Transversal Covers and Packings

by

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Abstract

A transversal cover (packing) is a set of gk points in k disjoint groups of size g and a minimum (maximum) collection of transversal subsets, called blocks, such that any pair of points not contained in the same group appear in at least (most) one block. A central question is to determine, for given g and k, the minimum (maximum) possible b, denoted $tc(k,g)$ ($tp(k,g)$).

Transversal covers are applicable to software testing, data compression and error free communication. The case $g = 2$ was previously solved. Asymptotic results are known for all g, but little was understood for small values of k. We develop constructions, yielding upper bounds by four methods: incomplete transversal designs, concatenation techniques, generalized Wilson's constructions and group divisible designs. We develop three general lower bounds: $tc(k,g) \geq \lceil (g \log k)/2 \rceil + g + 1$, using a construction; $k \leq \frac{1}{2}\binom{2\lfloor \frac{b}{g} \rfloor + \delta_{b,g}}{\lfloor \frac{b}{g} \rfloor}$ from the study of intersecting set-systems; and $k \leq \left\lfloor \left(\binom{b}{\frac{b}{g}-(g-2)} - \sum_{i=2}^{n} \binom{n}{i} \binom{b-n}{\frac{b}{g}-g-(i-2)} \right) / g \binom{\frac{b}{g}}{g-2} \right\rfloor$, when all points appear equally often, using a set packing argument. In addition, we investigate lower bounds for small k that reduce or eliminate the gap between lower and upper bounds.

Transversal packings are significantly less studied, but Abdel-Ghaffar and Abbadi found the maximum k admitting $b > g$, upper bounds and some optimality results, and discussed an application to optimal disk allocation. We develop constructions, yielding lower bounds, from incomplete transversal designs, Wilson's method, and techniques applied to an equivalent matching problem for graphs. Upper bounds are derived from coding theory, standard packing arguments, and consideration of sets of disjoint blocks.

We give tables and figures of the values obtained from these bounds for both transversal covers and packings, in the two appendices.

To the memory of Onion.

to cover

to be sure cover cover all over

Samuel Beckett, *Malacoda, Echo's Bones*

I was of course inadequately covered, but whose fault was that?

Samuel Beckett, *Molloy*

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Contents

Chapter 1

Introduction

Motivated by transversal covers' utility, much recent discussion, and the lack of investigation of small parameter values, we discuss this natural extension of transversal designs to larger block sizes, covering conditions replacing strict uniform pair occurrence. The complementary problem, transversal packings, is a natural investigation also with some applications. The object of this thesis is to calculate both upper and lower bounds for both these objects, produce good instances, optimal if possible, and investigate their structure. As a consequence, we have also produced algorithms which recursively and directly construct these objects.

While it would seem the appropriate time to discuss the history of these problems, the various different methods used to approach these topics would require the reader to be able to translate between these disparate viewpoints. Considering the time it has taken to make these translations ourselves during the last four years as we encountered each new viewpoint, I would not expect this of any interested reader. We want the reader to proceed through the history comfortably, understanding beforehand why seemingly inapplicable results are mentioned in connection to transversal covers and packings. Therefore, we will plunge directly into definitions and notation, explaining the variety of perspectives on this subject and establishing their equivalence.

1.1 Definitions and Notation

1.1.1 Miscellaneous Notation

We begin with some common notation found in this work. We use log for $\log_2$; logarithms to any other base will be explicitly noted as such. $\lfloor x \rfloor$ will denote the largest integer $\leq x$. $\lceil x \rceil$ will denote the smallest integer $\geq x$. If x has a distribution of values, $\overline{x}$ will refer to the mean of the distribution. When we remove x from a set, we use $\hat{x}$ for emphasis.

A *g-set* is a set of cardinality g. It will usually be either $\{0, 1, \ldots, g-1\}$ or $\{1, 2, \ldots, g\}$ and which of these is used should be clear from the context. A g-set will sometimes be referred to as a *g-ary alphabet*. This is usually when the particular set is not one of the above.

1.1.2 General Incidence Structures

Many incidence structures will be mentioned and used throughout this thesis. These definitions come from either [7] or [12].

Definition 1.1. Let K be a subset of positive integers. A *pairwise balanced design* ($PBD(v, K)$) of order v with block sizes from K is a pair $(V, \mathcal{B})$, where V is finite set, of cardinality v, and $\mathcal{B}$ is a family of subsets (blocks) of V which satisfy the following: if $B \in \mathcal{B}$ then $|B| \in K$ and every pair of distinct elements of V occurs in exactly one block. If $K = \{k\}$ then we will denote this as $PBD(v, k)$.

A PBD can also be called a $(v, K, 1)$-design, where the 1 refers to the fact that every pair appears exactly once amongst the blocks. When every pair appears at least once amongst the blocks we will refer to the structure as a $(v, K, 1)$-cover. When every pair appears at most once amongst the blocks, we will refer to the structure as a $(v, K, 1)$-packing. These will sometimes be called *standard* or *pairwise* packings.

Definition 1.2. Let k, g and n be positive integers. A *group divisible design* of order ng (k-*GDD* of type g^n) is a triple $(V, \mathcal{G}, \mathcal{B})$ where V is a finite set of cardinality ng, $\mathcal{G}$ is a partition of V into n groups of size g, and $\mathcal{B}$ is a family of subsets, called blocks, of V which satisfy the properties:

1. If $B \in \mathcal{B}$ then $|B| = k$;
2. every pair of distinct elements of V occurs in exactly one block or one group but not both; and
3. $n \geq 1$.

Definition 1.3. Let $\mathcal{B}$ be a set of blocks of some incidence structure. A *resolution class* is a collection of blocks which partitions the point set of the incidence structure.

A k-*GDD* of type g^n is called *resolvable* and denoted k-*RGDD* of type g^n if its blocks can be partitioned into resolution classes.

Definition 1.4. A *transversal design* of order g, block size k, denoted $TD(k, g)$ is a k-*GDD* of type g^k. $TD(k)$ will denote the set of all $g \in \mathbb{N}$ such that there exists a $TD(k, g)$.

Definition 1.5. [12] An *incomplete transversal design* ($ITD(k, n; b_1, b_2, \ldots, b_s)$ $0 \leq b_i$, $\sum_{i=1}^{s} b_i \leq n$) is a quadruple $(V, \mathcal{G}, \mathcal{H}, \mathcal{B})$, where

1. V is a set of kn elements;
2. $\mathcal{G}$ is a partition of V into k groups, each of size n;
3. $\mathcal{H}$ is a set of disjoint subsets $H_1, H_2, \ldots, H_s$ of V, called holes, with the property that, for each $1 \leq i \leq s$ and each $G \in \mathcal{G}$, $|G \cap H_i| = b_i$;
4. $\mathcal{B}$ is a collection of k-subsets of V called blocks; and
5. every unordered pair of elements from V is

- contained in a hole, and contained in no blocks; or
- contained in a group, and contained in no blocks; or
- contained in neither a hole nor a group, and contained in exactly one block.

A $TD(k,g)$ is equivalent to $k-2$ mutually orthogonal latin squares ($MOLS$) of order g. Incomplete transversal designs are equivalent to $k-2$ incomplete, or holey, $MOLS$ of order n with holes of size $b_1,\ldots,b_s$.

Definition 1.6. An *existential array* ($EA(c;g_1,\ldots,g_r)$) is a $r\times c$ array, where row i has entries from a g_i-ary alphabet, and given any two columns, j and k, there exists at least one row, i, where the symbols in the two columns differ.

It is clear that $r \leq \prod_{i=1}^{r} g_i$, and that this is achievable.

There are a number of parameters associated with any incidence structure. In any incidence structure v will denote the number of points, b will be the number of blocks in the structure, r_x will denote the replication number of x, i.e., the number of blocks incident with the point x, and k_B will denote the size of block B, the number of points with which it is incident. $\lambda_{x,y}$ will denote the number of blocks on which the pair of points x and y occur together. If this is a constant on the entire incidence structure it will simply be referred to as λ. And finally $\mu_{A,B}$ will be the number of points that blocks A and B have in common, in other words, viewing A and B as sets, $\mu_{A,B}=|A\cap B|$. If this is constant it will be denoted as simply μ.

1.1.3 Transversal Covers

We now define our objects of study.

Definition 1.7. Let k, g and $n\leq g$ be positive integers. A *transversal cover* ($TC(k,g:n)$) is a triple $(X,\mathcal{G},\mathcal{B})$ where $|X|=kg$, $\mathcal{G}=\{G_1,G_2,\ldots,G_k\}$ is a partition of X into k sets of size g, $\mathcal{B}$ is a collection of subsets of X, called blocks or transversals, each block has size k and intersects each G_i in exactly one point, and each pair of points of X not in the same G_i occurs in at least one block. Further, there is a set of at least n disjoint blocks in $\mathcal{B}$. The smallest number of blocks possible in a $TC(k, g : n)$ is denoted by $tc(k, g : n)$. The largest number of groups possible in a transversal cover with b blocks is denoted by $kc(b, g : n)$.

Example 1.1. Let $V = \{0, 1, 2, 3, 4, 5, 6, 7\}$ partitioned into groups $G_1 = \{0, 1\}, G_2 = \{2, 3\}, G_3 = \{4, 5\}, G_4 = \{6, 7\}$. Then the following blocks form a transversal cover:

$$\{0, 2, 4, 6\}, \{1, 3, 5, 6\}, \{1, 3, 4, 7\}, \{1, 2, 5, 7\}, \{0, 3, 5, 7\}$$

A $TC(k, g : n)$ with g^2 blocks is obviously a transversal design. We will call a $TC(k, g : n)$ with the n disjoint blocks removed an incomplete transversal cover or $ITC(k, g : n)$. It is clear that

$$tc(k, g : i) \leq tc(k, g : j) \leq tc(k, g : i) + j - i, \text{ for any } 1 \leq i < j \leq n. \tag{1.1}$$

Treating transversal covers as $b \times k$ arrays of elements from a g-ary alphabet, allows easy translation among the many ways that these objects have been viewed and approached in the literature. The array is formed by placing the same g-ary alphabet on each group and then listing the blocks explicitly as the rows of the array. The groups become the columns and a set of disjoint blocks becomes a set of rows with pairwise Hamming distance k. With this in mind, we define:

Definition 1.8. A *covering array* ($CA(k, g : n)$) is an array with k columns of values from a g-ary alphabet such that given any two columns, i and j, and for all ordered pairs of elements from a g-ary alphabet, (g_1, g_2), there exists a row, r, such that $a_{i,r} = g_1$ and $a_{j,r} = g_2$. Further, there is a set of at least n rows that pairwise differ in each column; they are disjoint.

It is obvious that row and column permutations, as well as permuting symbols within each column, leave the covering conditions unchanged.

Example 1.2. The transversal cover from Example 1.1, with the 2-ary alphabet $\{0, 1\}$ placed on each group, yields the following covering array:

$$\begin{array}{cccc} 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{array}$$

Before showing the third equivalent formulation of the problem we make two new definitions:

Definition 1.9. A *g-partition* , $\mathcal{A}$, of a b-set, B, is a collection of subsets $\{A_i\}_{i=1}^{g}$ of B such that $A_i \cap A_j = \emptyset$ for $i \neq j$ and $\cup_{i=1}^{g} A_i = B$.

Definition 1.10. A family of g-partitions, $\{\mathcal{A}^i\}_{i=1}^{k}$ of a b-set B is called *t-independent* if whenever one chooses t distinct g-partitions and one part from each, then the intersection of these parts is non-empty. In other words for all $i_1 < i_2 < \cdots < i_t$,

$$A_{j_1}^{i_1} \cap A_{j_2}^{i_2} \cap \cdots \cap A_{j_t}^{i_t} \neq \emptyset$$

for any set $\{j_1, \ldots j_t\}$ where $1 \leq j_l \leq g$ for all $1 \leq l \leq t$.

We can see that a covering array is equivalent to a set of 2-independent g-partitions of a b-set. Each column defines a g-partition of a b-set in the obvious way. Then the covering conditions of the transversal cover imply 2-independence. When speaking in terms of transversal covers or covering arrays one sometimes refers to strength t instead of t-independence. This thesis will not address the question of t-independence for $t \geq 3$.

Example 1.3. To present the transversal cover from Example 1.1 we label the blocks, in order, $B = \{a, b, c, d, e\}$ then the family of 2-independent 2-partitions of B is

$$\begin{aligned} A^1 &= \{\{a, e\}, \{b, c, d\}\} \\ A^2 &= \{\{a, d\}, \{b, c, e\}\} \\ A^3 &= \{\{a, c\}, \{b, d, e\}\} \\ A^4 &= \{\{a, b\}, \{c, d, e\}\} \end{aligned}$$

We may require more structure on a transversal cover, in particular that each point is in an equal number of blocks.

Definition 1.11. A *point-balanced transversal cover* $(PBTC(k, g : n))$, is a transversal cover such that r_x is constant for every $x \in X$, i.e. every point appears equally often. The smallest number of blocks possible in an $PBTC(k, g : n)$ is denoted by $pbtc(k, g : n)$.

1.1.4 Transversal Packings

Definition 1.12. Let k, g and $n \leq g$ be positive integers. A *transversal packing* $(TP(k, g : n))$ is a triple $(X, \mathcal{G}, \mathcal{B})$ where $|X| = kg$, $\mathcal{G} = \{G_1, G_2, \ldots, G_k\}$ is a partition of X into k sets of size g, $\mathcal{B}$ is a collection of subsets of X, called transversals or blocks, each block has size k and intersects each G_i in exactly one point, and each pair of points of X not in the same G_i occurs in at most one block. Further, there is a set of at least n disjoint blocks in $\mathcal{B}$. The largest number of blocks possible in a $TP(k, g : n)$ is denoted by $tp(k, g : n)$. The largest number of groups possible in a transversal packing with b blocks will be denoted by $kp(b, g : n)$.

Example 1.4. Let $V = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14\}$ partitioned into groups $G_1 = \{0, 1, 2\}, G_2 = \{3, 4, 5\}, G_3 = \{6, 7, 8\}, G_4 = \{9, 10, 11\}, G_5 = \{12, 13, 14\}$.

Then the following blocks form a transversal packing:

$$\{0,3,6,9,12\},\{0,4,8,8,10\},\{1,3,7,11,14\},$$
$$\{2,4,6,10,14\},\{2,5,7,9,13\},\{1,5,8,10,12\}$$

Viewing the transversal packing as a $b \times k$ array of values from a $g-$set, in exactly the same way as for transversal covers give the equivalent object:

Definition 1.13. A *packing array* $(PA(k, g : n))$ is an array with k columns of values from a g-ary alphabet such that given any two columns, i and j, and for all ordered pairs of elements from a g-ary alphabet, (g_1, g_2), there is at most one row, r, such that $a_{i,r} = g_1$ and $a_{j,r} = g_2$. Further, there is a set of at least n rows that pairwise differ in each column: they are disjoint.

Row and column permutations, as well as permuting symbols within each column, leave the packing condition intact.

Example 1.5. The transversal packing from Example 1.4, with the 3-ary alphabet $\{0, 1, 2\}$ placed on each group, yields the following covering array:

$$\begin{array}{ccccc}
0 & 0 & 0 & 0 & 0 \\
0 & 1 & 2 & 2 & 1 \\
1 & 0 & 1 & 2 & 2 \\
2 & 1 & 0 & 1 & 2 \\
2 & 2 & 1 & 0 & 1 \\
1 & 2 & 2 & 1 & 0
\end{array}$$

1.2 Historical Remarks

1.2.1 Transversal Covers

We restate the two most common questions:

1. What is the minimum b given k $(tc(k, g : n))$?

2. What is the maximum k given b ($kc(b, g : n)$)?

These two questions are equivalent and can be translated into each other. Indeed,

$$tc(k, g : n) = \min\{b | kc(b, g : n) \geq k\}$$

and

$$kc(b, g : n) = \max\{k | tc(k, g : n) \leq b\}.$$

Performing these inversions can be difficult. Asymptotically, they can often be algebraically inverted; but, for finite values the inversion is usually algorithmic and computational. In Chapter 3, we will see this difficulty in action. We develop two lower bounds on $tc(k, g : n)$, one of which is derived from an upper bound on $kc(k, g : n)$. Even though this lower bound is better, it is less useful because of the difficulty inverting it.

In 1928, Sperner asked the question: what is the largest cardinality of a family, $\{A_i\}_{i=1}^m$, of subsets of an n-set with the property that $A_i \subset A_j$ never holds for $i \neq j$? He showed that

$$m \leq \binom{n}{\lfloor \frac{n}{2} \rfloor} \tag{1.2}$$

and that this maximum is attainable [39]. This question is equivalent to asking for the largest anti-chain of subsets of an n-set where the partial ordering is given by inclusion. In our language, with the addition of two new points, one always in each set and one never in any set A_i, (equivalent to adding two new rows to the corresponding covering array: one with all zeros and one with all ones), we have the result that if $f(t) = \binom{t}{\lceil \frac{t}{2} \rceil}$, then we conclude that

$$tc(k, 2 : 2) = \min\{t : f(t) \geq k\} + 2. \tag{1.3}$$

Independently, Katona [22] and Kleitman and Spencer [23] solved the analogous problem where

$$A_i \cap A_j,\ A_i \cap A_j^c,\ A_i^c \cap A_j \text{ and } A_i^c \cap A_j^c$$

are all non-empty. They showed that

$$m \leq \binom{n-1}{\lceil \frac{n}{2} \rceil - 1}$$

and again that this was attainable. If $g(t) = \binom{t-1}{\lceil \frac{t}{2} \rceil - 1}$, then this translates into

$$tc(k, 2:1) = \min\{t : g(t) \geq k\}. \tag{1.4}$$

Rényi, who also solved the above problem for even b, first asked an important generalization of this question: If, instead of a 2-partition of the n-set into A_i and A_i^c, we g-partition the set and require that any two parts from two different partitions intersect, then what is the maximum number of partitions [36].

The next mention of transversal covers, by Poljak and Rödl, made reference to both covering arrays and maximal g-partitions [30]. In this article, they proved a number of results, including:

$$\begin{aligned} tc(k, g_1 g_2 : 1) &\leq tc(k, g_1 : 1)\, tc(k, g_2 : 1) & (1.5)\\ tc(k_1 k_2, g : 1) &\leq tc(k_1, g : 1) + tc(k_2, g : 1) & (1.6)\\ tc(k, g : 1) &\leq \binom{g}{2} tc(k, 2 : 1), & (1.7) \end{aligned}$$

if g is a prime power then

$$tc(g k_1 k_2 \cdots k_g, g : 1) \leq \sum_{i=1}^{g} tc(k_i, g : 1), \tag{1.8}$$

if g is a prime power and α_i are non-negative integers $(1 \leq i \leq j)$ then

$$tc(\prod_{i=2}^{j}[(g+1)^{g^{i-2}} g^{\frac{g^{i-1}-1}{g-1}}]^{\alpha_i}, g:1) \leq \alpha_j g^j + \alpha_{j-1} g^{j-1} + \cdots + \alpha_1 g + \alpha_0 \tag{1.9}$$

if $tc(k_1, g_1 : 1) \leq b$ and $tc(k_2, g_2) \leq g_1$ then

$$tc(k_1 k_2, g_2 : 1) \leq b. \tag{1.10}$$

These results have the corollaries that for k large

$$tc(k, g:1) \leq \frac{g^2 \log(k)}{\log(g+1)} \text{ if } g \text{ is a prime power and} \tag{1.11}$$

$$tc(k, g:1) \leq \frac{4g^2 \log(k)}{2g} \text{ when } g \text{ is composite.} \tag{1.12}$$

They also proved the lower bound on b: If $f(t,g) = \binom{t-1}{\lfloor \frac{t}{g} - 1 \rfloor}$ then

$$tc(k, g:1) \geq \min\{t | f(t,g) \geq k\}. \tag{1.13}$$

In this article, they also relate $tc(k, g:1)$ to graph theoretical parameters. If $\overline{\omega}(G)$ is the minimum number of independent sets such that every pair of non-adjacent vertices is contained in at least one of them [6, 48] then

$$\overline{\omega}(kK_g) = tc(k, g:1). \tag{1.14}$$

Also, if there exists a $TC(k, g:1)$ with gr blocks, having an additional property (called the permutation property) then

$$\dim(kK_g) \leq r. \tag{1.15}$$

These transversal covers are not necessarily point-balanced, even though they satisfy the necessary condition on b. In a later paper, Poljak, Pultr and Rödl [29] improved these results, showing for k sufficiently large and g arbitrary

$$tc(k,g:1) \leq \frac{\frac{9g}{8}^2 \log(k)}{\log(\frac{9g}{8})}, \tag{1.16}$$

also proving a lower bound for k large:

$$tc(k,g:1) \geq \frac{g(\log_e(k) + \log_e(g))}{\log_e(g) + 1}. \tag{1.17}$$

Later Poljak and Tuza [31] proved some additional results. This paper was the first to make use of more than one disjoint block, specifically, $n = g$ for $g \geq 3$. They showed

$$tc(k,g:1) + g \ \geq \ tc(k,g:g) \geq tc(k,g:1) \tag{1.18}$$

$$tc(k_1 k_2, g:g) \ \leq \ tc(k_1, g:g) + tc(k_2, g:g). \tag{1.19}$$

Their other results are more simply stated as upper and lower bounds on the maximum k achievable given b (the equivalent formulation discussed above).

$$kc(b,g:1) \leq \frac{1}{2}\binom{\lfloor \frac{2b}{g} \rfloor}{\lfloor \frac{b}{g} \rfloor} = \mathcal{O}\left(4^{b/g} b^{-1/2}\right) \tag{1.20}$$

which translates to a lower bound on b, and

$$kc(b,g:1) \geq \frac{2e^{\frac{b}{2g^2}+1}}{g} \tag{1.21}$$

which translates into

$$tc(k,g:1) \leq 2g^2(\log_e(k) - \log_e(\frac{2e}{g})). \tag{1.22}$$

When g is a prime power and $(g^2-g)|(b-g)$

$$k \geq g^{\frac{b-g}{g^2-g}}, \tag{1.23}$$

a direct application of Inequality 1.19, and finally when $g=3$

$$k \geq \frac{1}{2}\binom{\lfloor \frac{b}{3} \rfloor}{\lfloor \frac{b}{6} \rfloor}. \tag{1.24}$$

Many of the above mentioned papers also consider and prove results about the more general problem of r-independent k-partitions of an n-set.

Körner and Simonyi in 1992 [25] started considering the asymptotic size of transversal covers. Motivated by one of Shannon's results, they define

$$q_g = \limsup_{b\to\infty} \frac{\log kc(b, g:1)}{b}. \tag{1.25}$$

In these terms, the results of Poljak and Tuza [31] are

$$\frac{1}{3} \leq q_3 \leq \frac{2}{3}. \tag{1.26}$$

Using binary sequences with no consecutive 1's and the Fibonacci numbers, Körner and Simonyi show that

$$0.409 \leq q_3. \tag{1.27}$$

The beauty of this result is the original and innovative construction used. Transversal covers that attain this bound always exist, but it is only useful for large b (and consequently for large k) and behaves badly for small parameter values. In this paper, the authors also reveal a new motivation and approach to transversal covers, Shannon's conception of a zero-error communication along a noisy channel. In brief, the problem is to find the maximum b such that any two columns are covered for a given set of pairs of letters, rather than all possible pairs. These pairs are represented by the edges of a graph of order g. Körner and Simonyi call solutions for this general class of problems Sperner Capacities. This motivation will be discussed along with other applications in Section 1.3.

Gargano, Körner and Vaccaro [15], published in the same year, gave another construction that yields slightly better values for q_3 and new bounds for higher g:

$$\begin{array}{lll} q_3 \geq 0.4833 & q_4 \geq 0.2556 & q_5 \geq 0.1823 \\ q_6 \geq 0.1116 & q_7 \geq 0.0870 & q_8 \geq 0.0628 \\ q_9 \geq 0.0541 & q_{10} \geq 0.0392 & q_{11} \geq 0.0351 \\ q_{12} \geq 0.0274 & q_{13} \geq 0.0253. & \end{array} \tag{1.28}$$

The results for $g \geq 4$ come from the very valuable *Two Word Lemma* (TWL) proved in their paper; the results show that a particular sets of words formed from concatenation of two prescribed subwords must satisfy the same coincidence conditions as the two subwords. If the subwords satisfy the condition that they cover all ordered pairs of letters from a g-ary alphabet, then the set of words formed from them will as well, forming the columns of a covering array. The result given above for q_3 is found using a four subword analogue of the TWL. Like Körner and Simonyi's [25] construction, this theorem and construction technique is elegant but restricted to large b and k. The TWL can be used for many of the Sperner capacity problems. Indeed, this paper concerns itself with Sperner capacities of cycles in addition to the complete graphs which yield results on transversal covers. In a paper published a year later [16], the same authors, using probabilistic techniques, were able to prove that

$$q_g = \frac{2}{g} \tag{1.29}$$

for every g. This can be reformulated:

$$\lim_{k \to \infty} \frac{tc(k, g:1)}{\log k} = \frac{g}{2}, \tag{1.30}$$

and thus solves completely the asymptotics of the transversal covers problem. In this paper, they also calculate the Sperner capacities of other graphs, including stars.

There are further results on general Sperner capacities in their 1994 paper [17].

Sloane reports on many results for $g = 3$ and small k: Östergård's result that $tc(5,3:1) \geq 11$; Applegate's result from integer programming that $tc(5,3:1) \leq 11$, showing that

$$tc(5,3:1) = 11 \tag{1.31}$$

(the first nontrivial optimal solution known); and Cook's integer programming demonstration that $tc(6,3:1) \leq 12$. Sloane himself offers a construction along the same lines as Inequality 1.7 that yields

$$kc(3a,3:1) \leq \binom{a}{\lceil \frac{a+1}{3} \rceil}, \tag{1.32}$$

and he gives a table of the best known values at the time of publication

$$\begin{array}{c|cccccccccccccccccc} k: & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 & \ldots \\ \hline tc(k,3:1) \leq & 9 & 9 & 9 & 11 & 12 & 15 & 15 & 15 & 15 & 15 & 15 & 18 & 18 & 18 & 18 & 21 & 21 & \ldots \end{array}. \tag{1.33}$$

Sloane's focus in his article was the binary 3-covering arrays, equivalent to the 3-independent 2-partitions from Definition 1.10 .

We conclude with the connection to combinatorial block designs. From a block design theoretic approach, transversal covers are natural extensions of transversal designs to standard covering requirements (that pairs appear not only once amongst the blocks but at least once). This extension allows the consideration of block sizes known to be impossible for transversal designs or not currently known to admit transversal designs. Transversal designs give the trivial lower bound $tc(k,g:n) \geq g^2$ ($tc(k,g:n) \geq g^2+1$ for k not admitting transversal designs) and much has been studied about them and their equivalent formulation: sets of mutually orthogonal latin squares [7]. Of particular note are the various transversal design constructions. The foremost is Wilson's construction and its generalizations [7, 13]:

Theorem 1.1. *Let* $\mathbf{T} = (V, \mathcal{A}, \mathcal{G})$ *be a* $TD(k+l:g)$ *with block set* $\mathcal{A} \subset \binom{V}{k+l}$ *and* *group set* $\mathcal{G} = \{G_1, \ldots, G_k, H_1, \ldots, H_l\}$. *Let* $\mathcal{S}$ *be an* s*-subset of* $H_1 \cup \cdots \cup H_l$ *and* $V_0 := G_1 \cup \cdots G_k$. *For each block* $A \in \mathcal{A}$ *we write*

$$A_0 := A \cap V_0,\ A' := A \cap \mathcal{S},\ u_A := |A'| = |A \cap \mathcal{S}|.$$

Assume

$$h_i := |\mathcal{S} \cap H_i| \in TD(k) \text{ for } i = 1, \ldots l,$$

and the existence of a positive integer m *satisfying the condition that for each block* $A \in \mathcal{A}$ *there is a* $TD[k; m + u_A]$ *with* u_A *disjoint blocks. Then one has*

$$mg + s \in TD(k). \tag{1.34}$$

What we can conclude from this history is that although the asymptotics, some good constructions for large parameters and one lower bound are known, very little else is understood about transversal covers. The asymptotic limit is not constructive and only one non-trivial optimal transversal cover was known at the time of Sloane's paper. Design theory offers a number of approaches that construct transversal covers for all parameter values, some of which are new and some are generalizations of constructions appearing in earlier papers.

1.2.2 Transversal Packings

We restate the two most common questions:

1. What is the maximum b given k, $(tp(k, g : n))$?

2. What is the maximum k given b $(kp(b, g : n))$?

These two questions are equivalent and can be translated into each other. Explicitly

$$tp(k, g : n) = \min\{b | kp(b, g : n) \geq k\}$$

and

$$kp(b, g : n) = \min\{k | tp(k, g : n) \geq b\}.$$

Again, as mentioned for transversal covers, these inversions are hard to perform.

Much less has been explicitly written about transversal packings. The rows of the corresponding $b \times k$ packing arrays form the maximal set of words from a partial maximum distance separable code (MDS code) with minimum Hamming distance $k-1$. Much has been studied concerning these codes and therefore, implicitly about transversal packings. There are a number of coding theory bounds applicable to transversal packings: the Plotkin, Singleton, Elias and Hamming bounds. The reader is referred to [26] or [46] for comprehensive books on coding theory or to [45] or [9] for an introductory level discussion. All these books have extensive bibliographies.

The large minimum distance of the code corresponding to transversal packings forces $g \leq b \leq g^2$. One explicit mention of these particular parameter values in codes has been made by Abdel-Ghaffar and Abbadi [2]. In order to obtain bounds on the sizes of partial MDS codes, they prove that

$$tp(k, g : 1) \geq g + 1 \;\Rightarrow\; k \leq \frac{g^2 + g}{2}, \tag{1.35}$$

or in other words

$$tp(\frac{g^2 + g + 2}{2}, g : 1) = g. \tag{1.36}$$

They also illustrate the equivalence between such codes and sets of pairwise orthogonal partial latin squares. The reader is referred again to work on latin squares [7, 12]. In a later paper, Abdel-Ghaffar solves the question for $n = 1$ and $b \leq 2g$ and presents some additional bounds [1].

Transversal packings are related to other packing structures and some bounds for these packing structures are applicable to transversal packings. Two important bounds for packings are the residual and derived bounds. The reader is referred to Mills and Mullin's survey on this subject [28].

Complete determination of either $tp(k, g : 1)$ or $tc(k, g : 1)$ for all k would determine, as a consequence, the number of mutually orthogonal latin squares of side g and solve the existence question for all transversal designs. This is known to be an exceedingly difficult problem.

