# Enumeration of Mullen compositions over finite fields

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## Compositions over finite fields

A composition of an element $\mathbf{s} \in \mathbb{F}_q$ with $m$ parts is an $m$-tuple $(\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_m)$ of nonzero elements of $\mathbb{F}_q$ such that $\mathbf{x}_1 + \mathbf{x}_2 + \cdots + \mathbf{x}_m = \mathbf{s}$.

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**Examples** of compositions over $\mathbb{Z}_5$.
$(1, 2, 3, 4)$ is a 4-composition of 0.
$(1, 2, 3, 4, 1)$ is a 5-composition of 1.
Let $c_m(s)$ be the number of $m$-compositions of $s \in \mathbb{Z}_k$. What is $c_m(s)$?

Since

$$\hat{c}_m(n) := [z^n]\left(z + z^2 + \cdots + z^{k-1}\right)^m$$

is the number of all $m$-tuples $(x_1, \ldots, x_m)$ satisfying

$$x_1 + x_2 + \cdots + x_m = n, \quad x_j \in \{1, 2, \ldots, k-1\},$$

we have

$$c_m(s) = \sum_{n \equiv s \pmod{k}} \hat{c}_m(n).$$

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$$c_m(s) = \sum_{n \equiv s \pmod{k}} \hat{c}_m(n).$$

This sum can be evaluated using the "multisection formula".

## Multisection formula

Let $\omega_k = \exp(2\pi i/k)$, and let $A(z) = \sum_{n\geq 0} a_n z^n$ be a generating function. Then

$$\sum_{n\equiv s \pmod{k}} a_n z^n = \frac{1}{k}\sum_{j=0}^{k-1} \omega_k^{-js} A\left(z\omega_k^j\right),$$

$$\sum_{n\equiv s \pmod{k}} a_n = \frac{1}{k}\sum_{j=0}^{k-1} \omega_k^{-js} A\left(\omega_k^j\right).$$

A useful formula:

$$1 + \omega_k^j + \omega_k^{2j} + \ldots + \omega_k^{(k-1)j} = k[j \equiv 0 \pmod{k}].$$

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## An example

The number of $m$-compositions of $s$ over $\mathbb{Z}_k$ is equal to

$$\begin{aligned}
c_m(s) &= \sum_{n \equiv s \pmod{k}} \hat{c}_m(n) \\
&= \frac{1}{k} \sum_{j=0}^{k-1} \omega_k^{-js} \left( \omega_k^{j} + \omega_k^{2j} + \ldots + \omega_k^{(k-1)j} \right)^m \\
&= \frac{1}{k} \left( (k-1)^m + \sum_{j=1}^{k-1} \omega_k^{-js} (-1)^m \right) \\
&= \frac{1}{k} \left( (k-1)^m + (-1)^m (k[s=0] - 1) \right)
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Locally restricted compositions over $\mathbb{Z}_k$ can be modeled by directed walks in a digraph and then enumerated by the transfer matrix method. In the following, $\varepsilon_s$ denotes a distinguished copy of the empty sequence.

**Example** 3-Mullen compositions over $\mathbb{Z}_k$ can be defined using walks in the following digraph $D$. The vertex set is $V(D) = \{\varepsilon_s\} \,\dot\cup\, \mathcal{R} \,\dot\cup\, \mathcal{T}$, where $\mathcal{R}$ is the set of all 3-Mullen compositions of length 3, and $\mathcal{T}$ is the set of all 2-Mullen compositions of length less than 3.

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- There is no arc between vertices in $\mathcal{T}$.
- There is an arc from $\varepsilon_s$ to every vertex in $\mathcal{R}$.
- There is an arc from a vertex $\mathbf{u} \in \mathcal{R}$ to a vertex $\mathbf{v} \in \mathcal{R} \,\dot\cup\, \mathcal{T}$ if and only if the concatenation $\mathbf{uv}$ is 3-Mullen.

# Walks in a digraph

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- The (integer) *weight* of a composition $\mathbf{v} = (v_1, \ldots, v_j)$ is $|\mathbf{v}| := v_1 + \cdots + v_j$ (sum as integers).
- The weight of an arc $\mathbf{uv}$ is defined to be $z^{|\mathbf{v}|}$, and the weight of a walk is the product of the weights of its arcs.

# The transfer matrix

The transfer matrix $T(z)$ is defined as follows. Its rows and columns are indexed by the vertices of $\mathcal{R}$. The $(\mathbf{u}, \mathbf{v})$ entry of $T(z)$ is defined to be $z^{|\mathbf{v}|}$ if $(\mathbf{u}, \mathbf{v})$ is an arc in $D$; otherwise it is zero.

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Write $m = 3a + b$ with $0 \leq b \leq 2$, and define the *finish vector* $\boldsymbol{\beta}_b(\mathbf{z})$ as the column vector whose $j^{\text{th}}$ entry is the sum of weights of all arcs from the $j^{\text{th}}$ vertex of $\mathcal{R}$ to a vertex in $\mathcal{T}$ of length $b$.

