Wave–Mean-Flow Interactions in a Forced Rossby Wave Packet Critical Layer

By L. J. Campbell

This paper describes the nonlinear critical layer evolution of a zonally localized Rossby wave packet forced in mid-latitudes and propagating horizontally on a beta plane in a zonal shear flow. The wave packet has an amplitude that varies slowly in the zonal direction. Numerical solutions of the governing nonlinear equations show that the wave–mean-flow interactions differ from those that would result with a monochromatic forcing. With the localized forcing, the net absorption of the disturbance at the critical layer continues for large time, because there is an outward flux of momentum in the zonal direction. Further insight into the mechanism for this and other aspects of the evolution of the critical layer is obtained through an approximate asymptotic analysis which is valid for large time.

1. Introduction

Our understanding of wave–mean-flow interactions in the atmosphere and ocean has been greatly advanced in the past few decades by theoretical studies of wave instabilities in shear flows. A mathematical difficulty that arises in the linear inviscid theory of waves in shear flows is that a singularity is generally present at the critical line where the mean flow velocity is equal to
the phase speed of the perturbation. In the critical layer centered at this line, wave-breaking and reflection may occur, often leading to turbulence. Critical layer interactions of planetary waves have been proposed as the basis of theories for a number of phenomena observed in the atmosphere and ocean, such as stratospheric sudden warmings [1] and the quasi-biennial oscillation [2]. It has also been suggested that the critical layer behavior of Rossby waves may be useful in understanding the dynamical balance that controls circulation in the Southern Ocean [3].

There have been a number of reports of observations in the atmosphere and ocean that agree qualitatively with the results of the theoretical studies, namely, the absorption of small-amplitude waves and the reflection and breaking of larger amplitude waves in the vicinity of the critical layer. An example illustrating the large-amplitude case is shown in the paper by Molteni et al. [4]. Their map of the Pacific North American pattern shows observations that could be interpreted as low-latitude reflection of a horizontally propagating Rossby wave packet at a critical latitude [5].

Theoretical critical layer studies have involved reintroducing to the critical layer the effects that are neglected in the steady linear inviscid theory. When viscosity [6] or nonlinearity [7, 8] is restored to the governing equation in the critical layer, the waves are assumed to be linear, inviscid, and steady in the regions above and below the critical layer and the inner (critical layer) solution is used to connect the solutions in the two regions.

An alternative solution procedure is to assume the amplitude of the disturbance evolves with time. Dickinson [9] and Warn and Warn [10] derived asymptotic solutions to the linearized barotropic vorticity equation to examine the case of a forced Rossby wave propagating horizontally toward a critical layer on a beta plane. Their results showed that the disturbance is absorbed at the critical layer and, consequently, the Reynolds stress is discontinuous across the critical layer. In addition, Warn and Warn observed that the neglected nonlinear terms would become nonnegligible for large time. In a nonlinear study [11] a few years later, Warn and Warn derived an analytic solution valid for large time outside the critical layer, as well as an equation describing the long-time evolution of the vorticity in the inner region, which they then solved numerically. In independent work around the same time, Stewartson [12] examined the case of waves with long zonal wavelength and also derived the equation for the vorticity in the critical layer. He showed that it was possible to derive an approximate solution to this equation for a certain special configuration of the flow domain. Both of these papers concluded that the jump in the Reynolds stress was modified for large time, eventually oscillating around zero, indicating reflection of the incident waves at the critical layer.

The aforementioned studies all dealt with waves which are periodic in the zonal (x) direction, but yet another way of removing the singularity is to employ a disturbance in the form of a spatially localized wave packet
rather than a monochromatic wave [13]. With the assumption of dispersive waves described by linear inviscid equations, one may assume the disturbance amplitude depends on $x$ and $t$, on a scale much larger than the wavelength. Accordingly, slow $x$ and $t$ variables are introduced and the amplitude of the disturbance can be written as $A(X, y, T)$, where $y$ is the meridional variable and $X = \mu x$ and $T = \mu t$ are the slow variables, with $\mu$ being a small parameter. An equation involving derivatives with respect to $X$ and $T$ is solved to determine the amplitude of the disturbance. If the nonlinear terms are included in the analysis [14], then an appropriate balance must be established between the nonlinear parameter and $\mu$, the wave packet parameter.

This idea leads naturally to the question of what effect a spatially localized forcing with a slowly varying amplitude would have in the time-dependent problem for forced waves. This is in fact a more realistic representation of the propagation of waves in the atmosphere or ocean. Employing a monochromatic periodic forcing can be justified only if the wavelength of the disturbance is assumed to be of the order of magnitude of the circumference of the earth. It was pointed out by Brunet and Haynes [5] that, while this assumption may be valid in the stratosphere, it is less realistic in the troposphere.

This was realized independently by Campbell and Maslowe and motivated a study on the effects of zonal localization on the linear critical layer [15]. In that study, the forcing took the form of a Rossby wave packet with a Gaussian amplitude modulation. An approximate solution was derived using multiple scaling and matched asymptotic expansions and modifications to the solution derived in the monochromatic case [10] were obtained. A numerical solution was also carried out and the results found to be in agreement with the analytic predictions, namely, that the packet was completely absorbed at the critical latitude, a steady state was eventually attained away from the critical layer and the streamline pattern was modified as a result of the spatial localization. The aim of the present paper is to extend this linear study by solving the fully nonlinear equations, thus taking into account all four of the effects mentioned above, namely, nonlinearity, time-dependence, viscosity, and spatial localization.

A nonlinear investigation of the evolution of forced Rossby wave packets was reported by Brunet and Haynes [5]. They discussed the absorption/reflection properties of the nonlinear critical layer, generalizing the discussion of Killworth and McIntyre [16] concerning periodic disturbances to the case of a wave packet of finite length in a domain unbounded in the longitudinal direction. They conjectured that, in the unbounded case, the critical layer could continue to act as an absorber, even after the nonlinear effects had come into play, because there could be an outward flux of wave activity in the $x$-direction, so that the wave-breaking region would increase in longitudinal extent with time. This is represented schematically in their Figure 1. They considered the special case of a forcing for which an analytic solution for the leading-order
streamfunction had been derived by Brown and Stewartson [17]. From this solution, they calculated the time-integrated absorptivity of the critical layer and found it to be bounded, as in the periodic case, indicating that the critical layer did not continue absorbing indefinitely, they had hypothesized initially. Brunet and Haynes then went on to describe numerical simulations using the shallow water equations of a wave packet propagating toward the equator from the northern mid-latitudes. The numerical results showed the packet being reflected back to high latitudes from a wave-breaking region near the equator; however, these reflections were observed at a later stage than in their corresponding periodic simulations.

The related problem of the critical layer for forced internal gravity waves propagating vertically in a density-stratified flow has also received a great deal of attention in the past 40 years. The pioneering study of the gravity wave critical layer carried out by Booker and Bretherton [18] showed that small-amplitude gravity waves are absorbed at their critical level, at least at the early (linear) stages of the development of the critical layer. Brown and Stewartson [19] extended this analysis into the nonlinear late-time regime and found evidence of wave reflection; however, their study was restricted to strictly periodic monochromatic waves. Bacmeister and Pierrehumbert [20] carried out nonlinear, time-dependent simulations of gravity waves forced by flow over a horizontally localized obstacle and discussed the interaction of the disturbance with the mean flow at its critical level. They noted that an absorbing state appeared to persist indefinitely in the critical layer because there was a horizontal divergence of momentum flux. This is the same scenario that was later suggested by Brunet and Haynes [5], but was not actually observed in their numerical simulations. Brunet and Haynes, in their paper, raised the question of why there was a difference between the results of the two papers with respect to the absorption/reflection behavior of the critical layer.

One of the motivations for this project was to seek an answer to that question. We have examined both problems (Rossby and gravity wave packets) using similar basic flows and configuration. The gravity wave packet problem is described in [21], so here the discussion is restricted to the case of Rossby wave packets. The numerical simulations described here show that the absorption/reflection behavior of the critical layer depends on the balance between the wave packet parameter and the nonlinear parameter. An asymptotic analysis has also been carried out; this provides further insight into the nonlinear critical layer evolution and the effect of the spatial localization. An overview of the paper is as follows. In the next section, the governing equations are described. The asymptotic analysis is presented in Section 3, beginning with a short summary of the results of earlier asymptotic studies. The results of the numerical simulations are discussed in Section 4 and finally, in Section 5, some concluding remarks are given.
2. The nonlinear critical layer equations

2.1. The governing equations

This paper deals with the evolution of a Rossby wave packet forced at some northern latitude in a zonal flow with velocity $\bar{u}(y)$ on a beta plane and propagating horizontally toward the equator. The governing equation for the flow, written in terms of nondimensional variables and parameters, is the barotropic vorticity equation

$$\nabla^2 \Psi_t + \Psi_x \nabla^2 \Psi_y - \Psi_y \nabla^2 \Psi_x + \beta \Psi_x - Re^{-1} \nabla^4 \Psi = 0,$$

where $\Psi(x, y, t)$ is the total streamfunction and the subscripts denote partial differentiation with respect to time and the two space variables. The various quantities have been made nondimensional on the basis of typical length scales $L_x$ and $L_y$, in the zonal and meridional directions, respectively, $U$, a typical velocity scale, and $\varphi$, the dimensional amplitude of the forcing. The longitudinal scale $L_x$ is assumed to be of the order of magnitude of the dimensional zonal wavelength of the disturbance at the forcing latitude. In the numerical simulations described in Section 4, where the nondimensional zonal wavenumber is 2 and the nondimensional wavelength is $\pi$ units, $L_x$ is equal to the dimensional wavelength divided by $\pi$. The Laplacian operator in (1) is nondimensional with the $x$-derivative in the operator multiplied by the factor $\delta = L_x^2 / L_y^2$ which is the square of the aspect ratio. The parameter $Re$ is the Reynolds number and $\beta$ is the (nondimensional) gradient of planetary vorticity.

The total streamfunction is written as the sum of a contribution from the steady basic flow and a time-dependent perturbation:

$$\Psi(x, y, t) = \bar{\psi}(y) + \varepsilon \psi(x, y, t).$$

The basic flow is taken to be the initial $x$-independent flow. It is assumed that $\bar{\psi} \sim O(1)$, so the parameter $\varepsilon$ gives a measure of the magnitude of the perturbation relative to that of the basic flow. The corresponding velocity components for the basic flow and for the disturbance are denoted by $\bar{u}(y) = -\bar{\psi}_y, \bar{u}(x, y, t) = -\psi_y$, and $v(x, y, t) = \psi_x$. In terms of the perturbation vorticity $\zeta(x, y, t) = \nabla^2 \psi$, equation (1) can be written as

$$\zeta_t + \bar{\psi}_x \zeta_x + (\beta - \bar{u}''')\psi_x + \varepsilon(\psi_x \zeta_y - \psi_y \zeta_x) - Re^{-1} \nabla^2 \zeta - Re^{-1} \varepsilon^{-1} \bar{u}''' = 0.$$

The primes denote differentiation with respect to $y$. Note that the last term in equation (3) must be included because the mean streamfunction $\bar{\psi}(y)$ used in the numerical solutions described here does not satisfy the viscous equation.

