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A composition of an element  $\mathbf{s} \in \mathbb{F}_q$  with m parts is an m-tuple  $(\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_m)$  of nonzero elements of  $\mathbb{F}_q$  such that  $\mathbf{x}_1 + \mathbf{x}_2 + \cdots + \mathbf{x}_m = \mathbf{s}$ .

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## Compositions over finite fields

# Since

$$\hat{c}_m(n) := [z^n] (z + z^2 + \dots + z^{k-1})^m$$

is the number of all *m*-tuples  $(x_1, \ldots, x_m)$  satisfying

$$x_1 + x_2 + \dots + x_m = n, \ x_j \in \{1, 2, \dots, k-1\},$$
 we have

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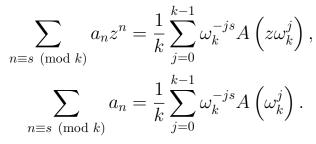
$$c_m(s) = \sum_{n \equiv s \pmod{k}} \hat{c}_m(n).$$

This sum can be evaluated using the "multisection formula".

# Multisection formula

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Let  $\omega_k = \exp(2\pi i/k)$ , and let  $A(z) = \sum_{n\geq 0} a_n z^n$  be a generating function. Then



A useful formula:

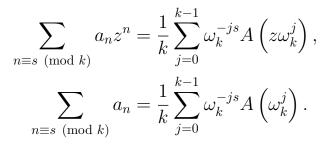
 $1 + \omega_k^j + \omega_k^{2j} + \ldots + \omega_k^{(k-1)j} = k[j \equiv 0 \pmod{k}].$ 

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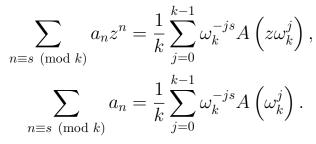
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 $1 + \omega_k^j + \omega_k^{2j} + \ldots + \omega_k^{(k-1)j} = k[j \equiv 0 \pmod{k}].$ 

The number of  $m\text{-}\mathrm{compositions}$  of s over  $\mathbb{Z}_k$  is equal to

$$c_m(s) = \sum_{n \equiv s \pmod{k}} \hat{c}_m(n)$$
  
=  $\frac{1}{k} \sum_{j=0}^{k-1} \omega_k^{-js} \left( \omega_k^j + \omega_k^{2j} + \dots + \omega_k^{(k-1)j} \right)^m$   
=  $\frac{1}{k} \left( (k-1)^m + \sum_{j=1}^{k-1} \omega_k^{-js} (-1)^m \right)$   
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## Locally restricted compositions

A composition is called *locally restricted* if any d consecutive parts satisfy certain restrictions for a given positive integer d.

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**Example** *d*-Carlitz compositions: there is no repeated part among any d + 1 consecutive parts. Locally restricted compositions over  $\mathbb{Z}_k$  can be modeled by directed walks in a digraph and then enumerated by the transfer matrix method. In the following,  $\varepsilon_s$  denotes a distinguished copy of the empty sequence.

• There is no arc between vertices in  $\mathcal{T}$ .

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- There is an arc from  $\varepsilon_s$  to every vertex in  $\mathcal{R}$ .
- There is an arc from a vertex u ∈ R to a vertex
   v ∈ R ∪ T if and only if the concatenation uv is
   3-Mullen.

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- ► The directed walks from \(\varepsilon\_s\) to a vertex in \(\mathcal{T}\) give sequences of elements of \(\mathcal{Z}\_k\) corresponding to 3-Mullen compositions over \(\mathcal{Z}\_k\).
- ► The (integer) weight of a composition **v** = (v<sub>1</sub>,...,v<sub>j</sub>) is |**v**| := v<sub>1</sub> + ··· + v<sub>j</sub> (sum as integers).
- The weight of an arc uv is defined to be z<sup>|v|</sup>, and the weight of a walk is the product of the weights of its arcs.

The transfer matrix T(z) is defined as follows. Its rows and columns are indexed by the vertices of  $\mathcal{R}$ . The  $(\mathbf{u}, \mathbf{v})$  entry of T(z) is defined to be  $z^{|\mathbf{v}|}$  if  $(\mathbf{u}, \mathbf{v})$  is an arc in D; otherwise it is zero. The transfer matrix T(z) is defined as follows. Its rows and columns are indexed by the vertices of  $\mathcal{R}$ . The  $(\mathbf{u}, \mathbf{v})$  entry of T(z) is defined to be  $z^{|\mathbf{v}|}$  if  $(\mathbf{u}, \mathbf{v})$  is an arc in D; otherwise it is zero. The *start vector*  $\boldsymbol{\alpha}(\mathbf{z})$  is defined to be the row vector whose  $j^{\text{th}}$  entry is the weight of the arc from  $\varepsilon_s$  to the  $j^{\text{th}}$  vertex of  $\mathcal{R}$ . The transfer matrix T(z) is defined as follows. Its rows and columns are indexed by the vertices of  $\mathcal{R}$ . The  $(\mathbf{u}, \mathbf{v})$  entry of T(z) is defined to be  $z^{|\mathbf{v}|}$  if  $(\mathbf{u}, \mathbf{v})$  is an arc in D; otherwise it is zero. The start vector  $\boldsymbol{\alpha}(\mathbf{z})$  is defined to be the row vector whose  $j^{\text{th}}$  entry is the weight of the arc from  $\varepsilon_s$  to the  $j^{\text{th}}$  vertex of  $\mathcal{R}$ . Write m = 3a + b with  $0 \le b \le 2$ ,

