Approximation of non-conservative hyperbolic systems based on different shock curve definitions

N. Chalmers[1,*], and E. Lorin[2]

[1] *Department of Applied Mathematics, University of Waterloo, Waterloo, Canada, N2L 3G1*
[2] *School of Mathematics and Statistics, Carleton University, Ottawa, Canada, K1S 5B6*

Abstract. The aim of this paper is to lay a theoretical framework for developing numerical schemes for approximating Non-Conservative Hyperbolic Systems (NCHSs). We first recall some key points of the theory of NCHSs, beginning with the definition non-conservative products proposed by Dal Maso, LeFloch, and Murat [14]. Next, we briefly introduce the vanishing viscosity solutions and shock curves derived from Bianchini and Bressan's center manifold technique [7], and their partial generalization recently proposed by Alouges and Merlet [5]. Approximation of these shock curves also proposed by Alouges and Merlet are then introduced and discussed. We then investigate the numerical implementation of these analytical approaches using Godunov-like schemes, which either use the approximate Shock curves of Alouges and Merlet directly in a Riemann solver, or use the framework of Dal Maso, LeFloch, and Murat, in combination with these approximate shock curves. To our knowledge, this work is the first attempt to numerically implement shock curves derived from Bianchini and Bressan's center manifold approach.

Key words: non-conservative hyperbolic systems, finite volume methods, distribution products

1 Introduction

This paper is devoted to the numerical approximation of Non-Conservative Hyperbolic Systems (NCHSs). Non-conservative hyperbolic systems arise in several areas, in particular in the study of compressible multi-phase/fluid flows and have various industrial applications, such as two-phase flows in nuclear power plant reactors, solid rocket motors, chemical plants, detonations, shallow water bi-fluid flows, and others [16], [34], [26], [31]. These systems have proven to be difficult to analyze and have been much less studied than Hyperbolic Systems of Conservation Laws (HSCL). Nevertheless, their wide range of applications have recently motivated large efforts to better understand these systems and their numerical approximation.

An example of a system of interest is the model developed by Deledicque and Papalexandris in [15]. Their system is a two-phase, two pressure system modeling the dynamics of fluids with a gaseous, g, and liquid, l, phase. Each phase is assigned a density ρ_α, pressure p_α, specific internal energy e_α, velocity u_α, and volume fraction ϕ_α, where $\alpha = g,l$. The governing equations consist of mass, momentum, and energy balance laws for each phase, plus a convection equa-

*Corresponding author. *Email addresses:* bnachalm@uwaterloo.ca (N. Chalmers)

tion for the solid volume fraction,

$$
\begin{cases}
\dfrac{\partial\phi_l\rho_l}{\partial t}+\dfrac{\partial\phi_l\rho_l u_l}{\partial x} & = & 0,\\
\dfrac{\partial\phi_l\rho_l u_l}{\partial t}+\dfrac{\partial(\phi_l\rho_l u_l^2+\phi_l p_l)}{\partial x} & = & p_g\dfrac{\partial\phi_l}{\partial x},\\
\dfrac{\partial\phi_l\rho_l E_l}{\partial t}+\dfrac{\partial(\phi_l u_l(\rho_l E_l+p_l))}{\partial x} & = & p_g u_l\dfrac{\partial\phi_l}{\partial x},\\
\dfrac{\partial\phi_g\rho_g}{\partial t}+\dfrac{\partial\phi_g\rho_g u_g}{\partial x} & = & 0,\\
\dfrac{\partial\phi_g\rho_g u_g}{\partial t}+\dfrac{\partial(\phi_g\rho_g u_g^2+\phi_g p_g)}{\partial x} & = & -p_g\dfrac{\partial\phi_l}{\partial x},\\
\dfrac{\partial\phi_g\rho_g E_g}{\partial t}+\dfrac{\partial(\phi_g u_g(\rho_g E_g+p_g))}{\partial x} & = & -p_g u_g\dfrac{\partial\phi_l}{\partial x},\\
\dfrac{\partial\phi_l}{\partial t}+u_l\dfrac{\partial\phi_l}{\partial x} & = & 0,
\end{cases}
$$

where $E_\alpha=e_\alpha+u_\alpha^2/2$ is the total specific energy for each phase. The following saturation condition is also assumed

$$\phi_g+\phi_l=1.$$

Together with the equations of state for p_g and p_l, this system of balance laws can be written as a non-conservative system and is thus a special case of the general system which we will consider. Note that most of multi-phase/fluid models (with one or two pressures) contain a non-conservative product. See for instance [31], [2], [35].

More generally, in this paper we will be interested in one dimensional NCHS,

$$\frac{\partial\mathbf{u}}{\partial t}+A(\mathbf{u})\frac{\partial\mathbf{u}}{\partial x}=\mathbf{0},\qquad \mathbf{u}\in\Omega\subseteq\mathbb{R}^n,\,(x,t)\in\mathbb{R}\times\mathbb{R}^+,\tag{1.1}$$

where Ω is open convex set and A is a smooth function $A:\mathbb{R}^n\to M_n(\mathbb{R})$. We assume that this system is *strictly hyperbolic*, that is, A has n real and distinct eigenvalues $\lambda_1(\mathbf{u})<\lambda_2(\mathbf{u})<\ldots<\lambda_n(\mathbf{u})$, $\forall\mathbf{u}\in\Omega$ with linearly independent eigenvectors. Recall that when A is the Jacobian matrix of some vector-valued function $\mathbf{f}:\mathbb{R}^n\to\mathbb{R}^n$, i.e. $A(\mathbf{u})=D\mathbf{f}(\mathbf{u})$, then this system reduces to a HSCL.

Our primary goal when studying hyperbolic systems is to completely describe solutions of the *Riemann problem* for (1.1)

$$\mathbf{u}(x,0)=\begin{cases}\mathbf{u}_L, & x<0,\\ \mathbf{u}_R, & x>0.\end{cases}$$

Because of the non-conservative term, $A(\mathbf{u})\mathbf{u}_x$, and the fact that products of distributions are not defined by the theory of distributions [32], we cannot rigorously define the notion of *weak solutions* for system (1.1) and we cannot derive a Rankine-Hugoniot Jump Condition, as in the conservative case. Finally, we cannot define, a priori, the notion of shock wave for NCHSs. Although this constitutes an old problem, 'recently' two distinct ways of overcoming this issue in the framework of NCHSs have been proposed. The first considered in this paper is due to Dal Maso, LeFloch and Murat (DLM) [14], [23]. Specifically, the authors propose a definition of non-conservative products, and hence they can define weak solutions of (1.1). They suggest to introduce a family of paths, $\psi:[0,1]\times\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}^n$ which satisfies the following properties,

$$\psi(0;\mathbf{u}_L,\mathbf{u}_R)=\mathbf{u}_L,\qquad \psi(1;\mathbf{u}_L,\mathbf{u}_R)=\mathbf{u}_R,$$

$\forall \mathbf{u}_L, \mathbf{u}_R \in \mathbb{R}^n$. They then define the non-conservative product, $A(\mathbf{u})\mathbf{u}_x$, not as a distribution, but as a bounded Borel measure which depends on this family of paths. This measure, denoted by $[A(\mathbf{u})\mathbf{u}_x]_\psi$, is defined as

$$[A(\mathbf{u})\mathbf{u}_x]_\psi(B) = \int_B A(\mathbf{u})\mathbf{u}_x \, dx,$$

when $\mathbf{u}$ is continuous on a Borel set B, and by

$$[A(\mathbf{u})\mathbf{u}_x]_\psi((x_0,t_0)) = \int_0^1 A(\psi(s;\mathbf{u}_L,\mathbf{u}_R)) \frac{\partial \psi}{\partial s}(s;\mathbf{u}_L,\mathbf{u}_R) \, ds,$$

when $\mathbf{u}$ has a jump discontinuity and $\mathbf{u}_L$ and $\mathbf{u}_R$ are the left and right limits of the discontinuity, respectively. It is important to note that this definition of a non-conservative product only applies to functions which are piecewise differentiable with finite jump discontinuities, and not for general distributions. However, a priori these functions are all we need in order to solve the Riemann problem for NCHSs. In fact, this product extends the definition of non-conservative products given by Volpert [36], which can be recovered in this framework by choosing the family of straight lines $\psi(s;\mathbf{u}_L,\mathbf{u}_R) = \mathbf{u}_L + s(\mathbf{u}_R - \mathbf{u}_L)$. This formulation of the non-conservative product allows us to define weak solutions of the system, and furthermore it allows us to generalize the Rankine-Hugoniot jump condition [14] to

$$\sigma(\mathbf{u}_R - \mathbf{u}_L) = \int_0^1 A(\psi(s;\mathbf{u}_L,\mathbf{u}_R)) \frac{\partial \psi}{\partial s}(s;\mathbf{u}_L,\mathbf{u}_R) \, ds.$$

Using this condition, it is possible to proceed as done in the conservative case and solve the Riemann problem by using shock waves, rarefaction waves, and contact discontinuities to separate at most $n+1$ constant states. The obvious drawback in this formulation is that the definition of the non-conservative product depends on the choice of path, ψ. Because of this, it is difficult, a priori, to select the paths that will give us the correct, physical solution. As LeFloch remarks in [23], appropriate paths could be chosen so that they parametrize viscous profiles. However, the question of how to determine the viscous profiles is made difficult since it involves finding bounded solutions of an ODE on an infinite domain. For a more complete discussion see [33].

Another approach for finding solutions to NCHSs was developed in the recent works of Bianchini and Bressan [7], and Alouges and Merlet [5] who partially generalized Bianchini and Bressan's work. In their very technical work, Bianchini and Bressan investigate the solutions of the following *viscous system*,

$$\mathbf{u}_t^\varepsilon + A(\mathbf{u}^\varepsilon)\mathbf{u}_x^\varepsilon = \varepsilon \mathbf{u}_{xx}^\varepsilon,$$

which is a parabolic regularization of the original system (1.1), with the specific viscosity matrix $B(\mathbf{u}) = I$. They define solutions of (1.1) as *vanishing viscosity* solutions of the viscous system, i.e. solutions to (1.1) are constructed as the limit of the solution to this viscous system as $\varepsilon \to 0$. In a very general setting, they show that these vanishing viscosity solutions are unique and, in particular, they describe the shock curves and viscous shock profiles associated to this viscosity matrix $B = I$. The work of Bianchini and Bressan was then generalized by Alouges and Merlet in [5], who extended their results to the case where the viscosity matrix B commutes with A. The authors also propose a definition of shock curves of non-conservative systems as solution of the following dynamical system

$$\begin{cases} (A(\mathbf{u}) - \sigma I) \dfrac{d\mathbf{u}}{d\sigma} = \mathbf{u} - \mathbf{u}_L, \\ \mathbf{u}(\lambda_i(\mathbf{u}_L)) = \mathbf{u}_L. \end{cases}$$

They prove that the shock curves given by this system are close to the shock curves deduced from the center manifold theory of Bianchini and Bressan, up to the third order. This result gives us a way for selecting the admissible discontinuities and therefore allows us to solve Riemann Problems in NCHSs. Moreover Alouges and Merlet prove that the shock curves defined by the system above also agree with the viscous shock profiles up to the third order. These shock curves therefore give us a close approximation of the viscous profiles which we can use for instance, as a path in the DLM theory. This leads us to investigate two interesting designs for numerical schemes. Note finally that Colombeau has proposed [12] to extend the set of distributions as a quotient algebra, allowing to define the product of "extended generalized functions". Within this framework, shock waves solutions of non-conservative hyperbolic systems can "easily" be defined as well as their discretization (for elasticity models in particular) [11], [13] using weak-strong formulations of the considered system. See also [25] for a more recent work on numerical schemes for non-conservative hyperbolic systems based on Colombeau's generalized functions. In this paper, Colombeau's approach will not be discussed, but some of its links with Bianchini & Bressan's, and LeFloch's works will be addressed in a forthcoming paper.

The question of implementing the DLM theory in a numerical solver has been investigated by several authors. Originally Toumi and Kumbaro proposed a path-based approach [35] to build a Roe solver for NCHSs. Later other authors, in particular Parés and Castro [9], [8], [27], and Rhebergen, Bokhove, and Van der Vegt, [29], [28] have investigated other numerical methods (Godunov, Discontinuous Galerkin, etc) for these systems which are based on DLM's path-theory. Note that many other approaches have been proposed to treat the non-conservative product, in particular in the multi-phase flow framework (see [18], for instance).