1.3 Applications

1.3.1 Transversal Covers

One of the reasons for the plentiful research on transversal covers (and related structures) has been their many and varied applications, both to other mathematics and perhaps more importantly, to problems in industry, including software testing and data compression. In fact, in recent years a number of individuals and groups have proposed their use in commercial projects or developed software that generates transversal covers for use in applications. We will discuss three of their major applications and mention some others.

Software Testing

One of the most discussed applications is the utility of transversal covers for designing test protocols. This application appeared in Sloane's 1993 paper although the example that he used was not software testing but testing switch settings (those switches on the backs of printers and modems). Recently, a proposal was made to IBM to implement an online program for generating software test protocols [27]. Williams and Probert[49] and Cohen *et al.* [11, 10] have developed such software. Williams and Probert are interested in the interaction between various hardware components that interact over a network. Cohen *et al.* have developed a system called AETG (Automatic Efficient Test Generator) which is already being used at Bellcore for unit, system and interoperability testing [11, 10].

Consider a new piece of software which has k inputs each taking one of g values. It is desirable to test all possible g^k input strings for software failure, but if k or g are large, this may be infeasible. One common solution to this time constraint is testing a large but feasible number of random input strings, cutting down on the number of test strings. This solution has some risk of incompleteness. Random test protocols can reduce the likelihood of failure to low levels, if the random numbers used are independent. However, most applications requiring many random numbers use pseudo-random number generators which often do not produce independent random numbers so the risk is large that the software will not be tested comprehensively.

However, if we use the smallest number of strings that contain between them, all possible input combinations for each t-set of input variables, then we have achieved some measure of comprehensiveness and also reduced the number of test strings. This minimal set is exactly the blocks of a strength t transversal cover. Cohen *et al.* did an empirical study of user interface software at Bellcore and discovered that most software faults were cause by "either incorrect single values or by an interaction of pairs of values"[11]. This evidence confirms the power of strength 2 transversal covers for this application. Another encouraging report from the Bellcore researchers was their system's detection of faults that had been missed by the standard testing protocols on software just about to be released on the market. Using transversal designs or covers in place of random test strings is often called *derandomization* and a good discussion of it can be found in Gopalakrishnan and Stinson's chapter in [12].

Example 1.6. To illustrate this application, we consider the program used by most universities when hiring recent graduates. The program asks four questions of the job candidate in order of importance: "Is your supervisor famous or unknown?"; "Do you have an external grant?"; "Do you have teaching experience?"; "Was your thesis passed with corrections or modifications?". To test this software against failure for all pair interactions, the test strings generated from the transversal cover given in Example 1.2 would be

test 1	famous	external grant	teaching experience	corrections
test 2	unknown	no grant	no teaching experience	corrections
test 3	unknown	no grant	teaching experience	modifications
test 4	unknown	external grant	no teaching experience	modifications
test 5	famous	no grant	no teaching experience	modifications

Both Williams and Probert and Cohen *et al.* were motivated by concrete problems. Often in these scenarios, there are pairs of input values (or hardware units in Williams and Probert's study) that are known to be independent and do not interact. Alternatively, early entry of some values may bypass the entry of other input variables entirely. The variables may not all take values from a set of fixed size. Most variables could interact in pairs but some may be known to interact in sets of three or more. Other complex relations between input variables or nodes on a network can be imagined. These additional relationships can lead to either a strengthening or a relaxation of the conditions required in test cases. The AETG system developed at Bellcore can take into account a wide range of constraints and relationships: differing group sizes, forbidden pairs, intensive interactions, complex relations, and developer required test sets. It is in part based upon a greedy algorithm [11, 10].

To maximize the effectiveness of such an online test case generator, we would ideally like to have software that could generate the best known transversal covers (this could be a module inside a more comprehensive system that could generate test sets for software that meet certain requirements, as discussed above). An excellent model for such an implementation is Colbourn and Dinitz's automated table generator for $MOLS$ [13]. The program knows all the constructions and is fed input of all known structures. It runs through all the constructions and generates new structures which it then adds to its database. These and any newly discovered structures are then fed back into the algorithm which proceeds recursively to generate more and more instances.

The ideal format would include a comprehensive algorithm which contains all known constructions, and a large database of known transversal covers. When new transversal covers are found, they could be sent by e-mail to the system, which would verify them and then cascade through all the constructions. The resulting new transversal covers would be given a construction stamp (a short description of how this particular cover was constructed: thus allowing the database to store only the cover's size and its construction method rather than the cover itself, saving large amounts of storage space and algorithmic efficiency). This software, coupled with some form of randomized search program, could find its own instances of transversal covers and yield improved results. Researchers could also request given covers for their particular needs and applications. Although a large and formidable project, an implementation of this kind would no doubt justify its own cost. We have already developed a simulated annealing algorithm and a limited implementation of the recursive algorithm described above for finding transversal covers.

Compressing Inconsistent Data

The second large application of these structures is in compression theory. Körner and Lucertini discuss the inability of Shannon theory to deal with the compression of contradictory data [24].

> In the Shannon theory (*sic*) information appears as a substance without shape and hence (*sic*) measurable by a scalar called entropy. Entropy is the limit of compressibility of information. Likewise, in the Shannon approach (*sic*) the communication link or storage device is characterized by the scalar called capacity that indicates the volume of information the link or storage device is able to safely transmit or store. [24]

Because information is shapeless, contradictory information cannot be represented. Hence Körner and Lucertini developed a new paradigm to cope with compressing inconsistent data.

If overlapping fragmentary observations of a large and complex system are made by many observers and some of the information gathered about the large system is contradictory, we may want to store all observations now and try to resolve the inconsistencies later. It may also be possible that the inconsistencies accurately represent the system. A large research project, like the Human Genome Project, is an apt example of such a system. The job is too large for any one research group or lab, so the genome is split up into fragments and these are distributed to different labs to be mapped. However, we must require that several labs sequence each section of the genome, so that we have reliable data. DNA taken from two cells of a single individual may differ due to different mutations in the two somatic cell lineages. The labs sequencing a given portion of the genome may arrive at different results, even if they are perfect at sequencing. They also may arrive at different data because of normal experimental error. In either of these two situations, we may want to maintain the contradiction, in the first case because the difference represents the actual state of the system (and we are interested in somatic mutation rate for example) or in the latter because we may not be able to resolve which lab is in error for some time and we need to store the data efficiently now.

The model that Körner and Lucertini developed to deal with this situation involves a large k-set (the genome in this case with k immensely large) and a family of functions, each with a restricted domain within the set. For each function, we have a set of range vectors for the possible observations made on the domain set (the observation alphabet). In the Human Genome Project, the function domains would probably be large, overlapping consecutive sets and the range is all vectors $\{A, C, G, T\}^{|\text{domain}|}$. The minimal number of complete descriptions of the whole system, such that each fragmentary observation occurs wholly in at least one of them, gives a good measure of the amount of information in the system. This number also represents the maximum compressibility of the data fragments.

If the observation fragments are all possible t-sets of observation values and their possible ranges to g^t is unrestricted, then a strength t transversal cover is the solution to the problem. Considering all t-sets is a simplified example of the overall question, but solutions to these problems may contribute to the answer of more realistic formulations and certainly give upper bounds. Körner and Lucertini survey many different families of maps and many different permitted range vectors and give the best results known as of 1994. Their survey is an excellent presentation of the current state of these problems and the methods used to solve them.

Zero-Error Noisy Channel Communication

The last, large application of transversal covers fits well into Shannon's information theory: in particular, as mentioned in Subsection 1.2.1, Shannon's zero-error noisy channel problem [25]. We can represent a series of noisy communication channels by a family of graphs. Each graph has the same vertex set representing the letters that can be sent across the channel. For a particular channel, an edge, (x, y), in the corresponding graph represents the fact that the probability of x and y being received as identical is zero (i.e. in this channel x and y are distinguishable). An edge missing denotes that the two letters incident to it could be indistinguishable at the receiving end of the channel. We do not know which channel we are going to be using, so in order to obtain error-free transmission, we must find the maximum set of code words that hits an edge in every possible graph. Formally this is: given any two code words, x and y, for every graph, G, there exists an edge $(x, y) \in G$ and an index i such that $u_i = x$ and $v_i = y$. This guarantees that these two codewords will remain distinguishable after reception.

The graphs could be any family of graphs on b vertices. The columns of a covering array correspond to a code for the worst case when the family of graphs is all possible one edge graphs on the b set. The covering conditions guarantee that any pair of words can be used distinguishably. In fact, what we actually need to use is the transversal cover with a set of g disjoint blocks removed, the ITC, because we don't need to cover the pairs (a, a) since these are clearly indistinguishable. The code derived from the ITC will transmit faithfully across any channel with any noise pattern as long as each channel has at least one edge. As discussed in Subsection 1.2.1, Körner and Simonyi [25], and Gargano, Körner, Vaccaro [16, 17] have solved Sperner capacity problems for general graphs and some specific families of graphs.

Others

Katona, in his paper solving the transversal covers problem for $g = 2$ and $n = 1$, showed that the solution determines the minimum length of any monotonically increasing truth function in disjunctive-normal form [22]. Körner and Lucertini also mention applications to testing of logic circuits, computer architecture design, random access communications and have references to other uses [24].

1.3.2 Transversal Packings

In their paper, Abdel-Ghaffar and Abbadi [2] use MDS codes to allocate large database files to multiple hard disk systems so that the retrieval time is optimal. Although their application used MDS codes and not partial MDS codes and only considered alphabets of prime power cardinality, the partial MDS codes corresponding to transversal packings could possibly be used for optimal disk allocation to a large number of disks, specifically at least g^{k-2}, where, in their terminology, g is the number of sets into which each attribute of the database is divided. If transversal packings were used for optimal disk allocation in this model, any search with at least two specified attributes would yield search time one.

In general, the codes corresponding to transversal packings could be useful for any of the standard applications for codes: error detection; error correction. One would imagine that transversal packings would be most useful in a situation where the error rate in the information channel was exceedingly high requiring that the minimum distance between codewords be large as it is in these partial MDS codes.

By such a brief mention of the coding applications of transversal packings, we do not wish to indicate that either transversal packings or their applications are limited or uninteresting.

In Subsection 4.1.1, we discus the structure of transversal packings that meet generalized Plotkin bounds. The duals of these transversal packings are resolvable block designs. The group structure of the transversal packings ensures that each resolution class has the same number of blocks and often also implies that the same number of each block size appear in each resolution class. This type of resolvable PBD is important for its application to experimental design. Resolvability gives unbiased estimates of error, some efficiency guarantees, and design management benefits [44]. The added condition of a constant number of blocks in each resolution class adds to the management benefits. Each block of experimental treatments may be sent out to separate labs and it is more efficient to use each lab as much as possible. Additionally, each lab may be dedicated to a fixed block size, so designs that cater to this constraint are useful. These structures are currently being investigated by the author and Peter Danziger.

1.4 Outline of Thesis

The determination of $tc(k, 2 : n)$ has been completely solved, independently, by Katona [22] and Kleitman and Spencer [23]. This thesis investigates the cases where $g \geq 3$. Most of the work previously done on this problem produces asymptotic results. Körner and Simonyi [25] developed a constructive technique as did Gargano, Körner and Vaccaro [15], but these constructions are only useful at extremely large values. Sloane reports a construction for $g = 3$ (a generalization of this appears herein) and presents the best known covers for small values of k. We develop techniques that can produce instances for any parameter sets, large and small. To this end, we have approached the subject from design theory.

In the first part of the thesis, we examine transversal covers and in the latter part, transversal packings. In each case, we are interested in both upper and lower bounds. The two chapters addressing transversal covers deal with upper bounds (constructed examples) and lower bounds. The chapters on transversal packings deal with upper bounds and then lower bounds (constructions).

In the second chapter, we present five constructive methods. The first uses incomplete transversal designs and is only applicable to a small range of cases. The second method is a generalization of one of Poljak and Tuza's constructions and demonstrates itself to be very effective. The third and fourth generalize Wilson's construction and use GDD's, respectively to construct transversal covers. We then briefly discuss how well these constructions perform asymptotically. The last section is a discussion of a simulated annealing algorithm we developed to find instances of transversal covers and the recursive algorithm that implements all the constructions.

The third chapter presents a number of lower bounds on the size of transversal covers. We derive three generally applicable lower bounds (valid for all parameter sets), one from the known asymptotic sizes and a construction, one an improvement on Poljak and Tuza's lower bound and the last using $PBTC$'s. There follows a section investigating lower bounds for very small TC's. In this section, we prove that many of the known small covers are, in fact, optimal.

The fourth chapter starts the consideration of transversal packings. Many coding theory bounds can be directly applied producing upper bounds on the sizes of transversal packings. Some of these can be modified to include information about sets of disjoint blocks. Finally, structural information of transversal packings can lead to some additional bounds. The fifth and final research chapter deals with constructing transversal packings. Several of the constructions used for transversal covers can be analogously applied to transversal packings. In addition to these, we consider the case when $b \leq 2g$ and solve a wide range of transversal packings with these parameters by translating the problem to colouring or decomposition problems on graphs.

Chapter 2

Transversal Covers: Constructions and Upper Bounds

In this chapter, we derive four constructions for transversal covers. The first is based on filling the holes of incomplete transversal designs. This construction has a severely limited range of applicability, but it can be used to construct an optimum cover for $g = 6$. The second construction, the blocksize recursive method, is a recursive concatenation technique that increases k for a fixed g. The third set of constructions is increases g and generalizes Wilson's construction. These three constructions appear in a forthcoming paper by the author and Eric Mendelsohn[42]. The last construction is a direct construction using GDD's and extending their blocks. At the end of the chapter, we discuss a computer implementation of these constructions, a simulated annealing algorithm for finding transversal covers, and the relative efficacy of these constructions.

2.1 Incomplete Transversal Designs

In this section, we use incomplete transversal designs to construct transversal covers. Incomplete transversal designs are good starting objects for the construction of transversal covers because most of the pairs (those not inside a hole) appear in only one block rather than at least one. If we can avoid covering these pairs again in the construction then we are producing a cover which is predominantly a design.

2.1.1 Presentation of the Method

Given an $ITD(k, g; b_1, b_2, \ldots, b_s)$, we fill the holes with covers of order b_i and length k.

Definition 2.1. A *filling* of an $ITD(k, g; b_1, b_2, \ldots, b_s)$ by $TC(k, b_i)$ is a set of bijections $\{f_{G,i}, f_{1,i}, f_{2,i}, \ldots, f_{k,i}\}$ $1 \leq i \leq s$. $f_{G,i}$ is a bijection between the groups of the ITD and the groups of the $TC(k, b_i)$. $f_{j,i}$ is a bijection between the points of $H_i \cap G_j$ and the points of the jth group of $TC(k, b_i)$. The images of all the blocks of the $TC(k, b_i)$ under these bijections are then added to the blocks of the ITD.

Theorem 2.1. *If there exists an $ITD(k, g; b_1, b_2, \ldots, b_s)$ then*

$$tc(k, g : i) \leq \min_{\substack{i_i + i_2 + \cdots + i_s = i \\ i_j \leq b_j}} \left(g^2 - \sum_{j=1}^{s} \left(b_j^2 + tc(k, b_j : i_j) \right) \right).$$

Proof. Fill the holes. The holes are disjoint, thus the union of the sets of disjoint blocks from the $TC(k, b_j : i_j)$ will also be a set of disjoint blocks. □

Counting the disjoint blocks in covers produced by this construction is not trivial, and at the moment, we know of no formula for the maximum number. The number of disjoint block might depend, not only on the individual structures used, but also on the bijections chosen in the filling. The result above, on the number of disjoint blocks, is therefore a lower bound on the total number.

2.1.2 Example

In particular, the existence of $ITD(4,6;2)$ and $ITD(6,10;2)$ [12] yield $tc(4,6) = 37$ and $tc(6,10) \leq 102$. $TC(4,6)$ is explicitly

$$\begin{pmatrix} 0&0&0&0&0&0&1&1&1&1&1&1&2&2&2&2&2&2&3&3&3&3&3&3&4&4&4&4&5&5&5&5&4&4&5&5&5 \\ 0&1&2&3&4&5&0&1&2&3&4&5&0&1&2&3&4&5&0&1&2&3&4&5&0&1&2&3&0&1&2&3&4&5&4&5&5 \\ 4&5&2&3&0&1&1&0&5&4&2&3&5&4&0&1&3&2&3&2&4&5&1&0&0&3&1&2&2&1&3&0&4&5&5&4&5 \\ 0&1&4&5&2&3&5&4&0&1&3&2&3&2&5&4&0&1&4&5&3&2&1&0&1&3&2&0&2&0&1&3&4&5&5&5&4 \end{pmatrix}^T$$

where the last five rows are the filled hole. We observe that this is actually a $TC(4,6:5)$: blocks 1, 8, 18, 27 and 35 are a maximal set of disjoint blocks. We have verified by computer that no more than five disjoint blocks arise no matter what bijections are used in the filling.

2.1.3 Limits on the Use of the Construction

Lemma 2.2. *[13] An $ITD(k,n;h)$ exists only if $h = n$ or $(k-1)h \leq n$. When $(k-1)h = n$, all blocks have exactly one point in the hole.* $\square$

This lemma holds for ITDs with more than one hole as well. Thus $k \leq n/h + 1$, where h is the size of the largest hole, and this lemma shows that the incomplete transversal design method can never be used to find transversal covers with k bigger than $n/2 + 1$, since a hole of size one implies the existence of a transversal design. In particular, this construction is only useful for non prime power values of g. When $(k-1)h < n$, it is known that there are at least two disjoint blocks, both disjoint from the hole [13]. So in this case we get

$$tc(k, g : i+2) \leq \min_{\{h \mid \exists ITD(k,g;h)\}} (g^2 - h^2 + tc(k, h : i)).$$

Filling a hole with a non-minimal cover produces a cover that is also clearly too big.

Even if we fill a hole with an optimal cover, there is still no guarantee of producing an optimum cover although it may be close to optimal. However, this method is still one of the best known for finding transversal covers with large groups and small blocks.

The CRC handbook [12] has a vast table of incomplete transversal designs which can be used to find transversal covers which have bigger k than the largest known block size of transversal designs with the same order: $ITD(7, 22; 3)$, $ITD(8, 36; 5)$, $ITD(7, 39; 4)$, $ITD(7, 54; 5)$, $ITD(8, 58; 2)$, and $ITD(8, 60; 4)$ to name just a few. The tables in Appendix A can be very useful in this construction for group sizes listed in the CRC tables.

2.2 Blocksize Recursive Method

2.2.1 Presentation of the Method

Using the sets of disjoint blocks to their fullest advantage, we can formulate a construction that is similar to Inequality 1.6 and Inequality 1.19 but better and more general.

Theorem 2.3. *Let $n, m \leq g$. Then,*

$$tc(k, g : n) \leq \min_{d|k}(tc(d, g : n) + tc(k/d, g : m) - m). \tag{2.1}$$

Proof. We construct a cover of the required size. The proof is easier if we represent the transversal cover as a covering array. Then, for a divisor, d, of k we write the k/d identical arrays of length d next to each other.

$$\boxed{TC(d, g : n)} \quad \boxed{TC(d, g : n)}$$

We index the columns in the following way: the ith column of the jth array is labeled by j_i. Since the arrays are covering arrays and they are all identical, all pairs of columns are covered except for pairs with the same subscript. Within these pairs it is evident that we have covered the pairs (k, k) for $0 \leq k \leq g-1$. Here is an example with $g = 3$. The first column of the first cover and the second column of the second cover are covered, but the two first columns are not except for the pairs (1,1), (2,2) and (3,3).

0	0		0	0	
0	1		0	1	
0	2		0	2	
1	0		1	0	
1	1	$\cdots$	1	1	$\cdots$
1	2		1	2	
2	0		2	0	
2	1		2	1	
2	2		2	2	
$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\ddots$

To cover these remaining pairs we put an $ITC(k/d, m : m)$ on the d sets of k/d columns with the same subscript. The pairs from the set of m disjoint blocks removed have been covered already, see Figure 2.2.1. □

Since $tc(k, g) \leq tc(l, g)$ for $k \leq l$, we may be able to generate a better $TC(k, g)$ by considering block sizes larger than k, say l, which have more divisors or where the transversal covers with block sizes of the divisors of l are better than the covers with block sizes of divisors of the original k. For example, to construct $TC(5, g)$, it may be better to construct $TC(6, g)$ since 6 has more divisors than 5, and drop one group. With this in mind, the theorem is more generally stated as

$$tc(k, g : n) \leq \min_{2 \leq i \leq \lceil \frac{k}{2} \rceil} (tc(i, g : n) + tc(\lceil k/i \rceil, g : m) - m). \tag{2.2}$$

This recursive procedure can, of course, be iterated.

Sometimes the incomplete transversal cover, $ITC(k/d, g : m)$, may have another

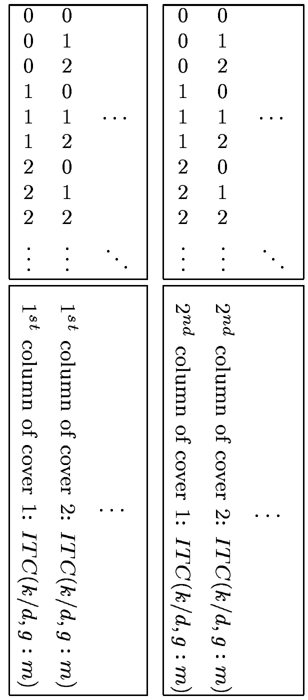


Figure 2.1: Columns extended to cover remaining pairs

set of i disjoint blocks. If $i \geq n$, we will use this set of disjoint blocks and have generated a $TC(k, g : i)$ instead. This extra set will always exist when we are using a transversal design of prime power order and $k \leq g$. For example, $ITC(3, 3 : 3)$, has an additional set of three disjoint blocks even after the first set has been removed. This will allow us to construct

$$tc(k, 3 : 3) \leq tc(3, 3 : 3) + tc(\lceil \tfrac{k}{3} \rceil, 3 : 1) - 3.$$

2.2.2 Example

This construction applied to a $TC(3, 3\!:\!1)$ and an $ITC(2, 3\!:\!3)$ produces a $TC(6, 3\!:\!3)$:

0	0	0	0	0	0
0	1	2	0	1	2
0	2	1	0	2	1
1	0	2	1	0	2
1	1	1	1	1	1
1	2	0	1	2	0
2	0	1	2	0	1
2	1	0	2	1	0
2	2	2	2	2	2

0	0	0	1	1	1
0	0	0	2	2	2
1	1	1	0	0	0
1	1	1	2	2	2
2	2	2	0	0	0
2	2	2	1	1	1

2.2.3 An Alternative Formulation

There is another way to view this construction. In Definition 1.6, we have defined the existential array which was used implicitly by Sloane several times to construct strength 3 covering arrays [38].

For the first row of an $EA(c, g_1, \ldots, g_r)$, we label the columns of the covering array corresponding to a $TC(g_1, g : 1)$ with the g_1-ary alphabet used in the first row of the EA. Likewise, for each of the rows of the EA and its alphabet, we label the columns of the covering array corresponding to an $ITC(g_i, g : g)$. Then, we replace each symbol in the EA with the column it indexes from the appropriate cover. The fact that any two columns in the EA have some row where they differ means that the columns from the covering array replacing the two values there will cover these two columns in the final array. The columns inserted into the first row of the EA guarantee that all pairs of columns contain the pairs (i, i). Theorem 2.1 implicitly uses an $EA(k; d, k/d)$. The existential array encompasses the r-fold iteration of this method.

2.3 Generalizing Wilson's Construction to Covers

2.3.1 Presentation of the Method

Theorem 2.4. *Let C be a $TC(k+l,t)$ with groups $G_1, G_2, \ldots, G_k, H_1, H_2, \ldots, H_l$. Let $\mathcal{S}$ be any subset of $H_1 \cup H_2 \cup \cdots \cup H_l$ of cardinality u, m be any nonnegative integer, and $h_i = |H_i \cap \mathcal{S}|$. For any block A of C let $u_A = |\mathcal{S} \cap A|$. Then*

$$tc(k, mt+u) \leq \sum_A (tc(k, m+u_A : u_A) - u_A) + \sum_{i=1}^{l} tc(k, h_i).$$

Proof. The proof is a straightforward generalization of Wilson's construction. □

There are many other generalizations of Wilson's construction that generate designs with holes, for example [13]. These constructions can similarly be extended to covers with holes.

2.3.2 MacNeish's Theorem for Covers

If $l = 0$ in the preceding theorem, this is exactly the same as multiplying groups of size t and m. In an obvious generalization of MacNeish's theorem, we get

Theorem 2.5.

$$tc(k, g : n) \leq \min_{\substack{2 \leq i \leq \lceil \frac{g}{2} \rceil \\ \max(1, \frac{n}{\lceil \frac{g}{i} \rceil}) \leq j \leq \min(n,i)}} \left((tc(k, i : j)\ tc(k, \lceil g/i \rceil : \lceil n/j \rceil) \right).$$

□

Not much need be said about this method. The complicated subscripts arise for two reasons. As in previous method, $tc(k, g_1 : n) < tc(k, g_2 : n)$, for $g_1 < g_2$. Thus we may benefit from constructing transversal covers with larger group sizes than g, when

g has few divisors. The strict inequalities above are true since we may assume, by relabeling, that one block is $(g_2-1, g_2-1, \ldots, g_2-1)$. This block may be removed, and every other occurrence of the letter g_2-1 arbitrarily changed to any letter from the g_1-ary alphabet. The second complication of indexing stems from the need to guarantee that the number of disjoint blocks is less than the alphabet size in each of the smaller covers.

2.3.3 Probabilistic Implementation

The essential problem in using Theorem 2.4 is that we need to know the cardinality of the intersection between each block and the set S, which requires substantially more information than just $tc(k+l,t)$. In the case $l=1$, however, we get, for $0 \le u \le t$

$$tc(k, mt+u) \le \sum_{A \mid A\cap S=\emptyset} tc(k, m:1) + \sum_{A \mid A\cap S\neq\emptyset} (tc(k, m+1:1)-1) + tc(k, u:1) \quad (2.3)$$

and we can reduce this problem to knowing the number of blocks that intersect S and the number of blocks that intersect S^c. Further, by calculating the expected number of blocks that intersect S, we can remove this aspect totally from the formula and express it as a recursion in the covering numbers alone.

We calculate the expected number of blocks that intersect a set S of size u, where S is a subset of one group, say G.

$$\begin{aligned}
\frac{1}{\binom{t}{u}} \sum_{S\subset G, |S|=u} \text{number of blocks intersecting } S &= \frac{1}{\binom{t}{u}} \sum_{\substack{S\subset G\\ |S|=u\}}} \sum_{x\in S} r_x \\
&= \frac{1}{\binom{t}{u}} \sum_{x\in G} r_x |\{S : |S| = u, x\in S\}| \\
&= \frac{1}{\binom{t}{u}} \binom{t-1}{u-1} \sum_{x\in G} r_x \\
&= \frac{u}{t} tc(k+1, t).
\end{aligned}$$

Therefore, there must exist a set S with $\mid \{A|A \cap S \neq \emptyset\} \mid \leq \frac{u}{t} tc(k+1,t)$. As $tc(k, m:1) \leq tc(k, m+1:1) - 1$, this value for $tc(k, mt+u)$ in Inequality 2.3, is minimized if $\mid \{A|A \cap S \neq \emptyset\} \mid$ is as small as possible. Using this expected value, and the fact that disjoint blocks in the original cover in Wilson's construction contribute blocks to disjoint point sets in the final cover, we can achieve:

$$tc(k, mt+u:n) \leq \min_{\substack{ij+l \geq n \\ 1 \leq l \leq u \\ 1 \leq i \leq t \\ 1 \leq j \leq m}} \quad tc(k, u:l) + \frac{t-u}{t} tc(k+1, t:i) tc(k, m:j)$$
$$+ \frac{u}{t} tc(k+1, t:i)(tc(k, m+1:j+1) - 1). \qquad (2.4)$$

If the variance is zero then each subset of a group with cardinality u must be incident with exactly $(u/g)tc(k+1, t:i)$ blocks. A variance of zero implies that each point is incident with $tc(k+1, t:i)/g$ blocks. For the variance to be zero, g must divide $tc(k+1, t:i)$ and r_x must be constant for all points x; in other words, we have a $PBTC$. In the few examples where $tc(k+1, t:i)/g$ is an integer, the covers have constant r_x, so it may not be possible to prove in general that the variance is greater than zero. We believe that optimal covers that are also point-balanced are rare. However, if $tc(k+1, t:i)$ is not a multiple of g then the variance must be positive and we can improve the bound.

2.3.4 Generalization of Sloane's Construction

By setting $m = 0$ in Theorem 2.4, we can achieve another explicit upper bound. When $m = 0$, the groups $G_1, G_2, \ldots, G_k$ contribute nothing and without loss of generality, $k = 0$. The construction then yields

Theorem 2.6. *If there is a GDD on u points with group sizes $h_1, h_2, \ldots, h_l$ and block set $\mathcal{B}$, then*

$$tc(k, u) \leq \sum_{B \in \mathcal{B}} (tc(k, |B| : |B|) - |B|) + \sum_{i=1}^{l} tc(k, h_i).$$

□

If we did not remove the disjoint blocks from the cover put on the blocks we would cover the pairs of these points too often. This motivates a generalization.

For group size g, to construct a $TC(k, g)$, take any PBD on g points (in Theorem 2.6 with $m = 0$ the PBD has a resolution class, in other words, a GDD). Then for each block, B, of this PBD, consider the $CA(k, |B|)$ with the $|B|$-ary alphabet being the points in block B. Vertical concatenation of these arrays forms a $CA(k, g)$. The transversal that covers the pair (i, j) in columns k and l is the transversal that covers this pair in the covering array placed on the block from the PBD that contains i and j.

However, as mentioned above, we cover the pairs (i, i), $0 \leq i \leq g-1$ each time that i appears on a block of the PBD. In Theorem 2.6, we avoided this overlap by only allowing the blocks from the resolution class to cover these pairs, and removing the disjoint blocks from the other subcovers that redundantly covered these pairs. But, in a given PBD we may not always have a resolution class. We can avoid overlap appropriately even without a resolution class:

Lemma 2.7. *In a PBD on v points, with at least one block of size less than v, there exists a set of distinct representatives for the point, i.e. for each point, x, there exists a block B_x with $x \in B_x$ and all the B_x distinct.*

Proof. For there to be a set of distinct representatives, it is necessary and sufficient that for each collection of n points, there are at least n blocks induced by them.

