

Verma modules for Yangians

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Abstract

We study the Verma modules $M(\mu(u))$ over the Yangian $Y(\mathfrak{a})$ associated with a simple Lie algebra \mathfrak{a} . We give necessary and sufficient conditions for irreducibility of $M(\mu(u))$. Moreover, regarding the simple quotient $L(\mu(u))$ of $M(\mu(u))$ as an \mathfrak{a} -module, we give necessary and sufficient conditions for finite-dimensionality of the weight subspaces of $L(\mu(u))$.

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1 Introduction

For a simple Lie algebra \mathfrak{a} over \mathbb{C} consider the corresponding Yangian $Y(\mathfrak{a})$; see Drinfeld [5, 6] and the definition in Section 2 below. Let n be the rank of the Lie algebra \mathfrak{a} . Given any n -tuple $\mu(u) = (\mu_1(u), \dots, \mu_n(u))$ of formal series

$$\mu_i(u) = 1 + \mu_i^{(0)}u^{-1} + \mu_i^{(1)}u^{-2} + \dots, \quad \mu_i^{(r)} \in \mathbb{C}, \quad (1.1)$$

the Verma module $M(\mu(u))$ over $Y(\mathfrak{a})$ is defined in a standard way as the quotient of $Y(\mathfrak{a})$ by a left ideal; see Section 2. The n -tuple $\mu(u)$ is called the highest weight of $M(\mu(u))$. A standard argument shows that $M(\mu(u))$ has a unique simple quotient $L(\mu(u))$. By a theorem of Drinfeld [6], every finite-dimensional irreducible representation of the Yangian $Y(\mathfrak{a})$ is isomorphic to $L(\mu(u))$ for an appropriate highest weight $\mu(u)$. Moreover, the same theorem gives necessary and sufficient conditions for the representation $L(\mu(u))$ to be finite-dimensional.

In this paper we are concerned with the necessary and sufficient conditions on the highest weight $\mu(u)$ for the Verma module $M(\mu(u))$ to be irreducible. Our first main result is the following.

Theorem 1.1. *The Verma module $M(\mu(u))$ over the Yangian $Y(\mathfrak{a})$ is reducible if and only if for some index $i \in \{1, \dots, n\}$ the series $\mu_i(u)$ is the Laurent expansion at $u = \infty$ of a rational function in u ,*

$$\mu_i(u) = \frac{P(u)}{Q(u)},$$

where $P(u)$ and $Q(u)$ are monic polynomials in u of the same degree.

This theorem will be proved in Section 3. We would like to note a rather unexpected difference between these reducibility conditions and those for the Verma modules over the Lie algebra \mathfrak{a} itself. It is well-known (see e.g. Dixmier [4, Chapter 7]) that the reducibility conditions for a Verma module over \mathfrak{a} involve arbitrary positive roots of \mathfrak{a} , while the conditions of Theorem 1.1 only involve the simple roots labelled by the indices $i = 1, \dots, n$.

Since the Yangian $Y(\mathfrak{a})$ contains the universal enveloping algebra $U(\mathfrak{a})$ as a subalgebra, we may regard both $M(\mu(u))$ and $L(\mu(u))$ as \mathfrak{a} -modules. Each of these modules admits a weight space decomposition with respect to the Cartan subalgebra of \mathfrak{a} . It is immediate from the definition of $M(\mu(u))$ that all its weight subspaces are infinite-dimensional, except for the one spanned by the highest vector. The following is our second main theorem.

Theorem 1.2. *All weight subspaces of the \mathfrak{a} -module $L(\mu(u))$ are finite-dimensional if and only if for each index $i \in \{1, \dots, n\}$ the series $\mu_i(u)$ is the Laurent expansion at $u = \infty$ of a rational function in u ,*

$$\mu_i(u) = \frac{P_i(u)}{Q_i(u)},$$

where $P_i(u)$ and $Q_i(u)$ are monic polynomials in u of the same degree.

We prove this theorem in Section 4. Note that by Drinfeld's theorem [6], the finite-dimensional modules $L(\mu(u))$ correspond to the case where for each $i = 1, \dots, n$ we have $P_i(u) = Q_i(u + d_i)$ for some positive integers d_i defined in the next section; see also [3, Theorem 12.1.11]. We also note similarity of Theorem 1.2 with the recent work of Billig and Zhao [2] where the finite-dimensionality conditions for the weight subspaces are found for a wide class of representations of the exp-polynomial Lie algebras.