In this paper, we will focus on the shock and approximate shock curves as defined by Alouges and Merlet. Our main scheme of interest will be a Godunov-like Scheme [20] using an exact Riemann solver:

$$V_j^{n+1} = V_j^n - \frac{\Delta t}{\Delta x}\left(G^{n,-}_{j+\frac{1}{2}} + G^{n,+}_{j-\frac{1}{2}}\right).$$

Castro, Parés, et. al. show in [9] that using the DLM definitions in a Godunov solver leads naturally to select

$$G^{n,-}_{j+\frac{1}{2}} = \int_0^1 A(\psi(s;V_j^n,V^n_{j+\frac{1}{2}}))\frac{\partial\psi}{\partial s}(s;V_j^n,V^n_{j+\frac{1}{2}})\,ds,$$

$$G^{n,+}_{j+\frac{1}{2}} = \int_0^1 A(\psi(s;V^n_{j+\frac{1}{2}},V^n_{j+1}))\frac{\partial\psi}{\partial s}(s;V^n_{j+\frac{1}{2}},V^n_{j+1})\,ds,$$

where $V^n_{j+\frac{1}{2}}$ is the value at $x=0$ of the solution to the Riemann Problem

$$V(x,0) = \begin{cases} V_j^n, & x<0, \\ V_{j+1}^n, & x>0. \end{cases}$$

Although the Godunov scheme is known to be very slow, at this point the goal of this paper is not to propose a fast and an accurate solver for NCHSs, but rather to propose a first (to our knowledge) numerical implementation of Bianchini & Bressan and Alouges & Merlet's shock and approximate shock curves in a finite volume solver. When implementing the shock curves of Alouges and Merlet, we have to solve Riemann Problems at each interface and select the fluxes $G^{n,\pm}_{j+1/2}$ dependent on the type of the wave solution (i.e. 1-shock and 2-shock, 1-shock and 2-rarefaction, etc.). This first scheme is then a Godunov scheme based on an exact Riemann solver. We will first apply this scheme to a hyperbolic system of conservation laws and compare their numerical solutions with solutions of reference to check that we recover correct

results.
We will then implement Alouges-Merlet's approximate shock curves in a DLM path-dependent scheme and compare the numerical solutions with the ones found by Parés' Godunov scheme, described above. To obtain a worth while comparison, we will apply these schemes to a truly non-conservative system. We show that, in this particular example, the numerical solution does indeed converge to the exact solution, seemingly overcoming the convergence problem for non-conservative systems as studied in [10], [21], [24], [21] and [1]. Another interesting result that we will show is that the two Godunov schemes we have constructed are in fact close in a sense that will be defined below.

The remainder of this paper is organized as follows. In Section 2, we present some key elements of the theories developed by Dal Maso, LeFloch and Murat, Bianchini and Bressan, and Alouges and Merlet. We then move to the numerical implementation of these approaches in Section 3. Comparisons of the numerical solutions obtained with these different approaches will then be presented. Section 4 is devoted to concluding remarks.

2 Non-Conservative Hyperbolic Systems

In this section we recall some important features of non-conservative hyperbolic systems. As mentioned in the introduction, the first fundamental difficulty that we must address are how to define weak solutions of these systems, and how to properly define the shock curves in order to solve the Riemann problem.

We begin this section by recalling some key elements of Dal Maso, LeFloch, and Murat's path-theory, in particular their definition of non-conservative products. We then give an overview of the vanishing viscosity solutions studied by Bianchini and Bressan. Finally, we present the approximate shock curves defined by Alouges and Merlet and recall when and how they approach the shock curves deduced from Bianchini and Bressan's center manifold approach. These shock curves will be referred in the following as *Bianchini and Bressan's shock curves*.

2.1 Dal Maso-LeFloch-Murat Non-Conservative Products

As we mentioned in the introduction, the main issue with non-conservative hyperbolic systems is the presence of the non-conservative term $A(\mathbf{u})\mathbf{u}_x$. As a product of distributions, it is not clear how this term should be defined, and thus we are unable to specify what discontinuity waves can be weak solutions. The idea proposed by Dal Maso, LeFloch, and Murat [14] was to regard this term not as a distribution, but as a bounded Borel measure. Let us quickly recall the principle. When $\mathbf{u}$ is smooth on a Borel set B, this measure is defined by

$$[A(\mathbf{u})\mathbf{u}_x](B) = \int_B A(\mathbf{u})\mathbf{u}_x\,dx.$$

The problem arises when $\mathbf{u}$ has a jump discontinuity, as:

$$\mathbf{u} = \begin{cases} \mathbf{u}_L, & x < 0, \\ \mathbf{u}_R, & x > 0. \end{cases}$$

Regarding $A(\mathbf{u})\mathbf{u}_x$ as a measure, we require that:

$$[A(\mathbf{u})\mathbf{u}_x] = C\delta_0,$$

where δ_0 is the Dirac measure at $x=0$ and C is a constant to be determined. If $A(\mathbf{u})$ were a Jacobian matrix, $A(\mathbf{u})=D\mathbf{f}(\mathbf{u})$,

$$\begin{aligned}[A(\mathbf{u})\mathbf{u}_x](\{0\}) &= \lim_{\varepsilon\to 0}\int_{-\varepsilon}^{\varepsilon} A(\mathbf{u})\mathbf{u}_x\,dx,\\ &= \lim_{\varepsilon\to 0}\int_{-\varepsilon}^{\varepsilon} (\mathbf{f}(\mathbf{u}))_x\,dx,\\ &= \mathbf{f}(\mathbf{u}_R)-\mathbf{f}(\mathbf{u}_L).\end{aligned}$$

So $C=\mathbf{f}(\mathbf{u}_R)-\mathbf{f}(\mathbf{u}_L)$ in this case. As a consequence the definition proposed by DLM is then to introduce a path ψ, such that $\psi(0)=\mathbf{u}_L$ and $\psi(1)=\mathbf{u}_R$, and define:

$$[A(\mathbf{u})\mathbf{u}_x]=\left(\int_0^1 A(\psi(s))\frac{\partial\psi}{\partial s}ds\right)\delta_0.$$

Then this gives a value to C, and does indeed recover the correct results in the case when A is a Jacobian matrix. This idea motivates the central theorem of Dal Maso-LeFloch-Murat that we recall here.

Theorem 2.1 (Dal Maso-LeFloch-Murat Non-Conservative Product). *Let $\psi:[0,1]\times\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}^n$ be a Lipschitz continuous family of paths which satisfies the following properties:*

1. $\forall \mathbf{u}_L,\mathbf{u}_R\in\mathbb{R}^n$,
$$\psi(0;\mathbf{u}_L,\mathbf{u}_R)=\mathbf{u}_L, \qquad \psi(1;\mathbf{u}_L,\mathbf{u}_R)=\mathbf{u}_R.$$

2. $\exists k>0$, *such that* $\forall \mathbf{u}_L,\mathbf{u}_R\in\mathbb{R}^n,\forall s\in[0,1]$,
$$\left|\frac{\partial\psi}{\partial s}(s;\mathbf{u}_L,\mathbf{u}_R)\right|\leqslant k|\mathbf{u}_L-\mathbf{u}_R|.$$

3. $\exists k>0$, *such that* $\forall \mathbf{u}_L,\mathbf{u}_R,\mathbf{v}_L,\mathbf{v}_R\in\mathbb{R}^n,\forall s\in[0,1]$,
$$\left|\frac{\partial\psi}{\partial s}(s;\mathbf{u}_L,\mathbf{u}_R)-\frac{\partial\psi}{\partial s}(s;\mathbf{v}_L,\mathbf{v}_R)\right|\leqslant k(|\mathbf{u}_L-\mathbf{u}_R|+|\mathbf{v}_L-\mathbf{v}_R|).$$

Then there exist a unique real-valued Borel measure, denoted $[A(\mathbf{u})\mathbf{u}_x]_\psi$, on $\mathbb{R}$ characterized by:

1. *If $\mathbf{u}$ is continuous on a Borel set, B, then,*
$$[A(\mathbf{u})\mathbf{u}_x]_\psi(B)=\int_B A(\mathbf{u})\mathbf{u}_x\,dx.$$

2. *If $\mathbf{u}$ is discontinuous at the point x_0 then,*
$$[A(\mathbf{u})\mathbf{u}_x]_\psi(\{x_0\})=\int_0^1 A(\psi(s;\mathbf{u}(x_0^-),\mathbf{u}(x_0^+)))\frac{\partial\psi}{\partial s}(s;\mathbf{u}(x_0^-),\mathbf{u}(x_0^+))\,ds.$$

This Borel measure is called the non-conservative product of $A(\mathbf{u})$ and $\mathbf{u}_x$.

Remark 2.1. Note that this non-conservative product is defined only for functions $\mathbf{u}$ which are piecewise differentiable and contain jump discontinuities, and it is therefore not defined for a general distribution. To use this product in the framework of NCHSs, we will regard the term, $A(\mathbf{u})\mathbf{u}_x$ as a non-conservative product which also depends on the variable t.

Remark 2.2. It is clear that in this framework, the non-conservative product defined above is dependent on the choice of path, ψ. The question of how one chooses such a family of paths is far from trivial. As noted in [21] and [1], when using this definition to design numerical schemes, poor choices in the family of paths can result in the scheme converging to the incorrect shock curves. In fact even an appropriate choice of paths implemented in a non-conservative scheme can lead to convergence to wrong solutions [1], [10] (and [21] to give elements of explanation for understanding this fundamental issue).

Using this non-conservative product, weak solutions are defined as follows.

Definition 2.1. We say that $\mathbf{u}$ is a weak solution of (1.1) if and only if

$$\int_{\mathbb{R}^+}\int_{\mathbb{R}} \mathbf{u}\phi_t + [A(\mathbf{u})\mathbf{u}_x]_\psi \phi \, dx dt = \mathbf{0},$$

as measures, for all test functions $\phi \in \mathbf{C}_0^1(\mathbb{R}\times\mathbb{R}^+)$.

Moreover, this weak formulation allows us to define a Rankine-Hugoniot Jump Condition as given by LeFloch in [23].

Theorem 2.2 (DLM Rankine-Hugoniot Jump Condition)**.** *Let ψ be the family of paths as in Theorem 2.1. Let $\mathbf{u}$ be a solution to* (1.1) *in the weak sense, with respect to this family of paths, and let $\mathbf{u}$ be smooth throughout a region D, except along a curve $x=\gamma(t)$ which divides D into two regions D_L and D_R, and along which $\mathbf{u}$ has a jump discontinuity. Then,*

$$\gamma'(t)(\mathbf{u}_R - \mathbf{u}_L) = \int_0^1 A(\psi(s;\mathbf{u}_L,\mathbf{u}_R))\frac{\partial \psi}{\partial s}(s;\mathbf{u}_L,\mathbf{u}_R)\, ds, \tag{2.1}$$

where,

$$\mathbf{u}_R(t) = \lim_{\substack{(x,t)\to(\gamma(t),t)\\ (x,t)\in D_R}} \mathbf{u}(x,t), \qquad \mathbf{u}_L(t) = \lim_{\substack{(x,t)\to(\gamma(t),t)\\ (x,t)\in D_L}} \mathbf{u}(x,t),$$

are the values of $\mathbf{u}$ at either side of the discontinuity.

Let us focus now on contact discontinuity curves. Suppose $\mathbf{u}_L$ and $\mathbf{u}_R$ are separated by a k-contact discontinuity. Then $\mathbf{u}_L$ and $\mathbf{u}_R$ lie on the same contact discontinuity curve $\mathbf{v}$. Then $\mathbf{v}$ solves

$$\begin{cases} \mathbf{v}'(\xi) = \mathbf{r}_k(\mathbf{v}(\xi)), \\ \mathbf{v}(0) = \mathbf{u}_L, \end{cases}$$

and furthermore

$$A(\mathbf{v}(\xi))\mathbf{r}_k(\mathbf{v}(\xi)) = \lambda_k(\mathbf{u}_L)\mathbf{r}_k(\mathbf{v}(\xi))$$

for all ξ, since λ_k remains constant along $\mathbf{v}(\xi)$. We can re-parametrize $\mathbf{v}$ so that $\mathbf{v}(1) = \mathbf{u}_R$. Then

$\psi(s;\mathbf{u}_L,\mathbf{u}_R)=\mathbf{v}(s)$ and we find

$$\begin{aligned}\int_0^1 A(\psi(s;\mathbf{u}_L,\mathbf{u}_R))\frac{\partial\psi}{\partial s}(s;\mathbf{u}_L,\mathbf{u}_R)\,ds &= \int_0^1 A(\mathbf{v}(s))\mathbf{v}'(s)\,ds,\\ &= \int_0^1 A(\mathbf{v}(s))\mathbf{r}_k(\mathbf{v}(s))\,ds,\\ &= \lambda_k(\mathbf{u}_L)\int_0^1 \mathbf{r}_k(\mathbf{v}(s))\,ds,\\ &= \lambda_k(\mathbf{u}_L)\int_0^1 \mathbf{v}'(s)\,ds,\\ &= \lambda_k(\mathbf{u}_L)(\mathbf{u}_R-\mathbf{u}_L).\end{aligned}$$

Hence, this discontinuity satisfies the DLM jump condition. Thus we can indeed use the definition of contact discontinuities as done in the conservative case. Furthermore, now that we have a jump condition we can define the shock curves.