Assume that there is no set of distinct representatives. Pick the smallest set of points, say S, of order n, that induce fewer than n blocks. Clearly $n \geq 2$. If we remove the points not in S from these blocks we get a PBD on n points with less than n blocks. When we remove these points we cannot get a singleton block. If we did, say the point was x, then consider the $n-1$ points in $S \setminus x$. Since $S \setminus x$ is a smaller set than S there must be at least $n-1$ blocks through these points. These account for all the blocks of the PBD on S, so there can be no additional block that is just $\{x\}$.

Since we have a PBD with more points than blocks, we have violated Fisher's inequality. So in the PBD on point set S, there is a block with n points on it. Now consider any $n-1$ subset, T, of S. The PBD induced by these points must have have at least $n-1$ blocks, and since the PBD induced by S has less than n blocks, both have exactly $n-1$ blocks. The induced structure on T must be a symmetric PBD with at least one block of size $n-1$. If there were blocks of more than one size, then by Ryser and Woodall's theorem, the other size would have to be one [7]. This was shown impossible above.

Thus, all the blocks in the induced PBD on T are of size $n-1$. Since λ in the induced design is the same as λ in the original design, $\lambda = n-1$ and for any point, $x \in S$, $r = n-1$ as well. Since $\lambda = r$, any point $y \notin S$ must be on these $n-1$ blocks. Since there are $n-1$ blocks with every point on them and $\lambda = n-1$, clearly these are all that we have in the original PBD. Considering that we must have at least one smaller block, this contradicts the assumption that no set of distinct representatives exists. $\square$

This means that if the PBD is not trivial, for each point we can pick a block that will represent that point. We put $ITC(k, g : g-1)$'s on each block where the removed disjoint blocks can be assumed to be $(i, i, \ldots, i)$, where i is any symbol other than the one represented by this block, and put $ITC(k, g : g)$'s on every other block. The removal of these disjoint blocks still leaves a TC because the only blocks lost are of the form $(i, i, \ldots, i)$. Each pair (i, i) will still be covered because there is a block that represents the point i and we have avoided overcovering. The TC placed on this block covers all of these pairs because only disjoint blocks representing the other points were removed from this TC.

A set of distinct representatives is not the only scheme that avoids overlapping to optimize this construction. We could have blocks representing more than one point, and thus we should require some covers with fewer disjoint blocks and some with more. The CA we put on each block would still have disjoint blocks for each non-represented point, which would be removed. However, the set of distinct representatives may be better since the sizes of $ITC(k, g : g-1)$ and $ITC(k, g : g)$ are no bigger than $ITC(k, g : n)$ for all other n. Thus we have:

Theorem 2.8. *Given a $(v, \{2, 3, \ldots, g-1\}, 1)$-design, and for each point x, a chosen block, B_x, with $x \in B_x$, we can construct a $TC(k, g)$. For each block, B, of the design, we define u_B to be the number of points on this block not represented by it. Then*

$$tc(k, g) \leq \sum_{B} tc(k, |B| : u_B) - u_B. \tag{2.5}$$

□

Optimization

The size of the resulting cover depends on the scheme of point representation. There may be better or worse methods to pick which blocks represent which of its points. As yet, it is unknown if there is a simple scheme of representations that yields the best cover. However, we can eliminate some schemes of representation.

Lemma 2.9. *A system of representatives that has a block representing all its points is not optimal.*

Proof. In the construction, we would put a cover on this block that had zero disjoint blocks, but all covers have at least one disjoint block. It is possible to shift the representation of one of these points to another block that would not then represent all of its points. This shift would reduce the number of disjoint blocks required for the transversal cover on the block to which we have shifted, while not increasing the number of disjoint blocks in the transversal cover used on the block from which we have shifted. If it were not possible for a point x on the first block to be so shifted, then this scheme of representation is simply the following: all blocks incident with x represent each of their points, with the exception of x which is represented by only one of them and all other blocks (a non-empty set by Fisher's inequality) represent none of their points. In this case, some other point on the block that represents all its points can be shifted to one of the blocks that represent none of their points. $\square$

It will be shown in Section 2.6.2 that $(v, \{2, 3, \dots, g-1\}, 1)$-covers yield better results than $(v, \{2, 3, \dots, g-1\}, 1)$-designs, for sufficiently large k, and the optimum incidence structure is known.

Sets of Disjoint Blocks

In general, it is a non-trivial task to find how many disjoint blocks the PBD construction yields. For $g = 3$, we give one illustrative example.

Example 2.1. Consider the case where $g = 3$ and $k = 6$. The blocks of the only PBD on three points are $\{\{0,1\},\{1,2\},\{2,0\}\}$ and we say that each block represents its first point. The blocks and their represented points are cyclically permuted so the covers placed on each one can be the same with the symbols cyclically permuted: letters 1 and 2 in the TC placed on the second block wherever letters 0 or 1, respectively, appeared in the TC placed on the first block. So if, for example, the block $(0,0,2,1,0,2)$ appeared in one of them then the other two would contain $(1,1,0,2,1,0)$ and $(2,2,1,0,2,1)$ which are clearly disjoint. Since this is true of each block, we get many different sets of disjoint blocks.

This generalization of Sloane's example suggests the following theorem which can guarantee the existence of some set of disjoint blocks:

Theorem 2.10. *In the construction from Theorem 2.5, if there exists a set of* n *blocks*

1. *all of the same size;*

2. *all representing the same number of points, s; and*

3. *for each block we can chose an ordering of the points, o_B such that the points that the block represents are the first points ($o_B(1), o_B(2) \ldots$) in the ordering and the ith point of the ordering for two different blocks are different,*

then the construction yields a $TC(k, g : n)$.

Proof. We pick the subTC to put on B_1 with the appropriate disjoint blocks (covering the points not represented by this block). We determine the TC's to put on the remaining disjoint blocks of our specified collection by using the orderings to generate maps between the points of two blocks: $o_B(i) \to o_{B'}(i)$, $1 \leq i \leq |B|$. The TC we put on block B' is the isomorphic image of the TC put on block B under the map defined above. Pick any block from the first TC. This block together with all its image blocks in the other TC's will be disjoint. In fact, we will have $tc(k, |B_1| : |B_1| - s) - |B_1| - s$ such sets of disjoint blocks. □

Additionally, by using only $TC(k, g : g)$ instead of $TC(k, g : n)$ as subcovers, we can achieve the maximum number of disjoint blocks. The construction is the same, but for each point x, a block of the form $\{x, x, \ldots, x\}$ remains. This block will be from the subTC placed on the block of the PBD that represents x. This method may not increase the number of total blocks because the $TC(k, g : g)$ may be as good as the TC's called for in the construction.

2.4 Group Divisible Design Construction

2.4.1 The Basic Construction

Of the three constructions previously discussed, two are recursive and one is direct. The direct ITD construction is severely limited in the range of values for which it is useful. Using group divisible designs, we can formulate another direct construction. The work in this section will appear in a paper co-authored with Alan Ling [41]. Because all pairs of points not on the same group are covered by blocks it is obvious that

Theorem 2.11. *If a k-GDD of type g^n exists then*

$$tc(n, g : 1) \leq \frac{g^2 n(n-1)}{k(k-1)}.$$

□

Proof. Arbitrarily extend each block to a transversal. □

Using the block size recursive construction we can show that

$$tc(k, g) \leq \left\lceil \frac{\log n}{\log(m+2)} \right\rceil (g^2 - g) + g.$$

Let m be the maximum number of idempotent $MOLS$ known of order g. Using Fisher's theorem on the number of blocks in a design [7], it is necessary, for the GDD construction to yield results better than already known, that

$$(g-1)n < \frac{g^2 n(n-1)}{k(k-1)} \leq \left\lceil \frac{\log n}{\log(m+2)} \right\rceil (g^2 - g) + g. \tag{2.6}$$

If g is a prime power then $m = g - 2$.

For g a prime power, we can show that for this construction to be better than what we already know it is necessary that $g + 2 \leq n \leq 2g$. Checking the divisibility conditions for $g \leq 7$ and comparing the results to the best covers known, the only possibilities that could come from this construction, for $g \leq 7$, are $n = g + 2$ which is dealt with in section 2.4.3.

2.4.2 Adding More Groups

The restrictions on the GDD's enabling them to yield better covers, are quite strong. In all the cases except those mentioned above, the number of blocks is far too large. However, if we can extend the covers by a significant number of groups then we may be able to produce covers better than we already know. If we start with a $RGDD$ then we can add a number of new groups.

Theorem 2.12. *The transversal cover constructed from a k-RGDD of type g^n can be extended by at least e groups, where*

$$e = \left\lfloor \frac{\left(\frac{g(n-1)}{k-1}\right) - 1}{g-1} \right\rfloor .$$

Proof. There are

$$\frac{g(n-1)}{k-1}$$

resolution classes each with ng/k blocks. Viewing the resulting transversal cover as a covering array on symbol set $\{0, 1, \ldots, g-1\}$, and defining

$$e = \left\lfloor \frac{\left(\frac{g(n-1)}{k-1}\right) - 1}{g-1} \right\rfloor$$

add e zeros to the rows of the array that correspond to the blocks of the first resolution class. In the first of the new e columns place the symbols $1, 2, \ldots, g-1$ on the rows of the next $g-1$ resolution classes, a different symbol for each class. On the rows of each additional resolution class place the symbols $1, \ldots, g-1$, so that every resolution class has at least one row which gets each symbol. This can be done since each resolution class has $ng/k \geq g$ blocks. On the second of the new e columns, place the symbols $1, 2, \ldots, g-1$ on the rows of the $g+1^{st}$ to $2g-1^{st}$ resolution classes, a different symbol for each class. On all the other rows, place the symbols $0, 1, \ldots, g-1$, so that every resolution class has at least one row which gets each symbol. On the ith new column place the symbols $1, 2, \ldots, g-1$ on the rows of the $(i-1)g - i + 3^{rd}$ to

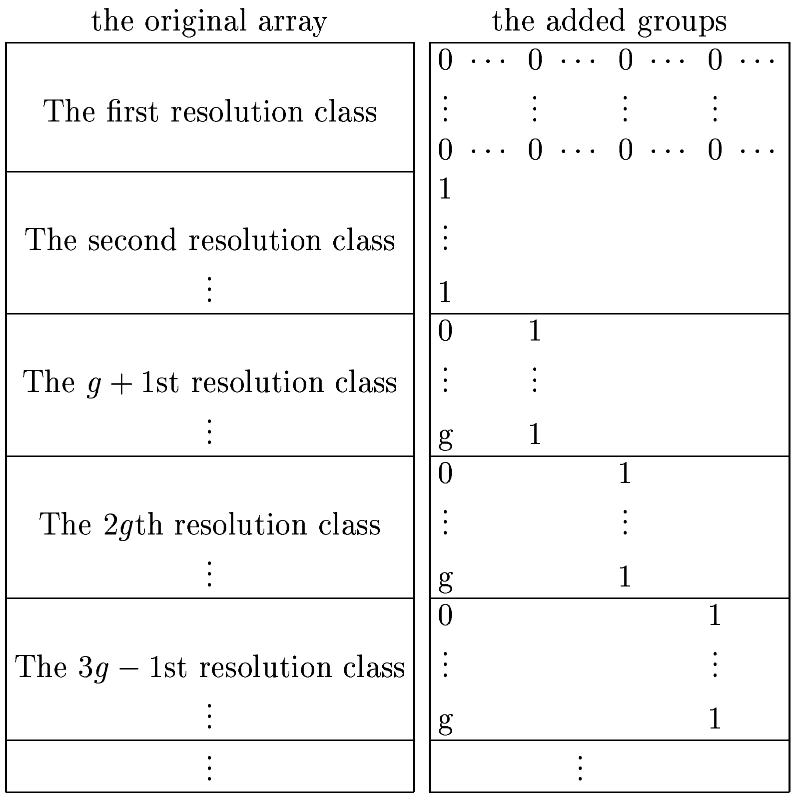


Figure 2.2: Method for Adding New Groups

$ig - i + 1^{st}$ resolution classes, a different symbol for each class. On all the other rows, again place the symbols $0, 1, \ldots, g-1$, so that every resolution class has at least one row which gets each symbol. See Figure 2.2. This is a covering array. $\square$

Theorem 2.13. *The transversal cover constructed from a k-RGDD of type g^n can be extended by e groups where e is the maximum integer such that*

$$tc(e, g : g) \le \left(\frac{g(n-1)}{k-1} - g \right) \frac{ng}{k} + g.$$

Proof. Again, we will think of the covers as covering arrays and extend by e columns. On the rows corresponding to the first g resolution classes we put the symbols $0, 1, \ldots, g-1$ in each of the e columns, one symbol per class. We have covered all pairs of columns, one from the original set and one from the new set, and because we started with a covering array, all pairs of columns from the original set of columns.

All we must do is now cover all pairs of columns from the new set. We have

$$\left(\frac{g(n-1)}{k-1} - g\right)\frac{ng}{k}$$

rows empty on the set of new columns. In each pair of new columns, we have covered the pairs of identical symbols (i, i) and so we place in the set of unfilled rows, the largest $ITC(e, g : g)$, that has fewer than this number of rows, which is exactly the e required. □

Theorem 2.14. *The transversal cover constructed from a k-RGDD of type g^n can be extended by e groups where e is the maximum integer such that*

$$tc(e, g : g) \leq \frac{g(n-1)}{k-1}.$$

Proof. On the rows from each resolution class, we will always place the same symbol. This will guarantee that we cover all pairs of columns, one from the original array and one from the new set of columns. Then to cover all pairs of columns, both from the new set, we put the largest covering array that has fewer than $\frac{g(n-1)}{k-1}$ rows, on the new columns treating each resolution class as a single column, and arbitrarily completing any empty cells. □

We do not actually need a resolvable GDD to use any of these methods of adding groups. All we actually require is that we can partition the blocks into classes such that, after they have been extended, (Theorem 2.4.1) they cover each point of the original GDD at least once. However, finding the sufficient conditions for this is hard. The condition of resolvability on the original GDD is sufficient. Unfortunately there are not many families of $RGDD$'s known. We do not know many $RGDD$'s with relatively large block size with respect to n. We have looked for $RGDD$'s that might give good covers using these theorems, but so far we have been unable to find any. The next family of GDD's, although not resolvable, has large block size and at least one new group can be added.

2.4.3 An Infinite Family from the Construction

As mentioned above, when $n = g + 2$, the divisibility conditions are met and the number of blocks is reasonably small. An affine plane of order q with one point removed is a q-GDD of type $(q-1)^{q+1}$. From this we can construct a $TC(q+1, q-1 : 1)$ with $q^2 - 1$ blocks. The blocks of this GDD can be partitioned into $q+1$ sets of $q-1$ blocks which are mutually disjoint. Each of these sets of blocks misses one group entirely, so when we are extending the GDD to a TC we can choose the new points of these blocks so as to maintain the property of them being sets of disjoint blocks. We can extend this TC by one group to get

Theorem 2.15. *If g is one less than a prime power then $tc(g+3, g : g) \leq g^2+2g$.* $\square$

This gives

$$\begin{aligned} tc(7,4:4) &\leq 24 \\ tc(9,6:6) &\leq 48 \\ tc(10,7:7) &\leq 63 \end{aligned} \tag{2.7}$$

where the previous constructions only give

$$\begin{aligned} tc(7,4:4) &\leq 28 \\ tc(9,6:6) &\leq 64 \\ tc(10,7:7) &\leq 91, \end{aligned} \tag{2.8}$$

but in a few cases we can take advantage of the structure of these GDDs to add more than one group. These GDD's come from the 1-rotational presentation of the affine plane [12, 8], that is, the q^2 points of the affine plane are ∞ and the points of $\mathbb{Z}_{q^2-1}$. The blocks are generated additively from two base blocks. The first is the short block $G_0 = \{\infty\} \cup \{a(q+1) : 0 \leq a \leq q-2\}$ and the second is $B_0 = \{d_1, \ldots, d_q\}$ where $d_1 = 0$ and $\mu^{d_i} = 1 + \mu^{u+(i-2)(q-1)}$ for $i = 2, \ldots, q$, μ a primitive element of $\mathbb{F}_{q^2}$ and u an integer not a multiple of $q+1$ [12]. The GDD is just this design with ∞ removed, the groups generated by the short base block, G_0, and the blocks generated additively from B_0. The points are labeled as shown, where columns represent groups.

$$\begin{array}{cccc} 0 & 1 & \cdots & q \\ q+1 & q+2 & \cdots & 2q+1 \\ \vdots & \vdots & & \vdots \\ q^2-q-2 & q^2-q-1 & \cdots & q^2-2 \end{array}.$$

To make the blocks transversals, we need to add a point to B_0. We will determine this point later. Now with this presentation, the sets of disjoint blocks mentioned earlier are $P_i = \{B_0 + k(q+1) + i\}_{k=0}^{q-2}$ where $j = 0, 1, \ldots, q$. But consider also the sets of blocks $Q_j = \{B_0 + k(q-1) + j\}_{k=0}^{q}$ where $j = 0, 1, \ldots, q-2$. If q is a power of 2 then $q-1$ and $q+1$ are relatively prime. Define the extension by two new groups in the following way. For any block B add symbol j in the first new column if $B \in Q_j$ and i in the second new column if $B \in P_i$. That $q-1$ and $q+1$ are relatively prime means that the last two columns are covered. This method will cover all pairs of columns, one from the original set and one from the new set of two as long as every point in $\mathbb{Z}_{q^2-1}$ appears in each of the Q_j. We have not yet extended the blocks, nor added symbols in the second column if $B \in P_i$ for $i \geq q-1$ (we did not need to put the symbols in the second new column from the first $q-1$ P_i, but we could have chosen any $q-1$ of the P_i in any order). These flexibilities may allow us to extend by more than two rows.

The base block, B_0, from the *GDD* contains q points and the pairwise differences cover every element of $\mathbb{Z}_{q^2-1}$ that are not multiples of $q+1$. The q positive differences that are multiples of $q-1$ will be covered, and the pairs of elements of B_0 whose differences are multiples of $q-1$ will generate the same point sets under development in the Q_j. In each Q_j, we will generate the same point set twice if $q = 4$ and at least three times if $q > 4$. Since we can still add a point to the B_0 to cover all the points by each Q_j, we need at least $q-2$ different point sets generated which is impossible when $q > 4$. Hence, this construction can only work if $q = 4$ and indeed, it does: use base block $B_0 = \{0, 2, 3, 11\}$ and add the point 1 to it, then each Q_j will cover each point at least once. The P_i cover each point exactly once and so we can add an additional two columns.

However, in this case, we can do even better by taking all possible sets of three P_i's (as mentioned above we can pick any $q-1$ of the P_i) and adding the following columns:

$$
\begin{array}{ccccccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 2 \\
1 & 1 & 1 & 1 & 0 & 2 & 1 & 0 & 0 & 0 & 2 \\
2 & 2 & 0 & 1 & 1 & 2 & 1 & 1 & 1 & 2 & 0 \\
0 & 0 & 2 & 1 & 2 & 2 & 1 & 2 & 2 & 1 & 1 \\
1 & 0 & 1 & 2 & 1 & 1 & 2 & 2 & 2 & 2 & 2 \\
2 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 2 & 1 \\
0 & 1 & 1 & 1 & 1 & 0 & 2 & 0 & 0 & 0 & 1 \\
1 & 2 & 1 & 0 & 1 & 2 & 2 & 1 & 1 & 1 & 0 \\
2 & 1 & 2 & 0 & 2 & 1 & 1 & 2 & 1 & 1 & 1 \\
0 & 1 & 0 & 2 & 2 & 1 & 2 & 1 & 2 & 2 & 2 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 2 & 0 & 0 \\
2 & 1 & 1 & 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 2 & 2 & 2 & 1 & 2 & 0 & 1 & 1 & 0 & 0 \\
1 & 2 & 2 & 2 & 2 & 0 & 1 & 2 & 0 & 1 & 1 \\
2 & 2 & 2 & 2 & 0 & 1 & 2 & 0 & 2 & 2 & 2
\end{array}
$$

The first column is from the Q_j and the remaining ten columns are from the ten possible triples of P_i's which are ordered lexicographically as the ten possible triples from a 5-set. These additional columns gives us that $tc(16, 3 : 1) \leq 15$ which is better than 17, the value obtained from other methods.

When q is an odd prime power, we can do the same sort of construction to try to get two additional groups. Because $q-1$ and $q+1$ are not relatively prime then we will not automatically get that the two new columns (from the P_i and Q_j) are automatically covered. We won't have filled in all of the columns determined by the P_i (the second new one). This flexibility may allow us still to succeed. Because we need all the points to be in each Q_j, a similar argument to that above shows that $q = 3$ and $q = 5$ are the only possibilities. When $q = 3$, we are constructing a cover with $g = 2$ which is uninteresting since we know what optimal covers for $g = 2$ look like. It is, however, worth mentioning that a total of 31 groups can be added which is the most possible and achieves the optimal cover with block size 35.

When $q = 5$, we have $B_0 = \{0, 2, 15, 16, 19\}$ to which we add the point 1. The Q_j each cover all the points and so we must only worry about covering all the pairs on the two new rows. This can be done with two new columns:

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 0 & 1 & 2 & 3 & 0 & 1 & 2 & 3 & 0 & 1 & 2 & 3 & 0 & 1 & 2 & 3 & 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 3 & 1 & 0 & 0 & 1 & 2 & 3 & 1 & 0 & 0 & 1 & 2 & 3 & 3 & 2 & 0 & 1 & 2 & 3 & 3 & 2 \end{pmatrix}^T$$

so we get that $tc(8, 4 : 2) \leq 24$ which is better than 27, the value from other methods. By this method, we can only extend by two groups because we cannot add another column from a different set of four P_i's. Some variation of these methods may get beyond two new columns for these restricted values of q.

2.5 A Recursive Algorithm and Simulated Annealing

2.5.1 A Recursive Algorithm

In Appendix A, the tables of upper bounds on $tc(k, g : n)$ are the results of all the above methods being applied to construct transversal covers with $3 \leq g \leq 7$ and $2 \leq k \leq 50$. A recursive algorithm implementing all the constructions was written for *Mathematica*. The values of $tc(k, 2 : i)$ and some additional results from simulated annealing and hand calculation were given as the starting point for the constructive program. The group multiplication, blocksize recursive method, and Wilson's generalization for $l = 1$ can be directly programmed.

The other generalizations of Wilson's construction are more complicated to implement. When $m \neq 0$, $l \neq 1$, this constructive method relies on knowing the replication numbers of points in the cover and on the intersections of blocks. Without storing all the covers generated, this method is of limited value. Additionally without searching for optimal values of the replications and intersections of blocks, the covers that it constructs may be bad. The time and space constraints of these facts led us to decide that the initial implementation of this method would only be done for the cases $m = 0$ and $l = 1$.

The necessary information for the case $m = 0$, as discussed above, is the systems of representatives for each PBD on g points. Not knowing, *a priori*, which PBDs and which system of representatives on them yield the best values. This necessitates investigation of each PBD and all systems of representatives on it. This process could be automated, but for the present consideration, hand calculation was easier. We also chose to investigate the $(v, K, 1)$-covers discovered to yield the best asymptotic result in Theorem 2.16. Additional features, all $(v, K, 1)$-covers, and full automation of the PBD construction, may be implemented in the future.

For each value of g, all PBDs were enumerated. The PBDs that were worse than others were eliminated from consideration. This elimination can be done whenever a set of blocks of small size, say two or three, can be replaced by one block of larger size, say p; since the PBD on p points, with only blocks of the small size, yields a constructive method for $tc(k, p : i)$ and our value of $tc(k, p : i)$ must be at least as small as the one yielded by this subPBD.

After these PBDs were eliminated, the systems of representatives were enumerated, again eliminating ones that were clearly non-optimal. This elimination is described in Corollary 2.9. However, systems of representatives were not discarded when they increased the number of disjoint blocks that the method produces. Once the search had been narrowed, we explored by hand the number of disjoint blocks that each individual PBD and system of representation generated. All these methods were input directly. The program first calculated the best upper bounds for $g = 3$ and used the smaller values calculated to calculate values of higher k and then higher

g. The last step for any set of values for the same k and g was to use the fact that

$$tc(k, g : i) \leq tc(k, g : j) \leq tc(k, g : i) + j - i$$

to lower some values of $tc(k, g : i)$ using $tc(k, g : j)$. This process is repeated a number of times since one change in value may force another which would, in turn, force others. This iteration was also used to take advantage of the fact that for $g_1 < g_2$, $tc(k, g_1 : n) < tc(k, g_2 : n)$. When $g + 1$ is a prime power and $TD(k, g + 1)$ exists, then $tc(k, g : g) \leq (g + 1)^2 - 1$. Since we can assume that one of the blocks is $(g, g, \ldots, g)$, we are able to remove that block. Every other block intersects the removed block. The g blocks, which intersected this block in the first group, are disjoint elsewhere so replacing the symbol g in these g blocks by the g different symbols from the smaller alphabet gives a cover with g disjoint blocks. This is an instance of the GDD construction.

Finally, the methods generating a cover with i disjoint blocks may actually create more, but determining the exact number is too expensive. The methods used just guarantees us at least i.

2.5.2 Simulated Annealing

The previously mentioned *ad hoc* results, used as starting values for the construction, are of two kinds. Those that we discovered by hand were

$$tc(6, 3 : 3) \leq 14,\ tc(7, 3 : 1) \leq 14,\ tc(13, 3 : 1) \leq 15,\ tc(14, 3 : 1) \leq 16.$$

We have also written a simulated annealing algorithm to find other instances of covers. This program is similar to standard simulated annealing in most regards. The actual simulated annealing loop moves to a neighbor incidence structure by randomly choosing one block and changing one letter in it to another. The "goodness" or "badness" of this move is determined by the number of pairs that remain uncovered. If this new structure is better, we use it. If it is worse, we use it with a probability decreasing exponentially with time (temperature) [19]. We have adapted an algorithm given to us by Peter Gibbons [18]. The significant change from a design search is that we do not know how many blocks a minimum cover has. We overcome this problem by finding a random, but probably very large, cover, which is used as an initial upper bound on the number of blocks; g^2 is used as an initial lower bound. The simulated annealing searches, by bisection, for a cover with the number of blocks half way between the upper and lower bound. If it finds one, this number of blocks is now the upper bound, if not, it is the lower bound and the process repeats with the new value halfway between the two bounds. In a completely automated system, the upper bounds from the recursive algorithm would be fed back to the simulated annealing program.

In the runs of this algorithm, we have opted to use a heuristic inspired by D. Ashlock in a slightly different context [3]. Ashlock developed genetic algorithms to find designs. Although larger population sizes are more likely to succeed, he found that running ten ecologies in parallel, with a population size of 60 would find designs faster than one ecology with a population size of 600. This is a result of the large standard deviation in search time. The algorithm described above halts after a certain number of trials. We have opted to use a small cutoff value but run many trials. Say that there is a $TC(6,3:3)$ with 11 blocks; any particular run of the simulated annealing may get caught in a local optimum and fail to find it. If we run the algorithm to find ten covers, nine of them may fail to find it, but one may succeed.

2.6 Previous Constructions and Asymptotic Upper Bounds

Besides developing these constructions to generate covers for small and medium parameter values, we are also interested in how they behave asymptotically. The limited range of use of the ITD construction means that this construction has no asymptotic behavior. It is unclear how to describe the asymptotic behavior of the GDD construction since that behavior depends on settling the existence of GDD's with large k with respect to the number of blocks. The asymptotic existence of GDD's by Wilson's theorem is not appropriate for this construction. Wilson's theorem states that for a fixed block size, k, there exists a size for which all GDD's with larger point sets exist. This is the very opposite of the kind of GDD useful in this construction, where we need large k with respect to the point size. The asymptotic behavior for transversal covers is known (Equation 1.30) but this is non-constructive. The asymptotic behavior discussed in this section is constructive asymptotics.

2.6.1 Block Size Recursive Method

If one starts with a maximum transversal design, the block size recursive method yields the following asymptotic bound in the case where g is a prime power:

$$tc(m, g : g) \leq \left(\left\lceil \frac{\log m}{\log g} \right\rceil (g^2 - g) \right) - g. \tag{2.9}$$

However, the Inequality 2.9 is not the best asymptotic bound derivable by this method. If $tc(m, g : g) = x$ then

$$tc(m^{i+1}, g : 1) \leq i(x - g) + x$$

which yields the asymptotic result

$$\frac{tc(k, g : 1)}{\log k} \leq \frac{x - g}{\log m} \tag{2.10}$$

which can be much better depending on the choice of m. For example, when $g = 3$, where the asymptotic limit is $\frac{tc(k,3:1)}{\log k} = 1.5$, Inequality 2.9 gives $\lim_{k\to\infty} \frac{tc(k,3:1)}{\log k} \leq 3.79$, but if we pick $m = 12$ in Inequality 2.10, we get $\lim_{k\to\infty} \frac{tc(k,3:3)}{\log k} \leq 3.3475$. Using

$tc(20,4;4) \leq 28$ we can construct $\lim_{k\to\infty} \frac{tc(k,4:4)}{\log k} \leq 5.553$. Neither of these values are as good as the values obtained by Gargano *et al.* (Inequalities 1.28). But, as we improve the known values for transversal covers, the asymptotic behavior of the construction will improve, possibly beyond the constructions described in [15].

In particular, if $tc(k_1, g:1) \approx \frac{g \log k_1}{2}$ and $tc(k_2, g:1) \approx \frac{g \log k_2}{2}$ then this construction produces a $tc(k_1k_2, g:1) \lessapprox \frac{g \log k_1 k_2}{2}$. However, one should not get too excited about this. This statement is rigorously examined in Section 3.1.1. There we show that this construction cannot generate covers meeting the asymptotic limit. However, if this construction starts with covers that are less than c times as big as the asymptotic limit then it produces covers that are this good as well. This construction will produce tighter asymptotic sizes as we know more good covers, which we expect even by randomized search.

2.6.2 Generalizations of Wilson's Construction

MacNeish's Theorem

If we use the analogue of MacNeish's theorem and have two covers $TC(k, g_1:1)$ and $TC(k, g_2:1)$ with

$$tc(k, g_1:1) \approx \frac{g_1}{2} \log k$$

and

$$tc(k, g_2:1) \approx \frac{g_2}{2} \log k$$

then we obtain $tc(k, g_1g_2:1) \approx \frac{g_1g_2}{4} \log^2 k$ which differs from the asymptotic value only by a factor of $\frac{\log k}{2}$. But, by only increasing the group size, we will eventually reach a g after which transversal designs always exist [7]. Constructions that only increase group size behave badly because for a fixed k, $tc(k,g:2) = g^2$ as long as $g \geq (k-2)^{14.8}$.