When $\varepsilon$ and $Re^{-1}$ are both assumed to be nonzero in equation (3), the relative importance of viscosity and nonlinearity is measured by the parameter $\lambda = k Re^{-1} \varepsilon^{-3/2}$, where $k$ is the wavenumber of the disturbance. This parameter is defined to be the cube of the ratio of the critical layer thickness according
to the viscous steady theory to that in the nonlinear theory [7]. In all the investigations described in this paper, $\varepsilon$ and $Re$ are chosen so that $\lambda \ll 1$; this is an appropriate assumption for geophysical flows.

If the nonlinear and viscous terms are neglected in (3), the resulting equation may be solved by the normal mode approach. The solution is assumed to take the form

$$
\psi(x, y, t) = \phi(y)e^{i(k(x-c)t)}
$$

with $k$ and $c$ real, and the amplitude $\phi(y)$ is given by the Rayleigh–Kuo equation

$$
\phi_{yy} + \left(-\delta k^2 + \frac{\beta - \bar{u}''}{\bar{u} - c}\right) \phi = 0,
$$

(4)

which is singular if $\bar{u} = c$ at some critical latitude $y = y_c$. This equation can be solved using the method of Frobenius to give the solutions

$$
\phi_A(y) = (y - y_c) + \frac{\beta - \bar{u}''}{2\bar{u}''} (y - y_c)^2 + \cdots,
$$

(5)

$$
\phi_B(y) = 1 + \cdots + \frac{\beta - \bar{u}''}{2\bar{u}''} \phi_A \log(y - y_c) + \cdots,
$$

(6)

where quantities evaluated at $y_c$ have been indicated with the subscript $c$ and the primes denote differentiation with respect to $y$. The general solution can be written as a linear combination

$$
\phi = a\phi_A + b\phi_B
$$

and the constants $a$ and $b$ determined from the boundary conditions. To accommodate the negative values of $(y - y_c)$, the logarithm is defined to be $\log(y - y_c) = \log|y - y_c| + i\theta$ for $y < y_c$, where $\theta$ is termed as the logarithmic phase shift. Thus, there is a discontinuity in the solution $\phi_B$ across the critical layer. Alternatively, one can write

$$
\phi = a^\pm \phi_A + b^\pm \phi_B,
$$

(7)

with the definition $\log(y - y_c) = \log|y - y_c|$, so that the discontinuity appears in the constants instead. This definition is used in Section 3. Lin [6] showed that $\theta = -\pi$, by including viscosity in the critical layer, carrying out an asymptotic analysis which showed that the critical layer thickness is $Re^{-1/3}$, and then taking the limit $Re \to \infty$. The phase change is found to be $-\pi$ in the time-dependent linear inviscid problem for forced waves [10] as well, but when nonlinearity is included [11], the phase change goes to zero for large time.

### 2.2. Wave–mean-flow interactions

In the nonlinear time-dependent problem given by Equation (3), the mean flow is modified in the vicinity of the critical layer as a result of the nonlinear interactions. The question arises as to how the “mean” flow should be defined if the forcing is a localized wave packet. In the case of a periodic forcing $e^{ikx}$, the mean is generally taken to be an average over a zonal wavelength, $2\pi/k$. At late time, the perturbation develops higher harmonics and a zero
Wavenumber component and may be written as a Fourier series

\[ \psi(x, y, t) = \sum_k \hat{\psi}_k(y, t)e^{ikx}. \]  

The total streamfunction can thus be written as

\[ \Psi(x, y, t) = \bar{\psi}(y) + \psi_0(y, t) + \varepsilon \psi_1(x, y, t), \]

where \( \bar{\psi}(y) \) is the initial streamfunction, \( \psi_0(y, t) = \varepsilon \hat{\psi}_{k=0}(y, t) \) is the wave-induced mean streamfunction and \( \psi_1(x, y, t) \) comprises the periodic part of the disturbance only, not including the zero wavenumber component. The wave-induced mean velocity is \( u_0(y, t) \). Substituting this expression for \( \psi_1 \) into (1) and averaging over a zonal wavelength gives

\[ \frac{\partial u_0}{\partial t} - \nu \frac{\partial^2 u_0}{\partial y^2} = -\varepsilon^2 \frac{\partial}{\partial y} (\bar{u}\bar{v}) + \nu \bar{u}'', \]

which is the usual equation for the evolution of the mean flow. The quantity \( \bar{u}\bar{v} \) is the zonally averaged meridional momentum flux; the overbar denotes the average over a wavelength. In the “quasi-linear” calculations of Geisler and Dickinson [22], Matsuno [1] and others, \( \varepsilon \) is set to zero in Equation (3) and the resulting linear equation is solved along with Equation (10) (without the viscous terms in both equations). This idea had been used earlier by Stuart [23]. This type of approximation could be justified only if the zero wavenumber component were to be much larger in magnitude than the terms corresponding to the higher harmonics, but that is not the case in general, as we shall see from the numerical solutions described in Section 4.

In the case of a localized forcing with an amplitude that varies slowly with \( x \), the mean cannot be taken over a single wavelength, because the disturbance has a different amplitude over each wavelength interval. The most appropriate definition of the “mean” in this case would be an average taken over a length \( L \) equal to the nondimensional length of the packet. The packet length, however, increases with time, as we shall see in Section 4, so \( L \) must be chosen to be of the same order of magnitude as the nondimensional length of the packet at the forced boundary, but large enough that the packet is completely contained within an interval of length \( L \) at all times. In the simulations described here, the forcing is centered at some wavenumber \( k \) and takes the form

\[ \psi(x, \mu x, y_1, t) = A(\mu x, t)e^{ikx} + \text{c.c.,} \]

where \( y_1 \) is the forcing latitude and \( \mu \) is a small parameter. With the Gaussian amplitude function used here, the packet length and, hence \( L \), is proportional to \( \mu^{-1} \). The zonally localized perturbation may be expressed as a Fourier integral

\[ \psi(x, y, t) = \int_{-\infty}^{\infty} \hat{\psi}(\kappa, y, t)e^{i\kappa x} d\kappa. \]
The wave-induced mean streamfunction corresponds to the zero wavenumber term in the integrand, i.e.,

$$\tilde{\psi}_0(y, t) = \frac{\varepsilon}{L} \int_{-\infty}^{\infty} \psi \, dx = \frac{\varepsilon}{L} \tilde{\psi}(0, y, t), \quad (13)$$

and one can also define the wave-induced mean velocity $\tilde{u}_0(y, t) = -\tilde{\psi}_0 y$. By averaging (1) over the length $L$, one obtains an equation for the evolution of the mean velocity,

$$\left( \frac{\partial}{\partial t} - \nu \frac{\partial^2}{\partial y^2} \right) \tilde{u}_0 = -\frac{\varepsilon^2}{L} \frac{\partial}{\partial y} \int_{-\infty}^{\infty} uv \, dx + \nu \tilde{u}'' \quad (14)$$

which is the analogue of (10).

It is important to note, however, that a quasi-linear approximation cannot be employed in the wave packet problem. In the monochromatic case, a quasi-linear calculation involves using just two wavenumbers, the zero wavenumber and the fundamental. However, in the wave packet problem, the forcing and, hence, the disturbance, even at the linear stage, is comprised of not just a single wavenumber, but a continuous spectrum of wavenumbers and the components corresponding to all these wavenumbers must be included.

In the wave packet problem, taking an average over the length of the packet gives us information about the zero wavenumber component of the disturbance only. However, it is of interest also to examine the evolution of the components of the disturbance corresponding to other wavenumbers. To do this, one can write the total streamfunction as

$$\Psi(x, y, t) = \tilde{\psi}(y) + \psi_0(\mu x, y, t) + \varepsilon \psi_1(x, \mu x, y, t). \quad (15)$$

This decomposition is analogous to (9), with the difference being that, in (15), the second and third terms on the right-hand side depend on the slow scale ($\mu x$). The function $\psi_0$ does not depend explicitly on the fast $x$ scale; it is a long wave comprised of contributions from wavenumbers $\kappa$ in a small neighborhood of the zero wavenumber, and $\psi_1$ is comprised of contributions from the higher wavenumbers.

3. Asymptotic analysis

3.1. Previous asymptotic analyses

Before proceeding to describe the numerical simulations, a special case shall be examined in which an approximate analytic solution may be derived. This involves taking the limit as the aspect ratio $\delta \to 0$ in Equation (3), i.e., making the long-wave assumption. In a semi-infinite region $y_1 \geq y > \infty$, the basic flow is assumed to have constant shear, i.e., $\bar{u}'$ is taken to be a constant that can, without loss of generality, be set to 1. For simplicity, $y$ and $t$ are defined
in terms of the dimensional variables \(y^*\) and \(t^*\) by \(y = y^*/(\beta L_y)\) and \(t = \beta U t^*/L_y\), so that the constant \(\beta\) is replaced by 1. In addition, the viscous terms in Equation (3) are ignored and the equation becomes

\[
\left( \frac{\partial}{\partial t} + y \frac{\partial}{\partial x} \right) \psi_{yy} + \psi_x + \varepsilon(\psi_x \psi_{yy} - \psi_y \psi_{yx}) = 0. \tag{16}
\]

If \(\varepsilon\) is also set to zero and a monochromatic forcing is applied at the boundary \(y = y_1\), it is then quite straightforward to derive asymptotic solutions, valid for \(t \gg 1\), both inside and outside the critical layer, by taking the Laplace transform in time of the linear equation. Warn and Warn [10] derived solutions in this manner in the regions where \(|y|t \gg 1\), \(|y|t \sim 1\), and \(|y|t \ll 1\), namely, the regions outside, at the boundary of, and inside the critical layer, respectively.

When the forcing is assumed to take the form of a wave packet, instead of a monochromatic wave, the solution procedure becomes more complicated. Campbell and Maslowe [15] solved the linear equation, with the boundary condition \(\psi = e^{-\mu^2 x^2} e^{ikx}\) at the northern boundary, \(y = y_1\). It was necessary to take a Fourier transform in \(x\), as well as a Laplace transform in time, and to make use of the fact that most of the contribution to the solution would come from a small neighborhood around the central wavenumber \(k\). In this way, solutions were derived for the outer and inner regions and the critical layer thickness was seen to be \(\mu^{-1}\). The outer solution was found to be

\[
\psi(x, y, t) = e^{-\mu^2 x^2} e^{ikx} \left\{ h_1(y) + h_2(y) \frac{e^{-ikyt}}{k^2 y t^2} + h_3(y) \frac{e^{-iky_1 t}}{k^2 y_1 t^2} \right. \\
+ \left. \frac{\mu^2}{k^2} \frac{2x - yt}{t} \left( h_2(y) e^{-ikyt} + h_3(y) e^{-iky_1 t} \right) + O(\mu^4) \right\} + \text{c.c.}, \tag{17}
\]

where

\[
h_1(y) = \frac{(-y)^{1/2} K_1(2[-y]^{1/2})}{(-y_1)^{1/2} K_1(2[-y_1]^{1/2})},
\]

\[
h_2(y) = -\frac{1}{2[y - y_1]^{1/2} K_1(2[y - y_1]^{1/2})},
\]

and

\[
h_3(y) = 2[y_1 - y]^{1/2} K_1(2[y_1 - y]^{1/2}),
\]

where \(K_1\) denotes the modified Bessel function of order 1. In the limit as \(\mu \to 0\), the forcing becomes monochromatic and the periodic solution of Warn and Warn [10] is obtained, comprising the first three terms only, without the factor \(e^{-\mu^2 x^2}\). In the critical layer, \(|y|t < O(1)\), the solution was found by
asymptotic matching and multiple scales to be
\[ \psi \sim \alpha_1 e^{-\mu^2 x^2} e^{ikx} \left\{ 1 - y \log |y| + \left( 1 - 2\gamma - \frac{i\pi}{2} \right)y \right. \\
+ \left( \text{sgn } y \right) \frac{E_2^{(sgn y)}(ik|y|t)}{ikt} + O(\mu^2) \right\} + \text{c.c.}, \quad (18) \]
where \( \alpha_1 = \{-y_1\}^{-1/2} K_1(2\{-y_1\}^{1/2})^{-1} \), \( \gamma \) is Euler’s constant and \( E_k^+(z) \) and \( E_k^-(z) \) are defined as
\[ E_k^\pm(z) = \int_1^\infty \frac{e^{\mp zu}}{u^k} du, \quad (19) \]
for \( \text{Re}(z) \geq 0 \) and \( \text{Re}(z) \leq 0 \), respectively.