Theorem 2.3 (DLM shock Curves)**.** *Suppose that the k-th field is genuinely nonlinear. Then given a left state $\mathbf{u}_L\in\Omega$ there exists a curve, $\mathcal{S}_k(\mathbf{u}_L)$, of right states that can be connected to $\mathbf{u}_L$ on the right by a k-shock wave.*

The proof of this theorem follows closely the arguments used to prove the existence of shock curves in the conservative case. See Chapter 1, Section 4, Theorem 4.1 in [19].

Using the definition of k-shock curves for genuinely nonlinear fields, we can then invoke entropy conditions to identify admissible shock waves. In particular we apply the Lax shock entropy condition to identify the admissible portion of the k-shock curves. Thus, given two states $\mathbf{u}_L$ and $\mathbf{u}_R$, sufficiently close, we can uniquely solve their Riemann problem as a composition of k-simple waves.

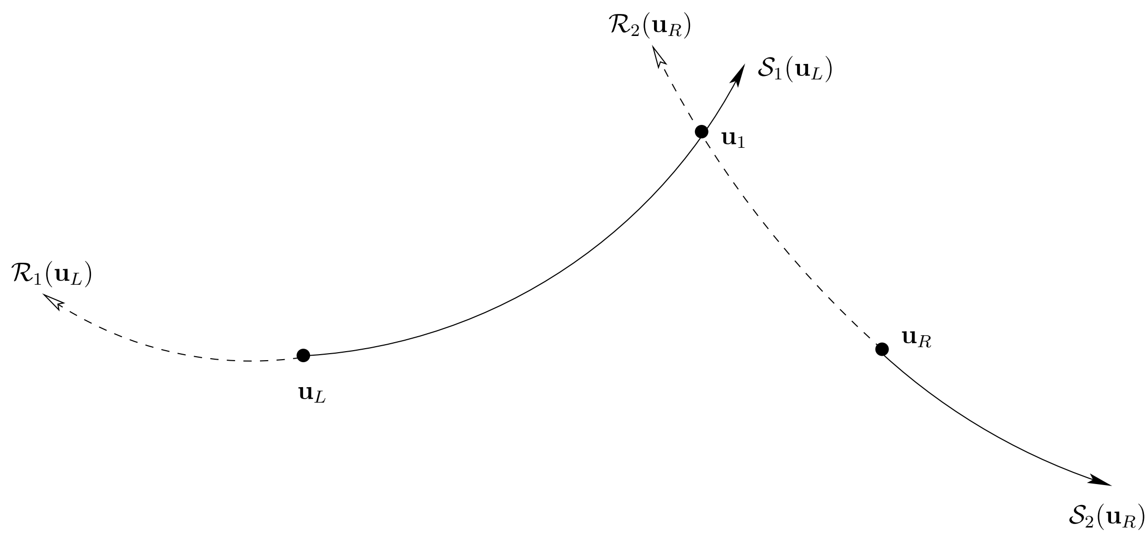

Figure 1: Example of shock and rarefaction curves in a 2×2 system. The entropy condition allows us to determine the admissible parts of the shock curves and we can then find the intersection to uniquely determine $\mathbf{u}_1$.

As we remarked earlier, when designing a numerical scheme, choosing the family of paths is not trivial and different choices of paths can lead to vastly different numerical solutions. It is clear that we need some way of determining what paths will give us physical, entropic solutions. To this end, let us consider the vanishing viscosity entropy condition and examine the viscous profiles. First, we introduce an admissible viscosity matrix $B(\mathbf{u})$ to the system.

$$\mathbf{u}_t^\varepsilon + A(\mathbf{u}^\varepsilon)\mathbf{u}_x^\varepsilon = \varepsilon(B(\mathbf{u}^\varepsilon)\mathbf{u}_x^\varepsilon)_x,$$

and we look for the viscous profiles $\mathbf{u}^\varepsilon(x,t) = \mathbf{v}(\frac{x-\sigma t}{\varepsilon})$. The resulting ODE is

$$(A(\mathbf{v}) - \sigma)\mathbf{v}' = (B(\mathbf{v})\mathbf{v}')'.$$

Next, let us suppose the vanishing viscosity limit of this viscous profile is a shock wave, i.e,

$$\lim_{\varepsilon\to 0}\mathbf{u}^\varepsilon(x,t) = \begin{cases} \mathbf{u}_L, & x<\sigma t, \\ \mathbf{u}_R, & x>\sigma t. \end{cases}$$

Then the viscous profile will have the form

$$\mathbf{u}^\varepsilon(x,t) = \begin{cases} \mathbf{u}_L, & x<\sigma t-\varepsilon, \\ \phi(\frac{x-\sigma t+\varepsilon}{2\varepsilon}), & \sigma t-\varepsilon \leqslant x \leqslant \sigma t+\varepsilon, \\ \mathbf{u}_R, & x>\sigma t+\varepsilon, \end{cases}$$

where ϕ is a smooth function with the properties $\phi(0) = \mathbf{u}_L$ and $\phi(1) = \mathbf{u}_R$. Considering $\mathbf{u}^\varepsilon$ as a measure we see that

$$\lim_{\varepsilon\to 0}[A(\mathbf{u}^\varepsilon)\mathbf{u}_x^\varepsilon] = \left(\int_0^1 A(\phi(s))\frac{\partial \phi}{\partial s}ds\right)\delta_{x-\sigma t},$$

with the convergence in the sense of measures. Thus, in order to obtain the vanishing viscosity solution, we choose our path, ψ to be precisely the viscous profile ϕ. A similar argument shows the same results when $\mathbf{u}^\varepsilon$ limits to a rarefaction wave or a contact discontinuity, or any composition of these simple waves. Notice that, as in the conservative case, the viscosity profiles will, in general, depend on the viscosity matrix, B. On the other hand, in the conservative case the shock curves are defined using only the Rankine-Hugoniot jump condition and thus, do not depend on the choice of B. Let us state all of these ideas formally.

Criterion 2.4 (Choice of Paths)**.** To obtain a vanishing viscosity entropic solution of the non-conservative system (1.1), we choose the family of paths, ψ, so that, $\psi(s;\mathbf{u}_L,\mathbf{u}_R)$ is a parametrization of the viscous profile connecting the states $\mathbf{u}_L$ and $\mathbf{u}_R$. This path will, a priori, depend on the viscosity matrix, $B(\mathbf{u})$.

Moreover (as proposed by LeFloch et. al. in [10]),

- If the k-th field is linearly degenerate, given the k-contact discontinuity curve $\mathcal{C}_k(\mathbf{u}_L)$, and given that $\mathbf{u}_R \in \mathcal{C}_k(\mathbf{u}_L)$, then the path $s \mapsto \psi(s;\mathbf{u}_L,\mathbf{u}_R)$ is a parametrization of the arc of $\mathcal{C}_k(\mathbf{u}_L)$ connecting $\mathbf{u}_L$ and $\mathbf{u}_R$.
- If the k-th field is genuinely nonlinear, given the k-rarefaction curve $\mathcal{R}_k(\mathbf{u}_L)$, and given that $\mathbf{u}_R \in \mathcal{R}_k(\mathbf{u}_L)$, then the path $s \mapsto \psi(s;\mathbf{u}_L,\mathbf{u}_R)$ is a parametrization of the arc of $\mathcal{R}_k(\mathbf{u}_L)$ connecting $\mathbf{u}_L$ and $\mathbf{u}_R$.

We make this choice in the construction of ψ because contact discontinuity and rarefaction curves are the same as in the conservative case, and are therefore not dependent on the choice of B.

2.2 Vanishing Viscosity Solutions of Bianchini and Bressan

In order to define solutions to general non-conservative systems, Bianchini and Bressan [7] consider a regularization of the system (1.1),

$$\mathbf{u}_t^\varepsilon + A(\mathbf{u}^\varepsilon)\mathbf{u}_x^\varepsilon = \varepsilon(B(\mathbf{u}^\varepsilon)\mathbf{u}_x^\varepsilon)_x. \tag{2.2}$$

More specifically, they consider the case where the viscosity matrix $B(\mathbf{u})$ is the identity matrix, I. The authors define solutions to (1.1) as the unique limits of solutions to this viscous system as $\varepsilon \to 0$. The details are well beyond the scope of this paper, but we will state their main results. For a very general $A(\mathbf{u})$ (no genuine nonlinearity assumptions, etc.), the authors solve the Riemann Problem for $\mathbf{u}_L$ and $\mathbf{u}_R$ sufficiently close and recover the classical succession of self-similar k-waves and characterize them.

The obvious shortcoming of this study is that vanishing viscosity solutions of this system for a more general viscosity matrix are not given by this theory. Moreover, the way in which the shock curves and viscous shock profiles are derived makes them very difficult in implement explicitly in a numerical scheme. Although the theory presented by Bianchini and Bressan is very interesting from the theoretical perspective, this difficulty prevents us from applying these results in numerical schemes.

Recently, in the paper by Alouges and Merlet [5] the results presented by Bianchini and Bressan are extended to the case where the admissible viscosity matrix $B(\mathbf{u})$ is assumed to commute with $A(\mathbf{u})$. The authors establish the same results as Bianchini and Bressan in this more general setting and also propose a new definition for shock curves in the non-conservative case. Before we state this definition let us present its motivation. Suppose for the moment that the system we are considering is in fact conservative and consider an admissible k-shock wave with left state $\mathbf{u}_L$, and right state $\mathbf{u}_R$, propagating with speed σ. If we consider $\mathbf{u}_R$ as a function of σ, the Rankine-Hugoniot jump condition writes

$$\mathbf{f}(\mathbf{u}_R(\sigma)) - \mathbf{f}(\mathbf{u}_L) = \sigma(\mathbf{u}_R(\sigma) - \mathbf{u}_L),$$

with $\mathbf{u}_R(\lambda_k(\mathbf{u}_L)) = \mathbf{u}_L$. Differentiating with respect to σ yields

$$\begin{cases} (A(\mathbf{u}_R) - \sigma I)\dfrac{d\mathbf{u}_R}{d\sigma} = \mathbf{u}_R - \mathbf{u}_L, \\ \mathbf{u}_R(\lambda_k(\mathbf{u}_L)) = \mathbf{u}_L. \end{cases} \tag{2.3}$$

Alouges and Merlet use this system to define an *approximate shock curve*.

Definition 2.2 (Alouges-Merlet Shock Curves [5])**.** A non-constant solution of (2.3) is called an approximate shock curve of the non-conservative system (1.1).

Notice that this differential equation is not classical since there is a degeneracy at the initial point $\mathbf{u}_R(\lambda_k(\mathbf{u}_L)) = \mathbf{u}_L$, for each k. To overcome this, the authors prove the following result:

Proposition 2.1. Suppose that the k-th field is genuinely nonlinear. Then equation (2.3) has a unique, non-trivial solution in the neighborhood of $\lambda_k(\mathbf{u}_L)$. Moreover, the non-trivial solution satisfies

$$\mathbf{u}_R(\sigma) = \mathbf{u}_L + \frac{2(\sigma - \lambda_k(\mathbf{u}_L))}{\nabla\lambda_k(\mathbf{u}_L)\cdot\mathbf{r}_k(\mathbf{u}_L)}\mathbf{r}_k(\mathbf{u}_L) + O(|\sigma - \lambda_k(\mathbf{u}_L)|^2).$$

The degeneracy is then overcome by adding the initial condition

$$\frac{d\mathbf{u}_R}{d\sigma}(\lambda_k(\mathbf{u}_L)) = \frac{2}{\nabla\lambda_k(\mathbf{u}_L)\cdot\mathbf{r}_k(\mathbf{u}_L)}\mathbf{r}_k(\mathbf{u}_L),$$

to the differential equation. In fact, we can extend the above result to include the second order terms which gives a more precise description of these shock curves.