Asymptotically, in the PBD construction we do know the optimal incidence structure to use. It turns out to be not a PBD but a cover. We are also able to derive the asymptotic efficiency of this construction.

Theorem 2.16. *In the PBD construction, covers yield better results than designs for sufficiently large k. The best result attainable is $tc(k, g) \approx g \log k$ which is double both the asymptotic limit and the lower bound from Inequality 3.1.*

Proof. For a given large k, all covers with group size $g' < g$ have close to the asymptotic number of blocks, $tc(k, g') \approx (g'/2) \log k$. Then the incidence structure that yields the best construction will be the one that minimizes

$$\sum_{\text{blocks, } B, \text{ in the incidence structure}} \frac{k_B}{2} \log k = \frac{\log k}{2} \sum_B k_B$$

so we need to minimize

$$\sum_B k_B = \sum_x r_x.$$

If there exists a point x with $r_x = 1$, then the PBD consists only of one block which is forbidden in our construction (the construction would yield $tc(k, g) = tc(k, g)$). If there are no points with $r = 1$, but there is a point, x, with $r_x = 2$, then the PBD consists of two blocks of size k and $g - k + 1$ intersecting in the point x, and all other blocks of size two. Then

$$\sum_B k_B = g + 1 + 2(k - 1)(g - k).$$

Now, $g + 1$ is constant so we need only minimize the second half, the quadratic in k. This parabola is concave downwards and the range of k is $2 \leq k \leq g - 1$ so the minimum values occur at 2 and $g - 1$. It can be seen that both of these values yield a degenerate projective plane for which

$$\sum_B k_B = 3(g-1).$$

When used in the construction, this gives

$$tc(k,g) \leq \frac{3(g-1)\log k}{2}.$$

If there are no points, x, with $r_x = 1$ or 2, then clearly

$$\sum_B k_B = \sum_x r_x \geq 3g.$$

so the best result obtainable with a PBD is the degenerate projective plane described above. But the following $(v, \{2, 3, \ldots, g-1\}, 1)$-cover yields $g \log k$.

$$\mathcal{B} = \{\{0, 1, \ldots, g-2\}, \{1, 2, \ldots, g-1\}, \{0, g-1\}\}.$$

The fact that for all x, $r_x \geq 2$ shows $\sum_B k_B = \sum_x r_x \geq 2g$. This demonstrates that $g \log k$ is, in fact, the best that can be done with this construction. □

Even though, asymptotically, this method is not optimal, it is still valuable. For small k it can yield good results, and together with the other generalizations of Wilson's construction, these are the only two methods that allow us to increase the group size.

2.7 The Difference Between $tc(k, g : n)$ and $tc(k+1, g : n)$

The block size recursive method tends to construct many covers with the same number of blocks for large ranges of k. Because of this, the difference between the upper bounds for consecutive k tends to be either zero or large. If we had an upper bound on what this difference could be, we could smooth out this gross step-like behavior in the upper bound.

Theorem 2.17.

$$tc(k+1, g : n) - tc(k, g : n) \leq \frac{3g^2 - 4g + (g \mod 2)}{4}.$$

Proof. We start with a $TC(k, g : n)$ and construct a cover with block size $k+1$ and no more than the desired number of new blocks. We choose one group, G, arbitrarily and for each $x \in G$, consider the families of blocks $\mathcal{B}_x$, the blocks through $x \in G$. There are g of these families and we will divide them into two collections, one with $g - i$ of the B_x and the other with i. Without loss of generality, these are the first $g - i$ families and the last i families, respectively. To each B_j $(1 \leq j \leq g - i)$, we will add one block from each of the B_k $(g - i + 1 \leq k \leq g)$. We additionally add $g - i - 1$ new blocks with arbitrary values on the other groups but at their intersection with G, containing the points $\{1, 2, \ldots, j-1, \hat{j}, j+1, \ldots, g-i\}$, i.e. every other point than j from the first $g - i$ points of G, each point in exactly one block. The blocks of each of the augmented B_j $(1 \leq j \leq g - 1)$ (which have more blocks than it started with), cover every point of the design. To the blocks remaining in each B_j $(g-i+1 \leq j \leq g)$, we need add only $g - 1$ new blocks to each one to ensure that each of these sets cover all the points. We can extend this enlarged cover by one group, the blocks from B_j intersecting the jth point of this new group.

We have added $(g-i)(g-i-1)+i(g-1) = g^2-ig-g+i^2$ new blocks. Minimizing this quadratic over integer values i $(0 \leq i \leq g)$, yields the desired maximum value of increase for extension by one group. $\square$

This bound is quadratic in g. It is notable that $tc(k+1, 2:n) - tc(k, 2:n) \leq 1$, and in the tables, $tc(k+1, 3:n) - tc(k, 3:n) \leq 2$. We conjecture that the actual value might be linear in g. Smaller values would have the effect of further reducing the large jumps that exist in the tables.

Conjecture 2.18. *The worst size difference between consecutive optimal covers,*

$$\max_k \left(tc(k+1, g:n) - tc(k, g:n)\right) \tag{2.11}$$

is linear in g and, in fact, may be as low as $g-1$.

2.8 Discussion

Although the constructive methods given here are valuable, we believe that the current bounds are not the best possible, both because other constructions may exist and recursive constructions do depend heavily upon starting values. Any new, smaller covers discovered will cascade throughout the methods presented here and improve most known values for higher k and g. It appears, by inspection, that the construction which produces the best overall results is the block size recursive method (see Appendix A). This method is also notable because given covers that are proportionally close to known lower bounds, it constructs covers that are equally close.

Certainly one interesting direction would be to find transversal covers, by simulated annealing or other randomized methods, that are close to the asymptotic bound. These would improve our knowledge of the constructive asymptotic behavior. It seems, at this time, that this is the best way to proceed to obtain good constructive asymptotics. This only requires finding one good cover to show that the constructed behavior for larger covers on the same group sizes is equally good.

We should like to make more use of Wilson's method in its full generality. The PBD construction needs to be better understood for small k, namely which PBD's work best and which systems of representation of the points are optimal. And as noted above, reducing the upper bound on $tc(k+1, g : n) - tc(k, g : n)$ would have a large impact, reducing the magnitude of jumps in the upper bound.

Chapter 3

Transversal Covers: Lower Bounds

The bulk of this chapter will appear in a paper with Lucia Moura [43]. In this chapter, we derive several lower bounds on the number of blocks in a transversal cover. In Section 3.1, we develop three general lower bounds. The first one (Theorem 3.1) is proved using the block size recursive construction and the known asymptotic results. The second bound (given in Inequality 3.3) is proved by studying set systems with some intersection properties, and it can be seen as a generalization of the results for $g = 2$. The third general lower bound (Corollary 3.9) is shown using a set packing argument and establishing a relationship between point-balanced transversal covers and standard transversal covers. Figures comparing these lower bounds and known upper bounds for $g = 3, 5, 7$ are given in Appendix B. The set packing argument is also used to derive bounds when we have more information on the replication numbers in the covers (Corollary 3.7 and Corollary 3.12).

In Section 3.2, we develop upper bounds on k for small fixed b, which lead to lower bounds on b for small k's. This proved useful in reducing or eliminating the gap between lower and upper bounds on the size of small covers. For instance, the results in Section 3.2.1 give, as a corollary, an alternative short proof of Applegate and Östergård's result that $tc(5, 3 : 1) = 11$, and the analysis in Section 3.2.2 implies, among other results, that $tc(6, 3 : n) = 12$ for all n and that $tc(7, 3 : 1) = 12$. This method is further expanded and a conjecture is made.

3.1 General Lower Bounds

In this section, three general lower bounds on $tc(k, g : n)$ are exhibited (Theorem 3.1, Inequality 3.3, and Corollary 3.9). They are illustrated by the figures in Appendix B.

3.1.1 The Block Size Recursive Construction Bound

One upper bound obtained from the block size recursive method is given in Theorem 2.1. This theorem, along with the asymptotic behavior demonstrated by Gargano *et al.* [16], is used to prove the following lower bound:

Theorem 3.1. *For any k, g and $n \geq g$, we have*

$$tc(k, g : n) \geq \left\lceil \frac{g \log k}{2} \right\rceil + n. \tag{3.1}$$

Proof. Assume the contrary, i.e., that there exists a k_0 for which

$$tc(k_0, g : n) \leq \frac{g \log k_0}{2} + n - a,$$

where a is a positive real number.

Then, using Theorem 2.1, we get

$$\begin{aligned} tc(k_0^i, g : n) &\leq i \left(\frac{g \log k_0}{2} + n - a \right) - (i-1)n \\ &= i \left(\frac{g \log k_0}{2} - a \right) + n. \end{aligned}$$

The numbers $tc(k_0^i, g : n)/\log k_0^i$, for $i \geq 1$, form a subsequence of the sequence of all

$tc(k, g : n)/\log k$, and so must have the same limit. Thus

$$\begin{aligned}
\frac{g}{2} &= \lim_{i\to\infty} \frac{tc(k_0^i, g : n)}{\log k_0^i} \\
&\le \lim_{i\to\infty} \frac{i(\frac{g\log k_0}{2} - a) + n}{\log k_0^i} \\
&= \lim_{i\to\infty} \frac{i(\frac{g\log k_0}{2} - a) + n}{i\log k_0} \\
&= \lim_{i\to\infty} \left(\frac{g}{2} - \frac{a}{\log k_0} + \frac{n}{i\log k_0}\right) \\
&= \frac{g}{2} - \frac{a}{\log k_0} \\
&< \frac{g}{2}.
\end{aligned}$$

This is the desired contradiction and we conclude that

$$tc(k, g : n) \ge \frac{g\log k}{2} + n, \text{ for all } k,$$

and since $tc(k, g : n)$ is an integer, we have

$$tc(k, g : n) \ge \left\lceil \frac{g\log k}{2} \right\rceil + n, \text{ for all } k.$$

□

If a k_0 existed with $tc(k_0, g : n) = \frac{g\log k_0}{2} + n$, then by setting $a = 0$ in the above equation, the covers achieving the asymptotic limit are constructible by the block size recursive method. However, we will show later that this is not possible.

3.1.2 The Set Systems Bound

The known lower bounds (which are also exact values) for $g = 2$ were proved by associating a system of intersecting sets and their complements to a transversal cover [39]. This idea can be generalized by a family of k 2-independent g-partitions of a b set [16]. Here, we introduce an equivalent definition using a more convenient notation for use in this section.

Definition 3.1. We say that a $k \times g$ array of subsets of $[1, b]$, say $\mathcal{M} = (A_i^j)$, is a (k, g, b) *intersecting array* if it satisfies the following properties:

1. sets in the same row form a partition of $[1, b]$, i.e., for any $1 \leq i \leq k$, the sets $A_i^1, A_i^2, \ldots, A_i^g$ form a partition of the set $[1, b]$;

2. sets in different rows intersect, that is, $A_i^j \cap A_{i'}^l \neq \emptyset$, for all $j, l, i \neq i'$.

In addition, we say that a (k, g, b) intersecting array $\mathcal{M} = (A_i^j)$ is n*-avoiding*, if there exists a subset L of $[1, b]$ with $|L| = n$ such that no pair of elements in L appear together in any of the sets A_i^j, for all i,j. Moreover, given an integer r, we say that a (k, g, b) intersecting array $\mathcal{M} = (A_i^j)$ is r*-uniform*, if $|A_i^j| = r$, for all i,j.

These arrays correspond to transversal covers with a set of n disjoint blocks. The proof of the following proposition is similar to the equivalence shown in Subsection 1.1.3:

Proposition 3.2. *For any k, g, b and $n \leq g$, we have that*

1. *there exists a $TC(k, g : n)$ with b blocks if and only if there exists an n-avoiding (k, g, b) intersecting array;*

2. *there exists a $PBTC(k, g : n)$ with b blocks if and only if there exists an n-avoiding $\frac{b}{g}$-uniform (k, g, b) intersecting array.*

□

Unfortunately, the maximum number of rows for such an array of sets is not easy to determine. However, we will develop an upper bound on their number of rows, producing a lower bound on the number of blocks of the corresponding transversal covers. In order to do so, we prove the following variation of a theorem by Katona [21] and Jaeger and Payan [20]. We follow the proof given by Füredi [14, Theorem 1.4]. Then we reduce the right-hand side by a factor of two, by imposing extra intersection conditions among A_i's, as well as among B_i's.

Theorem 3.3. *Let $A_1, \ldots, A_k$ and $B_1, \ldots, B_k$ be finite sets such that*

$$A_i \cap B_i = \emptyset,\ A_i \cap B_j \neq \emptyset,\ A_i \cap A_j \neq \emptyset \text{ and } B_i \cap B_j \neq \emptyset, \text{ for all } i \neq j.$$

Then,

$$\sum_{i=1}^{k} \frac{1}{\binom{|A_i|+|B_i|}{|A_i|}} \leq \frac{1}{2}.$$

Proof. Let $X = (\cup_{i=1}^{k} A_i) \cup (\cup_{i=1}^{k} B_i)$, $|A_i| = a_i$, $|B_i| = b_i$ and $|X| = n$. Let π be a permutation of X. We say the type of π is the set $\{i |\ A_i$ precedes B_i in $\pi\}$. We claim that π has at most one type. Indeed, suppose on the contrary that π has two types i and j, for $i \neq j$. Assume w.l.o.g. that $x_i = \max_{x \in A_i}\ x \leq \max_{x \in A_j} x$. Then, each element of A_i precedes each one of B_j, yielding $A_i \cap B_j = \emptyset$ which is a contradiction.

Now, let π be a permutation of some type i, and let us define $\tilde{\pi}$ to be the permutation of X that inverts the order of π. So $\tilde{\pi}$ makes every element of B_i precede every element of A_i. We claim that $\tilde{\pi}$ has no type. Indeed, suppose by contradiction that $\tilde{\pi}$ has type j. Let $y_i = \max_{x \in B_i} x$, and $x_i = \max_{x \in A_j}$. If $y_i \geq x_i$, then $A_i \cap A_j = \emptyset$, and if $x_j \geq y_i$ then $B_i \cap B_j = \emptyset$, which are both contradictions.

Moreover, we observe that $\pi_1 \neq \pi_2$ if and only if $\tilde{\pi}_1 \neq \tilde{\pi}_2$. Now, counting the permutations of type i, we get $\binom{n}{a_i+b_i} \times (n - a_i - b_i)! a_i! b_i! = \frac{n!}{\binom{a_i+b_i}{a_i}}$. Therefore, summing up all permutations π of some type and their corresponding $\tilde{\pi}$'s, we have

$$2 \sum_{i=1}^{k} \frac{n!}{\binom{a_i+b_i}{a_i}} \leq n!$$

$\square$

Now, we are able to deduce the following corollary.

Corollary 3.4. *Let $A_1, \ldots, A_k$ and $B_1, \ldots, B_k$ be finite sets with $|A_i| + |B_i| \le c$, $|A_i| \le a \le c/2$, and such that*

$$A_i \cap B_i = \emptyset,\ A_i \cap B_j \ne \emptyset,\ A_i \cap A_j \ne \emptyset \text{ and } B_i \cap B_j \ne \emptyset, \text{ for all } i \ne j.$$

Then,

$$k \le \frac{1}{2}\binom{c}{a}.$$

Proof. It's easy to see that

$$\binom{c}{a} \ge \binom{c}{|A_i|} \ge \binom{|A_i| + |B_i|}{|A_i|}, \text{ for all } i.$$

Thus,

$$\frac{k}{\binom{c}{a}} \le \sum_{i=1}^{k} \frac{1}{\binom{|A_i|+|B_i|}{|A_i|}} \le \frac{1}{2},$$

from which the result follows. □

This leads to the following upper bound on k for fixed number of blocks which is close to but slightly better than Inequality 1.20. Let us first denote

$$\delta_{t,g} = \begin{cases} 1 & \text{if } t \equiv -1 \pmod g, \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 3.5. *For any k, g, and $n \ge g$, we have*

$$k \le \frac{1}{2}\binom{2\left\lfloor \frac{tc(k,g:n)}{g} \right\rfloor + \delta_{tc(k,g:n),g}}{\left\lfloor \frac{tc(k,g:n)}{g} \right\rfloor} \le \frac{1}{2}\binom{\left\lfloor \frac{2tc(k,g:n)}{g} \right\rfloor}{\left\lfloor \frac{tc(k,g:n)}{g} \right\rfloor}. \tag{3.2}$$

Proof. Let $\mathcal{M} = (A_i^j)$ be the (k, g, b) intersecting array corresponding to a $TC(k, g : n)$ with $tc(k, g : n)$ blocks. For every i, $1 \le i \le g$, let x_i be the index of the set with minimum cardinality and y_i be the one with the next larger cardinality among the A_i^j, $1 \le j \le g$. Then, since the minimum must be no larger than the mean, for each

$1 \leq i \leq k$, we have $|A_i^{x_i}| \leq \left\lfloor \frac{tc(k,g:n)}{g} \right\rfloor$. Similarly, $|A_i^{y_i}| \leq \left\lfloor \frac{tc(k,g:n)-|A_i^{x_i}|}{g-1} \right\rfloor$. Thus,

$$
\begin{aligned}
|A_i^{y_i}| + |A_i^{x_i}| &\leq \left\lfloor \frac{tc(k,g:n) - |A_i^{x_i}|}{g-1} \right\rfloor + |A_i^{x_i}| \\
&= \left\lfloor \frac{tc(k,g:n) + (g-2)|A_i^{x_i}|}{g-1} \right\rfloor \\
&\leq \left\lfloor \frac{tc(k,g:n) + (g-2) \left\lfloor \frac{tc(k,g:n)}{g} \right\rfloor}{g-1} \right\rfloor .
\end{aligned}
$$

Let q and l be the quotient and remainder of the division of $tc(k,g:n)$ by g. So, $tc(k,g:n) = qg + l$, with $0 \leq l \leq g-1$. Then,

$$
\begin{aligned}
|A_i^{y_i}| + |A_i^{x_i}| &\leq \left\lfloor \frac{qg + l + (g-2)q}{g-1} \right\rfloor \\
&= \left\lfloor \frac{2(g-1)q + l}{g-1} \right\rfloor = 2q + \left\lfloor \frac{l}{g-1} \right\rfloor \\
&= 2 \left\lfloor \frac{tc(k,g:n)}{g} \right\rfloor + \delta_{tc(k,g:n),g}.
\end{aligned}
$$

We observe that $A_1^{x_1}, \ldots, A_k^{x_k}$ and $A_1^{y_1}, \ldots, A_k^{y_k}$ satisfy the conditions of Corollary 3.4 for $a = \left\lfloor \frac{tc(k,g:n)}{g} \right\rfloor$ and $c = 2 \left\lfloor \frac{tc(k,g:n)}{g} \right\rfloor + \delta_{tc(k,g:n),g}$. □

The previous theorem can be alternatively expressed as a lower bound on $tc(k,g:n)$. If we define

$$
h_g(t) := \frac{1}{2} \binom{2 \left\lfloor \frac{t}{g} \right\rfloor + \delta_{t,g}}{\left\lfloor \frac{t}{g} \right\rfloor},
$$

then,

$$
tc(k,g:n) \geq \min\{t : h_g(t) \geq k\}. \tag{3.3}
$$

3.1.3 Comparison Between the Two Previous Lower Bounds

We now ask when each of these lower bounds is better than the other. The largest that the lower bound from Inequality 3.1 can be for a given g and k is $\lceil \frac{g \log k}{2} \rceil + g$.

Using this value for b, if

$$\frac{1}{2}\binom{\left\lfloor\frac{2b}{g}\right\rfloor}{\left\lfloor\frac{b}{g}\right\rfloor} < k,$$

then the bound from Inequality 3.3 is strictly better. When $\lfloor\frac{b}{g}\rfloor \geq 3$, or equivalently, $b \geq 3g$, then

$$\begin{aligned}\frac{1}{2}\binom{\left\lfloor\frac{2b}{g}\right\rfloor}{\left\lfloor\frac{b}{g}\right\rfloor} &< \frac{.625\ 2^{\left\lfloor\frac{2}{g}\left\lceil\frac{g\log k}{2}\right\rceil\right\rfloor+1}}{2} \\ &= .625\ 2^{\left\lfloor\frac{2}{g}\left\lceil\frac{g\log k}{2}\right\rceil\right\rfloor} \\ &\leq .625\ 2^{\frac{2}{g}\left\lceil\frac{g\log k}{2}\right\rceil} \\ &\leq .625\ 2^{\log k+\frac{2}{g}} \\ &= .625k\ 2^{\frac{2}{g}}\end{aligned}$$

which for $g \geq 3$ implies

$$\frac{1}{2}\binom{\left\lfloor\frac{2b}{g}\right\rfloor}{\left\lfloor\frac{b}{g}\right\rfloor} < .625 \times 1.59k < k.$$

So, when $b \geq 3g$, Inequality 3.3 is strictly better. For $g \geq 3$, we have $b \geq g^2 \geq 3g$. Therefore, Inequality 3.3 is always strictly larger than Inequality 3.1 in the region where either bound gives any information at all, i.e., the range of k for which either bound is larger than g^2.

An obvious question to ask is why Inequality 3.1 is discussed and proved at all. It is in a far more tractable form and is therefore more useful. This bound is often very close to Inequality 3.3, and both bounds converge to the known asymptotic limit. So, having Inequality 3.1 in such a useful algebraic form permits quick and useful calculations.

The fact that, for $g \geq 3$, Inequality 3.3 is always better than the bound from Inequality 3.1, independently of n, allows for the following strengthening of the bound from Inequality 3.1:

$$tc(k, g : n) \geq \left\lceil \frac{g \log k}{2} \right\rceil + g + 1. \tag{3.4}$$

3.1.4 Set Packing Bounds

Point Balanced Transversal Cover Lower Bound

Throughout this subsection, we use the convention that $\binom{x}{y} = 0$, whenever $x < y$, $x < 0$ or $y < 0$.

Let r_{min} and r_{max} denote the minimum and maximum replications of elements of a transversal cover.

Theorem 3.6. *In a $TC(k, g : n)$ with b blocks, we have*

$$\sum_{x \in X} \binom{r_x}{r_{max} - g + 2} \leq \binom{b}{r_{max} - g + 2} - \sum_{i=2}^{n} \binom{n}{i} \binom{b-n}{r_{max} - g + 2 - i} \tag{3.5}$$

Proof. Let $\mathcal{M} = (A_i^j)$ be the (k, g, b) intersecting array corresponding to the given $TC(k, g : n)$. We claim that any $(r_{max} - (g-2))$-subset of $[1, b]$ cannot appear in more than one of the A_i^j, $1 \leq i \leq k$, $1 \leq j \leq g$. Indeed, suppose that two distinct sets $A_{i_1}^{j_1}$ and $A_{i_2}^{j_2}$ contain a set $P \subseteq [1, b]$ with $|P| = r_{max} - (g-2)$. Then, we know that $i_1 \neq i_2$ and that $A_{i_1}^{j_1} \cap A_{i_2}^{j} \neq \emptyset$ for all $1 \leq j \leq g$. Since $A_{i_2}^1, \ldots, A_{i_2}^g$ is a partition of $[1, b]$, we have that

$$\begin{aligned} r_{max} &\geq |A_{i_1}^{j_1} \cap [1, b]| = |A_{i_1}^{j_1} \cap \left(A_{i_2}^1 \cup \ldots \cup A_{i_2}^g\right)| \\ &= |(A_{i_1}^{j_1} \cap A_{i_2}^1) \cup \ldots \cup (A_{i_1}^{j_1} \cap A_{i_2}^g)| \geq (g-1) + |P| = r_{max} + 1, \end{aligned}$$

which is a contradiction. Therefore every $(r_{max} - (g-2))$-subset of $[1, b]$ occurs in at most one of the A_i^j, $1 \leq i \leq k$, $1 \leq j \leq g$.

Let $L \subset [1, b]$ be the n-set that is avoided by $\mathcal{M}$. Counting, on one hand, the number of $(r_{max} - (g-2))$-subsets of $[1, b]$ that are covered by the A_i^j's, and, on the other hand, the total number of $(r_{max} - (g-2))$-subsets of $[1, b]$ that avoid L, we obtain the theorem. $\square$

The previous theorem does not give a general upper bound on k. It does imply an upper bound on k for point-balanced transversal covers.

Corollary 3.7. *If there exists a $PBTC(k, g : n)$ with b blocks, then*

$$k \leq \left\lfloor \frac{\binom{b}{\frac{b}{g}-(g-2)} - \sum_{i=2}^{n} \binom{n}{i} \binom{b-n}{\frac{b}{g}-g-(i-2)}}{g\binom{\frac{b}{g}}{g-2}} \right\rfloor. \tag{3.6}$$

Proof. A $PBTC(k, g : n)$ with b blocks has $r_x = r_{max} = \frac{b}{g}$, for all $x \in X$. $\square$

General Transversal Cover Bound

We want this bound to yield a bound on arbitrary transversal covers so we need to relate $PBTC$s to TCs. We have:

Proposition 3.8. *For any integers k, g, $n \leq g$, we have*

$$\frac{pbtc(k, g : n)}{g} + g(g-1) \leq tc(k, g : n) \leq pbtc(k, g : n).$$

Proof. The second inequality follows from Definitions 1.7 and 1.11. All we have to prove is the first inequality. Let $(X, \mathcal{G}, \mathcal{B})$ be a $TC(k, g : n)$ with $tc(k, g : n)$ blocks. Let $R = \max_{x \in X}\ r_x$. We claim we can construct a $PBTC(k, g : n)$ with gR blocks. Indeed, just add $gR - tc(k, g : n)$ blocks to $(X, \mathcal{G}, \mathcal{B})$ so that $r_x = R$ for every $x \in \mathcal{B}$. For each group $G_i \in \mathcal{G}$, arbitrarily select $R - r_x$ of the new blocks to contain x, for any $x \in G_i$. The result is clearly a $TC(k, g : n)$ with uniform replication R and gR blocks. Thus $pbtc(k, g : n) \leq gR$. Now we just have to find an upper bound on the size of the maximum replication R. In the original transversal cover $r_x \geq g$, for all $x \in X$. So, $R \leq tc(k, g : n) - (g-1)g$. Then,

$$pbtc(k, g : n) \leq g\left(tc(k, g : n) - (g-1)g\right),$$

which implies the first inequality. $\square$

This result and the set packing bound leads to the following general bound for transversal covers.

Corollary 3.9. *Let* $f(b) = \left\lfloor \frac{\binom{b}{\frac{b}{g}-(g-2)} - \sum_{i=2}^{n} \binom{n}{i}\binom{b-n}{\frac{b}{g}-g-(i-2)}}{g\binom{\frac{b}{g}}{g-2}} \right\rfloor$. *Then,*

$$tc(k, g : n) \geq \min\{t : k \leq f(t \cdot g)\} + g(g-1). \tag{3.7}$$

Proof. The proof is a combination of Corollary 3.7 (rewritten as a lower bound on $pbtc(k, g : n)$) and Proposition 3.8. $\square$

Now, we want to get another upper bound on k when we have some information on the replication numbers. We will need the following well known results, which are probably folklore:

Lemma 3.10. *All but the two outermost diagonals of Pascal's triangle are concave up, i.e., if* $f_i(j) = \binom{j}{i}$, *then* $f_i(j-1) > \frac{f_i(j-2)+f_i(j)}{2}$, *for* $i \geq 2$.

Proof. Since $f_i(j) = f_{i-1}(j-1) + f_i(j-1)$, we see that all diagonals but the first ($i = 0$) are strictly increasing. So if $i \geq 2$ then

$$\begin{aligned}
f_i(j) - f_i(j-1) &= f_i(j-1) + f_{i-1}(j-1) - f_i(j-1) \\
&= f_{i-1}(j-1) \\
&> f_{i-1}(j-2) \\
&= f_i(j-2) + f_{i-1}(j-2) - f_i(j-2) \\
&= f_i(j-1) - f_i(j-2).
\end{aligned}$$

$\square$

Lemma 3.11. *If $\sum_{i=1}^{n} a_i = a$ then $\sum_{i=1}^{n} f_j(a_i)$ is minimized when the a_i are as equal as possible. Namely $(n(\lfloor a/n \rfloor + 1) - a$ of the a_i's are equal to $\lfloor a/n \rfloor$ and $a - n\lfloor a/n \rfloor$ of them are equal to $\lceil a/n \rceil$.*

Proof. If any two of the a_i differ by more than one, say $a_i - a_{i'} > 1$, then we can replace a_i by $a_i - 1$ and $a_{i'}$ by $a_{i'} + 1$ and we will have made $\sum_{i=1}^{n} f_j(a_i)$ smaller since f_j is a concave up function. $\square$

Since the values of Pascal's triangle on the left hand side of the Inequality 3.5 are all on the $(r_{max} + g - 2)$-th diagonal we know a lower bound for the left hand side yielding the following corollary:

Corollary 3.12. *If $r_{max} - r_{min} \leq g - 2$ in a $TC(k, g : n)$ with b blocks, then*

$$k \leq \frac{x}{y} + 1,$$

where

$$\begin{aligned} x = & \binom{b}{r_{max} - g + 2} - \left(\sum_{i=2}^{n} \binom{n}{i} \binom{b-n}{r_{max} - g + 2 - i} \right) - \binom{r_{max}}{r_{max} - g + 2} - \\ & \left((g-1) \left(\left\lfloor \frac{b - r_{max}}{g-1} \right\rfloor + 1 \right) - b + r_{max} \right) \binom{\left\lfloor \frac{b-r_{max}}{g-1} \right\rfloor}{r_{max} - g + 2} - \\ & \left(b - r_{max} - (g-1) \left\lfloor \frac{b - r_{max}}{g-1} \right\rfloor \right) \binom{\left\lceil \frac{b-r_{max}}{g-1} \right\rceil}{r_{max} - g + 2} \end{aligned}$$

and

$$y = \left(g \left(\left\lfloor \frac{b}{g} \right\rfloor + 1 \right) - b \right) \binom{\left\lfloor \frac{b}{g} \right\rfloor}{r_{max} - g + 2} + \left(b - g \left\lfloor \frac{b}{g} \right\rfloor \right) \binom{\left\lceil \frac{b}{g} \right\rceil}{r_{max} - g + 2}.$$

Proof. Without loss of generality, the first group contains a point x with $r_x = r_{max}$. The sum of the rest of the replication numbers from this group is a constant $b - r_{max}$, so we can apply Lemma 3.11 for $a = b - r_{max}$. On each other group, the replication numbers sum to a constant b, so each of the other $k-1$ groups can be minimized similarly. Algebraic manipulation produces the desired equation. $\square$

3.2 Upper Bounds on k for Small Fixed b

The lower bounds developed in Sections 3.1.1 and 3.1.2 are useless for values of k between the last k for which a transversal design exists and the first k for which these bounds are bigger than g^2 (approximately 2^{2g-2}). One way to find more information for small values of k is to investigate the maximum k for a fixed number of blocks. The case with g^2+1 blocks is the most straightforward.