One can observe from these solutions that the phase change is still \(-\pi\) as in the monochromatic case and that there is a discontinuity in the total meridional momentum flux across the critical layer. Thus, it would appear that the qualitative behavior of the wave packet solution is not significantly different from that of the corresponding monochromatic solution. This observation was supported by the numerical results of Campbell and Maslowe [15] for the more general linear problem in which the aspect ratio \( \delta \) is nonzero. In contrast, the nonlinear numerical solutions with the wave packet forcing that will be described in Section 4 yield several features that are completely different from the nonlinear strictly periodic case. This suggests that it would be interesting to extend the analytic investigations described in [15] to include the nonlinear effects, and thus provide a better understanding of the results of the numerical experiments. The simplest way to do this would be to carry out an analysis along the lines of the nonlinear solution of Warn and Warn [11], using the linear wave packet solutions (17) and (18) as the starting point.

3.2. Asymptotic solution in the outer region

To solve the nonlinear equation (16) with the wave packet forcing, an \( O(\varepsilon) \) term is added to the linear solution (17), i.e., the solution is written as
\[ \psi \sim \psi^{(0)} + \varepsilon \psi^{(1)} + \mu^2 \psi^{(2)}, \quad (20) \]
where
\[ \psi^{(0)} = \phi^{(0)}(y, t) e^{-\mu^2 x^2} e^{ikx} + \text{c.c.} \quad (21) \]
denotes the \( O(1) \) terms in (17) and (18) and \( \psi^{(2)} \) denotes the \( O(\mu^2) \) wave packet terms. The nonlinear term \( \psi^{(1)} \) can be found by substituting this expression into the nonlinear equation (16); we find that \( \psi^{(1)} \) must satisfy
\[ \left( \frac{\partial}{\partial t} + y \frac{\partial}{\partial x} \right) \psi^{(1)}_{yy} + \psi^{(1)}_x = -\left( \psi^{(0)} \psi^{(0)}_{yy} - \psi^{(0)}_y \psi^{(0)}_{yy} \right). \quad (22) \]
This suggests that $\psi^{(1)}$ take the form

$$
\psi^{(1)} = \phi^{(1)}(y, t)e^{-2\mu^2x^2}e^{2ikx} + \bar{\phi}^{(1)}(y, t)e^{-2\mu^2x^2} + \text{c.c., (23)}
$$

so that

$$
\left( \frac{\partial}{\partial t} + 2iky \right) \phi^{(1)}_{yy} + 2ik\phi^{(1)} = -ik\left( \phi^{(0)}\phi^{(0)}_{yy} - \phi^{(0)}_{y}\phi^{(0)}_{y} \right) \tag{24}
$$

and

$$
\bar{\phi}^{(1)}_{yyt} = -ik\left( \phi^{(0)*}\phi^{(0)}_{yy} - \phi^{(0)*}_{y}\phi^{(0)}_{y} \right), \tag{25}
$$

where the asterisks denote complex conjugates. In the outer region, $|y|t \gg 1$ and the largest terms in the expressions on the right hand side of equations (24) and (25) are $O(t)$; to this order,

$$
\left( \frac{\partial}{\partial t} + 2iky \right) \phi^{(1)}_{yy} + 2ik\phi^{(1)} \sim -k^2t \frac{h_1(y)h_2(y)}{y}e^{-ikyt} \tag{26}
$$

and

$$
\bar{\phi}^{(1)}_{yyt} \sim -k^2t \frac{h_1^*(y)h_2(y)}{y}e^{-ikyt}. \tag{27}
$$

In the inner region, $|y|t \ll 1$ and so the leading-order terms on the right-hand side of Equations (24) and (25) are of order $O(y^{-2})$ and $O(ty^{-1})$. Two inner equations are obtained. The solution of the four equations for $\phi^{(1)}$ and $\bar{\phi}^{(1)}$ in the outer and inner regions is described in Ref. [24]: the $\phi^{(1)}$ equations are solved by taking a Laplace transform in time of each equation; $\bar{\phi}^{(1)}$ is found by direct integration of Equation (27) and the corresponding inner equation. The solution to each of the equations takes a similar form to the corresponding solution obtained by Warn and Warn [11], but each one is multiplied by the factor of $e^{-2\mu^2x^2}$. Since the long-wave assumption has been made here and the mean velocity has been set to $\bar{u} = y$, some terms are simplified; for example, factors of $(\beta - \bar{u}'')/\bar{u}'$ in the solutions in [11] are simply unity in the solution presented here. However, the qualitative behavior of the solutions is not significantly different from what it would be without these simplifying assumptions. It is found that the $O(\varepsilon)$ solution in the outer region is of the form

$$
\psi^{(1)}(x, y, t) \sim e^{-2\mu^2x^2}e^{2ikx}\left\{ G_1(y) + O\left( \frac{1}{t} \right) \right\} + e^{-2\mu^2x^2}\left\{ i\frac{h_1^*(y)h_2(y)}{ky^2t}e^{-ikyt} + Q_1(t) + Q_2(y) \right\} + \text{c.c., (28)}
$$
while the $O(\varepsilon)$ inner solution takes the form

$$\psi^{(1)}(x, y, t) \sim e^{-2\mu^2 x^2} e^{2ikx} \{O(t)\}$$

$$+ e^{-2\mu^2 x^2} \left\{ -\frac{E_2^{(\text{sgny})}(ik|y|t)}{y} + ikt \log |y| + Q_1(t) + Q_2(y) \right\} + \text{c.c.} \quad (29)$$

Explicit expressions for the $\phi^{(1)}$ terms are not required for the present analysis; it suffices to note that the leading-order terms of $G_1$ are $O(y^{-1})$ for $y \to 0$. To determine the functions $Q_1$ and $Q_2$ in the outer solution, we must match with the corresponding terms in the inner solution. The term containing the exponential integral, in the last line of Equation (29), can be expanded as

$$\frac{E_2^{(\text{sgny})}(ik|y|t)}{y} \sim \left\{ (\text{sgn } y)ikt \log(k|y|t) - (\text{sgn } y)\frac{\pi}{2}kt - \frac{1}{y} + \frac{k^2yt^2}{2} + O(y^2t^3) \right\} \quad (30)$$

and, for this expression to be nonsingular at $y = 0$, one must choose

$$Q_1(t) \sim -(\text{sgn } y)\frac{\pi}{2}kt \quad (31)$$

and

$$Q_2(y) \sim -\frac{1}{y}. \quad (32)$$

Identical expressions for the corresponding terms in the outer solution are obtained by matching the outer solution (28) to the inner solution as $y \to 0$.

These solutions are valid for $x, t \leq O(\mu^{-1})$, this being the assumption that was made to derive the $O(\mu^2)$ terms in the linear solution of Campbell and Maslowe [15]. However, the expansions break down in the critical layer region for larger $t$; for these large values of $t$, we can derive a nonlinear solution in a similar fashion to Warn and Warn [11]. Expressions (28) and (29) correspond to their early time solutions and can be used as initial conditions for our analysis.

The question now arises as to the relative magnitude of the two parameters $\mu$ and $\varepsilon$. In the linear wave packet problem [15], the critical layer thickness was found to be $O(\mu)$; in the nonlinear periodic problem [11], it is $O(\varepsilon^{1/2})$. At this point, it is appropriate to define the parameter $\Lambda$ to be the ratio of these two thicknesses, i.e., $\Lambda = \mu \varepsilon^{-1/2}$. There are three cases to consider:

1. $\Lambda \ll 1, \mu \ll \varepsilon^{1/2}$, in which the nonlinear terms are dominant over the wave packet terms;
2. $\Lambda \approx 1, \mu \approx \varepsilon^{1/2}$, in which the nonlinear terms and wave packet terms are of the same order of magnitude in the outer region and the wave packet critical layer is approximately the same as the nonlinear critical layer;
3. $\Lambda \gg 1, \mu \gg \varepsilon^{1/2}$, with the wave packet terms being dominant over the nonlinear terms.
In case (1), the variation of the wave packet envelope with $x$ would be so slow that, on each interval $2(n - 1)\pi \leq x \leq 2n\pi$, $n = 0, \pm 1, \pm 2, \ldots$, one could obtain a periodic solution as done by Warn and Warn [11], with the wave packet terms appearing only at higher order. This case is discussed in more detail in the Appendix. In case (3), the nonlinear terms would be sufficiently small that the solutions (28) and (29) derived already would remain valid for long time even in the critical layer. Of the three cases, the second is of the most interest in the present context and the rest of this section concerns this balance.