Proposition 2.2. For a genuinely nonlinear k-field, the unique and non-trivial solution to (2.3) satisfies:

$$\mathbf{u}_R(\sigma)=\mathbf{u}_L+\frac{2(\sigma-\lambda_k(\mathbf{u}_L))}{\nabla\lambda_k(\mathbf{u}_L)\cdot\mathbf{r}_k(\mathbf{u}_L)}\mathbf{r}_k(\mathbf{u}_L)+\frac{4(\sigma-\lambda_k(\mathbf{u}_L))^2}{(\nabla\lambda_k(\mathbf{u}_L)\cdot\mathbf{r}_k(\mathbf{u}_L))^2}D\mathbf{r}_k(\mathbf{u}_L)\cdot\mathbf{r}_k(\mathbf{u}_L)$$
$$+O(|\sigma-\lambda_k(\mathbf{u}_L)|^3).$$

Proof. Let us expand $\mathbf{u}_R(\sigma)$ as

$$\mathbf{u}_R(\sigma)=\mathbf{u}_L+\frac{2(\sigma-\lambda_k(\mathbf{u}_L))}{\nabla\lambda_k(\mathbf{u}_L)\cdot\mathbf{r}_k(\mathbf{u}_L)}\mathbf{r}_k(\mathbf{u}_L)+\frac{1}{2}R(\mathbf{u}_L)(\sigma-\lambda_k(\mathbf{u}_L))^2+O(|\sigma-\lambda_k(\mathbf{u}_L)|^3),$$

where $R(\mathbf{u}_L)=\frac{d^2\mathbf{u}_R}{d\sigma^2}(\mathbf{u}_L)$ is to be determined. Using this expression, we can expand $A(\mathbf{u}_R(\sigma))$ around $\sigma=\lambda_k(\mathbf{u}_L)$ to obtain:

$$A(\mathbf{u}_R(\sigma))=A(\mathbf{u}_L)+\frac{2(\sigma-\lambda_k(\mathbf{u}_L))}{\nabla\lambda_k(\mathbf{u}_L)\cdot\mathbf{r}_k(\mathbf{u}_L)}DA(\mathbf{u}_L)\cdot\mathbf{r}_k(\mathbf{u}_L)+O(|\sigma-\lambda_k(\mathbf{u}_L)|^2).$$

Let us next perform the same expansion around $\sigma=\lambda_k(\mathbf{u}_L)$ in system (2.3) and use the above expressions to obtain,

$$\Bigg[A(\mathbf{u}_L)+\frac{2(\sigma-\lambda_k(\mathbf{u}_L))}{\nabla\lambda_k(\mathbf{u}_L)\cdot\mathbf{r}_k(\mathbf{u}_L)}DA(\mathbf{u}_L)\cdot\mathbf{r}_k(\mathbf{u}_L)+O(|\sigma-\lambda_k(\mathbf{u}_L)|^2)-\lambda_k(\mathbf{u}_L)I$$
$$-(\sigma-\lambda_k(\mathbf{u}_L))I\Bigg]\left(\frac{2(\sigma-\lambda_k(\mathbf{u}_L))}{\nabla\lambda_k(\mathbf{u}_L)\cdot\mathbf{r}_k(\mathbf{u}_L)}\mathbf{r}_k(\mathbf{u}_L)+R(\mathbf{u}_L)+O(|\sigma-\lambda_k(\mathbf{u}_L)|^2)\right)$$
$$=\frac{2}{\nabla\lambda_k(\mathbf{u}_L)\cdot\mathbf{r}_k(\mathbf{u}_L)}\mathbf{r}_k(\mathbf{u}_L)+O(|\sigma-\lambda_k(\mathbf{u}_L)|^2).$$

Expanding and re-arranging, we obtain:

$$\frac{2}{\nabla\lambda_k(\mathbf{u}_L)\cdot\mathbf{r}_k(\mathbf{u}_L)}[A(\mathbf{u}_L)-\lambda_k(\mathbf{u}_L)I]\mathbf{r}_k(\mathbf{u}_L)+\left[\frac{2}{\nabla\lambda_k(\mathbf{u}_L)\cdot\mathbf{r}_k(\mathbf{u}_L)}DA(\mathbf{u}_L)\cdot\mathbf{r}_k(\mathbf{u}_L)-2I\right]$$
$$\cdot\frac{2(\sigma-\lambda_k(\mathbf{u}_L))}{\nabla\lambda_k(\mathbf{u}_L)\cdot\mathbf{r}_k(\mathbf{u}_L)}\mathbf{r}_k(\mathbf{u}_L)+[A(\mathbf{u}_L)-\lambda_k(\mathbf{u}_L)I]R(\mathbf{u}_L)(\sigma-\lambda_k(\mathbf{u}_L))$$
$$+O(|\sigma-\lambda_k(\mathbf{u}_L)|^2)=0.$$

To satisfy this equation, the zeroth and first order terms must vanish. It is clear that the first term on the left (the zeroth order term) vanishes since $\mathbf{r}_k(\mathbf{u}_L)$ is an eigenvector of $A(\mathbf{u}_L)$ associated to the eigenvalue $\lambda_k(\mathbf{u}_L)$. For the first order terms to vanish, we must have that

$$\frac{2}{\nabla\lambda_k(\mathbf{u}_L)\cdot\mathbf{r}_k(\mathbf{u}_L)}\left[\frac{2}{\nabla\lambda_k(\mathbf{u}_L)\cdot\mathbf{r}_k(\mathbf{u}_L)}DA(\mathbf{u}_L)\cdot\mathbf{r}_k(\mathbf{u}_L)-2I\right]\mathbf{r}_k(\mathbf{u}_L)+[A(\mathbf{u}_L)-\lambda_k(\mathbf{u}_L)I]R(\mathbf{u}_L)=0. \tag{2.4}$$

In order to determine $R(\mathbf{u}_L)$, let us consider the identity

$$[A(\mathbf{u}_R(\sigma))-\lambda_k(\mathbf{u}_R(\sigma))I]\mathbf{r}_k(\mathbf{u}_R(\sigma))=0.$$

Differentiating this with respect to σ yields:

$$\left[DA(\mathbf{u}_R(\sigma))\frac{d\mathbf{u}_R}{d\sigma}-\left(\nabla\lambda_k(\mathbf{u}_R(\sigma))\cdot\frac{d\mathbf{u}_R}{d\sigma}\right)I\right]\mathbf{r}_k(\mathbf{u}_R(\sigma))$$
$$+[A(\mathbf{u}_R(\sigma))-\lambda_k(\mathbf{u}_R(\sigma))I]D\mathbf{r}_k(\mathbf{u}_R(\sigma))\cdot\frac{d\mathbf{u}_R}{d\sigma}=0,$$

and evaluating it at $\sigma=\lambda_k(\mathbf{u}_L)$, we obtain:

$$\left[\frac{2}{\nabla\lambda_k(\mathbf{u}_L)\cdot\mathbf{r}_k(\mathbf{u}_L)}DA(\mathbf{u}_L)\mathbf{r}_k(\mathbf{u}_L)-2I\right]\mathbf{r}_k(\mathbf{u}_L)$$
$$+\frac{2}{\nabla\lambda_k(\mathbf{u}_L)\cdot\mathbf{r}_k(\mathbf{u}_L)}[A(\mathbf{u}_L)-\lambda_k(\mathbf{u}_L)I]D\mathbf{r}_k(\mathbf{u}_L)\cdot\mathbf{r}_k(\mathbf{u}_L)=0.$$

Comparing this expression with (2.4), we find that

$$R(\mathbf{u}_L)=\frac{d^2\mathbf{u}_R}{d\sigma^2}(\mathbf{u}_L)=\frac{4}{(\nabla\lambda_k(\mathbf{u}_L)\cdot\mathbf{r}_k(\mathbf{u}_L))^2}D\mathbf{r}_k(\mathbf{u}_L)\cdot\mathbf{r}_k(\mathbf{u}_L).$$

which completes the proof.

□

It is clear that when A is a Jacobian matrix, this definition will recover the correct shock curves. Alouges and Merlet prove that these approximate shock curves agree with the ones found by the vanishing viscosity process of Bianchini and Bressan up to the third order near a given left state. Thus (2.3) gives us a simple way to approximate the shock curves as described by Bianchini and Bressan, that is this gives us a way to approximate 'true' vanishing viscosity solutions of the Riemann problem in the non-conservative case, which are very complex to implement numerically [5]. Moreover, since (2.3) is independent of $B(\mathbf{u})$, the approximate solutions are also B-independent.

Another point of interest is that these approximate shock curves coincide with the viscous shock profiles to the third order near a given left state. To see this, let us consider the viscous system (2.2) and let us examine the k-shock profiles, which have the form $\mathbf{u}^\varepsilon(x,t)=U(\frac{x-\sigma t}{\varepsilon};\sigma)=U(\xi;\sigma)$. Then these shock profiles will solve the following system, $\forall\sigma$

$$\begin{cases}(A(U)-\sigma I)U_\xi=(B(U)U_\xi)_\xi,\\ U(-\infty;\sigma)=\mathbf{u}_L,\\ U(\xi;\lambda_k(\mathbf{u}_L))\equiv\mathbf{u}_L.\end{cases}$$

The k-shock curve, $\mathcal{S}_k(\mathbf{u}_L)$, is defined by these profiles by $\mathbf{u}_R(\sigma)=U(+\infty;\sigma)$. Integrating this system along the profiles gives

$$\sigma(\mathbf{u}_R(\sigma)-\mathbf{u}_L)=\int_{\mathbb{R}}A(U)U_\xi\,d\xi,$$

and differentiating with respect to σ and integrating by parts we obtain

$$(A(\mathbf{u}_R(\sigma))-\sigma I)\frac{d\mathbf{u}_R}{d\sigma}=\mathbf{u}_R(\sigma)-\mathbf{u}_L+\int_{\mathbb{R}}A(U)_\xi U_\sigma-A(U)_\sigma U_\xi\,d\xi.$$

So that we recover system (2.3) up to the term

$$R(U,\sigma)=\int_{\mathbb{R}}A(U)_\xi U_\sigma-A(U)_\sigma U_\xi\,d\xi.$$

Now we note that if A is indeed a Jacobian matrix then $R(U,\sigma)$ vanishes. Also, if the shock curves and the shock profiles coincide then $R(U,\sigma)$ will again vanish. As noted earlier, approximate shock curves defined by (2.3) do indeed recover the correct shock curves up to the third order so $R(U,\sigma)=O(|\sigma-\lambda_k(\mathbf{u}_L)|^3)$. This tells us that these approximate shock curves are in fact also close to viscous shock profiles. This result is interesting since it gives us a way to approximate the viscous shock profiles which, as explained in the previous section, is a piece of information useful to solve the Riemann problem using the DLM path-theory.

2.2.1 Reversibility of Alouges-Merlet Approximate Shock Curves

Recall that in the conservative case, when a state $\mathbf{u}_R$ lies on a k-shock curve of a state $\mathbf{u}_L$, i.e. $\mathbf{u}_R \in \mathcal{S}_k(\mathbf{u}_L)$, then from the Rankine-Hugoniot jump condition we know immediately that $\mathbf{u}_L$ will lie on the k-shock curve of the state $\mathbf{u}_R$. In their paper, Alouges and Merlet point out that it is unclear whether this property will hold in the non-conservative case using the approximate shock curves. Here, we establish the following result:

Theorem 2.5. *Let $\mathcal{S}_k(\mathbf{u}_L)$ be the approximate k-Shock Curve of the state $\mathbf{u}_L$ for the system* (1.1). *Suppose that $\mathbf{u}_R \in \mathcal{S}_k(\mathbf{u}_L)$, that is, suppose that a state $\mathbf{u}_R$ can be connected to $\mathbf{u}_L$ on the right by a k-shock wave traveling with speed σ. Furthermore, let $\tilde{\mathbf{u}}_L(\sigma) \in \mathcal{S}_k(\mathbf{u}_R)$ be the state which can be connected to $\mathbf{u}_R$* on the left *by a k-shock wave, again traveling with speed σ. Then either $\tilde{\mathbf{u}}_L(\sigma) \equiv \mathbf{u}_L$, $\forall \sigma$ or $\tilde{\mathbf{u}}_L(\sigma) = \mathbf{u}_L + O(|\sigma-\lambda(\mathbf{u}_L)|)$.*

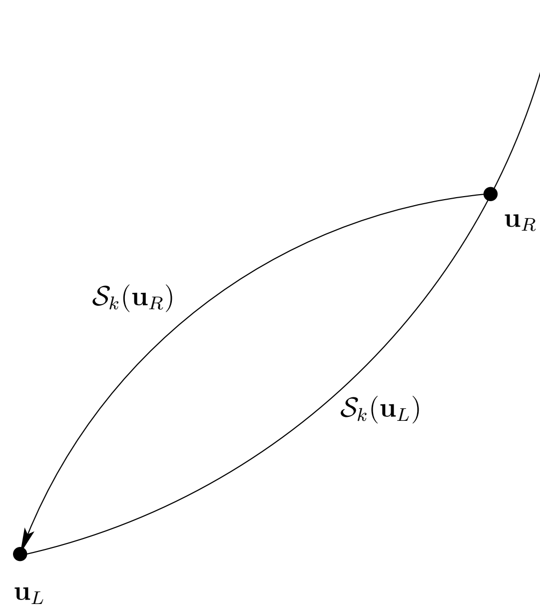

Figure 2: Reversibility of approximate shock curves of Alouges and Merlet: if the state $\mathbf{u}_R$ lies on the k-shock curve of $\mathbf{u}_L$ then $\mathbf{u}_L$ lies on the k-shock curve of $\mathbf{u}_R$.

