of $\zeta(s)$ does not vanish for $s > 1$. Montgomery (1983) showed that the partial sums always have zeros for $s > 1$ (for large enough n).
- Montgomery (1973) made the "pair-correlation" conjecture about the zeros of $\zeta(s)$ assuming RH. The conjecture suggests a relationship between $\zeta(s)$ and random Hermitian operators.

Fame and Fortune

- RH is one of the famous "Hilbert Problems" (1900). Originally Hilbert thought it one of the simpler ones. Later he was reported to say that if he came back to life several centuries after death his first question would be "Has anyone proved the Riemann hypothesis?"
- The Clay Institute (2000) offers one million dollars for a solution.
- There are numerous papers starting "Assume RH". A proof that RH is true will also prove 500 other theorems at the same time. There are many important consequences of RH.
- If you plan to prove RH you should first visit website of failed proofs http://www.secamlocal.ex.ac.uk/people/staff/mrwatkin/zeta/RHproofs. As Pólya said: Before climbing the Matterhorn you should visit the graves of failed climbers.

The Theory of Numbers has always been regarded as one of the most obviously useless branches of Pure Mathematics. The accusation is one against which there is no valid defence; and it is never more just than when directed against the parts of the theory which are more particularly concerned with primes.
A science is said to be useful if its development tends to accentuate the existing inequalities in the distribution of wealth, or more directly promotes the destruction of human life. The theory of prime numbers satisfies no such criteria. Those who pursue it will, if they are wise, make no attempt to justify their interest in a subject so trivial and so remote, and will console themselves with the thought that the greatest mathematicians of all ages have found in it a mysterious attraction impossible to resist.

[G. H. Hardy]

A few references

1. Bernhard Riemann, “Gesammelte Mathematische Werke” (reprint of 2nd ed.), Dover 1953.
2. Harold M. Edwards, “Riemann's Zeta Function”, Academic Press 1974 (reprinted by Dover).
3. Detlef Laugwitz, “Bernhard Riemann 1826–1866”, Birkhäuser 1999.
4. Peter Borwein, Stephen Choi, Brendan Rooney and Andrea Weirathmueller, “The Riemann Hypothesis”, Springer 2008.