3.2.1 Transversal Covers with g^2+1 Blocks

When there are g^2+1 blocks in a $TC(k, g : n)$, the following fact is clear: in each group G_i there is one point, say x_i, incident with $g+1$ blocks and all other points are incident with g blocks. Clearly, $\lambda_{x_i,x_j} = 2$ for any $i \neq j$, and $\lambda_{x,y} = 1$ for all other x,y not in the same group. If we define a_j to be the number of blocks incident with j of the x_i, then we get

$$\sum_{j=0}^{k} a_j = g^2 + 1,$$

and counting the number of times each x_i appears

$$\sum_{j=0}^{k} j a_j = (g+1)k.$$

Finally, counting the number of times two x_i's appear together on a block we get

$$\sum_{j=0}^{k} a_j \frac{j(j-1)}{2} = \sum_{\{i,j\}\subseteq[1,k]} \lambda_{x_i,x_j} = k(k-1)$$

which leads to

$$\sum_{j=0}^{k} j^2 a_j = \sum_{j=0}^{k} a_j j(j-1) + \sum_{j=0}^{k} a_j j = k(2k+g-1).$$

From this we can calculate that the mean, over all blocks, of the number j of x_i's on each block is $\overline{j} = \frac{(g+1)k}{g^2+1}$.

We now pick a block A containing j of the x_i's. Then counting the flags (x, B) with $x \in A$ and $x \in B$ we get

$$\sum_{B \neq A} \mu_{A,B} = jg + (k-j)(g-1) = kg - k + j,$$

and counting triples (x, y, B) with $x \neq y$, $x, y \in A$ and $x, y \in B$ we get

$$\sum_{B \neq A} \mu_{A,B}(\mu_{A,B} - 1) = j(j-1).$$

We then obtain the mean value of $\mu_{A,B}$, for $B \neq A$,

$$\overline{\mu}_A = \frac{kg - k + j}{g^2},$$

and further

$$\begin{aligned} 0 &\leq \sum_{B \neq A} (\mu_{A,B} - \overline{\mu}_A)^2 \\ &= g^2 \overline{\mu}_A^2 - 2\overline{\mu}_A(kg - k + j) + j(j-1) + kg - k + j, \end{aligned}$$

which reduces after some calculation to

$$0 \leq j^2(g^2-1) - j2k(g-1) + g^2 k(g-1) - k^2(g-1)^2.$$

The first two derivatives with respect to j are $2j(g^2-1) - 2k(g-1)$, and $2(g^2-1)$. So, this parabola is concave up in j; moreover, we know there exists a block A containing

j of the x_i's with $0 \le j \le \overline{j} = \frac{(g+1)k}{g^2+1}$. So, we have an upper bound on

$$j^2(g^2-1) - j2k(g-1) + g^2k(g-1) - k^2(g-1)^2, \text{ for } 0 \le j \le \overline{j}$$

which is the larger of its values at $j = 0$ or $j = \overline{j} = \frac{k(g+1)}{g^2+1}$ which are, respectively

$$g^2k(g-1) - k^2(g-1)^2 \text{ and,}$$
$$\frac{-k^2(g-1)^2(g^2+3)}{(g^2+1)^2} + k(g-1).$$

After some elementary calculation we find that the first is the larger and we get

$$k \le \frac{g^2}{g-1} = g + 1 + \frac{1}{g-1}$$

since k is an integer and $g \ge 3$ we get

$$k \le g + 1. \tag{3.8}$$

Therefore, we can conclude that $tc(k, g : n) \ge g^2 + 2$, for all $k \ge g + 2$ and $g \ge 3$.

The range for k from Inequality 3.8 is the same as the range of k for transversal designs. The consequences of Inequality 3.8, for g a prime power, are that we can only have transversal covers with $g^2 + 1$ blocks where we already have a transversal design. This does not mean that such covers do not exist, but, clearly for these k, $tc(k, g : 1) = g^2$ so minimum covers with g^2+1 blocks must have a set of more than one disjoint block. From this and the fact that transversal designs with g a prime power and $k \le g$ have g disjoint blocks, the only possible optimal covers with exactly g^2+1 blocks are $TC(g+1, g : n)$ where $n \ge 2$. For g not a prime power, there may be other optimal covers with $g^2 + 1$ blocks and $k \le g$. An example of this is $tc(4, 6 : 5) = 37$. Another possibility is $TC(6, 10 : 1)$. We know that $100 \le tc(6, 10 : 1) \le 102$. We suspect that there are a great many covers with $g^2 + 1$ blocks for g composite and k slightly bigger than the maximum k for this order of transversal designs. Applegate used integer programming to first prove that $tc(5,3:1) = 11$ [38]; our result offers a short alternative proof of that. It is also worth mentioning that the corresponding bound for $g = 2$ and $b = 5$ is actually $k \leq g + 2 = 4$ and a cover attaining this bound exists [23]. The only other optimal cover for $g = 2$ with $g^2 + 1 = 5$ blocks is $TC(3,2:2)$.

3.2.2 Transversal Covers with $g^2 + 2$ Blocks

Although a more complex case, a similar analysis of $g^2 + 2$ blocks also gives some information. Now on each group there can be either two points with replication $g+1$ or one point with replication $g+2$, all of the rest having replication g.

We now pick a block, say A, with i points with replication $g+1$ and j points with replication $g+2$. Then counting the flags (x, B) with $x \in A$ and $x \in B$ we get

$$\sum_{B \neq A} \mu_{A,B} = ig + j(g+1) + (k-i-j)(g-1) = kg - k + i + 2j,$$

and counting triples (x, y, B) with $x, y \in A$ and $x, y \in B$ we get

$$\sum_{B \neq A} \mu_{A,B}(\mu_{A,B} - 1) \leq i(i-1) + 2j(j-1) + 2ij.$$

We then obtain the mean value of $\mu_{A,B}$, for $B \neq A$

$$\overline{\mu}_A = \frac{kg - k + i + 2j}{g^2 + 1},$$

and further

$$\begin{aligned} 0 &\leq \sum_{B \neq A} (\mu_{A,B} - \overline{\mu}_A)^2 \\ &\leq -\frac{(kg - k + i + 2j)^2}{g^2 + 1} + i^2 + 2j^2 + 2ij + kg - k. \end{aligned} \tag{3.9}$$

The following lemma will be useful in analyzing the implications of this result.

Lemma 3.13. *For $g \geq 4$, a transversal cover with $k = g + 2$ and $g^2 + 2$ blocks must have at least one block with at least g points on it with replication number g.*

Proof. We will show that if every block has fewer than g points with replication number g, then we derive a contradiction. Assume that every block has at most $g - 1$ such points. Now each group has at least $g - 2$ points with replication number g. So there are at least $\binom{g+2}{2}(g-2)^2$ pairs of such points. There are g^2+2 blocks with at most $g - 1$ such points, making at most $(g^2 + 2)\binom{g-1}{2}$ such pairs represented in the cover. In a cover all pairs must be represented, but for $g \geq 4$, $\binom{g+2}{2}(g - 2)^2 > (g^2 + 2)\binom{g-1}{2}$ which is a contradiction. □

The following theorem summarizes some implications of Inequality 3.9.

Theorem 3.14. *If $k \geq g + 2$ and $g \geq 3$ then*

$$tc(k, g : n) \geq g^2 + 3,$$

with the only exception being $tc(5, 3 : 1) = 11$.

Proof. **Case 1. $g \geq 4$.** It is enough to prove the result for a transversal cover with $k = g + 2$. By Section 3.2.1, we know that such a transversal cover has at least $g^2 + 2$ blocks. Suppose that the transversal cover has exactly $g^2 + 2$ blocks. We will derive a contradiction. By Lemma 3.13, there must exist a block with at least g points of replication number g. On this block $i + j \leq 2$. All six pairs of i and j that satisfy this inequality contradict Inequality 3.9. Therefore, for $g \geq 4$ and $k \geq g+2$, we have $tc(k, g : n) \geq g^2 + 3$.

Case 2. $g = 3$**.** First, we analyze the case $k = 5$. As noted in Section 3.2.1, we know $tc(5, 3 : 1) = 11$. So, we concentrate on $n \geq 2$. We assume there exists a transversal cover with 11 blocks, and will derive a contradiction.

We claim there is no point with replication number $5 = g + 2$. Suppose there is such point. There are at least 14 pairs of points with replication number $3 = g$. If there were no more than one pair of such points on each block, the cover would represent only 11 pairs which is impossible. We conclude that there exists a block with at least three such points on it. On this block $i + j \leq 2$ and it is easy to check that the only i and j that satisfy this inequality and Inequality 3.9 is $i = j = 0$. Such a cover must have a block consisting completely of points with replication number 3 and by simple counting, all other blocks must have exactly one of these points. We then conclude that no such cover has a point with replication 5.

Thus each group must have one point with replication number 3 and two with replication number 4. Again using Inequality 3.9 we can see that the only admissible i are 0,3,4, and 5. If a_i is the number of blocks with i points with replication number 4, then we get

$$a_1 = a_2 = a_5 = 0, \; \sum_{i=0}^{5} a_i = 11, \; 10a_0 + a_3 = 10. \tag{3.10}$$

From this it easy to see that $a_0 = 1$ and $a_4 = 10$. Recalling that $n = 2$, we conclude that we must be able to complete, w.l.o.g., the following transversal cover

groups \ blocks	1	2	3	4	5	6	7	8	9	10	11
1	1	1	1	2	3	2	3	2	3	2	3
2	1	2	3	1	1	?	?	?	?	?	?
3	1	2	3	2	3	1	1	?	?	?	?
4	1	2	3	2	3	?	?	1	1	?	?
5	1	2	3	2	3	?	?	?	?	1	1

However, simple case enumeration shows that it is not possible. This shows that for $k \geq 5$ and $n \geq 2$, $tc(k, 3 : n) \geq 12$.

Now we analyze the case $k = 6$. Similar calculations show that there must exist a block with at least three points of replication number 3. This implies that $i + j \leq 3$ which together with Inequality 3.9 produces a contradiction. This means that for $k \geq 6$, $tc(k, 3 : n) \geq 12$. □

The lower bound given by Theorem 3.14 implies that $tc(5, 3 : 2) = tc(5, 3 : 3) = tc(6, 3 : n) = tc(7, 3 : 1) = 12$, since it coincides with known covers.

3.2.3 Similar Bounds for Larger b

When $b \geq g^2 + 3$, this method of proof gets much harder because we would have to consider points with replication numbers $g, g + 1, g + 2, g + 3$ and possibly higher. However, if we know some results about some blocks in the transversal cover then we can still obtain some results.

More Calculations

Lemma 3.15. *If for every group, G, there exists $x_G, y_G \in G$ $x_G \neq y_G$ and $r_{x_G} = r_{y_G} = g$, then there exists a block in the cover with at least four points with replication number $r_x = g$.*

Proof. Consider one such point in a particular column of the associated covering array. This is in exactly g rows. There are at least $g + 1$ remaining columns. By the covering condition there must be at least $2g + 2$ points with replication number g in the remaining columns of these g rows. So there must be a row with

$$\left\lceil \frac{2g + 2}{g} \right\rceil + 1 \geq 4 \tag{3.11}$$

such points in it. $\square$

Lemma 3.16. *In a $TC(k, g : n)$, with $k \geq g + 2$ and $b = g^2 + z < g^2 + g$, if there exists a block with at most two points on it with replication number greater than g then*

$$tc(k, g : n) \geq g^2 + g \text{ if } k \geq g + 3 \text{ and} \tag{3.12}$$

$$tc(g + 2, g : n) \geq g^2 + g - 1. \tag{3.13}$$

Proof. Let A be the block with at most two points with replication number greater than g Say those two points have replication number $g + a$ and $g + b$ respectively and $a \leq b$. Then

$$\sum_{B \neq A} \mu_{A,B} = k(g - 1) + a + b \tag{3.14}$$

$$\overline{\mu} = \frac{k(g - 1) + a + b}{g^2 + z - 1} \text{ and} \tag{3.15}$$

$$\sum_{B \neq A} \mu_{A,B}(\mu_{A,B} - 1) \leq a \leq z. \tag{3.16}$$

So we can conclude that

$$0 \leq \sum_{B \neq A} (\mu_{A,B} - \overline{\mu})^2 = \frac{-(k(g - 1) + a + b)^2}{g^2 + z - 1} + k(g - 1) + 2a + b. \tag{3.17}$$

The domain of the right hand side are the lattice points of the triangle with vertices $(a, b) = (0, 0), (0, z), (z, z)$.

It is easy, but tedious, to check that the only possible values of a and b that let Inequality 3.17 be at least zero are $a = b = 0$ and additionally the variance is zero. $\square$

Lemma 3.17. *In a $TC(k, g : n)$, if there exists a block with at most three points on* *it with replication number greater than g and $k \geq g+2$ then*

$$tc(k, g : n) \geq g^2 + g - 1. \tag{3.18}$$

Proof. We only need to prove that $tc(g+2, g : n) > g^2 + g - 2$ since all covering arrays with more columns can be truncated to have $k = g+2$, and the same number of blocks. We will assume that $b = g^2 + g - 2 = k(g-1)$ and derive a contradiction. This proof is exactly the same as the proof above except that we now have possibly three points on the block with replication number larger than g, say $g+a$, $g+b$ and $g+c$ where $0 \leq a \leq b \leq c \leq g-2$. In this case we have

$$\begin{aligned}
\sum_{B \neq A} \mu_{A,B} &= k(g-1) + a + b + c && (3.19)\\
\overline{\mu} &= \frac{k(g-1) + a + b + c}{g^2 + g - 3} && (3.20)\\
\sum_{B \neq A} \mu_{A,B}(\mu_{A,B} - 1) &\leq 2a + b \text{ and} && (3.21)\\
0 \leq \sum_{B \neq A} (\mu_{A,B} - \overline{\mu})^2 &\leq \frac{-((g+2)(g-1) + a + b + c)^2}{g^2 + g - 3} \\
&\quad +(g+2)(g-1) + 3a + 2b + c. && (3.22)
\end{aligned}$$

The domain of this function are the lattice points of the tetrahedron defined by the points $(a, b, c) = (0, 0, 0), (0, 0, g-2), (0, g-2, g-2), (g-2, g-2, g-2)$. Lemma 3.16 already deals with the case where $a = 0$. This leaves only the interior, three edges, three faces and one vertex to check. The three partial derivatives of the right hand side of Inequality 3.22 can never be simultaneously zero, which suffices for the interior. That the right hand side is negative on the rest of the boundary follows from simple but tedious calculations. □

We introduce the general method of solution here. We assume that $k = g+2$ and that there are $g^2 + g - 2$ blocks. We will derive a contradiction showing that there must be at least one more block. We always know that there will exist a block, A, with at least four points with replication g. We use this block to perform the analogous calculations as above, assuming that the replication numbers (r_x) of points in A increase from left to right. For the block A, we define a vector a of length $g+2$ and the a_i equal to the "excess" replication numbers ($r_x - g$) of points, $x \in A$. Then we have

$$
\begin{aligned}
\sum_{B \neq A} \mu_{A,B} &= k(g-1) + \sum_{i=1}^{g+2} a_i && (3.23)\\
\overline{\mu} &= \frac{k(g-1) + \sum_{i=1}^{g+2} a_i}{g^2+g-3} && (3.24)\\
\sum_{B \neq A} \mu_{A,B}(\mu_{A,B}-1) &\leq \sum_{i=1}^{g+2} (g-2-i+1)a_i \text{ and} && (3.25)\\
0 \leq \sum_{B \neq A} (\mu_{A,B} - \overline{\mu})^2 &\leq \frac{-((g+2)(g-1) + \sum_{i=1}^{g+2} a_i)^2}{g^2+g-3} \\
&\quad +(g+2)(g-1) + \sum_{i=1}^{g+2} (g+2-i+1)a_i. && (3.26)
\end{aligned}
$$

We will attempt to show that the right hand side of Inequality 3.26 is negative and derive a contradiction on the number of blocks. The contradiction will generate a lower bound on the number of blocks possible. In certain cases, it will not be possible to show that the right hand side of Inequality 3.26 is negative. In these case we will show that the left hand side is positive and the right hand side is still less than it.

We need one more lemma before we state the main result. This lemma will consider the number of points with replication g in A and show that, under certain conditions, the replications numbers on A imply the existence of another block with more points with replication g. In these cases, we do not eliminate the block under consideration, but consider the other block which behaves better for our calculations.

Lemma 3.18 (Reduction Lemma). *Let A be the block in the transversal cover* *with the most points with replication g, say m. Then the sum:*

$$\sum_{i=1}^{g+2} a_i < (m-2)g - (m-1). \tag{3.27}$$

Proof. Suppose not. A point with replication r in A is represented in a as the value $r-g$. There must be at least $g-(g-2-(r-g))-1 = r-g+1 = a_i+1$ points in this group with $r=g$, i.e. no "excess." Performing the same calculation as in Lemma 3.15 we see that if there are i points on the block with $r=g$ then

$$\left\lceil \frac{\sum_{j=i+1}^{g+2}(a_j+1)+(i-1)(2)}{g} \right\rceil + 1 \leq i \tag{3.28}$$

which is violated if

$$\sum_{j=1}^{g+2} a_j \geq (i-2)g-(i-1). \tag{3.29}$$

□

These allow us to prove the following theorem:

Theorem 3.19.

$$\textit{For } 2 \leq g \leq 5,\ tc(g+2, g:n) \ = \ g^2+g-1 \tag{3.30}$$

$$\textit{and for } g=6,7\ tc(g+2, g:n) \ \geq \ g^2+g-17. \tag{3.31}$$

Proof. Consider the following base blocks for the cases $2 \leq g \leq 5$:

$$0111 \tag{3.32}$$

$$01221 \tag{3.33}$$

$$012232 \tag{3.34}$$

$$0123224\ . \tag{3.35}$$

We develop them in the following way. Produce g blocks by applying the permutation $(0)(1,2,\ldots,g-1)$ to each block and then develop these as circulants. Add a final block of all zeros. Because each of these blocks covers each difference at each separation, they produce the desired covering arrays.

To show the lower bounds for $2 \leq g \leq 7$, we simply apply the lemmas. We assume that A is the block with the most points with replication g, which has, by Lemma 3.15 at least four points. The case $g = 2$ was done by Katona and Kleitman and Spencer (see Equation 1.4). Östergård solved $g = 3$ (Equation 1.31). In the case $g = 4$ and $g = 5$, just apply Lemma 3.17. In the case $g = 6$, we need to apply both Lemma 3.17 and Lemma 3.18. All the possible choices for A can either be eliminated by Lemma 3.17 or direct calculation that the variance is negative or be reduced to such cases by Lemma 3.18. For $g = 7$, we use Lemma 3.17, Lemma 3.18 to similarly reduce or eliminate the number of cases that need to be checked. After this enumeration and elimination, we have 20 remaining cases where the variance calculation yields a positive number. The blocks, A, that admit a positive variance are represented by the vectors a:

0	0	0	0	0	1	1	1	1
0	0	0	0	0	2	2	2	2
0	0	0	0	0	2	2	2	3
0	0	0	0	0	2	3	3	3
0	0	0	0	0	3	3	3	3
0	0	0	0	0	3	3	3	4
0	0	0	0	0	3	4	4	4
0	0	0	0	0	4	4	4	4
0	0	0	0	1	1	1	1	1
0	0	0	0	1	1	1	1	2
0	0	0	0	1	1	1	1	3
0	0	0	0	1	1	1	2	2
0	0	0	0	1	1	1	2	3
0	0	0	0	1	1	1	3	3
0	0	0	0	1	1	2	2	2
0	0	0	0	1	1	2	2	3
0	0	0	0	1	1	2	3	3
0	0	0	0	1	2	2	2	2
0	0	0	0	1	2	2	2	3
0	0	0	0	2	2	2	2	2

We eliminate these cases by noting that the variance cannot be zero because $\overline{\mu}$

is not an integer. In fact, from Equation 3.24, $\overline{\mu}$ is $1+\frac{1+\sum a_i}{53}$. In these cases the variance must be at least $\frac{(1+\sum a_i)(52-\sum a_i)}{53}$ and that each of these blocks produces a variance too small. Therefore $tc(9,7:1) > 54$. $\square$

This type of analysis fails for $g = 8$ because although we can eliminate or reduce a great number of cases, we do have cases where the variance is higher than the minimal possible variance. The investigation of these remaining cases could yield structural contradictions or we may find that the lower bound on the variance is actually larger and thus derive the same contradiction. In any case, by elimination and reduction we gain information about the types of blocks that must occur in transversal covers with $g^2 + g - 2$ if they exist.

Examples and a Conjecture

The construction used in the preceding proof is very interesting. The transversal covers meeting the bound are formed from a base block which has all its symbols other than 0 permuted to produce a total of $g-1$ base blocks and the final transversal cover is generated by the circulants of these blocks. The best formulation of such a base block is a list of length $g+2$ on symbols $\mathbb{Z}_{g-1} \cup \infty$. We consider the block to be cyclically ordered. The conditions that the block must satisfy are that all elements of $\mathbb{Z}_{g-1}$ must appear as a difference between some pair of elements for each separation of positions in the list. More than one example (not immediately non-isomorphic) of such a block exists for $3 \leq g \leq 5$ as was checked by an exhaustive search. We do not know if they produce isomorphic transversal covers. We have checked exhaustively that no such base blocks exist for $g = 6, 7, 8$ or 9 so this method will not produce the desired covers for larger groups. The simulated annealing has similarly failed to find covers with these parameters and desired sizes. They may exist, but be extremely difficult to find.

The possibility that there might be a general base block method for generating transversal covers is exciting. A quick, but inexact, calculation will justify that it may be unlikely. This sort of base block construction produces $1+k(g-1)$ blocks. If we want this construction to be better than the block size recursive method for prime powers then we need

$$1+k(g-1) \leq \frac{g(g-1)\log k}{\log g} \tag{3.36}$$

and for large g this reduces to

$$k\log g \leq g\log k \tag{3.37}$$

which clearly does not hold. The known asymptotic limit implies that if some other different base block method were to be known which just developed in $\mathbb{Z}_g$, not using circulants, then it would have to have about $\frac{\log k}{2}$ points. This is a strange number for the number of points and might discourage one from accepting the possibility of such a construction method being useful, but we believe this may yet be possible.

In Theorem 3.19, both the methods used to establish upper and lower bounds stop working after $g = 5, 7$ respectively. Either these covers have at least g^2+g-1 blocks or they have fewer. We believe that not having been able to find any with either the base block method or the simulated annealing is strong evidence for the following conjecture:

Conjecture 3.20.

$$tc(g+2, g : n) \geq g^2+g-1. \tag{3.38}$$

This conjecture is related to our previous Conjecture 2.18. This conjecture implies that although the minimum difference of cover sizes for consecutive k may be linear in g, it is certainly at least $g-1$ (the values that seem true from the tables) at least when g is a prime power.

3.2.4 Transversal Covers with $g^2 + z$ Blocks, $z \le g - 2$

Another way to obtain upper bounds for k when there are few blocks is to note that when $b \le g^2 + g - 2$ then $r_{max} - r_{min} \le g - 2$ and so we can use the set packing method described in Section 3.1.4.

Corollary 3.21. *In a $TC(k, g : n)$ with $b = g^2 + z$, where $z \le g - 2$, if $r_{max} = g + z$, then*

$$k \le \left\lfloor \frac{\binom{g^2+z}{z+2} - \left(\sum_{i=2}^{n} \binom{n}{i}\binom{g^2+z-n}{z+2-i}\right) - \binom{g+z}{z+2} - (g-1)\binom{g}{z+2}}{(g-z)\binom{g}{z+2} + z\binom{g+1}{z+2}} + 1 \right\rfloor. \tag{3.39}$$

Proof. Corollary 3.12. □

If $r_{max} \le g + z$, then $z + 2$ sets still cannot appear more than once and so we also get

$$k \le \left\lfloor \frac{\binom{g^2+z}{z+2} - \sum_{i=2}^{n} \binom{n}{i}\binom{g^2+z-n}{z+2-i}}{(g-z)\binom{g}{z+2} + z\binom{g+1}{z+2}} \right\rfloor.$$

However, this bound is much weaker than the one in Inequality 3.39. Another way to deal with the case $r_{max} \le g + z$ is to use the maximum upper bound achieved over $g + 1 \le r_{max} \le g + z$, namely

$$k \le \max_{g+1 \le r_{max} \le g+z} f(r_{max}), \tag{3.40}$$

where

$$f(r) = \left\lfloor \frac{\binom{g^2+z}{r-g+2} - \left(\sum_{i=2}^{n} \binom{n}{i}\binom{g^2+z-n}{r-g+2-i}\right) - \binom{r}{r-g+2} - (r-z-1)\binom{g}{r-g+2} - (z-r+g)\binom{g+1}{r-g+2}}{(g-z)\binom{g}{r-g+2} + (z)\binom{g+1}{r-g+2}} + 1 \right\rfloor$$

By inspection of this value for small g, it seems that the largest value is produced when $r_{max} = g + z$ and so it is likely that the bound in Inequality 3.39 is true even when $r_{max} \le g + z$. For $z = 1$, this bound is not as good as the same case explored in Section 3.2.1. For $z \ge 2$, the bound from Inequality 3.40 is sometimes better than the set systems bound from Inequality 3.2; but as z increases for a fixed g, a point is reached where Inequality 3.2 is better. The following table shows all the values for which the bound from Inequality 3.40 is stronger than the one on Inequality 3.2, for $g \leq 10$ and $n = 1$:

g	z	Inequality 3.40	Inequality 3.2
7	2	793	1716
8	2	1072	6435
9	2	1411	24310
9	3	20417	24310
10	2	1816	92378
10	3	27790	92378

Increased knowledge of the distribution of r_x in small covers would likely improve this bounds since $r_{max} < g + z$ improves the Inequality 3.40.

3.3 Discussion

We have derived three lower bounds for transversal covers in general. Even though the bound derived from the block size recursive construction is always worse than the set systems bound, its form is more suitable for calculations and so is still useful. The set packing bound from Inequality 3.7 is stronger for smaller k's, while the set systems bound is stronger for larger values of k (Appendix B). Although the previously published constructions [38, 15] and the constructions shown here are effective, no known covers (except transversal designs) attain these general bounds, which indicate that more research needs to be done. Some known covers do attain the less general bounds calculated in Section 3.2.

The lower bounds calculated here can help us in two ways. The first gives us some idea of how well our constructions are performing, both in particular cases and asymptotically. The lower bounds also give us access to the structure of covers. The set packing bound depends heavily on the replication numbers of the covers, and thus explicitly gives structural information about covers that attain the bounds generated. The point balanced version further restricts the structure. In this same vein, the calculation of upper bounds on k when we hold b fixed, as noted in Section 3.2.2, can eliminate certain structures which would imply a contradiction in Inequality 3.9 or similar equations for other fixed b.

Chapter 4

Transversal Packings: Upper Bounds

When we consider the complementary problem of packing instead of covering, the natural goals are finding the largest b given a fixed k, $tp(k, g : n)$ or finding the largest k possible for a fixed number of blocks $kp(b, g : n)$. Again, we shall talk mainly in terms of $tp(k, g : n)$ and we want to find both upper and lower bounds on these values. Because transversal packings are an inverse problem, constructions will contribute lower bounds instead of upper bounds as in the transversal covering case, although the structure forced by meeting some of the upper bounds will allow us to state existence results.

In this chapter, several upper bounds on transversal packings are derived. The first set of bounds consists of applications of standard coding theory bounds to transversal packings. Bounds in the second set are analogues of block design packing bounds. The third set comes from an observation on the constraints that sets of disjoint blocks place on these structures. We will also compare the relative utility of these bounds.

4.1 Coding Theory

If we view the transversal packing as a packing array, the packing conditions do not allow two rows to intersect in more than one point. Thus the rows of a $PA(k, g : n)$ is a maximum code on a g-ary alphabet with word length k and Hamming distance at least $k - 1$. The set of n disjoint blocks form a set of n codewords with mutual Hamming distance k. Since transversal packings are codes, we can apply the many coding theory bounds.

4.1.1 Plotkin Bound

One of the most potent coding theory bounds is the Plotkin bound [9].

Theorem 4.1 (Plotkin bound). *Let $\theta = 1 - \frac{1}{g}$, and suppose that $d > \theta k$. Then a g-ary code with length k and minimum distance d has at most $\frac{d}{d-\theta k}$ codewords.* □

Direct Application of the Plotkin Bound

In our application, we have $d = k - 1$ and the necessary condition in Theorem 4.1 becomes $k - 1 > k(1 - \frac{1}{g})$, or $k > g$. The Plotkin bound will only be useful in this range. The full statement of this direct application is

Corollary 4.2. *For $k > g$,*

$$tp(k, g : 1) \leq \frac{g(k-1)}{k-g}. \tag{4.1}$$

□

When equality is reached in the Plotkin bound, strong implications on the structure of the code arise [9]. In this case, the Plotkin bound must be an integer; the number of words must be a multiple of g; any two code words must intersect; and each symbol must appear equally often in each column. Let us examine these consequences for packing arrays.

We will dualize the transversal packing: we now consider the block set as a set of points, and the points of the original transversal packing are now considered as blocks. A new point, B, is said to be on a new block, x, if $x \in B$ in the original transversal packing.