Proof. Let us consider a genuinely nonlinear k-th field and consider the approximate k-shock curve of a fixed left state. Let us denote this curve by $\mathbf{u}_1(\xi)$. Then $\mathbf{u}_1(\xi)$ satisfies

$$\begin{cases} (A(\mathbf{u}_1)-\xi I)\dfrac{d\mathbf{u}_1}{d\xi} = \mathbf{u}_1 - \mathbf{u}_L, \\ \mathbf{u}_1(\lambda_k(\mathbf{u}_L)) = \mathbf{u}_L. \end{cases} \tag{2.5}$$

Let us select a point on this curve, say $\mathbf{u}_1(\sigma)$, for some σ. We wish to determine the state $\tilde{\mathbf{u}}_L(\sigma)$ which will lie on the k-shock curve of $\mathbf{u}_1(\sigma)$, that we denote by $\mathbf{u}_2(\tau;\sigma)$. Then $\mathbf{u}_2(\tau;\sigma)$ satisfies

$$\begin{cases} (A(\mathbf{u}_2)-\tau I)\dfrac{\partial \mathbf{u}_2}{\partial \tau}=\mathbf{u}_2-\mathbf{u}_1(\sigma), \\ \mathbf{u}_2(\lambda_k(\mathbf{u}_1(\sigma));\sigma)=\mathbf{u}_1(\sigma). \end{cases} \tag{2.6}$$

In Figure 3, we have depicted the construction of these curves, $\mathbf{u}_1(\xi)$ and $\mathbf{u}_2(\tau;\sigma)$. Let us

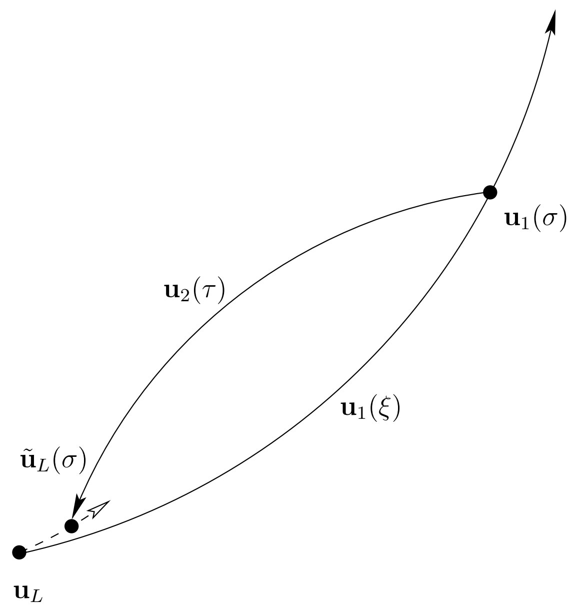

Figure 3: Graphical depiction of the k-shock curve of the left state $\mathbf{u}_L$ and the k-shock curve of the right state $\mathbf{u}_1(\tilde{\sigma})$.

integrate (2.5) from $\lambda_k(\mathbf{u}_L)$ to σ to obtain

$$\begin{aligned} \int_{\lambda_k(\mathbf{u}_L)}^{\sigma} (A(\mathbf{u}_1)-\xi I)\frac{d\mathbf{u}_1}{d\xi}d\xi &= \int_{\lambda_k(\mathbf{u}_L)}^{\sigma} \mathbf{u}_1(\xi)-\mathbf{u}_L\, d\xi, \\ \int_{\lambda_k(\mathbf{u}_L)}^{\sigma} A(\mathbf{u}_1)\frac{d\mathbf{u}_1}{d\xi}d\xi &= \int_{\lambda_k(\mathbf{u}_L)}^{\sigma} \xi\frac{d\mathbf{u}_1}{d\xi}+\mathbf{u}_1(\xi)-\mathbf{u}_L\, d\xi, \\ \int_{\lambda_k(\mathbf{u}_L)}^{\sigma} A(\mathbf{u}_1)\frac{d\mathbf{u}_1}{d\xi}d\xi &= \sigma(\mathbf{u}_1(\sigma)-\mathbf{u}_L). \end{aligned}$$

Similarly, we integrate (2.6) from $\lambda_k(\mathbf{u}_1(\sigma))$ to σ to obtain

$$\int_{\lambda_k(\mathbf{u}_1(\sigma))}^{\sigma} A(\mathbf{u}_2)\frac{\partial \mathbf{u}_2}{\partial \tau}d\tau=\sigma(\mathbf{u}_2(\sigma;\sigma)-\mathbf{u}_1(\sigma)).$$

The point we are interested in is $\tilde{\mathbf{u}}_L(\sigma)=\mathbf{u}_2(\sigma;\sigma)$. Adding these two equations, we obtain

$$\int_{\lambda_k(\mathbf{u}_L)}^{\sigma} A(\mathbf{u}_1)\frac{d\mathbf{u}_1}{d\xi}d\xi+\int_{\lambda_k(\mathbf{u}_1(\sigma))}^{\sigma} A(\mathbf{u}_2)\frac{\partial \mathbf{u}_2}{\partial \tau}d\tau=\sigma(\tilde{\mathbf{u}}_L(\sigma)-\mathbf{u}_L).$$

Note that this entire expression depends on the parameter σ. Let us differentiate this equation with respect to σ, to obtain

$$A(\mathbf{u}_1(\sigma))\frac{d\mathbf{u}_1}{d\xi}(\sigma)+A(\mathbf{u}_2(\sigma;\sigma))\frac{\partial\mathbf{u}_2}{\partial\tau}(\sigma;\sigma)-$$
$$A(\mathbf{u}_2(\lambda_k(\mathbf{u}_1(\sigma));\sigma))\frac{\partial\mathbf{u}_2}{\partial\tau}(\lambda_k(\mathbf{u}_1(\sigma));\sigma)\left(\nabla\lambda_k(\mathbf{u}_1(\sigma))\cdot\frac{d\mathbf{u}_1}{d\xi}(\sigma)\right)+$$
$$\int_{\lambda_k(\mathbf{u}_1(\sigma))}^{\sigma}\frac{d}{d\sigma}\left[A(\mathbf{u}_2)\frac{\partial\mathbf{u}_2}{\partial\tau}\right]d\tau=\tilde{\mathbf{u}}_L(\sigma)-\mathbf{u}_L+\sigma\frac{d\tilde{\mathbf{u}}_L}{d\sigma}(\sigma).$$

Using the initial condition $\mathbf{u}_2(\lambda_k(\mathbf{u}_1(\sigma));\sigma)=\mathbf{u}_1(\sigma)$ and the fact that $\tilde{\mathbf{u}}_L(\sigma)=\mathbf{u}_2(\sigma;\sigma)$, this reads:

$$A(\mathbf{u}_1(\sigma))\frac{d\mathbf{u}_1}{d\xi}(\sigma)+A(\tilde{\mathbf{u}}_L(\sigma))\frac{\partial\mathbf{u}_2}{\partial\tau}(\sigma;\sigma)-$$
$$A(\mathbf{u}_1(\sigma))\frac{\partial\mathbf{u}_2}{\partial\tau}(\lambda_k(\mathbf{u}_1(\sigma));\sigma)\cdot\left(\nabla\lambda_k(\mathbf{u}_1(\sigma))\cdot\frac{d\mathbf{u}_1}{d\xi}(\sigma)\right)+$$
$$\int_{\lambda_k(\mathbf{u}_1(\sigma))}^{\sigma}\frac{d}{d\sigma}\left[A(\mathbf{u}_2(\tau;\sigma))\frac{\partial\mathbf{u}_2}{\partial\tau}\right]d\tau=\tilde{\mathbf{u}}_L(\sigma)-\mathbf{u}_L+\sigma\frac{d\tilde{\mathbf{u}}_L}{d\sigma}(\sigma). \quad (2.7)$$

Let us examine the integral term in this expression. Expanding and integrating by parts we obtain

$$\begin{aligned}\int_{\lambda_k(\mathbf{u}_1(\sigma))}^{\sigma}\frac{d}{d\sigma}\left[A(\mathbf{u}_2)\frac{\partial\mathbf{u}_2}{\partial\tau}\right]d\tau&=\int_{\lambda_k(\mathbf{u}_1(\sigma))}^{\sigma}A(\mathbf{u}_2)_\sigma\frac{\partial\mathbf{u}_2}{\partial\tau}+A(\mathbf{u}_2)\frac{\partial^2\mathbf{u}_2}{\partial\tau\partial\sigma}\,d\tau\\&=A(\mathbf{u}_2(\sigma;\sigma))\frac{\partial\mathbf{u}_2}{\partial\sigma}(\sigma;\sigma)-A(\mathbf{u}_2(\lambda_k(\mathbf{u}_1(\sigma));\sigma))\frac{\partial\mathbf{u}_2}{\partial\sigma}(\lambda_k(\mathbf{u}_1(\sigma));\sigma)\\&\qquad+\int_{\lambda_k(\mathbf{u}_1(\sigma))}^{\sigma}A(\mathbf{u}_2)_\sigma\frac{\partial\mathbf{u}_2}{\partial\tau}-A(\mathbf{u}_2)_\tau\frac{\partial\mathbf{u}_2}{\partial\sigma}d\tau\\&=A(\tilde{\mathbf{u}}_L(\sigma)\frac{\partial\mathbf{u}_2}{\partial\sigma}(\sigma;\sigma)-A(\mathbf{u}_1(\sigma))\frac{\partial\mathbf{u}_2}{\partial\sigma}(\lambda_k(\mathbf{u}_1(\sigma));\sigma)+R(\mathbf{u}_2,\sigma).\end{aligned}$$

Here we used the notation $R(\mathbf{u}_2,\sigma)=\int_{\lambda_k(\mathbf{u}_1(\sigma))}^{\sigma}A(\mathbf{u}_2)_\sigma\frac{\partial\mathbf{u}_2}{\partial\tau}-A(\mathbf{u}_2)_\tau\frac{\partial\mathbf{u}_2}{\partial\sigma}d\tau$. Inserting this expression into (2.7), we obtain

$$A(\tilde{\mathbf{u}}_L(\sigma))\left(\frac{\partial\mathbf{u}_2}{\partial\tau}(\sigma;\sigma)+\frac{\partial\mathbf{u}_2}{\partial\sigma}(\sigma;\sigma)\right)+A(\mathbf{u}_1(\sigma))\left[\frac{d\mathbf{u}_1}{d\xi}(\sigma)-\frac{\partial\mathbf{u}_2}{\partial\tau}(\lambda_k(\mathbf{u}_1(\sigma));\sigma)\right.$$
$$\left.\cdot\left(\nabla\lambda_k(\mathbf{u}_1(\sigma))\cdot\frac{d\mathbf{u}_1}{d\xi}(\sigma)\right)-\frac{\partial\mathbf{u}_2}{\partial\sigma}(\lambda_k(\mathbf{u}_1(\sigma));\sigma)\right]+R(\mathbf{u}_2,\sigma)=\tilde{\mathbf{u}}_L(\sigma)-\mathbf{u}_L+\sigma\frac{d\tilde{\mathbf{u}}_L}{d\sigma}(\sigma).$$

Finally, note that $\frac{d\tilde{\mathbf{u}}_L}{d\sigma}=\frac{\partial\mathbf{u}_2}{\partial\tau}(\sigma;\sigma)+\frac{\partial\mathbf{u}_2}{\partial\sigma}(\sigma;\sigma)$ and notice that the term within the square braces is simply the derivative of the initial condition in (2.6). Thus, this term vanishes and we obtain

$$(A(\tilde{\mathbf{u}}_L)-\sigma I)\frac{d\tilde{\mathbf{u}}_L}{d\sigma}+R(\mathbf{u}_2,\sigma)=\tilde{\mathbf{u}}_L(\sigma)-\mathbf{u}_L. \quad (2.8)$$

This system is similar to the Shock Curve system proposed by Alouges and Merlet, the difference being the additional term $R(\mathbf{u}_2,\sigma)$. Since clearly $R(\mathbf{u}_2,\sigma)$ is $O(|\sigma-\lambda_k(\mathbf{u}_L)|)$ we can apply the existence and uniqueness result for these systems established by Alouges and Merlet in their paper to conclude that either $\tilde{\mathbf{u}}_L(\sigma)\equiv\mathbf{u}_L$, $\forall\sigma$ or $\tilde{\mathbf{u}}_L(\sigma)$ is a smooth curve with $\tilde{\mathbf{u}}_L(\sigma)=\mathbf{u}_L+O(|\sigma-\lambda_k(\mathbf{u}_L)|)$.

□

Note that if $R(\mathbf{u}_2,\sigma)\equiv 0$, which is guaranteed if $A(\mathbf{u})$ is a Jacobian matrix, then (2.8) is the defining system for the k-shock curve at $\mathbf{u}_L$. By the existence and uniqueness results of Alouges and Merlet we can conclude that either $\tilde{\mathbf{u}}_L(\sigma)\equiv\mathbf{u}_L$, $\forall\sigma$ or $\tilde{\mathbf{u}}_L(\sigma)$ coincides completely with the k-shock curve of $\mathbf{u}_L$. It is clear that the latter case is not possible since by construction $\mathbf{u}_2(\sigma,\sigma)$ cannot coincide with the point $\mathbf{u}_1(\sigma)$, so we must have that $\tilde{\mathbf{u}}_L(\sigma)\equiv\mathbf{u}_L$.

Remark 2.3. Although the proof states that it is possible to have $\tilde{\mathbf{u}}_L(\sigma)\not\equiv\mathbf{u}_L$, we are not able to provide an example of a non-conservative system for which this occurs. For the non-conservative system considered below the Alouges-Merlet Shock Curves seem reversible, i.e. $\tilde{\mathbf{u}}_L(\sigma)\equiv\mathbf{u}_L$.