2 Definitions and preliminaries

As before, we let \mathfrak{a} denote a finite-dimensional simple Lie algebra over \mathbb{C} and let n be the rank of \mathfrak{a} . Let $A = (a_{ij})$ be the Cartan matrix of \mathfrak{a} . The positive integers d_1, \dots, d_n are determined by the condition that they are coprime and the product DA is a symmetric matrix, where $D = \text{diag}(d_1, \dots, d_n)$. Following Drinfeld [6] (see also [3, Chapter 12]), we define the *Yangian* $Y(\mathfrak{a})$ as the associative algebra with generators $e_i^{(r)}, h_i^{(r)}, f_i^{(r)}$ where $i = 1, \dots, n$ and $r = 0, 1, 2, \dots$, and the following defining relations

$$\begin{aligned} [h_i^{(r)}, h_j^{(s)}] &= 0, & [e_i^{(r)}, f_j^{(s)}] &= \delta_{ij} h_i^{(r+s)}, & (2.1) \\ [h_i^{(0)}, e_j^{(s)}] &= d_i a_{ij} e_j^{(s)}, & [h_i^{(0)}, f_j^{(s)}] &= -d_i a_{ij} f_j^{(s)}, \\ [h_i^{(r+1)}, e_j^{(s)}] - [h_i^{(r)}, e_j^{(s+1)}] &= \frac{1}{2} d_i a_{ij} (h_i^{(r)} e_j^{(s)} + e_j^{(s)} h_i^{(r)}), \\ [h_i^{(r+1)}, f_j^{(s)}] - [h_i^{(r)}, f_j^{(s+1)}] &= -\frac{1}{2} d_i a_{ij} (h_i^{(r)} f_j^{(s)} + f_j^{(s)} h_i^{(r)}), \\ [e_i^{(r+1)}, e_j^{(s)}] - [e_i^{(r)}, e_j^{(s+1)}] &= \frac{1}{2} d_i a_{ij} (e_i^{(r)} e_j^{(s)} + e_j^{(s)} e_i^{(r)}), \\ [f_i^{(r+1)}, f_j^{(s)}] - [f_i^{(r)}, f_j^{(s+1)}] &= -\frac{1}{2} d_i a_{ij} (f_i^{(r)} f_j^{(s)} + f_j^{(s)} f_i^{(r)}), \\ \text{Sym} [e_i^{(r_1)}, [e_i^{(r_2)}, \dots, [e_i^{(r_m)}, e_j^{(s)}] \dots]] &= 0, \\ \text{Sym} [f_i^{(r_1)}, [f_i^{(r_2)}, \dots, [f_i^{(r_m)}, f_j^{(s)}] \dots]] &= 0, \end{aligned}$$

where the last two relations hold for all pairs $i \neq j$ with $m = 1 - a_{ij}$, and Sym denotes symmetrization with respect to the indices r_1, \dots, r_m .

The Yangian $Y(\mathfrak{a})$ admits a filtration defined by setting the degree of $e_i^{(r)}, h_i^{(r)}$ and $f_i^{(r)}$ to be equal to r . For any nonnegative integer r we let $Y(\mathfrak{a})_r$ denote the subspace of $Y(\mathfrak{a})$ spanned by the monomials of degree at most r in the generators. We shall denote the associated graded algebra by $\text{gr } Y(\mathfrak{a})$.

The universal enveloping algebra $U(\mathfrak{a})$ can be identified with the subalgebra of $Y(\mathfrak{a})$ generated by the elements $e_i^{(0)}, h_i^{(0)}, f_i^{(0)}$ where $i = 1, \dots, n$, so that $U(\mathfrak{a})$ coincides with $Y(\mathfrak{a})_0$. Let us set $h_i = h_i^{(0)}$. The linear span \mathfrak{h} of the elements h_1, \dots, h_n is a Cartan subalgebra of the Lie algebra \mathfrak{a} . We take the elements $e_i^{(0)}$ to be the simple root vectors of \mathfrak{a} with respect to \mathfrak{h} and set $e_{\alpha_i} = e_i^{(0)}$ and $f_{\alpha_i} = f_i^{(0)}$ for $i = 1, \dots, n$, where $\alpha_1, \dots, \alpha_n$ denote the simple roots. Let Δ^+ denote the set of positive roots.