If the number of blocks in the transversal packing meets the Plotkin bound then the dual is in fact a resolvable $PBD(b, b/g, 1)$. This gives a new existence corollary:

Corollary 4.3. *If there exists a resolvable $PBD(v, \kappa, 1)$ then*

$$tp\left(\frac{v-1}{\kappa-1}, \frac{v}{\kappa} : 1\right) = v. \tag{4.2}$$

□

For a comprehensive survey on these objects see [12]. In particular, the standard 1-factorization of the complete graph, K_{2g}, gives

$$tp(2g-1, g : 1) = 2g. \tag{4.3}$$

The Plotkin Bound Modified to Consider Sets of Disjoint Blocks

The straightforward application of the Plotkin bound to transversal packings does not take into account the sets of n disjoint blocks. We can modify the bound to use this information

Theorem 4.4. *A $TP(k, g : n)$ with b blocks and $n > 1$ satisfies*

$$b^2(\frac{k}{g} - 1) + b(1-k) + n(n-1) < 0. \tag{4.4}$$

Proof. Let N be the number of ordered pairs of different symbols in the columns of a $PA(k, g : n)$. Since pairs of rows from the set of n disjoint blocks never intersect (i.e. never have the same symbol in the same position) and any other pair of rows can share a common symbol in the same position in at most one place, we get

$$N \geq b(b-1)(k-1) + n(n-1). \tag{4.5}$$

On the other hand, we can determine this value, N, in another way. Let $r_{i,j}$ be the number of times the symbol i appears in column j. There are $b - r_{i,j}$ rows where a different symbol occurs. So the contribution of this column to N is

$$\sum_{i=1}^{g} r_{i,j}(b - r_{i,j}). \tag{4.6}$$

We know $\sum_i^g r_{i,j} = b$, which implies that

$$\sum_{i=1}^{g} r_{i,j}(b - r_{i,j}) = b^2 - \sum_{i=1}^{g} r_{i,j}^2 \leq b^2 - \frac{b^2}{g} \tag{4.7}$$

with equality if each each symbol appears equally often. This gives

$$N = \sum_{j=1}^{k} \sum_{i=1}^{g} r_{i,j}(b - r_{i,j}) \leq kb^2(1 - \frac{1}{g}). \tag{4.8}$$

These two bounds on N give

$$kb^2 \left(1 - \frac{1}{g}\right) \geq b(b-1)(k-1) + n(n-1) \tag{4.9}$$

which reduces to

$$b^2((k/g) - 1) + b(1-k) + n(n-1) \leq 0 \tag{4.10}$$

with equality if each symbol appears equally often in each column and each pair of rows (except when both are from the set of disjoint blocks) share exactly one symbol in one position.

Again, we can give a characterization of the packings that meet this bound. All pairs of rows not explicitly forbidden to do so must share exactly one symbol in exactly one column. In addition, each point has the same replication number. Consideration of the dual structure shows that equality can never be reached in this bound.

The dual structure would be a resolvable $PBD(v, \kappa, 1)$ with a hole of order n. Since each point would appear exactly the same number of times (the number of resolution classes), we have only one block size and the fact that pairs within the hole do not appear in any block imply that this structure cannot exist. This completes the proof. □

Bi-Regular Plotkin Bound

Equality in Corollary 4.2 implies that all the point replication numbers are the same and therefore, the number of blocks is a multiple of the group size. This is a strong restriction, We would like to know some bounds that integrate the possibility that the replication numbers vary. A simple case of this is when there are only two replication numbers, r_1 and $r_2 = r_1 + 1$.

If a transversal packing has such a structure then a derivation similar to the proof of the Plotkin bounds yields

Theorem 4.5. *If, in a transversal packing, $b = ug + v$ where $0 \leq v < g$, then*

$$k((g-v)u^2 + v(u+1)^2) \leq b^2 - b - n^2 + n + kb. \tag{4.11}$$

□

Abdel-Ghaffar derives a similar bound for the case $n = 1$ in his paper on mutually orthogonal partial latin squares [1]. If equality is attained, then the dual structure has different block sizes and these structures can exist. In fact, for $n = 1$, these structures are a subclass of a particular design:

Definition 4.1. A *restricted resolvable design* ($R_r RP(p,k)$) is a resolvable design on p points with block sizes r and $r+1$ such that each point appears k times in the design.

Pullman [32] and Stanton *et al.* [40] have solved the case where $r=1$, and Rees [33, 34, 35] solved the case $r=2$. However, the dual of a transversal packing must have a constant number of blocks in each resolution class. Only $R_r RP$'s with this condition apply to transversal packings. The dual structures meeting this requirement are a direction of future research.

Although this bound is generated by consideration of a structure with two replication numbers, this bound applies in general to transversal packings that may have more than two replication numbers. In the proof of Theorem 4.4, we need a bound on the sum $\sum_{i=1}^{g} r_{i,j}^2$ subject to knowing $\sum_{i=1}^{g} r_{i,j}$. We know that

$$\sum_{i=1}^{g} r_{i,j}^2 \geq \frac{(\sum_{i=1}^{g} r_{i,j})^2}{g} = \frac{b^2}{g}, \tag{4.12}$$

with equality if each $r_{i,j}$ is equal to $(\sum_{i=1}^{g} r_{i,j})/g$. This follows from the proof of the Plotkin bound. If we know that this constancy is not possible, as in this case because b/g is not an integer, then we can replace this inequality by the one that assumes that each $r_{i,j}$ is $\lfloor b/g \rfloor$ or $\lceil b/g \rceil$. Inequality 4.12 is useful in general. If the replication numbers are more varied then the bound derivable from this method will be tighter.

4.1.2 Other Bounds from Coding Theory

The Singleton bound from coding theory gives $tp(k,g:n) \leq g^2$ [37], which we already know. However, the sphere packing or Hamming bound can be applied to transversal packings [37].

Theorem 4.6.

$$tp(k, g : n) \leq \frac{g^k}{\sum_{i=0}^{t} \binom{k}{i} (g-1)^i}, \quad where\ t = \left\lfloor \frac{k-2}{2} \right\rfloor. \tag{4.13}$$

□

The related Elias bound also gives a bound for transversal packings.

Theorem 4.7. *Let $\theta = (g-1)/g$. If m is a positive number satisfying $m < \theta k$ and $m^2 - 2\theta km + \theta k(k-1) > 0$ then*

$$tp(k, g : n) \leq \frac{\theta k(k-1) g^k}{(m^2 - 2\theta km + \theta k(k-1)) \sum_{i=0}^{m} \binom{k}{i} (g-1)^i}. \tag{4.14}$$

□

4.2 Residual and Derived Bounds

Some of the techniques developed to give upper bounds on PBD packings also give upper bounds for transversal packings, in particular, the point residue method. To fully explain this method we introduce a new object.

Definition 4.2. A *transversal packing of type* $g_1 g_2 \cdots g_k$ is the same as a transversal packing except that the group sizes may vary. We denote this structure by $TP(g_1 g_2 \cdots g_k : n)$ and the optimal number of blocks by $tp(g_1 g_2 \cdots g_k : n)$.

Although we use this definition only here, varying the group sizes is a very useful and interesting variation on either the packing or cover problem. The variable group sizes could come directly from one of the many applications, see Section 1.3. In particular, transversal packings with this more general structure would be useful for disk allocation of large database files where different fields have variable sizes. Williams and Probert[49] and Cohen *et al.* [11, 10] have investigated the analogous generalization for transversal covers, including their applications. With this new definition we have

Theorem 4.8. *For any group, g_i, in a $TP(g_1g_2\cdots g_k : n)$*

$$tp(g_1g_2\cdots g_k : n) \leq \left\lfloor \frac{g_i}{g_i-1} tp(g_1\cdots g_i-1\cdots g_k : n-1)\right\rfloor. \tag{4.15}$$

Proof. Let $x \in G_i$. By removing this point and all the blocks through it we have a packing on the same structure with one less point in this group, and consequently less than $tp(g_1\cdots g_i-1\cdots g_k : n-1)$ blocks remaining. By taking the union of these block sets over each point in G_i we will have $g_i tp(g_1\cdots g_i-1\cdots g_k : n-1)$ blocks and each one will have been repeated exactly g_i-1 times. Since the bound must be an integer the result follows. □

This is similar to bounds derived for other packing incidence structures, see [28]. This bound can be iterated to yield

$$tp(k,g:n) \leq \left\lfloor \frac{g}{g-1}\left\lfloor \frac{g}{g-1}\cdots\left\lfloor \frac{g}{g-1}tp(k,g-1:1)\right\rfloor\cdots\right\rfloor\right\rfloor. \tag{4.16}$$

Transversal packings are in fact ordinary packings of a vk set with k sets, so we get:

Theorem 4.9. *If $g \geq k$ then*

$$tp(k,g:n) \leq D(kg,k,2) - k(D(g,k,2)) \tag{4.17}$$

and if $k > g$ then

$$tp(k,g:n) \leq \frac{D(kg,g,2)-k}{D(k,g,2)} \tag{4.18}$$

where $D(v,k,t)$ denotes the standard packing number for a v-set by k-subsets where every t-set appears at most once [28].

Proof. The first bound follows from the fact that transversal packing with a k packing of the groups is still a packing. The second arises from putting a g packing on each block of the transversal packing and then adding the set of groups to yield a standard packing. $\square$

Lastly if we remove all the points from one block we get

$$tp(k, g : n) \leq tp(k, g-1 : n-1) + k(g-1) + 1 \tag{4.19}$$

but this bound is probably very bad and almost certainly performs worse than Inequality 4.16.

4.3 Disjoint Block Bound

If we have a point in a $TP(k, g : m)$ with replication number $n > m$, then by deleting the group that this point is on we have a $TP(k-1, g : n)$. Therefore, bounds on the sizes of transversal packings can be translated into bounds on the admissible replication numbers for transversal packings with block size one larger. To this end, we calculate the maximum number of groups possible in a transversal packing with n disjoint blocks and at least $g+1$ blocks.

4.3.1 The Maximum k Admitting n Disjoint Blocks

Theorem 4.10.

$$kp(g+1, g : n) = \frac{g(g+1)}{2} - \frac{n(n-1)}{2}. \tag{4.20}$$

Proof. We will build the desired transversal packing, row by row, considering it as a packing array. The first n rows of the packing array are the row of all 0's, the row of all 1's and so on up to the row of all $(n-1)$'s. Since, within each column, we can permute the symbols, when we add each new row, i, (except the last) the symbols in this row can be all the symbols that we have used in previous rows and at most one new symbol.

Define $l_{i,n}$ to be the largest number of previously used symbols that can be used in building row i if we are building a transversal packing with a set of n disjoint blocks. Clearly $l_{n+1,n} = n$. After the n disjoint blocks have been laid down, we can add each symbol in them at most one time in each new row. These numbers satisfy the recursion: $l_{i+1,n} = l_{i,n} + i$. For, if we think of all the old symbols added in the left-most part of each new row and the new symbols (symbol i, in row $i+1$) only in the right-most section of the new row, then we see that at most the first $l_{i,n}$ symbols of row i will be old symbols and the remaining $k - l_{i,n}$ will be the new symbol $i-1$. Thus the right-hand side of the packing array under construction will look like a large set of disjoint blocks and the only intersections will be on the left-hand side. When building the new row $i+1$, we use new symbol i; we are able to add at most i of the old symbols to this row underneath the right hand part. This portion, along with the first $l_{i,n}$ positions in this new row, is the largest possible set of positions in which we could conceivably place old symbols.

Iterating, we see that $l_{i,n} = \frac{i(i-1)}{2} - \frac{n(n-1)}{2}$ and $l_{g,n} = \frac{g(g-1)}{2} - \frac{n(n-1)}{2}$. If we are going to be able to add a $g+1^{st}$ row, we will not be able to use any more new symbols, and so we must have a set of at most g right-most columns, restricted to which, the array so far constructed, consists of disjoint rows. So $k - l_{g,n} \leq g$, or

$$k \leq \frac{g(g+1)}{2} - \frac{n(n-1)}{2}. \tag{4.21}$$

To see that there is an array achieving this bound, we use the following algorithm for adding the old symbols to each new row. In row i, the first $l_{i-1,n}$ columns will have the old symbol, $i-2$ (new in the last row, $i-1$). The next $i-1$ columns will have each of the old symbols, $0, 1, \ldots, i-2$, once. The remaining $\frac{g(g+1)}{2} - \frac{n(n-1)}{2} - l_{i,n}$

columns of this row will have the new symbol, $i-1$. The last, $g+1^{st}$ row, follows the same pattern but will have no columns remaining for new symbols (which is good seeing as there aren't any). Each row intersects every other in at most one point and thus satisfies the packing conditions. □

Corollary 4.11. *In a $TP(k,g:n)$ with more than g blocks we have*

$$k \leq \frac{g(g+1)}{2} - \frac{n(n-1)}{2}. \tag{4.22}$$

□

Equation 1.36 is equivalent to this result for $n=1$.

4.3.2 Bounds on b

Theorem 4.12. *If, in a $TP(k,g:n)$ with more than g blocks, we define $r_{max} = \max_{x\in V} r_x$, then*

$$k-1 \leq \frac{g(g+1)}{2} - \frac{r_{max}(r_{max}-1)}{2}. \tag{4.23}$$

Proof. Remove any group which has a point achieving r_{max}.
This yields a $TP(k-1,g:\max(n,r_{max}))$ □

Observing that if a $TP(k,g:n)$ has more than mg blocks, it must have a point with replication number at least $m+1$, we have

Corollary 4.13. *For i positive*

$$kp(mg+i,g:n) \leq \frac{g(g+1)}{2} - \frac{m(m-1)}{2} + 1, \tag{4.24}$$

or alternatively, noting that $tp(k, g : n)$ is non-increasing in k, we have

$$tp\left(\frac{g(g+1)}{2} - \frac{m(m-1)}{2} + 2, g : n\right) \leq mg. \tag{4.25}$$

□

One notable consequence of this is the fact that

$$tp(g+2, g : n) \leq g^2 - g. \tag{4.26}$$

4.4 Transversal Packings with Large Sets of Disjoint Blocks

When the transversal packing has a set of g disjoint blocks, Theorem 4.10 states that if $k \geq g+1$ then $tp(k, g : g) = g$. If g is a prime power, then for $k \leq g$, $tp(k, g : g) = g^2$. If g is not a prime power, then the construction method from Theorem 4.10 clearly allows at least $2g$ blocks to be constructed.

When there is a set of $g-1$ disjoint blocks, the transversal packing is considerably constrained. In particular, remembering the construction technique of adding new rows to a packing array from the proof of Theorem 4.10, we see that every row besides the set of $g-1$ disjoint rows can have each of the first $g-1$ symbols, $0, 1, \ldots, g-2$, occurring at most once. The symbol $g-1$ must appear at least $k-g+1$ times in each row. These rows can never intersect in more than one coordinate position. This means that the last $b-g+1$ rows of the packing array with the first $g-1$ symbols replaced by 0's and the last symbol, $g-1$, replaced by 1, are the incidence vectors of a standard packing on k points with block size at most $k-g+1$. This gives us a bound

$$tp(k, g, g-1) \leq g - 1 + D(k, k-g+1, 2). \tag{4.27}$$

This bound cannot be achieved in general because we have only forbidden the rows from intersecting more than once on the last symbol. Replacing the 0's from the set of the first $g-1$ symbols and maintaining the packing conditions cannot always be done. The simplest example is $PA(4,3,2)$. The bound gives us $tp(4,3,2) \leq 8$, but we can only replace the 0's in the incidence matrix of the affine plane on four points for the rows corresponding to four of the blocks, not all six. In fact, this does give the true value $tp(4,3,2) = 6$.

However, when $\sqrt{g-1}$ is a prime power, the packing used is a projective plane and the replacement can be done.

Theorem 4.14. *If p is a prime power then*

$$tp(p^2+p+1, p^2+1, p^2) = 2p^2 + p + 1. \tag{4.28}$$

Proof. The first p^2 rows of the packing array are the p^2 constant rows with symbols $0, 1, \ldots, p^2-1$. The remaining p^2+p+1 rows have the symbol p^2 wherever a 1 appears in the incidence matrix of the projective plane of order p. Now each row is empty in p^2 cells and each column empty in the same number. Form the bipartite graph with the rows and columns as the bipartitions and an edge wherever a cell is empty. The maximum degree is p^2 and so we can edge colour this graph with colours $0, 1, \ldots, p^2-1$. We put symbol k into row $g-1+i$ and column j, when the edge between vertices i and j in the two partitions of the graph is coloured k. □

We end by noting a few small results. By Theorem 4.10, the maximum k such that a transversal packing can have more than g blocks is $2g-1$.

Theorem 4.15.

$$\textit{For } g \geq 3,\ tp(2g-1, g, g-1) \ = \ g+1, \tag{4.29}$$

$$\textit{and for } g > 4,\ tp(2g-2, g, g-1) \ = \ g+1. \tag{4.30}$$

Proof. Under the conditions stated and following the proof of Theorem 4.10, it is easy to see that only one additional block can be added. □

As mentioned above, $tp(4,3,2) = 6$, and by the same method of adding rows it is easy to check that $tp(6,4,3) = 7$.

4.5 Comparison of Upper Bounds

We have derived a number of upper bounds on the size of transversal packings. How do these bounds compare? We have tested all of these bounds except the ones that depend on $D(v,k,2)$. In these tests, the dominant (smallest) bounds are the Plotkin and the disjoint block bounds. We calculated the bounds for all g up to seven and the full range of n, and all k where transversal packings with more than g blocks could exist.

The sphere packing bounds only yield numbers below g^2 for relatively large k by which point the Plotkin bound or the disjoint block bounds are already smaller. The residual bound must be iterated k times and therefore, this bound is probably more useful for smaller k, although we have not found instances yet where this bound is better than any others. Even for small k, the iterations quickly yield values bigger than g^2.

The Plotkin bounds are overwhelmingly the best when k or n is small. However, the Plotkin bounds do have some weaknesses. In particular, they are still above $g+1$ after the point at which this value is no longer attainable. The Plotkin bounds also do not work well for transversal packings with large n.

Since the packing numbers, $D(v, k, 2)$, have only been investigated for small k and the transversal packings bound that depend on it require large k, we cannot currently calculate the values of these bounds. We do not know whether these bounds will be useful when more is discovered about $D(v, k, 2)$.

Chapter 5

Transversal Packings: Constructions and Lower Bounds

Because the size of transversal packings is non-increasing as k increases, constructions yield lower bounds instead of upper bounds as with transversal covers. Some of the constructions used for transversal covers can be used to build transversal packings, with packings as ingredients instead of covers. We review these methods and present a direct construction using matchings in graphs. Finally, we present a set of recursive constructions based on this matching problem.

5.1 Analogues from Transversal Cover Constructions

Both the incomplete transversal design method and the generalizations of Wilson's construction construct transversal packings. The ingredient designs must be either designs or packings instead of covers or designs, but the constructions translate directly.

If, instead of filling the hole of an incomplete transversal design with a transversal cover, we fill it with a transversal packing, we get

Theorem 5.1. *If there exists an* $ITD(k, g; b_1, b_2, \ldots, b_s)$ *then*

$$tp(k, g : i) \geq \max_{\substack{i_i + i_2 + \cdots + i_s = i \\ i_j \leq b_j}} \left(g^2 - \sum_{j=1}^{s} b_j^2 + tp(k, b_j : i_j) \right).$$

Proof. Fill the holes. The holes are disjoint and thus the sets of disjoint blocks from the $TP(k, b_j : i_j)$ will also be disjoint. □

The $ITD(4, 6; 2)$ and $ITD(6, 10; 2)$ yield $tp(4, 6 : 5) \geq 34$ and $tp(6, 10) \geq 98$. If a transversal packing has $g^2 - 1$ blocks then there will be exactly one point from each group which has replication number $g - 1$. Simple counting shows that the pairs of points not covered are exactly these pairs. Adding the block containing these k points will complete the packing to a transversal design. This shows that $tp(4, 6 : 5) = 34$ and $tp(6, 10) = 98$ or 100 but cannot be 99.

The same restrictions (Lemma 2.2) on the admissible k apply to this packing construction. Also similar to the case of covers, when $(k - 1)h < n$, there at least two disjoint blocks, both disjoint from the hole in the ITD [13]. In this case we get

$$tp(k, g : i + 2) \geq \max_{\{h \mid \exists ITD(k, g; h)\}} (g^2 - h^2 + tp(k, h : i)).$$

All the generalizations of Wilson's construction also apply to transversal packings. Besides noting that the ingredient structures must be designs or packings, we will just review the results.

Theorem 5.2. *Let* C *be a* $TP(k + l, t)$ *with groups* $G_1, G_2, \ldots, G_k, H_1, H_2, \ldots, H_l$. *Let* $\mathcal{S}$ *be any subset of* $H_1 \cup H_2 \cup \cdots \cup H_l$ *of cardinality* u, m *be any nonnegative integer, and* $h_i = |H_i \cap \mathcal{S}|$. *For any block* A *of* C, *let* $u_A = |\mathcal{S} \cap A|$. *Then*

$$tp(k, mt+u) \geq \sum_{A} (tp(k, m+u_A : u_A) - u_A) + \sum_{i=1}^{l} tp(k, h_i).$$

□

Theorem 5.3. *In analogy to Theorem 2.5 we get*

$$tp(k, g : n) \geq \max_{\substack{2 \leq i \leq \lfloor \frac{g}{2} \rfloor \\ \max(1, \frac{n}{\lfloor \frac{g}{i} \rfloor}) \leq j \leq \min(n,i)}} \left((tp(k, i : j)\ tp(k, \lfloor g/i \rfloor : \lceil n/j \rceil) \right), \tag{5.1}$$

and in analogy to Inequality 2.4 we get

$$\begin{aligned} tp(k, mt+u : n) \geq \max_{\substack{ij+l \geq n \\ 1 \leq l \leq u \\ 1 \leq i \leq t \\ 1 \leq j \leq m}} \quad & tp(k, u : l) + \frac{t-u}{t} tp(k+1, t : i) tp(k, m : j) \\ & + \frac{u}{t} tp(k+1, t : i)(tp(k, m+1 : j+1) - 1). \end{aligned} \tag{5.2}$$

□

Theorem 5.4. *Given a $(v, \{2, 3, \ldots, g-1\}, 1)$-design, and for each point x, a chosen block, B_x, with $x \in B_x$, we can construct a $TP(k, g)$. For each block, B, of the design, we define u_B to be the number of points on this block not represented by it. Then*

$$tp(k, g) \geq \sum_{B} tp(k, |B| : u_B) - u_B. \tag{5.3}$$

□

5.2 Direct Construction for $b \leq 2g$

5.2.1 Formulation of Transversal Packings as a Graph Problem

If $b \leq 2g$, consideration of the Plotkin bound from the last chapter implies that this transversal packing will achieve its maximum when the only two replication numbers are two and one. In this case, there will be $b-g$ points in each group with replication number two, and $2g-b$ points with replication number one. The dual of this structure is a packing of pairwise edge disjoint $(b-g)$-matchings into a $K_b - K_n$ (which can also be viewed as $K_{b-n} \vee I_n$), a kind of restricted resolvable design if the packing is a decomposition.

With this in mind, we ask the question: Given $g \leq b \leq 2g$ and $1 \leq n \leq g$, what is the largest number, k, of disjoint $(b-g)$-matchings that can be packed into a $K_b - K_n$? When we dualize this packing we will have $tp(k, g : n) \geq b$. We have a number of results about these packings.

5.2.2 Results on the Solution of the Graph Problem

From previous results about the decompositions of graphs into cycles [12] we have the following result.

Definition 5.1. We call parameters b, n, g, cycle admissible if the following conditions hold:

$$\begin{aligned} b \equiv n &\equiv 1 \pmod 2 \\ (b-n)(b+n-1) &\equiv 0 \pmod{4(b-g)} \\ (2g-b)(b-1) &\geq n(n-1). \end{aligned}$$

Theorem 5.5. *We can fully decompose $K_b - K_n$ into $\frac{(b-n)(b+n-1)}{2(b-g)}$ $(b-g)$-matchings* *when $2 \le b-g \le 7$ for all cycle admissible b and n and for any other cycle admissible parameter sets where $v \equiv n \pmod{4(b-g)}$.* □

Theorem 5.6. *If $b(2g-b-1) \ge n(n-1)$, then we can pack*

$$\left\lfloor \frac{(b-n)(b+n-1)}{2(b-g)} \right\rfloor \tag{5.4}$$

$(b-g)$-matchings into $K_b - K_n$, which is the maximum possible.

Proof. Arbitrarily remove

$$\frac{(b-n)(b+n-1)}{2} - \left\lfloor \frac{(b-n)(b+n-1)}{2(b-g)} \right\rfloor (b-g) \tag{5.5}$$

edges from the graph. If $b(2g-b-1) \ge n(n-1)$, then the maximum degree in the graph is $b-1$, so there will be a

$$\left\lfloor \frac{(b-n)(b+n-1)}{2(b-g)} \right\rfloor \tag{5.6}$$

edge colouring since this number is at least b. An augmenting path algorithm makes this an equitable colouring. The colour classes are now the desired set of disjoint $(b-g)$-matchings. □

Since $b \le 2g$ and using the existence of the 1-factorization of K_{2m} and near 1-factorization of K_{2m+1}, we get the following corollary:

Corollary 5.7. *For $n=1$ the maximum number of $(b-g)$-matchings, $\frac{b(b-1)}{2(b-g)}$ is always achievable.* □

Abdel-Ghaffar has also shown this corollary [1]. This value meets the generalized Plotkin bound and thus we know the exact sizes of transversal packings for these parameters. Abdel-Ghaffar and Abbadi [2] asked and answered the question: when does a transversal packing have more than g blocks? We have extended this definite result to include sets of disjoint blocks in Theorem 4.10. Theorem 5.6, and the generalized Plotkin bound, Theorem 4.5, give us the necessary and sufficient conditions for a transversal packing to have more than $g+1$ blocks:

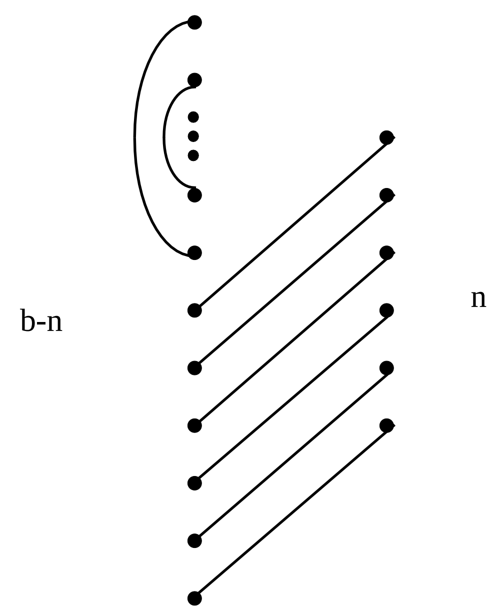
Corollary 5.8.

$$kp(g+2, g:n) = \frac{g^2+3g+2-n^2+n}{4}. \tag{5.7}$$

□

Theorem 5.9. *We can always pack at least $b-n$ $(b-g)$-matchings into $K_b - K_n$.*

Proof. Cyclically permute the $b-n$ vertices on the left, while keeping the n vertices on the right fixed.



□

When $b-n$ is an odd prime and $2g-b-n \geq 0$, we can pack in an additional

$$\frac{2g-b+n-1}{2}\left\lfloor\frac{b-n}{b-g}\right\rfloor \tag{5.8}$$

$(b-g)$-matchings by simply decomposing the $b-n$ cycles remaining after the removal of the matchings given in the proof. The condition $2g - b - n \geq 0$ guarantees that the cycle is large enough to contain a $(b-g)$-matching.

5.3 A Recursive Construction

5.3.1 Presentation of the Method

If we start with two transversal packings with the same k, and $b \leq 2g$ in each packing, then we can use the graph theory translation to construct larger transversal packings.

Theorem 5.10. *Suppose we have two graphs $K_{b_1} - K_{n_1}$ and $K_{b_2} - K_{n_2}$, both with k disjoint copies of $(b_i - g_i)$-matchings inside them. Suppose additionally that $b_1 \geq b_2$, and*

$$g_1 + g_2 \geq b_1 + \frac{n_1 n_2}{b_1}. \tag{5.9}$$

Then we can construct a packing of $k + \left\lfloor \frac{b_1 b_2 - n_1 n_2}{b_1 + b_2 - g_1 - g_2} \right\rfloor$ $(b_1 + b_2 - g_1 - g_2)$-matchings into $K_{b_1+b_2} - K_{n_1+n_2}$.

Proof. Take the disjoint union of the two graphs and the union, in pairs of the existing k matchings. We must pack the remaining bipartite $K_{b_1,b_2} - K_{n_1,n_2}$ into matchings of the appropriate size. Since bipartite graphs can be Δ edge coloured, the condition stipulated guarantees that we can do this to obtain the maximum. □

Theorem 5.11. *Suppose we have two graphs $K_{b_1} - K_{n_1}$ and $K_{b_2} - K_{n_2}$, both with k disjoint copies of $(b_i - g_i)$-matchings inside them. Suppose additionally that $b_1 \geq b_2$ and*

$$g_1 + g_2 \geq b_1. \tag{5.10}$$

Then we can construct a packing of $k + \left\lfloor \frac{b_1 b_2}{b_1 + b_2 - g_1 - g_2} \right\rfloor$ $(b_1 + b_2 - g_1 - g_2)$-matchings into $K_{b_1+b_2} - K_{\max n_i}$.

Proof. The proof is the same except that we no longer remove the edges connecting the two holes from the bipartite graph. □

In the situation of Theorem 5.10, we can pack in at least as many matchings as the maximum degree over all regular subgraphs which are large enough to contain the matching. If $b_2 - n_2$ is larger than the desired matching, we can clearly pack in b_1 additional matchings. In fact, because the graphs are nearly complete bipartite graphs, we can probably do much better in general.

These theorems translate into the transversal packing numbers in the following way. In all cases, $g_i \leq b_i \leq 2g_i$ and $n_i \leq g_i$.

Corollary 5.12. *Under the hypotheses of Theorem 5.10 we have*

$$tp\left(k + \left\lfloor \frac{b_1 b_2 - n_1 n_2}{b_1 + b_2 - g_1 - g_2} \right\rfloor, g_1 + g_2, n_1 + n_2\right) \geq tp(k, g_1, n_1) + tp(k, g_2, n_2). \quad (5.11)$$

□

Corollary 5.13. *Under the hypotheses of Theorem 5.11 we have*

$$tp\left(k + \left\lfloor \frac{b_1 b_2}{b_1 + b_2 - g_1 - g_2} \right\rfloor, g_1 + g_2, \max n_i\right) \geq tp(k, g_1, n_1) + tp(k, g_2, n_2). \quad (5.12)$$

□

When $g_1 = g_2$ the hypotheses from Theorem 5.11 are always met, and we get

Corollary 5.14.

$$tp\left(k + \left\lfloor \frac{b_1 b_2}{b_1 + b_2 - 2g} \right\rfloor, 2g, \max n_i\right) \geq tp(k, g, n_1) + tp(k, g, n_2). \quad (5.13)$$

□

If a transversal packing in the range of parameters, $b \leq 2g$, has any replication numbers greater than two, then in the dual this corresponds to some of the resolution classes having complete graphs larger than an edge as components. But this will not affect the constructions given above as long as the hypotheses are met.