3 Design of Numerical Schemes

We outline the design of some numerical schemes for approximating non-conservative hyperbolic systems using the above analysis. A natural first choice of scheme is a Godunov-like scheme which utilizes an "exact" Riemann solver. The first scheme we propose is a Godunov scheme which utilizes the approximate Shock curves defined by Alouges and Merlet. Then, we describe a Godunov scheme which utilizes Dal Maso, LeFloch, and Murat's path-theory in combination with Alouges and Merlet's approximate shock curves.

3.1 Alouges-Merlet Shock Curve-based Godunov Scheme

Before describing the Godunov scheme for non-conservative systems, let us quickly recall its principle for HSCLs, as it will be useful to justify and to understand its non-conservative versions. We first discretize the space-time domain by a grid of points (x_j,t_n), $j\in\mathbb{Z}$, $n\in\mathbb{Z}^+$, with uniform spatial step, Δx and a non-uniform time step Δt_n. The Godunov scheme is a *finite volume* scheme, that is we approximate the solution of the system by a piecewise constant function U, defined by

$$U(x,t)=U_j^n \qquad \text{for } x\in(x_{j-\frac{1}{2}},x_{j+\frac{1}{2}}),t\in(t_n,t_{n+1}),$$

where the $x_{j\pm 1/2}$ are the cell interface positions, i.e. $x_{j\pm1/2}=x_j\pm\frac{\Delta x}{2}$. The values U_j^n are the averages of the exact solution in the cell $(x_{j-1/2},x_{j+1/2})$, defined by

$$U_j^n=\frac{1}{\Delta x}\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}}\mathbf{u}(x,t_n)\,dx,$$

where $\mathbf{u}$ is the exact solution to the system. Now suppose that at time t_n, we are given an approximation, V_j^n, of these averages, U_j^n. Then the Godunov scheme is constructed as follows: first we solve exactly for all $j\in\mathbb{Z}$ the problem

$$\mathbf{u}_t+\mathbf{f}(\mathbf{u})_x=\mathbf{0}, \tag{3.1}$$

$$\mathbf{u}(x,0)=V_j^n, \quad x\in(x_{j-\frac{1}{2}},x_{j+\frac{1}{2}}).$$

This initial profile is just a superposition of Riemann problems, each of which has an entropic solution, $\mathbf{w}_{j+1/2}\left(\frac{x-x_{j+1/2}}{t};V_j^n,V_{j+1}^n\right)$, at the interface of the j-th and $(j+1)$-th cell. Moreover, these local solutions will not interact for Δt_n small enough. Specifically, we enforce a C.F.L. condition

$$\frac{\Delta t_n}{\Delta x}\max_{k,j}|\lambda_k(V_j^n)|\leqslant\frac{1}{2}.$$

We integrate system (3.1) over $(x_{j-1/2},x_{j+1/2})\times(0,\Delta t_n)$ to obtain

$$\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}}\mathbf{u}(x,\Delta t_n)-\mathbf{u}(x,0)\,dx+\int_0^{\Delta t_n}\mathbf{f}(\mathbf{u}(x_{j+\frac{1}{2}},t))-\mathbf{f}(\mathbf{u}(x_{j-\frac{1}{2}},t))\,dt=\mathbf{0}.$$

Using the definition of the cell averages, and the fact that the solution $\mathbf{u}(x,t)$ is a superposition of solutions to the Riemann problems at the $(j-\frac{1}{2})$-th and $(j+\frac{1}{2})$-th interfaces, we can write this as

$$\begin{aligned}\Delta x(V_j^{n+1}-V_j^n)+\int_0^{\Delta t_n}&\left[\mathbf{f}\left(\mathbf{w}_{j+\frac{1}{2}}(0;V_j^n,V_{j+1}^n)\right)-\mathbf{f}(V_j^n)\right]\\&+\left[\mathbf{f}(V_j^n)-\mathbf{f}\left(\mathbf{w}_{j-\frac{1}{2}}(0;V_{j-1}^n,V_j^n)\right)\right]dt=\mathbf{0}.\end{aligned}$$

Note that $\left(\mathbf{f}(V_j^n)\right)_j$ are added for convenience. We can write this as

$$V_j^{n+1}=V_j^n-\frac{\Delta t}{\Delta x}\left(G^{n,+}_{j-\frac{1}{2}}+G^{n,-}_{j+\frac{1}{2}}\right),\tag{3.2}$$

where

$$\begin{aligned}G^{n,-}_{j+\frac{1}{2}}&=\mathbf{f}(V^n_{j+\frac{1}{2}})-\mathbf{f}(V_j^n),\\G^{n,+}_{j-\frac{1}{2}}&=\mathbf{f}(V_j^n)-\mathbf{f}(V^n_{j-\frac{1}{2}}),\end{aligned}$$

and $V^n_{j+1/2}=\mathbf{w}_{j+1/2}(0;V_j^n,V_{j+1}^n)$ and $V^n_{j-1/2}=\mathbf{w}_{j-1/2}(0;V_{j-1}^n,V_j^n)$ are the values at the interfaces $x=x_{j\pm1/2}$. Naturally in this case the Godunov scheme is conservative.

Let us now develop the Godunov scheme in the non-conservative case. Clearly we cannot derive such a concise formulation of this scheme since no such function $\mathbf{f}(\mathbf{u})$ exists. Suppose again that at time t_n we are given an approximation, V_j^n, of the cell averages, U_j^n. Then we solve exactly for all $j\in\mathbb{Z}$ the problem

$$\mathbf{u}_t+A(\mathbf{u})\mathbf{u}_x=\mathbf{0},\tag{3.3}$$

$$\mathbf{u}(x,0)=V_j^n,\qquad x\in(x_{j-\frac{1}{2}},x_{j+\frac{1}{2}}).$$

In order to find unique, entropic, solutions for this superposition of Riemann Problems we make use of the Alouges-Merlet shock curves described in the previous section. At the interfaces, we can then determine an entropic solution, $\mathbf{w}_{j+1/2}\left(\frac{x-x_{j+1/2}}{t};V_j^n,V_{j+1}^n\right)$, to the Riemann Problem. Using this exact solution we can update the approximations, V_j^n, by

$$V_j^{n+1}=\int_{x_{j-\frac{1}{2}}}^{x_j}\mathbf{w}_{j-\frac{1}{2}}\left(\frac{x-x_{j-\frac{1}{2}}}{\Delta t_n};V_{j-1}^n,V_j^n\right)dx-\int_{x_j}^{x_{j+\frac{1}{2}}}\mathbf{w}_{j+\frac{1}{2}}\left(\frac{x-x_{j+\frac{1}{2}}}{\Delta t_n};V_j^n,V_{j+1}^n\right)dx.$$

This scheme can be rewritten in the form (3.2) where

$$G^{n,-}_{j+\frac{1}{2}}=\frac{1}{\Delta t_n}\left(\int_{x_j}^{x_{j+\frac{1}{2}}}\left[\mathbf{w}_{j+\frac{1}{2}}\left(\frac{x-x_{j+\frac{1}{2}}}{\Delta t_n};V_j^n,V_{j+1}^n\right)-V_j^n\right]dx\right),$$

$$G^{n,+}_{j-\frac{1}{2}} = \frac{1}{\Delta t_n}\left(\int_{x_{j-\frac{1}{2}}}^{x_j}\left[V_j^n - \mathbf{w}_{j-\frac{1}{2}}\left(\frac{x-x_{j-\frac{1}{2}}}{\Delta t_n};V_{j-1}^n,V_j^n\right)\right]dx\right).$$

In order to give more explicit formulas of the interfacial fluxes let us consider now 2×2 systems. However in principle what follows is still valid for general $n\times n$ systems.

3.1.1 Flux terms in 2×2 Systems

As it is well known, Riemann problems for 2×2 systems have different kinds of solution: 1-discontinuity waves and 2-discontinuity waves, or 1-discontinuity waves and 2-rarefaction waves, or ... etc.† Let us, for instance, assume that at the interface between the j-th and $(j+1)$-th cell, the solution consists of a 1-discontinuity wave and a 2-discontinuity wave traveling with speeds σ_1 and σ_2, respectively. The solution to the Riemann problem has then the form,

$$\mathbf{w}_{j+\frac{1}{2}}\left(\frac{x}{t};V_j^n,V_{j+1}^n\right) = \begin{cases} V_j^n, & \frac{x}{t}<\sigma_1, \\ V^*, & \sigma_1<\frac{x}{t}<\sigma_2, \\ V_{j+1}^n, & \frac{x}{t}>\sigma_2, \end{cases}$$

where V^* is the intermediate state determined by the intersection of the 1-shock and 2-shock curves (or contact discontinuity curves if the fields are linearly degenerate). Before calculating the general form of these fluxes, let us demonstrate how these flux terms are calculated. Let us assume for simplicity that $\sigma_1\leqslant 0$ and $\sigma_2\geqslant 0$. The interface flux becomes

$$\begin{aligned} G^{n,+}_{j+\frac{1}{2}} &= \frac{1}{\Delta t_n}\left(\int_{x_{j+\frac{1}{2}}}^{x_{j+1}}\left[V_{j+1}^n - \mathbf{w}_{j+\frac{1}{2}}\left(\frac{x-x_{j+\frac{1}{2}}}{\Delta t_n};V_j^n,V_{j+1}^n\right)\right]dx\right), \\ &= \int_0^{\frac{\Delta x}{2\Delta t_n}}\left[V_{j+1}^n - \mathbf{w}_{j+\frac{1}{2}}\left(\xi;V_j^n,V_{j+1}^n\right)\right]d\xi, \\ &= \int_0^{\sigma_2} V_{j+1}^n - V^*\,d\xi + \int_{\sigma_2}^{\frac{\Delta x}{2\Delta t_n}} V_{j+1}^n - V_{j+1}^n\,d\xi, \\ &= \sigma_2(V_{j+1}^n - V^*), \end{aligned}$$

where, in the second line above, we have used the change of variables $\xi = \frac{x-x_{j+1/2}}{\Delta t_n}$. Similarly, we calculate

$$G^{n,-}_{j+\frac{1}{2}} = \sigma_1(V^* - V_j^n).$$

Thus, we have calculated the flux terms for this specific case. More generally ($\sigma_1,\sigma_2\in\mathbb{R}$), the interfacial fluxes $G^{n,\pm}_{j+1/2}$ are given by

$$\begin{aligned} G^{n,+}_{j+\frac{1}{2}} &= \frac{1+\mathrm{sgn}(\sigma_2)}{2}\sigma_2(V_{j+1}^n - V^*) + \frac{1+\mathrm{sgn}(\sigma_1)}{2}\sigma_1(V^* - V_j^n), \\ G^{n,-}_{j+\frac{1}{2}} &= \frac{1-\mathrm{sgn}(\sigma_2)}{2}\sigma_2(V_{j+1}^n - V^*) + \frac{1-\mathrm{sgn}(\sigma_1)}{2}\sigma_1(V^* - V_j^n). \end{aligned}$$

Let us now consider the case where, at the interface located in $x_{j+1/2}$, the solution consists of a 1-rarefaction wave and a 2-discontinuity wave, traveling with velocity σ_2. The solution to

†The solution can obviously consists of only a single k-simple wave. In this case we choose any of the flux terms which contain this k-simple wave, since every case is reduced to the same flux.

the Riemann problem then has the form,

$$\mathbf{w}\left(\frac{x}{t};V_j^n,V_{j+1}^n\right)=\begin{cases}V_j^n, & \frac{x}{t}<\lambda_1(V_j^n),\\ \mathbf{v}\left(\frac{x}{t}\right), & \lambda_1(V_j^n)<\frac{x}{t}<\lambda_1(V^*),\\ V^*, & \lambda_1(V^*)<\frac{x}{t}<\sigma_2,\\ V_{j+1}^n, & \frac{x}{t}>\sigma_2,\end{cases}$$

where V^* is the intermediate state determined by the intersection of the 1-rarefaction and 2-shock (or 2-Contact discontinuity) curves, and $\mathbf{v}(x/t)$ is the section of the 1-rarefaction curve linking V_j^n and V^*. Again, before presenting the general form of the flux terms in this case, let us show a sample calculation. For the sake of the notations, let us assume for instance that $\sigma_2\geqslant 0$, $\lambda_1(V_j^n)\leqslant 0$, and $\lambda_1(V^*)\geqslant 0$. We repeat the above process and obtain

$$\begin{aligned}G_{j+\frac{1}{2}}^{n,+}&=\frac{1}{\Delta t_n}\left(\int_{x_{j+\frac{1}{2}}}^{x_{j+1}}\left[V_{j+1}^n-\mathbf{w}_{j+\frac{1}{2}}\left(\frac{x-x_{j+\frac{1}{2}}}{\Delta t_n};V_j^n,V_{j+1}^n\right)\right]dx\right),\\
&=\int_0^{\frac{\Delta x}{2\Delta t_n}}\left[V_{j+1}^n-\mathbf{w}_{j+\frac{1}{2}}\left(\xi;V_j^n,V_{j+1}^n\right)\right]d\xi,\\
&=\int_0^{\lambda_1(V^*)}V_{j+1}^n-\mathbf{v}(\xi)\,d\xi+\int_{\lambda_1(V^*)}^{\sigma_2}V_{j+1}^n-V^*\,d\xi,\int_{\sigma_2}^{\frac{\Delta x}{2\Delta t_n}}V_{j+1}^n-V_{j+1}^n\,d\xi\\
&=\lambda_1(V^*)V_{j+1}^n-\int_0^{\lambda_1(V^*)}\mathbf{v}(\xi)\,d\xi+\sigma_2(V_{j+1}^n-V^*)-\lambda_1(V^*)(V_{j+1}^n-V^*),\\
&=\sigma_2(V_{j+1}^n-V^*)-\int_0^{\lambda_1(V^*)}\mathbf{v}(\xi)-V^*\,d\xi,\end{aligned}$$

where again we have used the change of variables $\xi=\frac{x-x_{j+1/2}}{\Delta t_n}$ in the second line. Similarly, we calculate

$$G_{j+\frac{1}{2}}^{n,-}=\int_{\lambda_1(V_{j+1}^n)}^0\mathbf{v}(\xi)-V_j^n\,d\xi.$$