To examine the long-time evolution of the flow, it is necessary to define the slow scales $T = \varepsilon^{1/2}t$ and $X = \mu x = \varepsilon^{1/2}x$ and the fast scale $\eta = yt$. Following Warn and Warn, the small transient terms of $O(e^{-iky\varepsilon t})$ are ignored in the rest of this section; however, the terms of $O(e^{-ikyt})$ must be retained, because their derivatives with respect to $y$ would be multiplied by a factor of $t$ and would thus be of an order of larger magnitude. The outer solution can then be written in terms of these new variables as

$$
\psi(x, y, \eta, T) \sim h_1(y)e^{-X^2}e^{ikx} + \varepsilon^{1/2}\left\{-(\text{sgn} y)\frac{\pi}{2}kTe^{-2X^2}\right\} \\
+ \varepsilon\left\{h_2(y)\left(\frac{2X - yT}{k^2T} + \frac{1}{k^2yT^2}\right)e^{-X^2}e^{-ik(x-\eta)}
+ G_1(y)e^{-2X^2}e^{2ikx} - \frac{e^{-2X^2}}{y}\right\} + O(\varepsilon^{3/2}\log \varepsilon) + \text{c.c.} \quad (33)
$$

Therefore, the long-time solution in the outer region must be of the form

$$
\psi \sim \psi^{(0)} + \varepsilon^{1/2}\psi^{(1/2)} + \varepsilon\psi^{(1)} + \varepsilon^{3/2}\log \varepsilon\psi^{(3/2)} + \varepsilon^{3/2}\psi^{(3/2)} + \cdots \quad (34)
$$

This leads to a sequence of equations which are solved in the Appendix. Their solutions are of the form

$$
\psi^{(0)} \sim A^{(0)}(x, T)\phi_A(y) + B^{(0)}(x, T)\phi_B(y) + H^{(0)}(X, y, T), \quad (35)
$$

$$
\psi^{(1/2)} \sim A^{(1/2)}(x, T)\phi_A(y) + B^{(1/2)}(x, T)\phi_B(y)
+ \tilde{A}^{(1/2)}(x, T)F_A(y) + \tilde{B}^{(1/2)}(x, T)F_B(y)
+ H^{(1/2)}(X, y, T), \quad (36)
$$

$$
\psi^{(1)} \sim \frac{\tilde{B}^{(1)}(x, T)}{y} + \frac{h_2(y)e^{-(X-\eta)T^2}e^{ik(x-\eta)}}{k^2yT^2} + \text{c.c.}
+ H^{(1)}(X, y, T), \quad (37)
$$

$$
\psi^{(3/2)} \sim A^{(3/2)}(x, T)\phi_A(y) + B^{(3/2)}(x, T)\phi_B(y)
+ H^{(3/2)}(X, y, T), \quad (38)
$$
where
\[ \phi_A(y) = y^{1/2} J_1 \{2y^{1/2}\} \sim y + O(y^2) \quad (39) \]
and
\[ \phi_B(y) = -\pi y^{1/2} \gamma_1 \{2y^{1/2}\} \sim 1 - y \log y + (1 - 2y) y + O(y^2 \log y), \quad (40) \]
\( J_1 \) and \( \gamma_1 \) being the Bessel functions of order 1, and the other functions are as defined in the Appendix.

With these solutions for the outer region, sufficient information has been obtained about the outer solution to be able to investigate the long-time inner solution up to \( O(\varepsilon^{1/2}) \). In particular, the outer solution obtained includes the modifications to Warn and Warn’s nonlinear periodic solution resulting from using a localized wave packet forcing instead. The most important of these arises in the term \( H(1/2)(X, y, T) \). This is shown in the Appendix to take the form \( H(1/2) \sim N(1/2)(X, T) + O(y) \), where \( N(1/2) \) varies like \( e^{-2X^2/T^2} \); thus, as \( T \to \infty \), the length of the packet increases without bound. In addition, there is a factor of \( e^{-(X-yT)^2} \) in the \( O(\varepsilon) \) solution, not present in the corresponding term of the monochromatic solution. This factor indicates that the disturbance propagates longitudinally in the direction of positive (negative) \( X \) above (below) the critical layer.

### 3.3. Asymptotic solution in the inner region

Inside the critical layer, an inner variable \( Y = \varepsilon^{-1/2} y \) is defined. The outer solution, written in terms of this variable, is then
\[ \psi \sim B^{(0)}(x, X, T) + N^{(0)}(X, T) \]
\[ + \frac{1}{2} \varepsilon^{1/2} \log \varepsilon \left\{ -Y B^{(0)}(x, X, T) + B^{(1/2)}(x, X, T) \right\} \]
\[ + \varepsilon^{1/2} \left\{ -Y \log Y B^{(0)}(x, X, T) + (1 - 2y) Y B^{(0)}(x, X, T) \right. \]
\[ + Y A^{(0)}(x, X, T) + \log Y B^{(1/2)}(x, X, T) + B^{(1/2)}(x, X, T) \]
\[ + Y M^{(0)}(X, T) + N^{(1/2)}(X, T) + \frac{B^{(1)}(x, X, T)}{Y} - \frac{e^{-2X^2}}{Y} \]
\[ + \frac{e^{-(X-yT)^2} e^{ik(x-yT)}}{k^2 Y T^2} \left\} + O(\varepsilon^{3/2} \log \varepsilon) + \text{c.c.}, \quad (41) \]
where \( N^{(0)}(X, T), N^{(1/2)}(X, T) \) are the functions defined in the Appendix and \( M^{(0)}(X, T) \) is the \( O(y) \) component of \( H^{(0)} \). Note that the linear effects of the localized wave packet forcing, i.e., the \( O(\mu^2) \) terms in (17), do not...
appear at this order, because they are $O(\varepsilon)$. From (41), it is seen that the total streamfunction in the inner region must be of the form

$$\Psi \sim \varepsilon \Psi^{(1)}(x, X, Y, T) + \varepsilon^{3/2} \log \varepsilon \Psi^{(3/2)\ast}(x, X, Y, T) + \cdots.$$  

(42)

Substituting this into (1) gives a series of inner equations that can be solved and the results matched to the early-time inner solution (18 and 29) as $T \to 0$ and to the late-time outer solution (41) as $Y \to \pm \infty$. At lowest order,

$$\Psi^{(1)\ast}_{YTT} + \Psi^{(1)}_{XYY} - \Psi^{(1)}_{YYx} = 0.$$  

(43)

The solution to equation (43) satisfying the initial condition (18) and the boundary condition (41) is

$$\Psi^{(1)} \sim -\frac{Y^2}{2} + B^{(0)}(x, X, T) + N^{(0)}(X, T) + \text{c.c.},$$  

(44)

where $B^{(0)} \to 2 \alpha_1 e^{-X^2} \cos kx$ and $N^{(0)} \to 0$ as $T \to 0$. Note that $\Psi^{(1)}$ is the leading-order total streamfunction and the first term on the right-hand side corresponds to the mean flow. Integrating Equation (43) with respect to $Y$, we obtain

$$\Psi^{(1)}_{YY} + \Psi^{(1)}_{Y} - \Psi^{(1)}_{YYX} = f^{(1)}(x, X, T),$$  

(45)

where $f^{(1)}(x, X, T)$ is the leading-order pressure gradient in the $x$-direction and, as $Y \to \pm \infty$, it approaches $-B^{(0)\pm}(x, X, T)$. The pressure gradient must be continuous across the critical layer, so $[B^{(0)}] = 0$, where the square brackets denote the jump across the critical layer. Consequently, $[N^{(0)}] = 0$.

The equations for $\Psi^{(3/2)\ast}$ and $\Psi^{(3/2)}$ are

$$\left( \frac{\partial}{\partial T} + Y \frac{\partial}{\partial x} \right) \Psi^{(3/2)\ast}_{YY} + \Psi^{(1)}_{X} \Psi^{(3/2)\ast}_{YYX} = 0$$  

(46)

and

$$\left( \frac{\partial}{\partial T} + Y \frac{\partial}{\partial x} \right) \Psi^{(3/2)}_{YY} + \Psi^{(1)}_{X} \Psi^{(3/2)}_{YYX} + \Psi^{(1)}_{X} = 0.$$  

(47)

The fact that $[B^{(0)}] = 0$ means that $[\tilde{B}^{(1/2)}] = 0$ as well, and so $\Psi^{(3/2)\ast}$ is also continuous across the critical layer. Equation (47) is the familiar Stewartson-Warn-Warn (SWW) critical layer equation, which was solved numerically by Warn and Warn [11] and solved analytically by Stewartson [12] for the special case $B^{(0)}(x) = \cos x$. In the present study, $B^{(0)}$ is a function of the slow scale $X$, as well as of $x$ and $T$, and the simplest choice of $B^{(0)}$ satisfying the initial and boundary conditions would be $B^{(0)}(x, X) = 2\alpha_1 e^{-X^2} \cos kx$, corresponding to the case solved by Stewartson (see Appendix A.2). However, even with this choice of $B^{(0)}$, analytic solution of the equation would
prove complicated and it would be easier to solve it numerically using (29) and (41) as initial and boundary conditions. However, given the fact that the numerical solution of the full nonlinear equation without the long-wave assumption has already been carried out here, no attempt shall be made to solve this simplified equation as well.

Without solving the equation, one can, however, use it to make some observations about the behavior of the critical layer with regard to the absorption and reflection of the disturbance. As in the periodic case, it is at this order that the discontinuity in the solution appears. Integrating the equation with respect to $Y$ gives

$$
\Psi_{YT}^{(3/2)} + Y\Psi_{YX}^{(3/2)} - \Psi_{X}^{(3/2)} + B^{0}_x\Psi_{YY}^{(3/2)} + YB^{0}_x = f^{(3/2)}(x, X, T),
$$

(48)

where $f^{(3/2)}(x, X, T)$ is the pressure gradient in the $x$-direction. This gives, on matching with the outer solution,

$$
-2\gamma B^{0}_T + A^{0}_T - B^{(1/2)}_x + M^{0}_T = f^{(3/2)}(x, X, T),
$$

(49)

which is the analogous result to that obtained by Warn and Warn [11, p. 59]. The requirement that the pressure gradient be continuous across the critical layer means that $f^{(3/2)} = -2\gamma B^{0}_T + M^{0}_T$. Also, $[A^{0}_T - B^{(1/2)}_x] = 0$ and $[M^{0}_T] = 0$. The pressure must go to zero as $x \to \pm \infty$, so $M^{0}_T = 0$, as in the periodic case. Thus, $M^{0}_T = 0$ and $N^{(1/2)}$ gives the leading-order component of the part of the disturbance centered at the zero wavenumber and corresponds to the function $\psi_0(\mu x, y, t)$ defined in Equation (15). The pressure gradient is $f^{(3/2)} = -2\gamma B^{0}_T$; so in the Stewartson limit, where $B^{0}_T$ is steady, there would be no pressure gradient.

Equation (48) is integrated with respect to $x$ from $-\infty$ to $+\infty$ and then again with respect to $Y$ across the critical layer. On matching with the outer solution, it is seen that $\Psi^{(3/2)}_Y \to N^{(1/2)}_T(X, T)$ and $\Psi^{(3/2)}_Y \to A^{0}_T$ as $Y \to \pm \infty$. Thus, the result of the integration is

$$
\left[ \int_{-\infty}^{\infty} N^{(1/2)}_T \, dx \right] = -\int_{-\infty}^{\infty} B^{0}_x[A^{0}_T] \, dx,
$$

(50)

which corresponds to Equation (14) derived in Section 2 for the evolution of the wave-induced mean flow. The integral on the right-hand side is the jump across the critical layer in the total momentum flux over the length of the packet. This term can be written as

$$
[F] = \int_{-\infty}^{\infty} \hat{F}(\kappa, T) \, d\kappa \sim -\int_{-\infty}^{\infty} i\kappa \hat{B}[\hat{A}] \, d\kappa,
$$

(51)

where $\hat{A}(\kappa, T)$ and $\hat{B}(\kappa, T)$ are the Fourier transforms of $A^{0}_T$ and $B^{0}_T$ and $\hat{F}(\kappa, T)$ is the momentum flux corresponding to the wavenumber $\kappa$. 
The analogue of (50) and (51) for the case of a monochromatic forcing is

$$[N_T^{(1/2)}] = -\overline{B_x^0[A(0)]} = -\sum_k i\kappa \hat{B}_\kappa [\hat{A}_\kappa^*] = \sum_k [\hat{F}_\kappa] = [F],$$

(52)

where the overbar denotes an average over a wavelength $2\pi/k$ and $\hat{F}_\kappa(T)$ is the Reynolds stress corresponding to the wavenumber $\kappa$. By examining this equation, Warn and Warn [11] argued that the jump in the Reynolds stress corresponding to the forced wavenumber $\kappa = k$ would eventually go to zero.