5.3.2 Examples

An example of Theorem 5.11 is given here. We start with two copies of a $TP(4,3:2)$, vertically concatenated.

0	0	0	0
1	1	1	1
0	1	2	2
2	2	0	1
2	0	1	2
1	2	2	0
3	3	3	3
4	4	4	4
3	4	5	5
5	5	3	4
5	3	4	5
4	5	5	3

We find the maximum number of 6-matchings (6 is the new $b-g$) in the resulting $K_{6,6}$ and fill in the array to the right. The edges in the matching correspond to two rows (which are the points in the graph) sharing the same symbol in a column.

0	0	0	0	0	0	0	0	0	0
1	1	1	1	1	1	1	1	1	1
0	1	2	2	2	2	2	2	2	2
2	2	0	1	3	3	3	3	3	3
2	0	1	2	4	4	4	4	4	4
1	2	2	0	5	5	5	5	5	5
3	3	3	3	0	5	4	3	2	1
4	4	4	4	1	0	5	4	3	2
3	4	5	5	2	1	0	5	4	3
5	5	3	4	3	2	1	0	5	4
5	3	4	5	4	3	2	1	0	5
4	5	5	3	5	4	3	2	1	0

Thus we get that $tp(10,6:2) \geq 12$ which is the best possible by the Plotkin bound.

To perform this construction with the same starting ingredients, but to achieve a set of four disjoint blocks, we do the same concatenation. However, we must pack 6-matchings into $K_{6,6} - K_{2,2}$. In this case, the conditions in Theorem 5.10 are not met but, as always, a maximal packing must exist and we know that the minimum degree is achievable. So we can extend the array by four, not five columns:

$$\begin{array}{cccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 1 & 2 & 2 & 2 & 2 & 2 & 2 \\
2 & 2 & 0 & 1 & 3 & 3 & 3 & 3 \\
2 & 0 & 1 & 2 & 4 & 4 & 4 & 4 \\
1 & 2 & 2 & 0 & 5 & 5 & 5 & 5 \\
3 & 3 & 3 & 3 & 2 & 3 & 4 & 5 \\
4 & 4 & 4 & 4 & 3 & 4 & 5 & 1 \\
3 & 4 & 5 & 5 & 4 & 5 & 0 & 2 \\
5 & 5 & 3 & 4 & 5 & 0 & 1 & 4 \\
5 & 3 & 4 & 5 & 0 & 1 & 2 & 3 \\
4 & 5 & 5 & 3 & 1 & 2 & 3 & 0
\end{array}$$

and we have found $tp(8, 6 : 4) \geq 12$. This is the largest $TP(8, 6 : 4)$ known. The Plotkin bound for these parameters is 18 which is probably too large.

5.4 Utility of the Constructions

We used *Mathematica* to program a recursive construction of the transversal packing lower bounds using many of these methods. We only considered $3 \leq g \leq 7$. The constructions that were programmed were

- The generalization of MacNeish's theorem (Theorem 5.1)
- The probabilistic implementation of the generalization of Wilson's construction (Inequality 5.2)
- The PBD construction (Theorem 5.4)
- The existence of an appropriate sets of $MOLS$

- The existence of transversal packings reaching equality in the Plotkin bound (Corollary 4.2)
- The removal of one group to obtain larger sets of disjoint blocks
- The construction obtained from Theorem 5.6
- The existence facts discussed in Section 4.4
- The knowledge about covers with $b = g + 1, g + 2$, from Corollary 5.8 and Theorem 4.10
- The monotonic behavior in k, g and n
- Theorem 5.10
- Theorem 5.11
- Theorem 5.9.

Except for the optimum existence results (Corollary 4.2, Corollary 5.8, Theorem 4.10, and the existence of $MOLS$), the relative utility of these constructions is not obvious. Appendix A contains tables of the constructions used to obtain the current lower bounds, and Corollary 5.8 and Theorem 4.10 obviously dominate the tables; but these existence results are for a very restricted size of transversal packings. The other constructions are more interesting. A brief scan of the tables shows that the constructions from Theorem 5.6 and Theorem 5.9 produce a large number of the best currently known lower bounds. The constructions most useful for constructing transversal packings with more than $2g$ blocks seem to be the Plotkin bound, Wilson's construction or the PBD construction (when g is not a prime power) or packings obtained from removing blocks or groups.

Theorem 5.10 and Theorem 5.11 are applicable only in the range $b \leq 2g$. For values outside this range, the duals of the bi-regular transversal packings (discussed in Subsection 4.1.1) are expected to dramatically improve these tables. There may also be analogues to Theorem 5.10 and Theorem 5.11 for larger block sets. Abdel-Ghaffar has previously solved the transversal packing problem for $g = 34$ and $n = 1$. He found $tp(6, 4 : 1) = 9$ [1].

Chapter 6

Conclusion

6.1 Summary of Results

For transversal covers, we have developed a number of constructions, which yield upper bounds on the sizes of these objects. We have filled the holes of incomplete transversal designs to obtain transversal covers for non prime power group sizes. In some cases, these covers are clearly optimal, or close to optimal. We have improved and extended a construction of Poljak and Rödl (Inequality 1.6) taking into account sets of disjoint blocks. We have also generalized and discussed the application of Wilson's construction to transversal covers, yielding a number of useful constructions. In addition, we have extended the blocks of group divisible designs to obtain transversal covers extending them by additional groups to improve this construction. The block size recursive method is the best in practice, and has the best potential of constructing covers that are close to the known asymptotic bounds.

Lower bounds have also been calculated, of which three are applicable in general. Although Inequality 3.4 is never as good as Inequality 3.3, it is more practical because it is in a directly calculable form and thus easier to manipulate algebraically. The bound from Corollary 3.9 is more useful for small k, but also gives us good bounds for transversal covers with constant replication number. The set packing argument is more generally applicable when we have additional information about the range of replication numbers. In addition to these generally applicable bounds, we have investigated lower bounds when the number of blocks is not much larger than g^2. We proved that a number of known transversal covers are, in fact, optimal, and made

Conjecture 3.20.

$$tc(g+2, g : n) \geq g^2 + g - 1. \tag{3.38}$$

For transversal packings, upper and lower bounds were also presented, the latter coming from constructions. The two dominant upper bounds were the generalization of the Plotkin bound (Theorem 4.5) and the bound derived from the consideration of sets of disjoint blocks (Theorem 4.10). Both the incomplete transversal design construction and the generalizations of Wilson's construction can be applied to transversal packings. Using the dual of the transversal packing, we formulated this problem as a problem of matchings in certain families of graphs. For a restricted range of parameters, we solved this problem and obtained two recursive constructions from this viewpoint.

Much research was conducted on transversal covers prior to this thesis. We contend that the two most important contributions that distinguish this work from previous investigations are the application of combinatorial block design constructions and structures to these problems, and the consideration of sets of disjoint blocks. Although the constructions obtained from design theory may not meet the asymptotic limit, they allow for the construction of transversal covers with small k, one of the gaps in the existing literature. However the success of viewing both these structures as designs does not detract from the power of the many viewpoints used to approach transversal covers and packings. They are best considered from as many viewpoints as possible. Each approach has its particular strength.

As this thesis demonstrates, the consideration of sets of disjoint blocks is pertinent. First, this consideration permits the constructions to be optimized by reducing excess point coverage for transversal covers and guarantees the maintenance of the packing conditions for transversal packings. In either case, removing a group from a transversal structure can produce a smaller structure with a large set of disjoint blocks. Restrictions on the size of structures with large n, can translate to restrictions on the size of the larger structure, producing bounds or non-existence results. For transversal packings, consideration of sets of disjoint blocks produced bounds in and of itself, showed the equivalence of some transversal packings to projective planes, and restricted the replication numbers that can appear in a packing.

6.2 Further Work and Conjectures

A great deal of possible future work remains to be done on both these structures. The utility of transversal covers motivates further research on them. The particulars of the applications also stimulate the investigation into a number of related structures. Transversal packings prompt a number of inquiries which are very intriguing despite lack of immediate application.

When generalizing Wilson's constructions, we were only able to fully state a simple recursion on the covering numbers. Using this construction in more generality, not just for $l = 1$ would greatly enhance our power of construction. This may require understanding more about the internal structure of transversal coverings or probabilistic argument more complex than the one discussed in Subsection 2.3.3. From Wilson's construction, we developed a PBD construction. We would like to comprehend why certain PBDs are more useful in this construction and better understand which systems of representation are optimal.

The lower bounds on transversal covers could be improved by using more than two of the partitions in the set systems bound. We have briefly examined this and recognize that this problem requires more sophistication than we have at the moment.

We would need to generalize the argument made about the types of permutations to include at least three sets instead of two. However, even the enumeration of the ways of ordering possibilities of the three sets in the permutation was difficult.

We would also like to prove both the conjectures previously stated:

Conjecture 2.18. *The worst size difference between consecutive optimal covers,*

$$\max_{k} \left(tc(k+1, g : n) - tc(k, g : n)\right) \tag{2.11}$$

is linear in g and, in fact, may be as low as $g - 1$.

Conjecture 3.20.

$$tc(g+2, g : n) \geq g^2 + g - 1. \tag{3.38}$$

One consequence of Conjecture 3.20 is the following: when g is a prime power, there is a jump of at least $g-1$ between the size of two consecutive transversal covers. This would also imply a similar behavior to that known for transversal packings: when g is a prime power, $tp(g+2, g : 1) \leq g^2 - g$. Conjecture 3.20 is related to Conjecture 2.18, because we believe that the difference in size between consecutive transversal covers is at most linear in g and is probably less than $g - 1$, the value achieved in Conjecture 2.18. The methods for attacking Conjecture 3.20 may be similar to those used to prove the same result for several small g.

The study of transversal packings led to two very interesting combinatorial problems. The first is the dual structure of a transversal packing which meets the generalized Plotkin bound, Theorem 4.5. This is a resolvable PBD on b points with a hole of size n where each resolution class has the same number, g, of blocks in it and the structure has k resolution classes: a subclass of restricted resolvable designs. Investigation by the author and Peter Danziger is underway for the case where $n = 1$ and the block sizes are restricted to k and $k + 1$.

The second noteworthy problem that has arisen from the study of transversal packings is the question of the maximum number of edge disjoint m-matchings that can be packed into a graph G? If $g \leq b \leq 2g$, $n \leq g$, $m = b-g$, and the graph G is K_b with the edges of a K_n removed, then the dual is a transversal packing. Techniques similar to cycle decomposition and packing of graphs would apply here. See [12] for more details of cycle systems. From observing the tables of upper and lower bounds on transversal packings, a proof that these coincide for $k \geq 2g-1$, or even for $b \leq 2g$, should be within our grasp. Lastly, for transversal packings, we would like to extend the graph constructions beyond the restriction that $b \leq 2g$.

When developing upper bounds for transversal packings, we mentioned the equivalent object but with variable group sizes. The analogue of these generalizations for transversal covers would be extremely useful from the point of view of applications. Either the software testing or the inconsistent data compression problems often require multiple alphabet sizes. Most instances of these problems are bound to have this property.

Another generalization that is motivated by applications is the following: In testing software we may know, *a priori*, that two inputs do not interact. In the data compression problem, we may pick the observation sets to preclude certain overlaps. This is equivalent to dropping the requirement that ordered pairs of points from two groups be covered. We can use a graph to describe which pairs of groups need be covered.

Given a graph, G, describing which groups must interact, what is the minimal number of blocks in the resulting generalization of transversal covers? This number falls between $tc(\omega(G), g:1)$ and $tc(\chi(G), g:1)$, so the question is already answered for cliques, interval graphs, bipartite graphs, complements of connected bipartite graphs, perfect graphs and any other graphs where $\omega(G) = \chi(G)$.

For both transversal covers and packings we would like to develop the automated systems described in Subsection 1.3.1 and Subsection 2.5.1. As mentioned before, the motivating example for this implementation would be Colbourn and Dinitz's recursive algorithm for generating complete and incomplete transversal designs. Their article on designing this program [13] cogently discusses the difficulties in such a project: including the automation of which recursions to use when given a new ingredient design and the data structures needed to save the results economically. However, as mentioned before, having such a system would immensely benefit people needing these objects for applications.

Appendix A

Tables of Upper and Lower Bounds.

We include here tables showing the values for the upper bounds on transversal covers and the upper and lower bounds on transversal packings. To see data on the lower bounds for transversal covers, see the graphs in Appendix B. These general lower bounds were not displayed in table form because they are uninformative for the range of the tables. In the transversal covers tables, we have calculated the upper bounds for $3 \leq g \leq 7$ and $k \leq 50$. In the tables for transversal packings, we have calculated upper and lower bounds for $3 \leq g \leq 7$ and the entire range of k for which packings with more than g blocks exist. In all cases we supply corresponding tables of the methods used to obtain these values and the values that are known to be optimal are emphasized.

k	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27
$tc(k,3:1)$	***9***	***11***	***12***	***12***	13	13	15	15	15	15	15	15	15	18	18	18	18	18	18	18	18	19	19	19
$tc(k,3:2)$	***10***	***12***	***12***	13	14	14	15	15	15	16	16	16	16	18	18	18	18	18	19	19	19	19	19	19
$tc(k,3:3)$	***11***	***12***	***12***	14	14	14	15	15	15	17	17	17	17	18	18	18	18	18	19	19	19	19	19	19

k	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27
$tc(k,4:1)$	***16***	***16***	***19***	21	23	24	25	28	28	28	28	28	28	28	28	28	28	31	31	31	31	31	32	32
$tc(k,4:2)$	***16***	***17***	20	21	23	24	25	28	28	28	28	28	28	28	28	28	28	31	31	31	31	32	33	33
$tc(k,4:3)$	***16***	18	20	22	23	24	25	28	28	28	28	28	28	28	28	28	28	31	31	31	31	33	33	33
$tc(k,4:4)$	***16***	19	20	22	23	24	25	28	28	28	28	28	28	28	28	28	28	31	31	31	31	33	33	33

k	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27
$tc(k,5:1)$	***25***	***25***	***25***	***29***	34	37	42	45	45	45	45	45	45	45	45	45	45	45	45	45	45	45	45	45
$tc(k,5:2)$	***25***	***25***	***26***	30	34	37	42	45	45	45	45	45	45	45	45	45	45	45	45	45	45	45	45	45
$tc(k,5:3)$	***25***	***25***	27	31	34	37	43	45	45	45	45	45	45	45	45	45	45	45	45	45	45	45	45	45
$tc(k,5:4)$	***25***	***25***	28	31	35	38	44	45	45	45	45	45	45	45	45	45	45	45	45	45	45	45	45	45
$tc(k,5:5)$	***25***	***25***	29	31	35	39	45	45	45	45	45	45	45	45	45	45	45	45	45	45	45	45	45	45

k	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27
$tc(k,6:1)$	***37***	46	46	48	48	48	62	67	67	69	69	69	69	76	76	76	76	76	76	76	76	76	76	76
$tc(k,6:2)$	***37***	46	46	48	48	48	62	67	67	69	69	69	69	76	76	76	76	76	76	76	76	76	76	76
$tc(k,6:3)$	***37***	46	47	48	48	48	62	67	67	69	69	69	69	76	76	76	76	77	77	77	77	77	77	77
$tc(k,6:4)$	***37***	46	48	48	48	48	62	67	67	69	69	69	69	76	76	76	76	78	78	78	78	78	78	78
$tc(k,6:5)$	***37***	46	48	48	48	48	62	67	67	69	69	69	69	76	76	76	76	78	78	78	78	78	78	78
$tc(k,6:6)$	38	46	48	48	48	48	62	68	68	70	70	70	70	76	76	76	76	78	78	78	78	78	78	78

k	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27
$tc(k,7:1)$	***49***	***49***	***49***	***49***	***49***	63	63	91	91	91	91	91	91	91	91	91	91	91	91	91	91	91	91	91
$tc(k,7:2)$	***49***	***49***	***49***	***49***	***50***	63	63	91	91	91	91	91	91	91	91	91	91	91	91	91	91	91	91	91
$tc(k,7:3)$	***49***	***49***	***49***	***49***	51	63	63	91	91	91	91	91	91	91	91	91	91	91	91	91	91	91	91	91
$tc(k,7:4)$	***49***	***49***	***49***	***49***	52	63	63	91	91	91	91	91	91	91	91	91	91	91	91	91	91	91	91	91
$tc(k,7:5)$	***49***	***49***	***49***	***49***	53	63	63	91	91	91	91	91	91	91	91	91	91	91	91	91	91	91	91	91
$tc(k,7:6)$	***49***	***49***	***49***	***49***	54	63	63	91	91	91	91	91	91	91	91	91	91	91	91	91	91	91	91	91
$tc(k,7:7)$	***49***	***49***	***49***	***49***	55	63	63	91	91	91	91	91	91	91	91	91	91	91	91	91	91	91	91	91

Table A.1: Upper Bounds for Transversal Covers for $4 \leq k \leq 27$.

k	28	29	30	31	32	33	34	35	36	37	38	39	40	41	42	43	44	45	46	47	48	49	50
$tc(k,3:1)$	20	20	20	20	20	20	20	20	20	21	21	21	21	21	21	21	21	21	21	21	21	22	22
$tc(k,3:2)$	21	21	21	21	21	21	21	21	21	21	21	21	21	21	21	21	21	21	21	21	21	23	23
$tc(k,3:3)$	21	21	21	21	21	21	21	21	21	21	21	21	21	21	21	21	21	21	21	21	21	23	23

k	28	29	30	31	32	33	34	35	36	37	38	39	40	41	42	43	44	45	46	47	48	49	50
$tc(k,4:1)$	32	32	32	34	34	34	34	34	35	35	35	35	35	36	36	36	36	36	37	37	37	37	37
$tc(k,4:2)$	33	33	33	35	35	35	35	35	36	36	36	36	36	37	37	37	37	37	38	38	38	38	38
$tc(k,4:3)$	33	34	34	35	35	36	36	36	36	37	37	37	37	38	38	38	38	38	39	39	39	39	39
$tc(k,4:4)$	33	35	35	35	35	36	36	36	36	37	37	37	37	38	38	39	39	39	39	39	39	40	40

k	28	29	30	31	32	33	34	35	36	37	38	39	40	41	42	43	44	45	46	47	48	49	50
$tc(k,5:1)$	45	45	45	49	49	49	49	49	49	51	51	51	51	51	51	55	55	55	55	55	55	55	59
$tc(k,5:2)$	45	45	45	49	49	49	49	49	50	52	52	52	52	52	52	56	56	56	56	56	56	56	60
$tc(k,5:3)$	45	45	45	49	49	49	49	49	51	53	53	53	53	53	53	57	57	57	57	57	57	57	60
$tc(k,5:4)$	45	45	45	49	49	49	49	49	52	54	54	54	54	54	54	57	57	57	57	57	57	57	61
$tc(k,5:5)$	45	45	45	49	49	49	49	49	53	54	54	54	54	55	55	57	57	57	57	57	57	57	61

k	28	29	30	31	32	33	34	35	36	37	38	39	40	41	42	43	44	45	46	47	48	49	50
$tc(k,6:1)$	76	76	76	79	79	79	79	79	79	88	88	88	88	88	88	88	88	88	88	88	88	88	88
$tc(k,6:2)$	76	76	76	79	79	79	79	79	79	88	88	88	88	88	88	88	88	88	88	88	88	88	88
$tc(k,6:3)$	77	77	77	79	79	79	79	79	79	88	88	88	88	88	88	88	88	88	89	89	89	89	89
$tc(k,6:4)$	78	78	78	79	79	79	79	79	79	88	88	88	88	88	88	88	88	88	90	90	90	90	90
$tc(k,6:5)$	79	79	79	79	79	79	79	79	79	88	88	88	88	88	88	88	88	88	90	90	90	90	90
$tc(k,6:6)$	80	80	80	80	80	80	80	80	80	88	88	88	88	88	88	88	88	88	90	90	90	90	90

k	28	29	30	31	32	33	34	35	36	37	38	39	40	41	42	43	44	45	46	47	48	49	50
$tc(k,7:1)$	91	91	91	91	91	91	91	91	91	91	91	91	91	91	91	91	91	91	91	91	91	91	91
$tc(k,7:2)$	91	91	91	91	91	91	91	91	91	91	91	91	91	91	91	91	91	91	91	91	91	91	91
$tc(k,7:3)$	91	91	91	91	91	91	91	91	91	91	91	91	91	91	91	91	91	91	91	91	91	91	91
$tc(k,7:4)$	91	91	91	91	91	91	91	91	91	91	91	91	91	91	91	91	91	91	91	91	91	91	91
$tc(k,7:5)$	91	91	91	91	91	91	91	91	91	91	91	91	91	91	91	91	91	91	91	91	91	91	91
$tc(k,7:6)$	91	91	91	91	91	91	91	91	91	91	91	91	91	91	91	91	91	91	91	91	91	91	91
$tc(k,7:7)$	91	91	91	91	91	91	91	91	91	91	91	91	91	91	91	91	91	91	91	91	91	91	91

Table A.2: Upper Bounds for Transversal Covers for $28 \leq k \leq 50$.

Abbreviation	Construction Method
a	Transversal designs exist
b	Generalization of MacNeish's theorem, Theorem 2.5
c	$tc(k, g : n) < tc(k, g + 1 : n)$
d	Wilson's construction, Inequality 2.4
e	PBD construction, Theorem 2.8
f	Simulated annealing
g	Group divisible design construction, Subsection 2.4.3
h	Circulant method, Theorem 3.19
i	Incomplete transversal design method, Theorem 2.1
j	Found by hand
k	Block size recursive method, Inequality 2.2
l	$tc(k, g : n) \leq tc(k + 1, g : n)$
m	$tc(k, g : n) \leq tc(k, g : n + 1)$
n	$tc(k, g : n) \leq tc(k, g : m) + n - m$ for $m < n$

Table A.3: Abbreviation List of Methods Constructing Transversal Covers.

k	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28
$tc(k,3:1)$	a	h	flm	f	fl	f	efklm	klm	klm	jl	l	l	g	klm	klm	klm	klm	klm	kl	kl	k	klm	klm	km	kl
$tc(k,3:2)$	n	flmn	fm	fn	flmn	fmn	eflm	lm	m	ln	ln	ln	n	klm	klm	lm	lm	m	klmn	klmn	klmn	lm	lm	m	eklmn
$tc(k,3:3)$	n	fl	f	fln	fl	f	efkl	kl	k	kln	kln	kln	n	kl	kl	kl	kl	k	kl	kl	kl	kl	kl	k	ekl

k	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28
$tc(k,4:1)$	a	a	h	fm	fm	fm	fm	klm	klm	klm	klm	klm	klm	klm	klm	klm	km	klm	klm	klm	klm	k	kl	kl	kl
$tc(k,4:2)$	a	n	fmn	f	fm	fm	fm	klm	klm	klm	klm	klm	klm	lm	lm	lm	m	lm	lm	lm	m	kn	klmn	klmn	klmn
$tc(k,4:3)$	a	fn	fm	fmn	fm	fm	fm	klm	klm	klm	klm	klm	klm	lm	lm	lm	m	lm	lm	lm	m	klmn	lm	lm	m
$tc(k,4:4)$	a	fn	f	f	f	f	f	kl	kl	kl	kl	kl	kl	kl	kl	kl	k	kl	kl	kl	k	kl	kl	kl	k

k	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28
$tc(k,5:1)$	a	a	a	h	fm	fm	m	klm	klm	klm	klm	klm	klm	klm	klm	klm	klm	klm	klm	klm	klm	klm	klm	klm	klm
$tc(k,5:2)$	a	a	n	fn	fm	fm	e	klm	klm	klm	klm	klm	klm	klm	klm	klm	klm	klm	klm	klm	klm	klm	lm	lm	lm
$tc(k,5:3)$	a	a	n	fmn	f	f	n	klm	klm	klm	klm	klm	klm	klm	klm	klm	klm	klm	klm	klm	klm	klm	lm	lm	lm
$tc(k,5:4)$	a	a	n	fm	fmn	n	n	klm	klm	klm	klm	klm	klm	klm	klm	klm	klm	klm	klm	klm	klm	klm	lm	lm	lm
$tc(k,5:5)$	a	a	n	f	f	n	kln	kl	kl	kl	kl	kl	kl	kl	kl	kl	kl	kl	kl	kl	kl	kl	kl	kl	kl

k	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28
$tc(k,6:1)$	m	dlm	m	clm	clm	m	cm	klm	km	klm	klm	klm	km	klm	klm	lm	lm	lm	lm	lm	lm	lm	lm	lm	lm
$tc(k,6:2)$	m	delm	e	clm	clm	m	cm	klm	km	klm	klm	klm	km	eklm	eklm	elm	elm	el	el	el	el	el	el	el	el
$tc(k,6:3)$	m	m	n	clm	clm	m	cm	klm	km	klm	klm	klm	km	lm	lm	lm	m	ln	ln	ln	ln	ln	ln	ln	ln
$tc(k,6:4)$	m	m	clmn	clm	clm	m	cm	klm	km	klm	klm	klm	km	lm	lm	lm	m	klmn	klmn	klmn	klmn	klmn	klmn	klmn	ln
$tc(k,6:5)$	i	em	clm	clm	clm	m	cm	kl	k	kl	kl	kl	k	elm	elm	elm	em	klm	klm	klm	klm	klm	klm	km	kln
$tc(k,6:6)$	n	e	cl	cl	cl	g	c	kln	kn	kln	kln	kln	kn	el	el	el	e	kl	kl	kl	kl	kl	kl	k	kln

k	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28
$tc(k,7:1)$	a	a	a	a	a	clm	m	klm	klm	klm	klm	klm	klm	klm	klm	klm	klm	klm	klm	klm	klm	klm	klm	klm	klm
$tc(k,7:2)$	a	a	a	a	n	clm	m	klm	klm	klm	klm	klm	klm	klm	klm	klm	klm	klm	klm	klm	klm	klm	klm	klm	klm
$tc(k,7:3)$	a	a	a	a	n	clm	m	eklm	eklm	klm	klm	klm	klm	klm	klm	klm	klm	klm	klm	klm	klm	klm	klm	klm	klm
$tc(k,7:4)$	a	a	a	a	n	clm	m	eklm	eklm	klm	klm	klm	klm	klm	klm	klm	klm	klm	klm	klm	klm	klm	klm	klm	klm
$tc(k,7:5)$	a	a	a	a	n	clm	m	eklm	eklm	klm	klm	klm	klm	klm	klm	klm	klm	klm	klm	klm	klm	klm	klm	klm	klm
$tc(k,7:6)$	a	a	a	a	n	clm	m	klm	klm	klm	klm	klm	klm	klm	klm	klm	klm	klm	klm	klm	klm	klm	klm	klm	klm
$tc(k,7:7)$	a	a	a	a	n	cl	g	ekl	ekl	kl	kl	kl	kl	kl	kl	kl	kl	kl	kl	kl	kl	kl	kl	kl	kl

Table A.4: Methods Used to Obtain Values for Upper Bounds for Transversal Covers for $4 \leq k \leq 28$.

k	29	30	31	32	33	34	35	36	37	38	39	40	41	42	43	44	45	46	47	48	49	50
$tc(k,3:1)$	kl	kl	kl	kl	kl	kl	kl	k	klm	klm	klm	klm	klm	klm	klm	klm	klm	klm	klm	km	kl	kl
$tc(k,3:2)$	eklmn	eklmn	eklmn	eklmn	eklmn	eklmn	eklmn	klmn	lm	lm	lm	lm	lm	lm	lm	lm	lm	lm	lm	m	klmn	klmn
$tc(k,3:3)$	ekl	ekl	ekl	ekl	ekl	ekl	ekl	kl	kl	kl	kl	kl	kl	kl	kl	kl	kl	kl	kl	k	kl	kl

k	29	30	31	32	33	34	35	36	37	38	39	40	41	42	43	44	45	46	47	48	49	50
$tc(k,4:1)$	kl	k	kl	kl	kl	kl	k	kl	kl	kl	kl	k	kl	kl	kl	kl	k	kl	kl	kl	kl	k
$tc(k,4:2)$	kln	kn	klmn	klmn	kln	kln	kn	klmn	kln	kln	kln	kn	kln	kln	kln	kln	kn	kln	kln	kln	kln	kn
$tc(k,4:3)$	kln	kn	klm	km	klmn	klmn	klmn	km	klmn	klmn	klmn	kmn	klmn	klmn	kln	kln	kn	klmn	klmn	klmn	kln	kn
$tc(k,4:4)$	kln	kln	kl	k	kl	kl	kl	k	kl	kl	kl	k	kl	k	kln	kln	kln	kl	kl	k	kln	kln

k	29	30	31	32	33	34	35	36	37	38	39	40	41	42	43	44	45	46	47	48	49	50
$tc(k,5:1)$	klm	km	klm	klm	klm	klm	klm	k	kl	kl	kl	kl	kl	k	kl	kl	kl	kl	kl	kl	k	kl
$tc(k,5:2)$	lm	m	lm	lm	lm	lm	m	kn	kln	kln	kln	kln	kln	kn	kln	kln	kln	kln	kln	kln	kn	klmn
$tc(k,5:3)$	lm	m	lm	lm	lm	lm	m	kn	kln	kln	kln	kln	kln	kn	klmn	klmn	klmn	klmn	klmn	klmn	kmn	kl
$tc(k,5:4)$	lm	m	lm	lm	lm	lm	m	kn	klmn	klmn	klmn	klmn	kln	kn	klm	klm	klm	klm	klm	klm	km	klmn
$tc(k,5:5)$	kl	k	kl	kl	kl	kl	k	kn	kl	kl	kl	k	kln	kn	kl	kl	kl	kl	kl	kl	k	kl

k	29	30	31	32	33	34	35	36	37	38	39	40	41	42	43	44	45	46	47	48	49	50
$tc(k,6:1)$	lm	m	klm	klm	klm	klm	klm	km	klm	klm	klm	klm	klm	klm	klm	klm	klm	klm	klm	klm	klm	klm
$tc(k,6:2)$	el	e	klm	klm	klm	klm	klm	km	klm	klm	klm	klm	klm	klm	klm	klm	klm	kl	kl	kl	kl	kl
$tc(k,6:3)$	ln	n	klm	klm	klm	klm	klm	km	klm	klm	klm	klm	klm	klm	klm	klm	km	kln	kln	kln	kln	kln
$tc(k,6:4)$	ln	n	klm	klm	klm	klm	klm	km	klm	klm	klm	klm	klm	klm	klm	klm	km	cklmn	cklmn	cklmn	cklmn	cklmn
$tc(k,6:5)$	kln	kln	kl	kl	kl	kl	kl	k	klm	klm	klm	klm	klm	klm	klm	klm	km	cklm	cklm	cklm	cklm	cklm
$tc(k,6:6)$	kln	kln	kln	kln	kln	kln	kln	kn	kl	kl	kl	kl	kl	kl	kl	kl	k	ckl	ckl	ckl	ckl	ckl

k	29	30	31	32	33	34	35	36	37	38	39	40	41	42	43	44	45	46	47	48	49	50
$tc(k,7:1)$	klm	klm	klm	klm	klm	klm	klm	klm	klm	klm	klm	klm	klm	klm	klm	klm	klm	klm	klm	klm	klm	km
$tc(k,7:2)$	klm	klm	klm	klm	klm	klm	klm	klm	klm	klm	klm	klm	klm	klm	klm	klm	klm	klm	klm	klm	klm	m
$tc(k,7:3)$	klm	klm	klm	klm	klm	klm	klm	klm	klm	klm	klm	klm	klm	klm	klm	klm	klm	klm	klm	klm	klm	m
$tc(k,7:4)$	klm	klm	klm	klm	klm	klm	klm	klm	klm	klm	klm	klm	klm	klm	klm	klm	klm	klm	klm	klm	klm	m
$tc(k,7:5)$	klm	klm	klm	klm	klm	klm	klm	klm	klm	klm	klm	klm	klm	klm	klm	klm	klm	klm	klm	klm	klm	m
$tc(k,7:6)$	klm	klm	klm	klm	klm	klm	klm	klm	klm	klm	klm	klm	klm	klm	klm	klm	klm	klm	klm	klm	klm	m
$tc(k,7:7)$	kl	kl	kl	kl	kl	kl	kl	kl	kl	kl	kl	kl	kl	kl	kl	kl	kl	kl	kl	kl	kl	k