Generalizing this process for arbitrary choices of σ_2, etc., the interfacial fluxes $G_{j+1/2}^{n,\pm}$ are given by

$$\begin{aligned}G_{j+\frac{1}{2}}^{n,+}=\frac{1+\operatorname{sgn}(\sigma_2)}{2}\sigma_2(V_{j+1}^n-V^*)-\frac{1+\operatorname{sgn}(\lambda_1(V^*))}{2}\int_0^{\lambda_1(V^*)}\mathbf{v}(\xi)-V^*\,d\xi\\
+\frac{1+\operatorname{sgn}(\lambda_1(V_j^n))}{2}\int_0^{\lambda_1(V_j^n)}\mathbf{v}(\xi)-V_j^n\,d\xi,\end{aligned}$$

$$\begin{aligned}G_{j+\frac{1}{2}}^{n,-}=\frac{1-\operatorname{sgn}(\sigma_2)}{2}\sigma_2(V_{j+1}^n-V^*)-\frac{1-\operatorname{sgn}(\lambda_1(V^*))}{2}\int_{\lambda_1(V^*)}^0\mathbf{v}(\xi)-V^*\,d\xi\\
+\frac{1-\operatorname{sgn}(\lambda_1(V_j^n))}{2}\int_{\lambda_1(V_j^n)}^0\mathbf{v}(\xi)-V_j^n\,d\xi.\end{aligned}$$

The remaining cases follow analogously from these.
An implementation of this scheme for 2×2 systems follows the blueprint:

1. At each cell interface $x_{j+1/2}$, we numerically compute the 1-rarefaction and 1-shock curves (or 1-Contact discontinuity curves for linearly degenerate fields) from the state V_j^n, and the 2-rarefaction and admissible 2-shock curve (or 2-Contact discontinuity curves for linearly degenerate fields) from the state V_{j+1}^n, by using their corresponding differential systems[‡]. Then an entropy condition is used to select admissible shock and rarefaction waves (Lax shock condition in our case).

2. We determine the intermediate state at the (unique) intersection of these curves and construct the flux terms $G_{j+1/2}^{n,\pm}$ using the equations described above.

3. We then update the solution to time t_{n+1} using (3.2).

It is clear that this procedure is extremely costly since at each cell interface we must integrate four curves numerically and find their point of intersection. This is the unfortunate drawback of using an exact Riemann solver. One possible way to reduce the complexity of this scheme is to replace the exact Riemann solver with an approximate or linearized one, as it is done in the Roe solver [30]. Although this is an extremely important point for real, physical applications, the question of how to reduce computational complexity is not the focus of this paper.

3.2 DLM Godunov Scheme using Alouges-Merlet Shock Curves

The question of how to implement the DLM theory numerically has been thoroughly studied, first by Toumi [35] then by Parés [27]. Specifically, in [9], Castro and Parés present a non-conservative (but path-conservative) Godunov scheme for non-conservative hyperbolic systems using a Riemann solver based on DLM's theory. The authors show that the scheme has the form (3.2), with the flux terms given by

$$G_{j+\frac{1}{2}}^{n,+} = \int_0^1 A(\psi(s;V_{j+\frac{1}{2}}^n,V_{j+1}^n))\frac{\partial\psi}{\partial s}(s;V_{j+\frac{1}{2}}^n,V_{j+1}^n)\,ds,$$

$$G_{j+\frac{1}{2}}^{n,-} = \int_0^1 A(\psi(s;V_j^n,V_{j+\frac{1}{2}}^n))\frac{\partial\psi}{\partial s}(s;V_j^n,V_{j+\frac{1}{2}}^n)\,ds,$$

where $\left(V_{j+1/2}^n\right)_j$ are the interface values of the solution to (3.3). In the case where the solution to the Riemann problem is discontinuous at an interface, we replace $V_{j+1/2}^n$ in $G_{j+1/2}^{+,n}$ by the right limit of the discontinuity, and by the left limit of the discontinuity in $G_{j+1/2}^{-,n}$. The path $\psi(s;\mathbf{u}_L,\mathbf{u}_R)$ is chosen to be the composition of the k-Simple curves which we use to solve the Riemann problem with left state $\mathbf{u}_L$, and right state $\mathbf{u}_R$. Using the assumptions stated above on the choice of paths, the only missing piece of information we need to fully construct this family of paths is how we select the shock curves. The approximate shock curves of Alouges and Merlet enable us to complete this family of paths.

The implementation of this scheme for 2×2 systems is entirely analogous to the blueprint detail above for the Godunov scheme using the Alouges-Merlet approximate shock curves:

1. At a cell interface, say $x_{j+1/2}$, we numerically compute the 1-rarefaction and 1-shock curves (or 1-contact discontinuity curves for linearly degenerate fields) at the state V_j^n, and the 2-rarefaction and admissible 2-shock curve (or 2-Contact discontinuity curves for linearly degenerate fields) at the state V_{j+1}^n by using their defining differential equations

[‡]In our implementation, a Runge-Kutta 4 method is used to numerically solve these differential systems.

(the shock curves we use are the Alouges-Merlet approximate shock curves). Then an entropy condition is used to select admissible shock and rarefaction waves (Lax shock condition in our case).

2. We determine the intermediate state at the (unique) intersection of these curves and construct the paths $\psi(s;V_j^n,V_{j+\frac{1}{2}}^n)$ and $\psi(s;V_{j+\frac{1}{2}}^n,V_{j+1}^n)$ as we described above.

3. We calculate the flux terms $G_{j+1/2}^{n,\pm}$ using the equations above.

4. We then update the solution to time t_{n+1} using (3.2).

Again, this scheme suffers from the same high levels of computational complexity as the Godunov scheme presented in the previous section. Indeed, in order to input the path ψ into the expressions for the flux terms, we again need to integrate the rarefaction/shock curves numerically.

Remark 3.1. The schemes presented above are all derived from the classical Godunov scheme (see [19] for instance). As a consequence the stability analysis can easily be deduced from the analysis of the Godunov scheme for conservative hyperbolic systems. The convergence analysis is more complex due to the non-conservativity of the scheme. In particular, the convergence of the scheme to the exact solution (for a fixed path) is not a priori guarantied. The origin of this issue is the complex connection between the numerical viscosity and the measure source term (supported by the discontinuity lines) in the path-conservative scheme equivalent equation [10], [1] and [21]. The approach developed in this paper does not a priori fix this important open problem. We however conjecture that for instance the use of the zero-diffusive reservoir technique developed in [3], [4], applied to the above Godunov-like scheme could fix this problem.

3.3 Numerical Experiments

3.3.1 The Shallow Water Equations

We first implement a Godunov scheme using the exact Riemann solver based on the Alouges-Merlet approximate shock curves, then the DLM Godunov scheme, again using the Alouges-Merlet approximate shock curves, in order to solve the shallow water equations. This system is in fact conservative and hence we expect numerical solutions produced by these schemes to agree with the exact one. This is due to the fact that in the conservative case the Alouges-Merlet approximate shock curves recover the correct shock curves, and the flux terms of the DLM Godunov scheme will reduce to the classical fluxes of the Godunov scheme for systems of conservation laws.

We consider the Riemann problem with left state $\mathbf{u}_L=(5,0)^T$ and right state $\mathbf{u}_R=(1,0)^T$. The reader can easily verify that for this system both the 1-field and 2-field are genuinely nonlinear. Hence, using the results from Section 2, we can construct the k-rarefaction and admissible k-shock curves for $k=1,2$. In Figure 4 we see that the 1-rarefaction and 2-shock curves intersect at the intermediate state $\mathbf{u}_1 \approx (2.54,10.22)^T$. Thus, the solution to this Riemann problem will consist of a 1-rarefaction wave separating $\mathbf{u}_L$ and $\mathbf{u}_1$, and a 2-shock wave separating $\mathbf{u}_1$ and $\mathbf{u}_R$.

In Figure 5, we represent the solutions obtained by the two schemes. We examine the order of convergence of these two schemes in Tables 1 and 2, where the solution of reference was obtained with a VFFC solver (see [17]) with CFL number 0.99 and $N=1280$. From these tables we can see that these schemes converge to the reference solution with, as expected, an order of convergence close to 1.

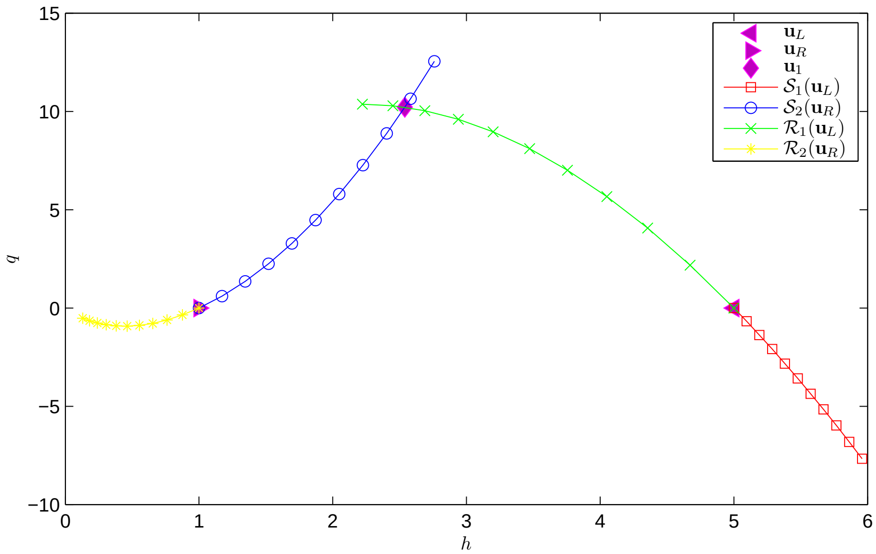

Figure 4: We consider the Riemann problem for the shallow water equations with left state $\mathbf{u}_L = (5,0)^T$, and right state $\mathbf{u}_R = (1,0)^T$. We plot the 1-shock and 1-rarefaction curves at the left state and the 2-shock and 2-rarefaction curves at the right state. We determine the intermediate state $\mathbf{u}_1 \sim (2.54, 10.22)^T$ at the intersection of the 1-rarefaction curve and 2-shock curve.

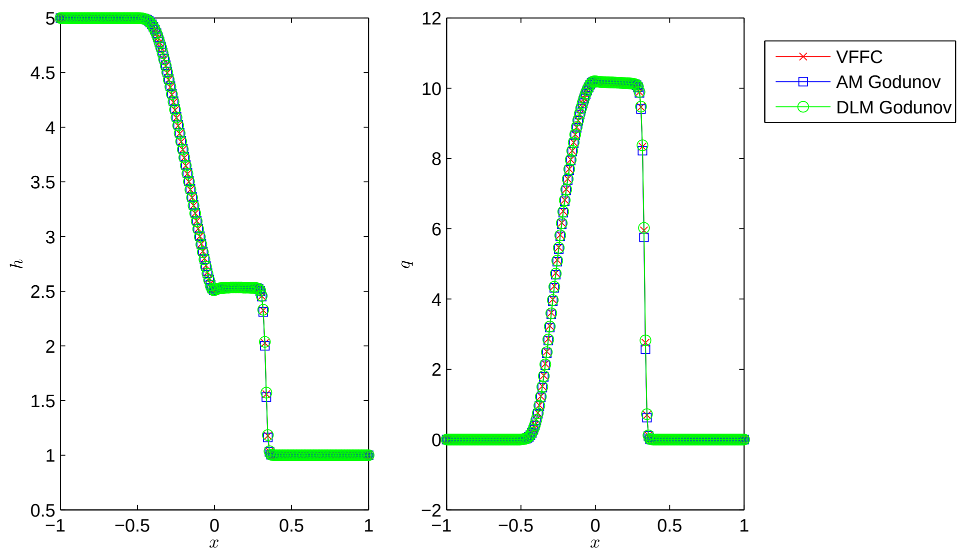

Figure 5: Comparison of numerical solutions of the Riemann problem for the Shallow Water Equations. A VFFC solver, a Godunov scheme based on Alouges-Merlet's approximate shock curves, and the DLM Godunov scheme based on Alouges-Merlet's approximate shock curves are used. Shown at $t = 0.05$, with $N = 200$.