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**Proposition** Let m = 3a + b, with  $0 \le b \le 2$ . Then  $\hat{C}_m(z) = \alpha(z)T^{a-1}(z)\beta_b(z)$  enumerates all directed walks of length m from  $\varepsilon_s$  to  $\mathcal{T}$ ,

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$$c_m(s) = \frac{1}{k} \sum_{j=0}^{k-1} \omega_k^{-js} \hat{C}_m\left(\omega_k^j\right)$$
$$\sim \frac{1}{k} \alpha(1) T^{a-1}(1) \boldsymbol{\beta}_b(1) \text{ as } m \to \infty$$
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$$c_m(s) = A \cdot B^m(1 + O(\theta^m)), \text{ as } m \to \infty,$$

where A, B, and  $\theta < 1$  are some positive constants.

**Corollary** Let 
$$g := |G|$$
 and  
 $g^{\underline{d}} := g(g-1) \cdots (g-d+1)$ . Then  
For *d*-Carlitz compositions,

$$A = \frac{1}{g}(g-1)^{\underline{d}}(g-1-d)^{-d}, \ B = g-1-d.$$

 For d-Carlitz weak compositions such that the first d parts are nonzero,

$$A = \frac{1}{g}(g-1)^{\underline{d}}(g-d)^{-d}, \ B = g - d.$$

For *d*-Mullen compositions,

$$A = \frac{1}{g}(g-1)^{\underline{d}}(g-d)^{-d}, \ B = g - d.$$

The exponential growth rate  ${\cal B}$  can sometimes be complicated.

► For the class of compositions over Z<sub>4</sub> such that the sum of any three consecutive parts is nonzero, we have

$$c_m(s) = \frac{3}{8}(1+\sqrt{2})^m (1+O(\theta^m)), \ m \to \infty,$$

where  $0 < \theta < 1$  is a constant.

**Bijection** For each *m*-composition  $\mathbf{u} = (u_1, u_2, \dots, u_m)$  over *G*, let  $\mathbf{v} = \phi(\mathbf{u})$  be a (weak) *m*-composition defined by

$$v_j = u_1 + \dots + u_j, 1 \le j \le m.$$

Then **u** is *d*-Mullen if and only if  $\phi(\mathbf{u})$  is *d*-Carlitz such that the first *d* parts of  $\phi(\mathbf{u})$  are all nonzero.

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Then  $\mathbf{u}$  is *d*-Mullen if and only if  $\phi(\mathbf{u})$  is *d*-Carlitz such that the first *d* parts of  $\phi(\mathbf{u})$  are all nonzero. **Corollary** For each nonzero element  $s \in G$ , there is a bijection between *d*-Mullen *m*-compositions of *s* and *d*-Mullen *m*-compositions of 1 (the identity element of *G*).

## Tables

m	$c_m(0)$	$c_m(0)/a_m$
11	5238	0.8602
12	16377	1.114
13	32196	0.9072
14	92133	1.075
15	194196	0.9388
16	524241	1.05
17	1156908	0.9596
18	3006279	1.033
19	6839406	0.9733
20	17332647	1.022
21	40234356	0.9824

Table : Comparison between exact counts and asymptotic counts  $a_m = \frac{3}{8}(1 + \sqrt{2})^m$  for compositions over  $\mathbb{Z}_4$  such that the sum of any 3 consecutive terms is nonzero.

# Tables

$\begin{bmatrix} m\\s \end{bmatrix}$	1	2	3	4	5	6	7	8	9	10
0	0	0	12	24	48	204	624	1680	5196	16008
1	1	3	6	21	69	192	573	1767	5262	15681
Number of 2-Mullen compositions over $\mathbb{Z}_5$ .										

$\begin{bmatrix} m\\ s \end{bmatrix}$	2	3	4	5	6	7	8	9	10
0	4	24	88	320	1248	5120	20728	82284	326296
1	6	18	72	320	1284	5120	20232	81738	329064
2	4	18	88	320	1236	5120	20728	81738	326296
3	6	24	72	320	1392	5120	20232	82284	329064
4	4	18	88	320	1236	5120	20728	81738	326296
5	6	18	72	320	1284	5120	20232	81738	329064

Number of 2-Carlitz weak compositions over  $\mathbb{Z}_6$ .

$\begin{bmatrix} m\\ s \end{bmatrix}$	2	3	4	5	6	7	8	9	10
0	4	12	32	80	280	812	2572	6644	23460
1	4	6	34	82	284	748	2498	7372	21522
2	2	12	32	80	274	866	2266	7484	21642
3	4	12	16	136	224	820	2480	7384	21432
4	2	12	32	80	274	866	2266	7484	21642
5	4	6	34	82	284	748	2498	7372	21522

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