For the mean flow to attain a steady state in the outer region, $[\Psi_T^{(3/2)}]$ and hence $[N_T^{(1/2)}]$ would have to be zero, so if $\hat{B}_\kappa[\hat{A}_\kappa^*] = 0$ for the higher harmonics, then it would have to be zero for $\kappa = k$ as well. Consequently, the jump $[F]$ in the overall Reynolds stress would be zero, implying perfect reflection of the incident disturbance.

With the spatially localized forcing, the situation would be different. In Appendix A.1, it was seen that $[N_T^{(1/2)}] \sim e^{-2X^2/T^2}$ for large time. The function $e^{-2X^2/T^2}$ is a Gaussian function whose nonzero length in the $X$-direction is proportional to $T$. For fixed $T$, it is $\sim O(1)$ for $X \sim O(1)$ and goes to zero as $X \to \pm \infty$. Its length increases with time and, in the limit as $T \to \infty$, it becomes infinite. A diagram showing the evolution of the function $e^{-2X^2/T^2}$ with time is given in Figure A.1. The form of $[N_T^{(1/2)}]$ means that the momentum flux jump $[F] = [N_T^{(1/2)}] \sim T^{-3} e^{-2X^2/T^2}$, which goes to zero as $T \to \infty$. Thus, although $[F]$ does approach zero as in the monochromatic case, it does so on the time scale that it takes for the length of the packet to become infinite. Indeed, the numerical solutions show that the packet increases in length with time and $[F]$ remains positive up to $t = 100$.

In the special case where the parameter $r$ (defined by the expression (A.30) in the Appendix) is zero, $B(0)$ is steady and so the packet length cannot increase without bound. At large time, $[N_T^{(1/2)}] \to 0$ and $[F] \to 0$, using the same reasoning as in the monochromatic case. In fact, it was noted in Appendix A.1 that, when $r = 0$, the function $\phi_B$ in (A.24) is zero for large $T$, so the discontinuity in the solution for $N_T^{(1/2)}$ would vanish for large time. Brunet and Haynes [5] showed, for a case in which $r = 0$, that the time-integrated absorptivity $\alpha(T) = \int_0^T [F(T')] dT'$ would be bounded, i.e., $[F]$ would go to zero for large time. Their reasoning could be applied to the configuration used here if $r = 0$ and a bound could be obtained in a similar manner.

So far in this section, it has been assumed that $\mu = \epsilon^{1/2}$. If it is assumed instead that $\mu \ll \epsilon^{1/2}$, then the late-time behavior of the solution is like that in the monochromatic case, at least to leading order. This situation is described in Appendix A.3. Finally, it must be noted that the conclusions of this section should be true in general, i.e., even without the long-wave assumption. The main difference would be that the solutions $\phi_A$ and $\phi_B$ would be given by (5) and (6) instead of (39) and (40). To leading order in $y$, the qualitative behavior
of the solutions would be unchanged. The rest of this paper is concerned with the general case.

4. Numerical experiments

4.1. Configuration and numerical methods

In the numerical simulations, the nonlinear equation (3) was solved on a rectangular domain with the wave source at the northern boundary, \( y = y_1 = 5 \). The forcing took the form

\[ \psi(x, y_1, t) = f(t)e^{-\mu^2x^2}e^{ikx}, \] (53)

with

\[ f(t) = \begin{cases} t/t_1 & t < t_1 \\ 1 & t \geq t_1 \end{cases}, \] (54)

with \( t_1 \) set to 5. The reason for this slow “switch-on” of the forcing was to prevent numerical instabilities from developing near the inflow boundary, which could occur if the time step used in the numerical approximation was not sufficiently small. In fact, in his numerical solution of the periodic problem, Béland [25] went further and set the linear terms to zero during the switch-on time; probably this procedure was needed because most of his simulations were completely inviscid. Here this precaution was found to be unnecessary provided the viscosity was nonzero. The basic flow was set to \( \tilde{u} = \tanh y \), so that the critical latitude was located at \( y_c = 0 \). At the southern boundary \( y = y_2 = -5 \), a time-dependent radiation condition similar to that developed by Béland and Warn [26] was applied, its purpose being to prevent any wave activity that might be transmitted beyond the critical layer from being reflected at the outflow boundary.

In choosing values for the various nondimensional parameters, it was important to ensure that the disturbance would be able to propagate toward the critical layer without amplification or attenuation and that the Rayleigh–Kuo criterion for stability, \( \beta - \tilde{u}'' \neq 0 \), would be satisfied everywhere in the domain [25]. Taking these requirements into consideration, a set of values of the parameters was chosen for a standard run and the results were used for comparison with those obtained by varying the different parameters. The values selected were \( \varepsilon = 0.02, \lambda = 0.01, \beta = 1.0, \) and \( \delta = 0.2 \). Because \( \lambda \ll 1 \), there was only a small amount of viscosity relative to the extent of nonlinearity. The viscosity was necessary to prevent the accumulation of wave activity at the high end of the wavenumber spectrum, as well as near the forced boundary. The forcing was centered at wavenumber \( k = 2 \) with \( \mu = 0.2 \), so the packet was approximately 20 nondimensional units long and there were several wave oscillations within the packet length. The asymptotic analysis indicates that the ratio \( \mu / \varepsilon^{1/2} \) determines the behavior of the wave packet critical layer with regard to the relative importance of absorption to reflection.
With this choice of $\varepsilon$ and $\mu$, this ratio was equal to $\sqrt{2}$. Most of the simulations shown here were carried out over an interval within the range $-20 \leq x \leq 40$. According to the asymptotic analysis, the packet length may increase with time; however, in practice, the solution domain is finite and the packet cannot become infinite. Moreover, the solution procedure requires that everything be zero at the boundaries. For this reason, the numerical simulations were all stopped at $t \sim O(100)$, before the solution became nonzero at the boundaries. The monochromatic calculations ($\mu = 0$) were carried out with the forcing at wavenumber 2 and all other parameters the same as in the wave packet case, but over the interval $0 \leq x \leq 2\pi$ instead and with periodic boundary conditions.

The solution of equation (3) was accomplished by taking a Fourier transform in the $x$ direction, with the nonlinear terms calculated using a pseudo-spectral method. The Fourier integrals were evaluated using a trapezoidal approximation and a variation of the fractional Fourier transform (FRFT) method (Bailey and Swarztrauber [27, 28]). This method is a generalization of the standard Fast Fourier Transform (FFT) algorithm that allows greater flexibility in the choice of the computational domain in both physical and wavenumber space. The application of the method to the present problem is described in [24].

The derivatives in the meridional direction were approximated using fourth-order compact finite differences. These methods have a number of advantages over other methods: they give a better representation of short length scales than traditional finite difference schemes and for that reason are comparable to spectral methods [29]. The finite difference mesh had nonuniform spacing: a very fine mesh in the critical layer region, a coarser mesh in the region below and intermediate spacing above. Derivatives on the nonuniform mesh were evaluated by first calculating them on a uniform mesh and then transforming them into derivatives on the nonuniform mesh. A description of the procedure is given in Ref. [30]. The choice of $\varepsilon = 0.02$ gives a nonlinear critical layer thickness of $\varepsilon^{1/2} \approx 0.141$ nondimensional units. By experimenting with the mesh spacing, reducing it until the computed solution was found to be independent of the mesh size, we found that a minimum of about 20 points was needed in the critical layer to accurately represent the evolution of the disturbance. The time derivatives were discretized using the second-order Adams–Bashforth method and the time step size was set to 0.02 nondimensional units, a value small enough to ensure stability of the computations.

### 4.2. Results of the standard run

Numerical simulations with the standard set of parameters were carried out over the nondimensional time interval from $t = 0$ to 100. Figures 1 and 2 show contour plots of the perturbation streamfunction at early time ($t = 10$) and late time ($t = 100$). With this configuration, the $x$-component of the group velocity of the packet is positive above the critical layer, so the packet propagates to the right, as seen in Figure 1a, and is incident on the critical layer in the region...
Figure 1. Wave packet forcing: perturbation streamfunction at (a) $t = 10$ and (b) $t = 100$. Parameters: $\varepsilon = 0.02$, $\lambda = 0.01$, $\mu = 0.2$, $k = 2$.

$-5 < x < 15$. At early time, it is completely absorbed at the critical layer, so there are no contours below the line $y = 5$. The corresponding contour plot for the case of monochromatic forcing (Figure 2a) also shows complete absorption of the incident waves at early time. In Figures 1b and 2b, it is seen that, at late time, there is still virtually no transmission of wave activity beyond the critical
layer and, in both cases, the direction of wave propagation above the critical layer appears to have changed. This would indicate that some reflection of the incident disturbance is taking place and that the northward-propagating reflected waves are superimposed on top of the incident waves. In Figure 1b,
the longitudinal extent of the packet has increased considerably both in the outer region and in the critical layer.

The simplest way to measure the extent of reflection or absorption of the disturbance is to examine the behavior of the meridional momentum flux \( F = -\int_{-\infty}^{\infty} uv \, dx \) in the vicinity of the critical layer. At early time, this was seen to be discontinuous across the critical layer, almost constant (negative) above and zero below, and consequently, the mean velocity was decelerated (became more negative) in the critical layer region. At each time step, the jump \([F]\) across the critical layer was calculated, normalized by the maximum value \([F]_{\text{max}}\) that it attained and the result plotted as a function of time. This is shown in Figure 3a for the wave packet forcing and in Figure 3b for the monochromatic forcing. When \( \varepsilon \) is set to zero for the duration of the calculation (the dashed line), the jump increases initially during the switch-on time, but eventually attains a steady state. In the nonlinear simulations (shown by the solid lines in Figures 3a and b), the early time evolution of \([F]\) is as in the linear case. According to the asymptotic analyses of the inviscid critical layer, one would expect the nonlinear regime to start around the time that \( t \sim O(\varepsilon^{-1/2}) \), i.e., around \( t = 10 \), although, with the inclusion of viscosity, it could be expected that it would be delayed slightly. In Figures 3a and b, the onset of the nonlinear regime is at approximately \( t = 11 \) and \( t = 13 \), respectively, when the solid and the dashed lines no longer coincide exactly. The difference in the evolution of the momentum flux between the periodic and the wave packet results then becomes apparent. In the periodic problem, the jump drops rapidly to zero and to negative values, indicating first reflection and then over-reflection of the waves, and then continues to oscillate between these states with an amplitude that decays with time. In contrast, with the wave packet forcing, although the jump in the momentum flux decreases, it remains positive all the way up to the end of the simulation. This means that, although there is some reflection, the extent of absorption is much larger and there is net absorption of momentum flux into the critical layer.