3.3.2 A Non-Conservative System

As expected, the two presented schemes recover the correct solutions where the studied system is a conservative system written in a non-conservative form. Let us now proceed to implement them in the truly non-conservative case. We consider the following non-conservative system,

$$u_t + u u_x + u v_x = 0,$$
$$v_t + v u_x + v v_x = 0.$$

N	h	Error 1, $e_1 = \|\|V - V_{ref}\|\|_2$	Order, $\frac{\Delta \log e_1}{\Delta \log h}$	Error 2, $e_2 = \|\|V - V_{ref}\|\|_\infty$	Order, $\frac{\Delta \log e_2}{\Delta \log h}$
40	5.12E-2	4.48E-1	–	3.50E-1	–
80	2.53E-2	1.96E-1	1.17	1.47E-1	1.23
160	1.26E-2	1.07E-1	0.87	7.80E-2	0.91
320	6.27E-3	4.82E-2	1.14	3.60E-2	1.11
640	3.13E-3	1.87E-2	1.34	1.42E-2	1.34

Table 1: Order of convergence the Godunov scheme using the Alouges-Merlet approximate shock curves. Riemann problem with left state $\mathbf{u}_L = (5,0)^T$ and right state $\mathbf{u}_R = (1,0)^T$. Order of convergence found with respect to the $\mathbf{L}^2$ norm, $||\cdot||_2$, and the $\mathbf{L}^\infty$ norm, $||\cdot||_\infty$. Data calculated at $t = 0.04$ and $x \in [-1,1]$.

N	h	Error 1, $e_1 = \|\|V - V_{ref}\|\|_2$	Order, $\frac{\Delta \log e_1}{\Delta \log h}$	Error 2, $e_2 = \|\|V - V_{ref}\|\|_\infty$	Order, $\frac{\Delta \log e_2}{\Delta \log h}$
40	5.12E-2	4.38E-1	–	3.42E-1	–
80	2.53E-2	1.91E-1	1.17	1.41E-1	1.25
160	1.26E-2	1.03E-1	0.86	7.48E-2	0.90
320	6.27E-3	4.62E-2	1.15	3.42E-2	1.12
640	3.13E-3	1.79E-2	1.37	1.32E-2	1.37

Table 2: Order of convergence of the Godunov schemes using Alouges-Merlet approximate shock curves as DLM's paths. Riemann problem with left state $\mathbf{u}_L = (5,0)^T$ and right state $\mathbf{u}_R = (1,0)^T$. Order of convergence is found with respect to the $\mathbf{L}^2$ norm, $||\cdot||_2$, and the $\mathbf{L}^\infty$ norm, $||\cdot||_\infty$. Data calculated at $t = 0.04$ and $x \in [-1,1]$.

This systems was studied by C. Berthon in [6] and finds its origin in bifluid flows. The eigenvalues of this system are $\lambda_1 = 0$ and $\lambda_2 = u+v$. Therefore, this system is strictly hyperbolic when $u \neq -v$. The 1-field is linearly degenerate for this system and therefore we will consider the 1-Contact discontinuity curve in our numerical solvers. The reader can also verify that the 2-field is genuinely nonlinear. We consider a Riemann problem with left state $\mathbf{u}_L = (4,3)^T$ and right state $\mathbf{u}_R = (2,0.5)^T$. The numerical solution is shown in Figure 6.

As expected, the numerical solutions produced by these two schemes are very close, even in this non-conservative case. As remarked in [21], a non-conservative form of the Godunov-like scheme used here can lead to the convergence of the numerical solution to the solution of an inhomogeneous system with a Borel measure source term with support on the line of discontinuity of the order of the entropy dissipation [6]. That is, in general "$\lim_h u_h \neq u_{\text{exact}}$". This is in particular what is observed in Abgrall-Karni [1], where examples of Parès' path conservative schemes converging to wrong solutions are exhibited (numerical paths do not converge to the chosen ones). However, from Figure 6 we see that in this case the measure source term is zero and indeed $\lim_h u_h = u_{\text{exact}}$. This is an interesting point as *we then have exhibited a non-trivial example where the non-conservative scheme is convergent to the exact solution*. The question of whether this scheme will converge to the exact solution for any non-conservative system is however still open and is the topic of an on-going research.

3.3.3 Equivalence of the schemes

In fact, the two Godunov schemes presented above are equivalent. In order to show it, we must verify that their flux terms are equivalent, i.e. these fluxes are just a single flux written

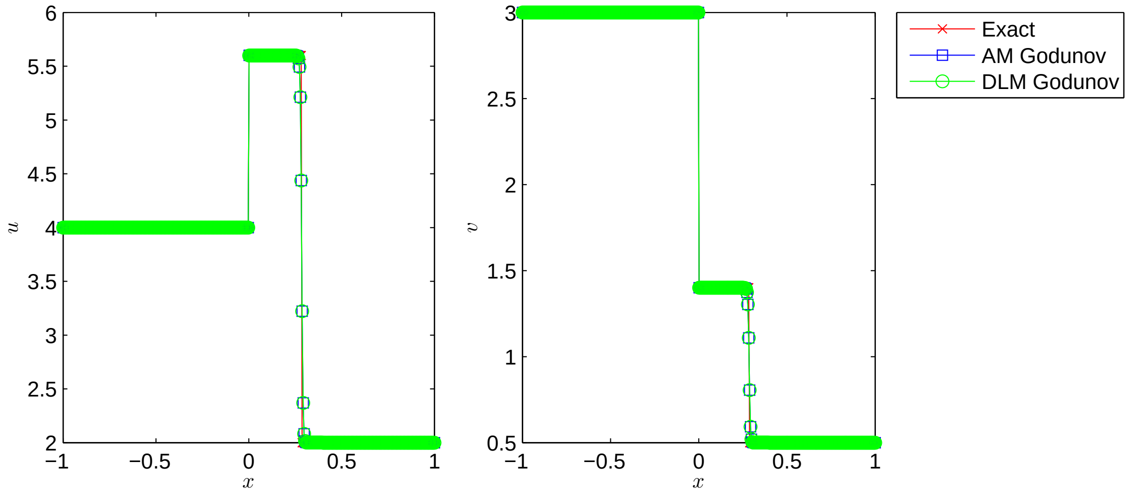

Figure 6: Comparison of the Alouges-Merlet shock curve-based Godunov Solver and the DLM Godunov solver, using the Alouges-Merlet shock curves, for the genuinely non-conservative system. The initial profile is a Riemann problem with $\mathbf{u}_L = (4,3)^T$ and $\mathbf{u}_R = (2,0.5)^T$. Shown at $t = 0.06$ with $N = 400$.

in different ways. To this end, let us consider an interface at which the Riemann problem contains a k-shock wave separating the states $\mathbf{u}^-$ on the left and $\mathbf{u}^+$ on the right, traveling with positive speed $\tilde{\sigma} > 0$. At this interface, the positive flux term of the Godunov scheme using the approximate shock curves of Alouges and Merlet will contain the term $\tilde{\sigma}(\mathbf{u}^+ - \mathbf{u}^-)$. Also, in the Godunov scheme based on DLM theory, the path ψ will contain the segment of the Alouges-Merlet approximate k-shock curve, parametrized by $\mathbf{u}(\sigma)$, which satisfies,

$$\begin{cases} A(\mathbf{u}(\sigma))\dfrac{d\mathbf{u}}{d\sigma} = \sigma\dfrac{d\mathbf{u}}{d\sigma} + \mathbf{u}(\sigma) - \mathbf{u}^-, \\ \mathbf{u}(\lambda_k(\mathbf{u}^-)) = \mathbf{u}^-. \end{cases}$$

So the positive flux for the DLM Godunov scheme will contain the term

$$\int_{\lambda_k(\mathbf{u}^-)}^{\tilde{\sigma}} A(\mathbf{u}(\sigma))\frac{d\mathbf{u}}{d\sigma}d\sigma.$$

Using the definition of the approximate shock curve yields,

$$\begin{aligned} \int_{\lambda_k(\mathbf{u}^-)}^{\tilde{\sigma}} A(\mathbf{u}(\sigma))\frac{d\mathbf{u}}{d\sigma}d\sigma &= \int_{\lambda_k(\mathbf{u}^-)}^{\tilde{\sigma}} \left(\sigma\frac{d\mathbf{u}}{d\sigma} + \mathbf{u}(\sigma) - \mathbf{u}^-\right) d\sigma \\ &= \left[\sigma(\mathbf{u}(\sigma) - \mathbf{u}^-)\right|_{\lambda_k(\mathbf{u}^-)}^{\tilde{\sigma}} \\ &= \tilde{\sigma}(\mathbf{u}^+ - \mathbf{u}^-). \end{aligned}$$

Thus, the positive flux term in the DLM Godunov scheme also contains the term $\tilde{\sigma}(\mathbf{u}^+ - \mathbf{u}^-)$. We can repeat this argument for shock waves which travel with negative speeds. Moreover, this argument can be used in a similar fashion to show that contributions in the flux terms due to contact discontinuities and rarefaction waves will be the same in both schemes. Finally, this tells us that the flux terms in both schemes are equivalent. Note that, the slight numerical differences are due to the fact that although these flux terms are theoretically the same, they are calculated numerically in entirely different ways. This remark does not, unfortunately, plead in favour of the presented Godunov-like scheme (Section 3.1) as the path-conservative Parès scheme is known in general to converge to wrong solutions.

4 Concluding remarks

In this paper, we have proposed different numerical approaches for solving non-conservative hyperbolic systems. The proposed schemes are designed using exact Riemann solvers which utilize the definitions recently proposed by Alouges and Merlet of approximate shock curves in non-conservative systems. These shock curves have proven to be useful in several ways. They are accurate approximations of the shock curves constructed by Bianchini and Bressan in their vanishing viscosity process, so that the solutions constructed using these approximate shock curves are accurate approximations of the vanishing viscosity solutions. Using this fact, we first constructed a Godunov scheme for non-conservative hyperbolic systems using an exact Riemann solver which implements these approximate shock curves. On the other hand, we also know that these approximate shock curves approximate the viscous profiles and thus are useful in the theory of non-conservative products introduced by Dal Maso, LeFloch, and Murat. This fact gives us the possibility to construct several numerical schemes for NCHSs using the framework proposed by Castro, Parés, et. al. An important result is that we are able to show that the two Godunov schemes we have constructed, which use exact Riemann solvers, are in fact equivalent numerical schemes. This result shows a subtle connection between these two different approaches to the analysis of NCHSs. An interesting result we have shown is that these numerical schemes, when applied to a particular non-conservative system, in fact converge to the exact solution. This is an interesting result as it seems that, in this case, these schemes overcome the problem of convergence of non-conservative schemes as studied by Hou and LeFloch [21], and Abgrall and Karni [1]. More generally, the convergence to the correct solution of non-conservative schemes approximating non-conservative hyperbolic systems is still an open problem. However, we conjecture that when the numerical viscosity matrix and the matrix A commute (which is typically satisfied for Roe or VFFC schemes with a "Karni-like" correction [22]), the numerical solution can converge to the exact solution, if a combination of Alouges&Merlet's shock curves is chosen as a path. This will be studied in a forthcoming paper.

The main issue that arises in the implementation of these numerical schemes is their extreme computational cost. We indeed have to determine numerically the rarefaction, shock, and contact discontinuity curves of each state, at each interface of the mesh. This process involves numerically solving up to $2n$ ODEs per state. It is clear that our primary objective for further research of these numerical schemes is to design a scheme which produces accurate numerical results but avoids this level of computational complexity (Roe- [30] and VFFC-type schemes [17]).

Extension of the proposed approach to $n\times n$ systems, although simple in principle becomes computationally challenging. Moreover, because the shock curves defined by Alouges and Merlet are merely approximations of the shock curves described by Bianchini and Bressan, it is still desirable to construct a numerical scheme which implements the true shock curves of Bianchini and Bressan directly. However, again due to the complexity of their derivation in [7], this is still challenging.

Another topic for future research is the question of how to extend these numerical schemes to the multidimensional case. Using the theory of Dal Maso, LeFloch, and Murat, Parés, Castro, et. al. addressed this equation in [8], so our first goal will be to use this framework to extend our one-dimensional numerical scheme to the multidimensional case. Similarly, we are also interested in the construction of high-order schemes.

Acknowledgment. The authors would like to thank the anonymous referees for their constructive comments.

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