Table A.5: Methods Used to Obtain Values for Upper Bounds for Transversal Covers for $29 \leq k \leq 50$.

k	4	5	6	7
$tp(k,3:1)$	***9***	***6***	***4***	***3***
$tp(k,3:2)$	***6***	***4***	***3***	***3***
$tp(k,3:3)$	***3***	***3***	***3***	***3***

k	4	5	6	7	8	9	10	11
$tp(k,4:1)$	***16***	***16***	***9***	***8***	***5***	***5***	***5***	***4***
$tp(k,4:2)$	***16***	8	***8***	***6***	***5***	***5***	***4***	***4***
$tp(k,4:3)$	***16***	8	6	***5***	***4***	***4***	***4***	***4***
$tp(k,4:4)$	***16***	***4***	***4***	***4***	***4***	***4***	***4***	***4***

k	4	5	6	7	8	9	10	11	12	13	14	15	16
$tp(k,5:1)$	***25***	***25***	***25***	***15***	***10***	***10***	***7***	***6***	***6***	***6***	***6***	***6***	***5***
$tp(k,5:2)$	***25***	***25***	15	11	***10***	***8***	***7***	***6***	***6***	***6***	***6***	***5***	***5***
$tp(k,5:3)$	***25***	***25***	15	11	8	***7***	***6***	***6***	***6***	***5***	***5***	***5***	***5***
$tp(k,5:4)$	***25***	***25***	11	***11***	***6***	***6***	***5***	***5***	***5***	***5***	***5***	***5***	***5***
$tp(k,5:5)$	***25***	***25***	***5***	***5***	***5***	***5***	***5***	***5***	***5***	***5***	***5***	***5***	***5***

k	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22
$tp(k,6:1)$	***34***	26	26	16	12	12	***12***	***12***	***9***	***8***	***8***	***7***	***7***	***7***	***7***	***7***	***7***	***7***	***6***
$tp(k,6:2)$	***34***	26	26	16	12	12	***12***	***10***	***8***	***8***	***7***	***7***	***7***	***7***	***7***	***7***	***7***	***6***	***6***
$tp(k,6:3)$	***34***	26	16	12	12	12	10	***9***	***8***	***7***	***7***	***7***	***7***	***7***	***7***	***6***	***6***	***6***	***6***
$tp(k,6:4)$	***34***	26	16	12	12	9	9	***8***	***7***	***7***	***7***	***7***	***6***	***6***	***6***	***6***	***6***	***6***	***6***
$tp(k,6:5)$	***34***	26	12	12	8	8	***7***	***7***	***6***	***6***	***6***	***6***	***6***	***6***	***6***	***6***	***6***	***6***	***6***
$tp(k,6:6)$	26	26	12	***6***	***6***	***6***	***6***	***6***	***6***	***6***	***6***	***6***	***6***	***6***	***6***	***6***	***6***	***6***	***6***

k	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29
$tp(k,7:1)$	***49***	***49***	***49***	***49***	***49***	***28***	***21***	14	***14***	***14***	***10***	***10***	***9***	***9***	***9***	***8***	***8***	***8***	***8***	***8***	***8***	***8***	***8***	***8***	***8***	***7***
$tp(k,7:2)$	***49***	***49***	***49***	***49***	28	21	14	14	***14***	***12***	***10***	***9***	***9***	***9***	***8***	***8***	***8***	***8***	***8***	***8***	***8***	***8***	***8***	***8***	***7***	***7***
$tp(k,7:3)$	***49***	***49***	***49***	***49***	28	21	14	14	12	***11***	***10***	***9***	***9***	***8***	***8***	***8***	***8***	***8***	***8***	***8***	***8***	***8***	***7***	***7***	***7***	***7***
$tp(k,7:4)$	***49***	***49***	***49***	***49***	28	14	14	12	12	***10***	***9***	***9***	***8***	***8***	***8***	***8***	***8***	***8***	***8***	***7***	***7***	***7***	***7***	***7***	***7***	***7***
$tp(k,7:5)$	***49***	***49***	***49***	***49***	14	14	11	11	***9***	***9***	***8***	***8***	***8***	***8***	***8***	***7***	***7***	***7***	***7***	***7***	***7***	***7***	***7***	***7***	***7***	***7***
$tp(k,7:6)$	***49***	***49***	***49***	***49***	14	10	10	***8***	***8***	***8***	***7***	***7***	***7***	***7***	***7***	***7***	***7***	***7***	***7***	***7***	***7***	***7***	***7***	***7***	***7***	***7***
$tp(k,7:7)$	***49***	***49***	***49***	***49***	***7***	***7***	***7***	***7***	***7***	***7***	***7***	***7***	***7***	***7***	***7***	***7***	***7***	***7***	***7***	***7***	***7***	***7***	***7***	***7***	***7***	***7***

Table A.6: Lower Bounds for Transversal Packings.

Abbreviation	Construction Method
a	$tp(k, g : n) > tp(k, g-1 : n)$
b	Generalization of MacNeish's theorem, Inequality 5.1
c	Transversal designs exist
d	Wilson's construction, Inequality 5.2
e	The PBD construction, Theorem 5.4
f	First recursive graph construction, Theorem 5.10
g	Second recursive graph construction, Theorem 5.11
h	Incomplete transversal design method, Theorem 5.1
i	Projective plane method, Theorem 4.14
j	Existence of appropriate resolvable design as dual, Corollary 4.3
k	$tp(k, g : n) \geq tp(k+1, g : n)$
l	$tp(k, g : n) \geq tp(k, g : n+1)$
m	Removal of group to increase n, see proof of Theorem 4.12
n	At least $g+2$ blocks possible, Corollary 5.8
o	Only g or $g+1$ blocks possible, Theorem 4.10 and Corollary 5.8
p	Graph construction, Theorem 5.9
q	Graph construction, Theorem 5.6
r	Construction of Abdel-Ghaffar [1]

Table A.7: Abbreviation List of Methods Constructing Transversal Packings.

k	4	5	6	7
$tp(k,3:1)$	cj	jp	o	o
$tp(k,3:2)$	mp	o	o	o
$tp(k,3:3)$	o	o	o	o

k	4	5	6	7	8	9	10	11
$tp(k,4:1)$	ckl	cj	klp	r	o	o	o	o
$tp(k,4:2)$	cl	mklp	mp	nq	o	o	o	o
$tp(k,4:3)$	cl	p	nq	o	o	o	o	o
$tp(k,4:4)$	cm	o	o	o	o	o	o	o

k	4	5	6	7	8	9	10	11	12	13	14	15	16
$tp(k,5:1)$	ckl	ckl	cj	j	klp	jp	nlq	o	o	o	o	o	o
$tp(k,5:2)$	ckl	cl	l	l	mp	q	nq	o	o	o	o	o	o
$tp(k,5:3)$	ckl	cl	m	l	q	nq	o	o	o	o	o	o	o
$tp(k,5:4)$	ckl	cl	mk	i	o	o	o	o	o	o	o	o	o
$tp(k,5:5)$	cmk	cm	o	o	o	o	o	o	o	o	o	o	o

k	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22
$tp(k,6:1)$	l	adekl	ael	ael	klp	klp	klp	gjp	q	nklq	ngq	o	o	o	o	o	o	o	o
$tp(k,6:2)$	l	adekl	ae	ae	mklp	mklp	mp	gq	nmklq	nmq	o	o	o	o	o	o	o	o	o
$tp(k,6:3)$	l	al	aml	amklp	mklp	p	q	gq	ngq	o	o	o	o	o	o	o	o	o	o
$tp(k,6:4)$	l	al	a	amklp	p	mkq	q	nfq	o	o	o	o	o	o	o	o	o	o	o
$tp(k,6:5)$	h	aeml	admklp	ap	nmkq	nq	o	o	o	o	o	o	o	o	o	o	o	o	o
$tp(k,6:6)$	ademk	ae	p	o	o	o	o	o	o	o	o	o	o	o	o	o	o	o	o

k	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27
$tp(k,7:1)$	ckl	ckl	ckl	ckl	c	j	j	klp	klp	jp	klq	q	nklq	ngklq	nq	o	o	o	o	o	o	o	o	o
$tp(k,7:2)$	ckl	ckl	ckl	cl	l	l	mklp	mklp	mp	q	mlq	nmklq	nmklq	nmq	o	o	o	o	o	o	o	o	o	o
$tp(k,7:3)$	ckl	ckl	ckl	cl	l	m	mklp	p	lq	q	gq	nmklq	ngq	o	o	o	o	o	o	o	o	o	o	o
$tp(k,7:4)$	ckl	ckl	ckl	cl	m	mklp	p	mkq	q	gq	nmkq	nq	o	o	o	o	o	o	o	o	o	o	o	o
$tp(k,7:5)$	ckl	ckl	ckl	cl	mklp	p	mkq	q	nmkq	nfq	o	o	o	o	o	o	o	o	o	o	o	o	o	o
$tp(k,7:6)$	ckl	ckl	ckl	cl	p	mkq	q	o	o	o	o	o	o	o	o	o	o	o	o	o	o	o	o	o
$tp(k,7:7)$	cmk	cmk	cmk	cm	o	o	o	o	o	o	o	o	o	o	o	o	o	o	o	o	o	o	o	o

Table A.8: Methods Used to Obtain Values for Lower Bounds for Transversal Packings.

k	4	5	6	7
$tp(k,3:1)$	***9***	***6***	***4***	***3***
$tp(k,3:2)$	***6***	***4***	***3***	***3***
$tp(k,3:3)$	***3***	***3***	***3***	***3***

k	4	5	6	7	8	9	10	11
$tp(k,4:1)$	***16***	***16***	***9***	***8***	***5***	***5***	***5***	***4***
$tp(k,4:2)$	***16***	14	***8***	***6***	***5***	***5***	***4***	***4***
$tp(k,4:3)$	***16***	13	8	***5***	***4***	***4***	***4***	***4***
$tp(k,4:4)$	***16***	***4***	***4***	***4***	***4***	***4***	***4***	***4***

k	4	5	6	7	8	9	10	11	12	13	14	15	16
$tp(k,5:1)$	***25***	***25***	***25***	***15***	***10***	***10***	***7***	***6***	***6***	***6***	***6***	***6***	***5***
$tp(k,5:2)$	***25***	***25***	23	13	***10***	***8***	***7***	***6***	***6***	***6***	***6***	***5***	***5***
$tp(k,5:3)$	***25***	***25***	22	12	10	***7***	***6***	***6***	***6***	***5***	***5***	***5***	***5***
$tp(k,5:4)$	***25***	***25***	21	***11***	***6***	***6***	***5***	***5***	***5***	***5***	***5***	***5***	***5***
$tp(k,5:5)$	***25***	***25***	***5***	***5***	***5***	***5***	***5***	***5***	***5***	***5***	***5***	***5***	***5***

k	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22
$tp(k,6:1)$	***34***	34	34	34	19	14	***12***	***12***	***9***	***8***	***8***	***7***	***7***	***7***	***7***	***7***	***7***	***7***	***6***
$tp(k,6:2)$	***34***	34	34	34	19	14	***12***	***10***	***8***	***8***	***7***	***7***	***7***	***7***	***7***	***7***	***7***	***6***	***6***
$tp(k,6:3)$	***34***	34	34	33	19	13	12	***9***	***8***	***7***	***7***	***7***	***7***	***7***	***7***	***6***	***6***	***6***	***6***
$tp(k,6:4)$	***34***	34	34	32	18	13	12	***8***	***7***	***7***	***7***	***7***	***6***	***6***	***6***	***6***	***6***	***6***	***6***
$tp(k,6:5)$	***34***	34	34	31	14	12	***7***	***7***	***6***	***6***	***6***	***6***	***6***	***6***	***6***	***6***	***6***	***6***	***6***
$tp(k,6:6)$	34	34	34	***6***	***6***	***6***	***6***	***6***	***6***	***6***	***6***	***6***	***6***	***6***	***6***	***6***	***6***	***6***	***6***

k	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29
$tp(k,7:1)$	***49***	***49***	***49***	***49***	***49***	***28***	***21***	15	***14***	***14***	***10***	***10***	***9***	***9***	***9***	***8***	***8***	***8***	***8***	***8***	***8***	***8***	***8***	***8***	***8***	***7***
$tp(k,7:2)$	***49***	***49***	***49***	***49***	47	26	19	15	***14***	***12***	***10***	***9***	***9***	***9***	***8***	***8***	***8***	***8***	***8***	***8***	***8***	***8***	***8***	***8***	***7***	***7***
$tp(k,7:3)$	***49***	***49***	***49***	***49***	46	25	18	15	14	***11***	***10***	***9***	***9***	***8***	***8***	***8***	***8***	***8***	***8***	***8***	***8***	***8***	***7***	***7***	***7***	***7***
$tp(k,7:4)$	***49***	***49***	***49***	***49***	45	24	17	15	13	***10***	***9***	***9***	***8***	***8***	***8***	***8***	***8***	***8***	***8***	***7***	***7***	***7***	***7***	***7***	***7***	***7***
$tp(k,7:5)$	***49***	***49***	***49***	***49***	44	23	16	14	***9***	***9***	***8***	***8***	***8***	***8***	***8***	***7***	***7***	***7***	***7***	***7***	***7***	***7***	***7***	***7***	***7***	***7***
$tp(k,7:6)$	***49***	***49***	***49***	***49***	43	22	15	***8***	***8***	***8***	***7***	***7***	***7***	***7***	***7***	***7***	***7***	***7***	***7***	***7***	***7***	***7***	***7***	***7***	***7***	***7***
$tp(k,7:7)$	***49***	***49***	***49***	***49***	***7***	***7***	***7***	***7***	***7***	***7***	***7***	***7***	***7***	***7***	***7***	***7***	***7***	***7***	***7***	***7***	***7***	***7***	***7***	***7***	***7***	***7***

Table A.9: Upper Bounds for Transversal Packings.

Abbreviation	Upper bound
a	Only g or $g+1$ blocks possible, Theorem 4.10 and Corollary 5.8
b	Disjoint block bound, Theorem 4.12
c	Hamming bound, Theorem 4.6
d	Plotkin bound, Theorem 4.4
e	Point residue bound, Theorem 4.8
f	Removing a block, Inequality 4.19
g	Plotkin bound, Theorem 4.5
h	$tp(k, g : n) \leq tp(k-1, g : n)$
i	$tp(k, g : n) \leq tp(k, g : n-1)$
j	The Elias bound, Theorem 4.7
k	Result found by hand
l	Incomplete transversal design method, Theorem 5.1

Table A.10: Abbreviations List of Methods for Transversal Packing Upper Bounds.

k	4	5	6	7
$tp(k,3:1)$	bdegh	bdg	a	a
$tp(k,3:2)$	k	a	a	a
$tp(k,3:3)$	a	a	a	a

k	4	5	6	7	8	9	10	11
$tp(k,4:1)$	bdh	bdgh	g	dg	a	a	a	a
$tp(k,4:2)$	bdhi	g	g	g	a	a	a	a
$tp(k,4:3)$	bdfhi	g	dgi	a	a	a	a	a
$tp(k,4:4)$	bdfhi	a	a	a	a	a	a	a

k	4	5	6	7	8	9	10	11	12	13	14	15	16
$tp(k,5:1)$	bdh	bdh	bdgh	dg	g	dgh	g	a	a	a	a	a	a
$tp(k,5:2)$	bdhi	bdhi	g	g	gi	g	gi	a	a	a	a	a	a
$tp(k,5:3)$	bdhi	bdhi	g	g	dgi	g	a	a	a	a	a	a	a
$tp(k,5:4)$	bdhi	bdfhi	g	g	a	a	a	a	a	a	a	a	a
$tp(k,5:5)$	bdhi	bdfhi	a	a	a	a	a	a	a	a	a	a	a

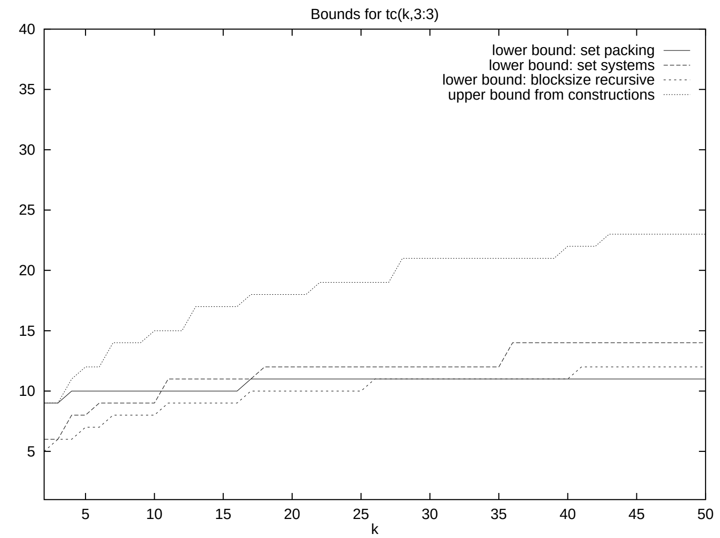
k	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22
$tp(k,6:1)$	l	h	h	h	g	g	g	dgh	g	g	gh	a	a	a	a	a	a	a	a
$tp(k,6:2)$	il	hi	hi	ghi	gi	gi	gi	g	g	ghi	a	a	a	a	a	a	a	a	a
$tp(k,6:3)$	il	hi	hi	g	gi	g	dgi	g	gi	a	a	a	a	a	a	a	a	a	a
$tp(k,6:4)$	il	hi	hi	g	g	gi	dgi	g	a	a	a	a	a	a	a	a	a	a	a
$tp(k,6:5)$	il	hi	hi	g	g	dg	a	a	a	a	a	a	a	a	a	a	a	a	a
$tp(k,6:6)$	i	hi	hi	a	a	a	a	a	a	a	a	a	a	a	a	a	a	a	a

k	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29
$tp(k,7:1)$	bdh	bdh	bdh	bdh	bdgh	dg	dg	g	g	dgh	g	gh	g	gh	gh	a	a	a	a	a	a	a	a	a	a	a
$tp(k,7:2)$	bdhi	bdhi	bdhi	bdhi	g	g	g	gi	gi	g	gi	g	ghi	ghi	a	a	a	a	a	a	a	a	a	a	a	a
$tp(k,7:3)$	bdhi	bdhi	bdhi	bdhi	g	g	g	gi	dgi	g	gi	gi	ghi	a	a	a	a	a	a	a	a	a	a	a	a	a
$tp(k,7:4)$	bdhi	bdhi	bdhi	bdhi	g	g	g	gi	k	g	g	ghi	a	a	a	a	a	a	a	a	a	a	a	a	a	a
$tp(k,7:5)$	bdhi	bdhi	bdhi	bdhi	g	g	g	g	g	gh	a	a	a	a	a	a	a	a	a	a	a	a	a	a	a	a
$tp(k,7:6)$	bdhi	bdhi	bdhi	bdfhi	g	g	g	a	a	a	a	a	a	a	a	a	a	a	a	a	a	a	a	a	a	a
$tp(k,7:7)$	bdhi	bdhi	bdhi	bdfhi	a	a	a	a	a	a	a	a	a	a	a	a	a	a	a	a	a	a	a	a	a	a

Table A.11: Methods Used to Obtain Values for Upper Bounds for Transversal Packings.

Appendix B

Graphs for Lower Bounds of Section 3.1



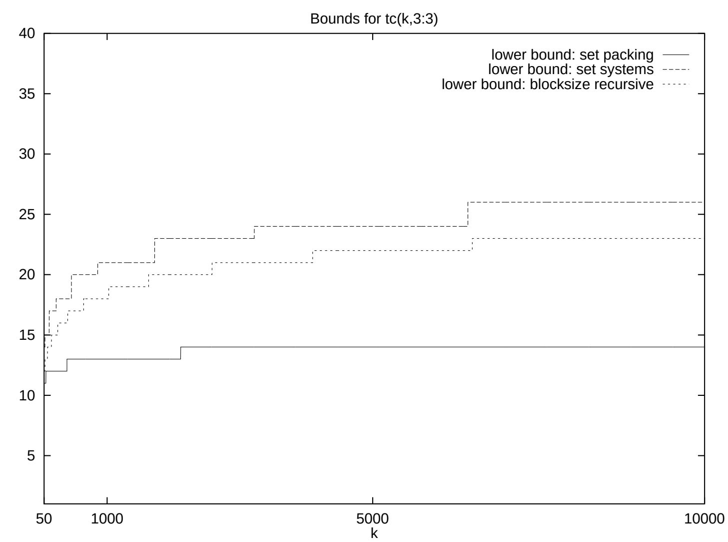
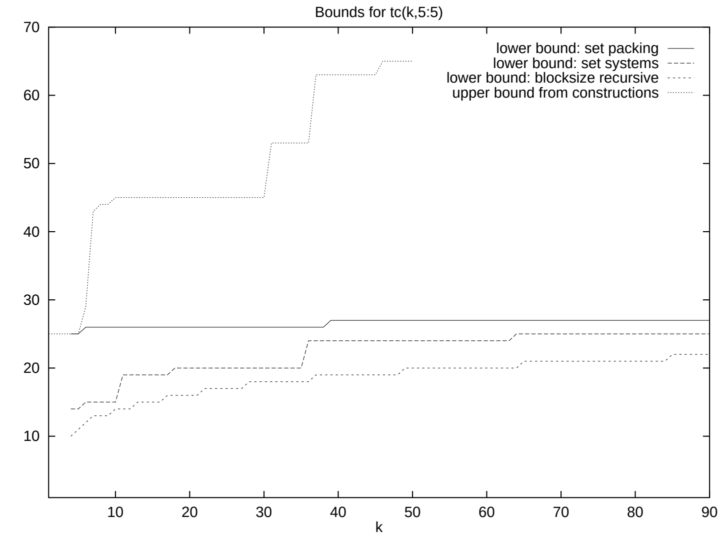


Figure B.1: Bounds on b for $g = n = 3$.



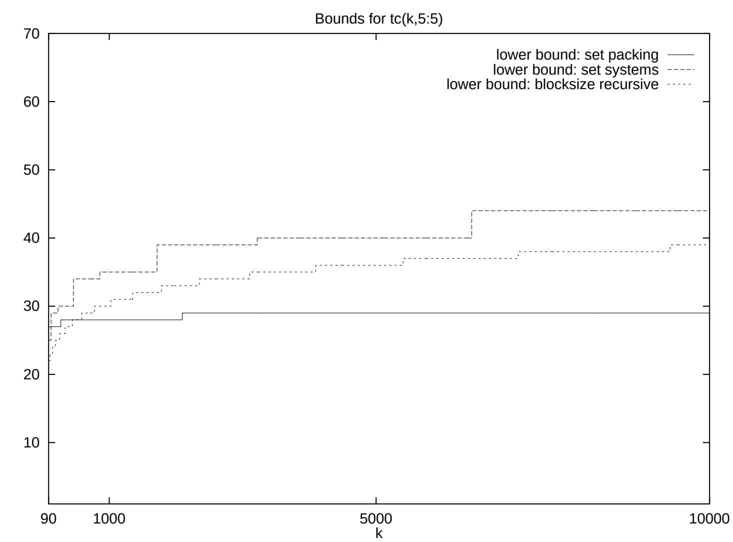
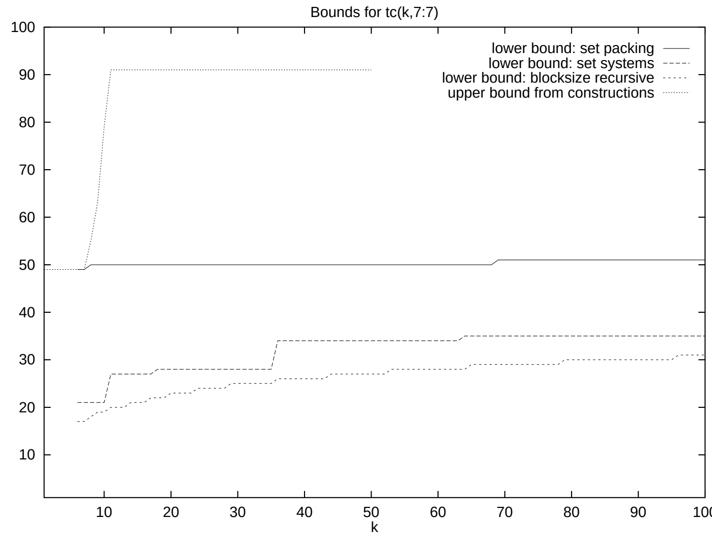


Figure B.2: Bounds on b for $g = n = 5$.



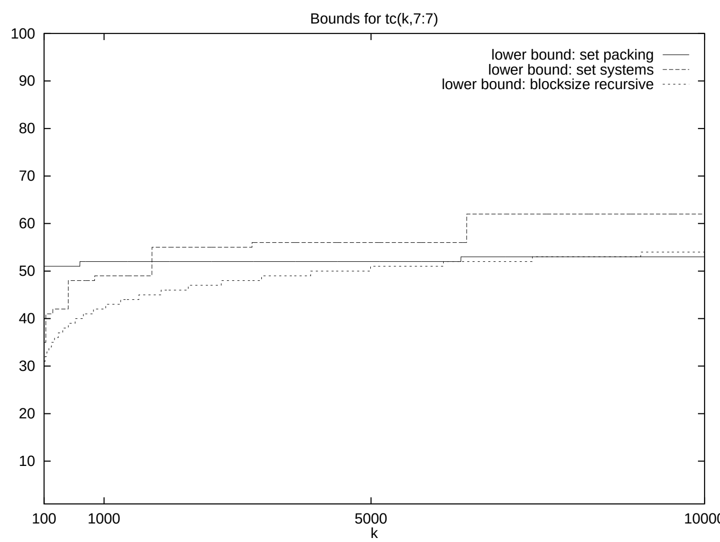


Figure B.3: Bounds on b for $g = n = 7$.

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$\hat{x}$	x removed from a set.	2
$\lceil x \rceil$	Smallest integer $\geq x$.	2
$\lfloor x \rfloor$	Largest integer $\leq x$.	2
$\overline{x}$	The mean value of x.	2
$\vee$	The join operator for two graphs.	107
A	Adenine.	22
$\mathcal{B}$	The block set of an incidence structure.	2
b	The number of blocks in an incidence structure.	4
C	Cytosine.	22
$CA(k, g : n)$	Covering array.	5
Δ	Maximum degree in a graph.	110
$D(v, k, t)$	Standard packing number.	96
$\dim(G)$	Dimension of a graph, G.	12
DNA	Deoxyribonucleic acid.	22
EA	Existential array.	4
g-partition	A partition of a set into g pairwise disjoint pieces.	6
g	Group size.	3
G	Guanine.	22
g-ary alphabet	A symbol set with g symbols.	2
g-set	A set with g elements.	2
GDD	Group divisible design.	3
ITD	Incomplete transversal design.	3
k_B	Size of block, B.	4
K_n	Complete graph on n vertices.	11
$K_{n,m}$	Complete bipartite graph.	110
$\mathcal{K}$	Set of block sizes in an incidence structure.	2
$kc(b, g : n)$	Largest k, fixing b, in a transversal cover.	5
$kp(b, g : n)$	Largest k, fixing b, in a transversal packing.	7

$\lambda_{x,y}$	Number of blocks on which both points x and y appear.	4
$\mu_{A,B}$	Number of points in the intersection of blocks, A and B.	4
MDS	Maximum distance separable code.	17
$MOLS$	Mutually orthogonal latin squares.	4
$PA(k, g : n)$	Packing array.	8
$PBD(v, \mathcal{K})$	Pairwise balanced design.	2
$PBTC(k, g : n)$	Point balanced transversal cover.	7
$pbtc(k, g : n)$	Smallest b, fixing k, in a point balanced transversal cover.	7
r_x	Number of blocks through x.	4
$RGDD$	Resolvable group divisible design.	3
$R_rRP(p, k)$	Restricted resolvable design.	94
T	Thymine.	22
$TC(k, g : n)$	Transversal cover.	4
$TD(k)$	The set of g such that there exists a $TD(k, g)$.	3
$TD(k, g)$	Transversal design.	3
$TP(k, g : n)$	Transversal packing.	7
$tc(k, g : n)$	Smallest b, fixing k in a transversal cover.	5
$tp(k, g : n)$	Largest b, fixing k, of a transversal packing.	7
$(v, \mathcal{K}, 1)$-cover	Pair covering incidence structure.	2
$(v, \mathcal{K}, 1)$-design	Pairwise balanced design.	2
$(v, \mathcal{K}, 1)$-packing	Pair packing incidence structure.	2
v	Number of points in an incidence structure.	4
$\mathcal{V}$	The point set of an incidence structure.	2
$\overline{\omega}(G)$	Minimum clique cover of the complement, $\overline{G}$.	11
$\omega(G)$	Maximum clique in a graph, G.	120
$\chi(G)$	Minimum vertex colouring of a graph, G.	120

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