In Figure 4a, the Fourier spectrum of the jump in the momentum flux across the critical layer is shown at \( t = 10 \) and \( t = 100 \). This is the quantity \([\hat{F}(\kappa, t)]\) defined in equation (51). Because \([\hat{F}] \approx 0 \) for \(|\kappa| > 4\), only the range of wavenumbers \(-5 \leq \kappa \leq 5\) is shown. The integral of \([\hat{F}]\) over the range of \( \kappa \) values gives \( 2\pi \) times the total momentum flux jump. At late time, there is a shift in the peak of the spectrum of the jump from \( \kappa = \pm 2 \) toward the zero wavenumber and, around the zero wavenumber, a nonzero component to the jump has developed. The corresponding picture for the case of monochromatic forcing (Figure 4b) shows that, as noted in Section 3, the only nonzero contribution to the momentum flux jump is that corresponding to the forced wavenumber \( \kappa = \pm 2 \). At late time, \([\hat{F}_\kappa]\) is still zero for \( \kappa = \pm 4 \) and for all the higher harmonics not shown in the graph. The magnitude of \([\hat{F}_\kappa]\) for \( \kappa = \pm 2 \) decreases from its early time linear value and oscillates around zero as shown in Figure 3. At \( t = 100 \), it is positive.
Figure 3. Variation of the jump in the (normalized) momentum flux with t. The solid lines show the results obtained with $\varepsilon = 0.02$; the dashed lines results obtained with $\varepsilon = 0$. (a) Wave packet forcing: $\mu = 0.2$ (b) Monochromatic forcing: $\mu = 0$. Other parameters: $\lambda = 0.01$, $k = 2$.

The Fourier spectra of the perturbation streamfunction and vorticity at $y_c$ and at time $t = 100$ are shown in Figure 5 for the case of wave packet forcing. In Figure 5a, it is seen that the maxima corresponding to the zero wavenumber and the second harmonics $\kappa = \pm 4$ are of the same order of magnitude. Thus,
the inclusion of the higher harmonics in the nonlinear simulations is at least as important as the representation of the mean flow distortion. There is a transfer of wave activity to higher wavenumbers, but even at this late time, the extent of the wavenumber domain \((-20 < \kappa < 20\) used in the calculations is sufficient because the amplitude of \(\hat{\zeta}(\kappa, y, t)\) is still effectively zero at \(\kappa = \pm 20\) (Figure 5b).
Further insight into the evolution of the disturbance can be obtained by examining the evolution of the different wavenumber components, as suggested by (15). This idea is taken further than (15) and the disturbance is decomposed into three parts, instead of two: (a) the contribution from the wavenumbers in the immediate vicinity of the zero wavenumber; (b) the part due to wavenumbers centered around the zero wavenumber, i.e., $1 < |\kappa| < 3$ and; (c), the part arising from the higher harmonics, i.e., $|\kappa| > 3$. These are evaluated at the critical latitude at early and late time and plotted separately in Figure 6. The
Figure 6. Wave packet forcing: perturbation streamfunction at $t = 10$ (dashed lines) and $t = 100$ (solid lines) decomposed into the part centered (a) at zero wavenumber, (b) at the forced wavenumber, and (c) the contribution from the higher harmonics. Parameters: $\varepsilon = 0.02$, $\lambda = 0.01$, $\mu = 0.2$, $k = 2$.

first of these corresponds to the term $\psi_0(\mu x, y, t)$ in expression (15) and its leading order part is the function $N^{(1/2)}(X, T)$ in the asymptotic solution. The corresponding quantity for the velocity is denoted as $u_0(\mu x, y, t)$ and is shown in Figure 7. The wave-induced mean velocity $\bar{u}_0(y, t)$ at that latitude is
obtained by averaging $u_0$ over the length $L$, which is taken to be the length of the computational domain. At early time, $\psi_0$ is proportional to $e^{-2\mu^2x^2}$; however, at late time, it increases in magnitude and length and is nonzero over the whole of the interval shown. All four graphs show that the packet increases in longitudinal extent with time, as predicted by the asymptotic analysis.

This is also seen in the contours of the relative vorticity $\tilde{\psi}_{yy} + \epsilon \nabla^2 \psi$ and the absolute vorticity $\beta y + \tilde{\psi}_{yy} + \epsilon \nabla^2 \psi$. These are shown in Figures 8–11 for both types of forcing. At early time, the relative vorticity contours have the characteristic cat’s eye form in the critical layer; later the cat’s eyes start to break down, as seen in Figures 8 and 9, and overturns in the absolute vorticity contours are also observed (Figures 10 and 11). With the wave packet forcing, at late time, the cat’s eyes spread out, particularly in the direction of positive $x$, and their breakdown takes place near the center of the packet (Figure 8). The overturning of the absolute vorticity contours also occurs in this central region. As the packet increases in length with time, the breaking and overturning region also increases.

4.3. The effect of varying $\mu$ and $\epsilon$

The asymptotic analysis indicates that the relative magnitude of the two small parameters $\mu$ and $\epsilon$ is important in determining the behavior of the solution at late time. To investigate the effect of varying the length of the packet at the forced boundary, the experiments described in the previous section were
repeated with $\mu = 0.1, 0.4,$ and $0.8$, while keeping $\varepsilon$ fixed at $0.02$. With these values of $\mu$, the thickness of the wave packet critical layer would be changed from its value in the standard run by factors of $1/2$, $2$, and $4$, respectively. The form of the forcing for each value of $\mu$ is shown in Figure 12. For $\mu = 0.1$, a longer computational domain ($-20 \leq x \leq 40$) was used than that shown. For
each $\mu$, the normalized momentum flux jump $[F]/[F]_{\text{max}}$ was calculated and plotted as a function of time. These graphs are shown in Figure 13. It is evident that the magnitude of $[F]$ relative to $[F]_{\text{max}}$ depends directly on the value of $\mu$. For small $\mu$, $[F]/[F]_{\text{max}}$ decreases at late time and eventually
attains a steady state. However, in contrast to the monochromatic case, it never actually decreases to zero and absorption continues to dominate. With large $\mu$, after an initial decrease at the end of the linear regime, $[F]/[F]_{\text{max}}$ starts to increase at late time and, by the end of the simulation, it has exceeded its linear value of 1. The reason this is possible is that the zonal extent of the
Figure 11. Monochromatic forcing: absolute vorticity in the critical layer at (a) $t = 10$ and (b) $t = 100$. Parameters: $\varepsilon = 0.02$, $\lambda = 0.01$, $k = 2$.

Packet in the outer region has increased considerably at this time. This is seen in the late-time plots of the perturbation streamfunction and the relative vorticity shown in Figures 14 and 15, respectively, for the case $\mu = 0.8$.

In the next set of experiments, $\varepsilon$ was varied while $\mu$ was kept fixed at a value of 0.2. To change the ratio $\Lambda$ by a factor of 1/2, $\varepsilon$ must be changed by a
Figure 12. Perturbation streamfunction at the forced boundary for different values of $\mu$. In all cases, $k = 2$. The $\mu = 0.2$ curve is that used in the standard run. The time evolution of the jump in the momentum flux for each of these values of $\mu$ is shown in Figure 13.

Figure 13. Variation of the jump in the (normalized) momentum flux with $t$ for different values of $\mu$ for fixed $\epsilon$. Parameters: $\epsilon = 0.02$, $\lambda = 0.01$, $\mu = 0.1, 0.2, 0.4, 0.8$, $k = 2$. The $\mu = 0.2$ curve shows the results of the standard run. The streamfunction and vorticity for the $\mu = 0.8$ run are shown in Figures 14 and 15, respectively. Results obtained using $\epsilon = 0.005, 0.02, 0.08$, and 0.32 are shown in Figure 16. With small $\epsilon$, there is greater net absorption, as seen from the relatively small decrease in the jump in the momentum flux from its linear value. In the limit as $\epsilon \rightarrow 0$, the steady state shown by the dashed line in
Figure 14. Large $\mu$ simulation: perturbation streamfunction at (a) $t = 10$ and (b) $t = 100$. Parameters: $\varepsilon = 0.02$, $\lambda = 0.01$, $\mu = 0.8$, $k = 2$.

Figure 3a would be obtained. For larger $\varepsilon$, the asymptotic analysis predicts that the nonlinear effects would dominate over the wave packet effects and, hence, that the results would more closely resemble those in the monochromatic case (see the Appendix). That is indeed the case shown by the $\varepsilon = 0.08$ and $\varepsilon = 0.32$ curves in Figure 16. The net absorption of the packet is greatly
reduced compared with the small $\varepsilon$ case. The streamlines and vorticity contours for the case $\varepsilon = 0.08$ are shown in Figures 17 and 18, respectively. It is interesting to note that with this value of $\varepsilon$, the packet does not increase in length as it did in the small $\varepsilon$ experiments; at $t = 100$ (Figure 17a), the disturbance is still completely contained in the region $-10 < x < 10$. 

Figure 15. Large $\mu$ simulation: relative vorticity in the critical layer at (a) $t = 10$ and (b) $t = 100$. Parameters: $\varepsilon = 0.02$, $\lambda = 0.01$, $\mu = 0.8$, $k = 2$. 
Figure 16. Variation of the jump in the (normalized) momentum flux with $t$ for different values of $\varepsilon$ for fixed $\mu$. Parameters: $\mu = 0.2$, $\lambda = 0.01$, $\varepsilon = 0.005$, 0.02, 0.08, 0.32, $k = 2$. The $\varepsilon = 0.02$ curve shows the results of the standard run. The streamfunction and vorticity for the $\varepsilon = 0.08$ run are shown in Figures 17 and 18, respectively.

One can also observe that in all cases, there is no significant transmission of wave activity to the south of the critical layer. For this reason, even in the large $\varepsilon$ experiments, the linear radiation condition continues to hold up well at late time.

5. Concluding remarks

In this paper, the effect of a slowly varying amplitude-modulated forcing on a nonlinear Rossby wave critical layer has been investigated. The numerical solutions showed that the spatial localization of the forcing reduces the reflection of the disturbance at the critical layer to the extent that absorption dominates even at large time. The jump in the total momentum flux over the length of the packet remains positive for large time, unlike in the monochromatic case. This is because the zonal length of the packet and the region over which it interacts with the basic flow increase with time. The extent to which this happens depends on the relative magnitude of $\mu$, the parameter that determines the initial length of the packet and $\varepsilon$, the nonlinear parameter that measures the amplitude of the forcing. As $\varepsilon$ is increased, there is greater net reflection from the critical layer and less longitudinal transmission within the critical layer and, with large enough $\varepsilon$, the packet remains confined within its original range of longitudes even at late time.
Figure 17. Large $\varepsilon$ simulation: perturbation streamfunction at (a) $t = 10$ and (b) $t = 100$. Parameters: $\varepsilon = 0.08$, $\lambda = 0.01$, $\mu = 0.2$, $k = 2$.

The results of the numerical solutions are supported by the approximate asymptotic solution that has been derived for the long-wave equation. The asymptotic analysis shows that, in the special case where the leading-order streamfunction in the critical layer is steady, the jump in the meridional momentum flux goes to zero with time, as it would in the monochromatic
Figure 18. Large $\varepsilon$ simulation: relative vorticity in the critical layer at (a) $t = 10$ and (b) $t = 100$. Parameters: $\varepsilon = 0.08$, $\lambda = 0.01$, $\mu = 0.2$, $k = 2$.

However, in general, when the streamfunction is not steady in the critical layer, the part of the disturbance centered at the zero wavenumber behaves like $e^{-x^2/t^2}$ and, thus, increases in length with time. The momentum flux jump only goes to zero in the limit $t \to \infty$, where the packet length has become infinite, which of course cannot happen in numerical simulations on a finite domain. In
addition, in the outer region, the transient terms are multiplied by a factor $e^{-\mu^2(x-\bar{u}(y)t)^2}$. This would indicate that the packet propagates outward in the longitudinal direction with a phase speed of $\bar{u}$.