**Proposition** Let $m = 3a + b$, with $0 \leq b \leq 2$. Then $\hat{C}_m(z) = \boldsymbol{\alpha}(z)T^{a-1}(z)\boldsymbol{\beta}_b(z)$ enumerates all directed walks of length $m$ from $\varepsilon_s$ to $\mathcal{T}$,

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$$\begin{aligned} c_m(s) &= \frac{1}{k}\sum_{j=0}^{k-1} \omega_k^{-js}\hat{C}_m\left(\omega_k^j\right) \\ &\sim \frac{1}{k}\boldsymbol{\alpha}(1)T^{a-1}(1)\boldsymbol{\beta}_b(1) \text{ as } m \to \infty \\ &= \frac{(k-1)(k-2)}{k}(k-3)^{m-2}. \end{aligned}$$

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**Main Theorem** Let $c_m(s)$ be the number of $m$-compositions of $s$ in a class of locally restricted compositions over $G$ defined by a digraph $D$. Under some aperiodic conditions on $D$, we have

$$c_m(s) = A \cdot B^m (1 + O(\theta^m)), \quad \text{as } m \to \infty,$$

where $A, B$, and $\theta < 1$ are some positive constants.

**Corollary** Let $g := |G|$ and $g^{\underline{d}} := g(g-1)\cdots(g-d+1)$. Then

- For $d$-Carlitz compositions,

$$A = \frac{1}{g}(g-1)^{\underline{d}}(g-1-d)^{-d},\ B = g-1-d.$$

- For $d$-Carlitz weak compositions such that the first $d$ parts are nonzero,

$$A = \frac{1}{g}(g-1)^{\underline{d}}(g-d)^{-d},\ B = g-d.$$

- For $d$-Mullen compositions,

$$A = \frac{1}{g}(g-1)^{\underline{d}}(g-d)^{-d},\ B = g-d.$$

The exponential growth rate $B$ can sometimes be complicated.

- For the class of compositions over $\mathbb{Z}_4$ such that the sum of any three consecutive parts is nonzero, we have

$$c_m(s) = \frac{3}{8}(1+\sqrt{2})^m \left(1 + O\left(\theta^m\right)\right), \ m \to \infty,$$

where $0 < \theta < 1$ is a constant.

**Bijection** For each $m$-composition $\mathbf{u} = (u_1, u_2, \ldots, u_m)$ over $G$, let $\mathbf{v} = \phi(\mathbf{u})$ be a (weak) $m$-composition defined by

$$v_j = u_1 + \cdots + u_j, 1 \leq j \leq m.$$

Then $\mathbf{u}$ is $d$-Mullen if and only if $\phi(\mathbf{u})$ is $d$-Carlitz such that the first $d$ parts of $\phi(\mathbf{u})$ are all nonzero.

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**Corollary** For each nonzero element $s \in G$, there is a bijection between $d$-Mullen $m$-compositions of $s$ and $d$-Mullen $m$-compositions of 1 (the identity element of $G$).

# Tables

| $m$ | $c_m(0)$ | $c_m(0)/a_m$ |
|---|---|---|
| 11 | 5238 | 0.8602 |
| 12 | 16377 | 1.114 |
| 13 | 32196 | 0.9072 |
| 14 | 92133 | 1.075 |
| 15 | 194196 | 0.9388 |
| 16 | 524241 | 1.05 |
| 17 | 1156908 | 0.9596 |
| 18 | 3006279 | 1.033 |
| 19 | 6839406 | 0.9733 |
| 20 | 17332647 | 1.022 |
| 21 | 40234356 | 0.9824 |

Table : Comparison between exact counts and asymptotic counts $a_m = \frac{3}{8}(1+\sqrt{2})^m$ for compositions over $\mathbb{Z}_4$ such that the sum of any 3 consecutive terms is nonzero.

# Tables

| $s$ \ $m$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 12 | 24 | 48 | 204 | 624 | 1680 | 5196 | 16008 |
| 1 | 1 | 3 | 6 | 21 | 69 | 192 | 573 | 1767 | 5262 | 15681 |

Number of 2-Mullen compositions over $\mathbb{Z}_5$.

| $s$ \ $m$ | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|
| 0 | 4 | 24 | 88 | 320 | 1248 | 5120 | 20728 | 82284 | 326296 |
| 1 | 6 | 18 | 72 | 320 | 1284 | 5120 | 20232 | 81738 | 329064 |
| 2 | 4 | 18 | 88 | 320 | 1236 | 5120 | 20728 | 81738 | 326296 |
| 3 | 6 | 24 | 72 | 320 | 1392 | 5120 | 20232 | 82284 | 329064 |
| 4 | 4 | 18 | 88 | 320 | 1236 | 5120 | 20728 | 81738 | 326296 |
| 5 | 6 | 18 | 72 | 320 | 1284 | 5120 | 20232 | 81738 | 329064 |

Number of 2-Carlitz weak compositions over $\mathbb{Z}_6$.

# Tables

| $s$ \ $m$ | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|
| 0 | 4 | 12 | 32 | 80 | 280 | 812 | 2572 | 6644 | 23460 |
| 1 | 4 | 6 | 34 | 82 | 284 | 748 | 2498 | 7372 | 21522 |
| 2 | 2 | 12 | 32 | 80 | 274 | 866 | 2266 | 7484 | 21642 |
| 3 | 4 | 12 | 16 | 136 | 224 | 820 | 2480 | 7384 | 21432 |
| 4 | 2 | 12 | 32 | 80 | 274 | 866 | 2266 | 7484 | 21642 |
| 5 | 4 | 6 | 34 | 82 | 284 | 748 | 2498 | 7372 | 21522 |

Table : Number of 2-Carlitz compositions over $\mathbb{Z}_6$.