The asymptotic analysis demonstrates the importance of the ratio $\Lambda = \mu/\varepsilon^{1/2}$. In the limit as $\Lambda$ goes to zero, the monochromatic nonlinear results are obtained. The analytic solution was re-examined with the assumption that $\mu \ll \varepsilon^{1/2}$ (see the Appendix). With this assumption, it was found that the leading-order term corresponding to the deformation of the basic flow, which was shown to be proportional to $e^{-2\mu^2x^2}$ at early time, would retain this form even at late time. In addition, the $e^{-\mu^2(x-\bar{u}(y)t)^2}$ factor mentioned above would then be simply replaced by $e^{-\mu^2x^2}$. The effects of the wave packet forcing would appear only in the higher order terms.

It is interesting to note the similarity between the results obtained here, especially those in which $\Lambda \sim O(1)$, and the results of Bacmeister and Pierrehumbert [20] for topographically forced gravity wave packets. They observed an outward (horizontal) flux of wave activity in the critical layer and prolonged absorption of the disturbance. Comparing with the results of Brunet and Haynes [5], one can observe that the large-amplitude (nonlinear) simulations with localized forcing described in the second part of their paper exhibit a greater degree of reflection than ours. In their large-amplitude simulations, the reflected disturbance is seen in the upper right-hand part of their computational domain [5, Figures 7(b) and (c)] and there is no evidence of propagation in the longitudinal direction within the critical layer. There are several considerations that must be taken into account, however, in making comparisons between their results and those presented here. The most important point to note is that their study employs a different model from the one that was considered here, namely, the shallow water equations for flow on a sphere. The disturbance is generated by means of a forcing term in each of the governing equations for the velocity components, rather than as a boundary condition. The forcing has a Gaussian amplitude and it appears to be localized in the meridional as well as in the zonal direction. Dissipation is represented by means of a sixth-order hyperdiffusion term in each of the governing equations for the velocity components of the flow. These differences mean that concepts from the $\beta$-plane study would have to be defined differently in the shallow water model, e.g., the critical layer thicknesses would not necessarily be $\varepsilon^{1/2}$, $\mu$, and $Re^{-1/3}$, and parameters such as $\Lambda$ and $\lambda$ would have to be defined accordingly. Thus, there is no reason to assume a priori that all the conclusions of the present study can be extended directly to the configuration used in [5].

However, it is reasonable to expect that in the shallow water model as well, the extent of reflection must depend on the various characteristics of the forcing and the configuration. In fact, Brunet and Haynes show that increasing the amplitude of the forcing results in greater net reflection, a result which is consistent with the conclusions of the present study and the case of monochromatic forcing. But they did not investigate the effect of varying the
zonal (or meridional) extent of their forcing region, which could potentially have had an effect on the amount of reflection. Another parameter that might affect the behavior of the critical layer in the shallow water model is the layer thickness $H$ (or, equivalently, the Froude number $Fr$). They report that varying $H$ did not produce significant differences, but it is not clear how large a range of values of $H$ was used. It would be of interest to know what happens in the limit as $H$ goes to zero in their model and whether the behavior of the critical layer would then more closely resemble what was seen in the present simulations. If it would, then one could indeed conclude that there is an inherent characteristic of the shallow water model that causes a greater degree of reflection than in the $\beta$-plane model.

The first part of Brunet and Haynes’ study deals with the solution of the inner equation (47) for a configuration in which the leading-order streamfunction $B^{(0)}$ is independent of time. They calculated a bound for the time-integrated absorptivity of the critical layer. As noted earlier, when $B^{(0)}$ is steady, the momentum flux jump would go to zero, as in the monochromatic case, and so the time-integrated absorptivity would indeed have to be bounded.

Another study that is related to the one described here is that of Enomoto and Matsuda [31]. They investigated the case in which the initial basic flow was assumed to vary in the zonal direction as well as meridionally. A different reflection mechanism was observed there; there was advection of vorticity, not by the wave-induced mean flow near the critical layer, but by the basic flow itself; consequently, reflection could occur even in the linear case, where there was no wave-induced mean flow. A possible future extension of the present study would be to employ a zonally varying basic flow in a configuration such as that described here. It would be interesting to see what effect that would have on the extent of reflection and absorption of the disturbance.

Acknowledgments

This paper is based on research that I did for my Ph.D. at McGill University. I wish to thank Professor S. A. Maslowe for his guidance and support. I also benefited from very helpful discussions with Dr. Gilbert Brunet and Dr. Tom Warn. I would like to acknowledge the Zonta International Foundation for support in the form of an Amelia Earhart Fellowship during the time that I was carrying out this research.

Appendix. The long-time outer solution

$A.1$. The case $\mu = \varepsilon^{1/2}$

Outside the critical layer, one may assume a solution of the form

$$\psi \sim \psi^{(0)} + \varepsilon^{1/2} \psi^{(1/2)} + \varepsilon^{1/2} \log \varepsilon \psi^{(1/2)^*} + \varepsilon \psi^{(1)} + \varepsilon^{3/2} \psi^{(3/2)} + \cdots \quad (A.1)$$
and obtain the following sequence of equations:

\[
\left( \frac{\partial}{\partial \eta} + \frac{\partial}{\partial x} \right) \psi^{(0)}_{\eta\eta} = 0, \tag{A.2}
\]

\[
yT \left( \frac{\partial}{\partial \eta} + \frac{\partial}{\partial x} \right) \psi^{(1/2)}_{\eta\eta} = - \left[ 2y \left( \frac{\partial}{\partial \eta} + \frac{\partial}{\partial x} \right) \psi^{(0)}_{\eta\eta} + T \left( \frac{\partial}{\partial T} + y \frac{\partial}{\partial X} \right) \psi^{(0)}_{\eta\eta} + 2 \psi^{(0)}_\eta \right], \tag{A.3}
\]

\[
\left( \frac{\partial}{\partial \eta} + \frac{\partial}{\partial x} \right) \psi^{(1/2)*}_{\eta\eta} = 0, \tag{A.4}
\]

\[
yT^2 \left( \frac{\partial}{\partial \eta} + \frac{\partial}{\partial x} \right) \psi^{(1)}_{\eta\eta} = - \left[ 2y T \left( \frac{\partial}{\partial \eta} + \frac{\partial}{\partial x} \right) \psi^{(1/2)}_{\eta\eta} + T^2 \left( \frac{\partial}{\partial T} + y \frac{\partial}{\partial X} \right) \psi^{(1/2)}_{\eta\eta} + 2 T \psi^{(1/2)}_{\eta\eta} \right] \]

\[
- \left[ y \left( \frac{\partial}{\partial \eta} + \frac{\partial}{\partial x} \right) \psi^{(0)}_{\eta\eta} + 2 T \left( \frac{\partial}{\partial T} + y \frac{\partial}{\partial X} \right) \psi^{(0)}_{\eta\eta} + 2 \psi^{(0)}_{\eta} + \psi^{(0)}_x \right], \tag{A.5}
\]

and so on.

These are the equations solved by Warn and Warn [11], but with additional terms involving the derivatives with respect to the slow scale \( X \). The solution proceeds as in their paper; at each order, terms on the right hand side that would lead to secular solutions are set to zero. Below is an outline of the solution procedure, highlighting the main differences and the additional terms and factors that result from using the wave packet forcing instead of a monochromatic forcing.

Equation (A.2) has solution

\[
\psi^{(0)} = F^{(0)}(x - \eta, X, y, T) + G^{(0)}(x, X, y, T), \tag{A.6}
\]

so, from equation (A.3), we must have

\[
\left( \frac{\partial}{\partial T} + y \frac{\partial}{\partial X} + \frac{2}{T} \right) F^{(0)}_{\eta\eta} = 0. \tag{A.7}
\]

This equation is readily solved by the method of characteristics to give

\[
F^{(0)} = \frac{f^{(0)}(x - \eta, X - yT, y)}{T^2}. \tag{A.8}
\]

Matching with the early time solution (33) as \( T \to 0 \) leads to \( f^{(0)} = 0 \), as in the periodic case.
Similarly, the next term, $\psi^{(1/2)}$, is found to be

$$\psi^{(1/2)} = F^{(1/2)}(x - \eta, X, y, T) + G^{(1/2)}(x, X, y, T)$$  \hspace{1cm} (A.9)

and, from the $O(\varepsilon)$ Equation (A.5), we find that $F^{(1/2)}$ takes the same form as $F^{(0)}$ above and so it must be zero as well, to match with the early time solution. From equation (A.5), we then obtain the solution for $G^{(0)}$.

$$G^{(0)} = C^{(0)}(x, X, T)\{ -y \}^{1/2}K_1(2\{ -y \}^{1/2}) + H^{(0)}(X, y, T).$$  \hspace{1cm} (A.10)

For $y > 0$, we can use the relation $K_1(z) = -\frac{1}{2}\pi(\mathcal{J}_1(iz) + i\mathcal{Y}_1(iz))$ and write

$$G^{(0)} = A^{(0)}(x, X, T)\phi_A(y) + B^{(0)}(x, X, T)\phi_B(y) + H^{(0)}(X, y, T),$$  \hspace{1cm} (A.11)

where $\phi_A(y) = y^{1/2}\mathcal{J}_1(2y^{1/2})$, and $\phi_B(y) = y^{1/2}\mathcal{Y}_1(2y^{1/2})$. Matching with the early time solution, we observe that, as $T \to 0$, $H^{(0)}$ must be zero and $A^{(0)}$, $B^{(0)}$, and $C^{(0)}$ must be proportional to $e^{-\mu^2x^2}e^{ikx}$.

Proceeding to the $O(\varepsilon^3/2)$ equation and setting to zero on the right hand side terms that would lead to secular solutions, it is found that $F^{(1)}$ also takes the form

$$F^{(1)} = \frac{f^{(1)}(x - \eta, X - yT, y)}{T^2}.$$  \hspace{1cm} (A.12)

Matching with the $O(\varepsilon)$ terms in the early time solution (33) gives

$$F^{(1)} = \frac{h_2(y)e^{-(X-yT)^2}e^{ik(x-\eta)}}{k^2yT^2}.$$  \hspace{1cm} (A.13)

At this order, an equation is obtained from which $G^{(1/2)}$ can be determined:

$$y\frac{\partial}{\partial x}G^{(1/2)}_{yy} + \frac{\partial G^{(1/2)}}{\partial x} = - \left\{ \left( \frac{\partial}{\partial T} + y\frac{\partial}{\partial X} \right) G^{(0)}_{yy} + \frac{\partial G^{(0)}}{\partial X} \right\}.$$  \hspace{1cm} (A.14)

The corresponding equation for $G^{(1/2)}$ in the periodic problem [11, p. 53] is also of this form, but without the terms for the dependence on the slow spatial scale. Two equations are now obtained:

$$y\frac{\partial}{\partial x}G^{(1/2)}_{yy} + \frac{\partial G^{(1/2)}}{\partial x} = - \left\{ A^{(0)}_T \phi_{Ayy} + B^{(0)}_T \phi_{Byy} \right\}.$$  \hspace{1cm} (A.15)

and

$$\left( \frac{\partial}{\partial T} + y\frac{\partial}{\partial X} \right) H^{(0)}_{yy} + \frac{\partial H^{(0)}}{\partial X} = 0.$$  \hspace{1cm} (A.16)

The general solution for $G^{(1/2)}$ is found to be

$$G^{(1/2)} = A^{(1/2)}(x, X, T)\phi_A(y) + B^{(1/2)}(x, X, T)\phi_B(y) + \tilde{A}^{(1/2)}(x, X, T)F_A(y) + \tilde{B}^{(1/2)}(x, X, T)F_B(y) + H^{(1/2)}(X, y, T),$$  \hspace{1cm} (A.17)
where \( \tilde{A}^{(1/2)} = A^{(0)}_T \) and \( \tilde{B}^{(1/2)} = B^{(0)}_T \) and, to leading order, \( F_A(y) \sim O(y \log y) \) and \( F_B(y) \sim \log y + O(y \log^2 y) \). In the limit as \( T \to 0 \), \( A^{(1/2)} \), \( B^{(1/2)} \), \( \tilde{A}^{(1/2)} \), \( \tilde{B}^{(1/2)} \) all go to zero and \( H^{(1/2)} \to -(\text{sgn } y) \pi kTe^{-2X^2} \).

The next step in the analysis of the outer region is to examine the \( O(\varepsilon^2) \) equation (this would be the next equation in the sequence (A.2)–(A.5)) to obtain more information about \( H^{(1/2)} \). The terms on the right hand side of the \( O(\varepsilon^2) \) equation give

\[
y \frac{\partial}{\partial x} G^{(1)}_{yy} + \frac{\partial G^{(1)}}{\partial x} = -\left\{ \left( \frac{\partial}{\partial T} + y \frac{\partial}{\partial X} \right) G^{(1/2)}_{yy} + \frac{\partial G^{(1/2)}}{\partial X} + J(G^{(0)}, G^{(0)}_{yy}) \right\},
\]

where \( J \) denotes the Jacobian, which is defined as \( J(f, g) = \frac{\partial f}{\partial x} \frac{\partial g}{\partial y} - \frac{\partial f}{\partial y} \frac{\partial g}{\partial x} \). This gives

\[
y \frac{\partial}{\partial x} G^{(1)}_{yy} + \frac{\partial G^{(1)}}{\partial x} = -\left\{ A^{(1/2)}_T \phi_{Ay} + B^{(1/2)}_T \phi_{Ay} + \tilde{A}^{(1/2)}_T F_{Ay} + \tilde{B}^{(1/2)}_T F_{By} + J(G^{(0)}, G^{(0)}_{yy}) \right\},
\]

from which \( G^{(1)} \) can be determined, and

\[
\left( \frac{\partial}{\partial T} + y \frac{\partial}{\partial X} \right) H^{(1/2)}_{yy} + \frac{\partial H^{(1/2)}}{\partial X} = 0.
\]

The solution of the \( G^{(1)} \) equation is given in [11] for the monochromatic problem. The exact form of \( G^{(1)} \) is not needed for the discussion in Section 3. Suffice it to say that \( G^{(1)} \) would take the form

\[
G^{(1)} \sim \frac{\tilde{B}^{(1)}(x, X, T)}{y} + H^{(1)}(X, y, T),
\]

where \( \tilde{B}^{(1)} \sim B^{(0)}_{TT} \) and that the form of \( H^{(1)}(X, y, T) \) could then be found by examining the next equation in the sequence, the \( O(\varepsilon^3/2) \) equation. The two terms in the expression for \( G^{(1)} \) above match to the terms \( y^{-1}e^{-2X^2}e^{2ikx} \) and \( y^{-1}e^{-2X^2} \) in the early time solution (33).

To solve for \( H^{(0)}(X, y, T) \) and \( H^{(1/2)}(X, y, T) \), a new variable \( \xi \equiv X/T \) is introduced in each of Equations (A.16) and (A.20). Writing \( H^{(1/2)}(X, y, T) \) as a function of \( \xi \) and \( y \) transforms Equation (A.20) into

\[
(y - \xi)H^{(1/2)}_{yy\xi} + H^{(1/2)}_\xi = 0,
\]

the solution of which takes the form

\[
H^{(1/2)} = \int (H_A(\xi)\phi_A(y - \xi) + H_B(\xi)\phi_B(y - \xi)) \, d\xi + h(y).
\]
In the limit of small \( y \), the term \( h(y) \) must be zero, to match with the early time solution. For small \( y \), \( H^{(1/2)} \) can be written as \( H^{(1/2)} \sim N^{(1/2)}(X, T) + O(y) \), where

\[
N^{(1/2)} = \int \left\{ H_A(\xi) \phi_A(-\xi) + H_B(\xi) \phi_B(-\xi) \right\} d\xi. 
\]  
(A.24)

To match with the early time solution, the jump in \( N^{(1/2)} \) across the critical layer must be proportional to \( e^{-2\xi^2} \). The solution of equation (A.16) also takes the form (A.23) and the \( y \)-independent term at this order \( N^{(0)}(X, T) \), say, would also be of the form (A.24), but it would be zero at early time. The time-evolution of the function \( e^{-2\xi^2} = e^{-2(X/T)^2} \) is shown in Figure A.1. This has important implications for the behavior of the packet at late time, as discussed in Section 3.3.

A final remark about the solution in this regime concerns the functions \( H_A \) and \( H_B \) in (A.23). The requirement that the mean flow be zero at \( y = y_1 \) means that these functions are related by

\[
H_B(\xi) = -\frac{\phi_A(y_1 - \xi)}{\phi_B(y_1 - \xi)} H_A(\xi). 
\]  
(A.25)

So in the limit as \( T \to \infty \) for finite \( X \) and \( \xi \to 0 \), \( H_B(\xi) \) would be zero if the location of the boundary \( y_1 \) were chosen so that \( \phi_A(y_1) = 0 \). This is equivalent to setting \( r = 0 \), where \( r \) is the parameter defined in Appendix A.2. The solutions (A.23) and (A.24) would then depend only on \( \phi_A \) and not on \( \phi_B \). Since it is in the function \( \phi_B \) that the singularity appears, \( N^{(1/2)} \) could, under those circumstances, be continuous across the critical layer.
A.2. The Stewartson limit

The circumstances under which the function $B^{(0)}$ would be steady were outlined by Warn and Warn [11]. They showed that the steady solution of Stewartson [12] would be obtained if the forcing boundary was at a certain distance from the critical layer. For our problem, this distance can be determined as follows.

The condition at the forced boundary requires that

$$A^{(0)}(X, T)\phi_A(y_1) + B^{(0)}(X, T)\phi_B(y_1) = (a^+\phi_A(y_1) + b\phi_B(y_1))e^{-X^2}e^{ikx} + \text{c.c.},$$

(A.26)

where $a^+ = a - i\pi$ and $b = 1$ are the Frobenius constants given in Equation (7). Thus, in terms of the Fourier transforms $\hat{A}^{\pm}(\kappa, T)$ and $\hat{B}(\kappa, T)$ of $A^{(0)}(x, X, T)$ and $B^{(0)}(x, X, T)$, we have, at the inflow and outflow boundaries:

$$\hat{A}^+ \phi_A(y_1) + \hat{B} \phi_B(y_1) = (\pi/\epsilon)^{1/2} \left\{ (a^+\phi_A(y_1) + b\phi_B(y_1))e^{-(\kappa-k)^2/4\epsilon} 
+ (a^{+*}\phi_A(y_1) + b^{*}\phi_B(y_1))e^{-(\kappa+k)^2/4\epsilon} \right\}.$$ (A.27)

and

$$\hat{A}^- = a^- \hat{B}.$$ (A.28)

From these relations, it is seen that the jump in $\hat{A}(\kappa, T)$ across the critical layer is given by

$$r[\hat{A}] = (\pi/\epsilon)^{1/2} \left\{ (1 - i\pi r) e^{-(\kappa-k)^2/4\epsilon} + (1 + i\pi r) e^{-(\kappa+k)^2/4\epsilon} \right\} - \hat{B}$$ (A.29)

if the parameter $r$ is defined, as in [11], by

$$r = \frac{\phi_A(y_1)}{a^-\phi_A(y_1) + \phi_B(y_1)}.$$ (A.30)

Note that with the configuration used in Section 3, i.e., with the long-wave assumption and the mean flow given by $\bar{u} = y$, $r$ does not depend on $\kappa$, although it would for the more general problem. The case $r = 0$ corresponds to the problems examined by Stewartson [12] and by Brunet and Haynes [5], where the leading order streamfunction remains steady in the critical layer, and with the forcing function used here, it leads to $B^{(0)} = \alpha_1 e^{-X^2} e^{ikx} + \text{c.c.}$

A.3. The case $\mu \ll \epsilon^{1/2}$

In Appendix A.1, the relation $\mu = \epsilon^{1/2}$ was assumed between the nonlinear and wave packet parameters. In this section, the case $\mu \ll \epsilon^{1/2}$ is examined. Let us consider, for example, the balance $\mu = \epsilon$. Then the appropriate slow variables, valid for $t \sim \epsilon^{-1/2}$, are $T = \epsilon^{1/2}t$ and $X = \mu x = \epsilon x$, with the fast scale still defined as $\eta = yt$. We obtain a set of equations similar to Equations
(A.2)–(A.5), the only difference being that, in the equations corresponding to (A.3) and (A.5), the terms involving derivatives with respect to \( X \) would appear instead in the equation of order \( \epsilon^{1/2} \) higher. All these terms still turn out to be zero, because the functions \( \psi^{(0)} \) and \( \psi^{(1/2)} \) are still independent of \( \eta \). Thus, the equations corresponding to (A.3) and (A.5) and the equations for \( H^{(0)} \) and \( H^{(1/2)} \) would not contain any \( X \) derivatives. The function \( H^{(0)} \) would still be zero, according to the matching conditions. \( H^{(1/2)} \) would simply be

\[
H^{(1/2)} = N^{(1/2)}(X, T) + y M^{(1/2)}(X, T),
\]

(A.31)

with \( N^{(1/2)}(X, T) = f(T)e^{-2X^2} \), where \( f(T) \to -\text{sgn} y \frac{k}{2}kT \) as \( T \to 0 \). Thus, \( N^{(1/2)}(X, T) \) would retain its initial shape for large time. Similarly, the transient terms in the \( F^{(1)} \) solution would just be proportional to \( e^{-X^2} \). The equation for \( F^{(1)} \) would be

\[
\left( \frac{\partial}{\partial T} + \frac{2}{T} \right) F^{(1)}_{\eta\eta} = 0,
\]

(A.32)

because in the case of monochromatic forcing, the solution of which is of the form

\[
F^{(1)} = \frac{f^{(1)}(x - \eta, X, y)}{T^2},
\]

(A.33)

When matched to the early time solution, this gives

\[
F^{(1)} = \frac{h_2(y)e^{-X^2}e^{ik(x-\eta)}}{k^2yT^2},
\]

(A.34)

which is simply the monochromatic solution multiplied by the slowly varying wave packet amplitude \( e^{-X^2} \). The modifications due to the slow \( X \) variation would only appear at higher order.

References


Carleton University

(Received September 10, 2002)