We shall need an analog of the Poincaré–Birkhoff–Witt theorem for the algebra $Y(\mathfrak{a})$ proved by Levendorskii [7]. Let $\alpha = \alpha_{i_1} + \dots + \alpha_{i_p}$ be a decomposition of a positive root $\alpha \in \Delta^+$ into a sum of simple roots such that

$$e_\alpha = [e_{\alpha_{i_1}}, [e_{\alpha_{i_2}}, \dots, [e_{\alpha_{i_{p-1}}}, e_{\alpha_{i_p}}] \dots]]$$

is a nonzero root vector corresponding to α , and

$$f_\alpha = [f_{\alpha_{i_1}}, [f_{\alpha_{i_2}}, \dots, [f_{\alpha_{i_{p-1}}}, f_{\alpha_{i_p}}] \dots]]$$

is a nonzero root vector corresponding to $-\alpha$. Given a decomposition of a nonnegative integer r into a sum of nonnegative integers $r = r_1 + \dots + r_p$ set

$$e_\alpha^{(r)} = [e_{i_1}^{(r_1)}, [e_{i_2}^{(r_2)}, \dots, [e_{i_{p-1}}^{(r_{p-1})}, e_{i_p}^{(r_p)}] \dots]]$$

and

$$f_\alpha^{(r)} = [f_{i_1}^{(r_1)}, [f_{i_2}^{(r_2)}, \dots, [f_{i_{p-1}}^{(r_{p-1})}, f_{i_p}^{(r_p)}] \dots]].$$

For any positive root α the images of the elements $e_\alpha^{(r)}$ and $f_\alpha^{(r)}$ in the r -th component of the graded algebra $\text{gr } Y(\mathfrak{a})$ are independent of the choice of partition of r : if $\tilde{e}_\alpha^{(r)}$ is an element obtained using a different partition, then $e_\alpha^{(r)} - \tilde{e}_\alpha^{(r)} \in Y(\mathfrak{a})_{r-1}$. The same property is shared by the elements $f_\alpha^{(r)}$. Given any total ordering on the set

$$\{e_\alpha^{(r)} \mid \alpha \in \Delta^+, r \geq 0\} \cup \{f_\alpha^{(r)} \mid \alpha \in \Delta^+, r \geq 0\} \cup \{h_i^{(r)} \mid i = 1, \dots, n, r \geq 0\},$$

the ordered monomials in the elements of this set form a basis of $Y(\mathfrak{a})$; see [7]. This implies that the associated graded algebra $\text{gr } Y(\mathfrak{a})$ is isomorphic to the universal enveloping algebra $U(\mathfrak{a} \otimes \mathbb{C}[x])$. The images of the elements $e_\alpha^{(r)}, f_\alpha^{(r)}$ and $h_i^{(r)}$ in the r -th component of $\text{gr } Y(\mathfrak{a})$ can be identified with $e_\alpha x^r, f_\alpha x^r$ and $h_i x^r$, respectively.

Given any n -tuple $\mu(u) = (\mu_1(u), \dots, \mu_n(u))$ of formal series (1.1) define the Verma module $M(\mu(u))$ to be the quotient of $Y(\mathfrak{a})$ by the left ideal generated by the elements $e_i^{(r)}$ and $h_i^{(r)} - \mu_i^{(r)}$ for $i = 1, \dots, n$ and $r \geq 0$. The image 1_μ of the element $1 \in Y(\mathfrak{a})$ in $M(\mu(u))$ is called the *highest vector* of $M(\mu(u))$ and the n -tuple $\mu(u)$ is its *highest weight*. By the Poincaré–Birkhoff–Witt theorem for the algebra $Y(\mathfrak{a})$, given any total ordering on the set $\{f_\alpha^{(r)} \mid \alpha \in \Delta^+, r \geq 0\}$, the ordered monomials

$$f_{\alpha^{(1)}}^{(r_1)} \cdots f_{\alpha^{(l)}}^{(r_l)} 1_\mu, \quad l \geq 0, \quad r_i \geq 0, \quad \alpha^{(i)} \in \Delta^+ \quad (2.2)$$

form a basis of $M(\mu(u))$. Regarded as an \mathfrak{a} -module, $M(\mu(u))$ has a weight space decomposition with respect to \mathfrak{h} ,

$$M(\mu(u)) = \bigoplus_{\eta} M(\mu(u))_{\eta}, \quad (2.3)$$

summed over n -tuples $\eta = (\eta_1, \dots, \eta_n)$, where

$$M(\mu(u))_{\eta} = \{y \in M(\mu(u)) \mid h_i y = \eta_i y, \quad i = 1, \dots, n\}.$$

The weight subspace $M(\mu(u))_{\mu^{(0)}}$ with $\mu^{(0)} = (\mu_1^{(0)}, \dots, \mu_n^{(0)})$ is one-dimensional and spanned by the highest vector 1_μ . All other nonzero weight subspaces correspond to the weights η of the form

$$\eta = \mu^{(0)} - k_1 \alpha_1 - \cdots - k_n \alpha_n, \quad (2.4)$$

where the k_i are nonnegative integers, not all of them are zero. These weight subspaces are infinite-dimensional.

The sum of all submodules of $M(\mu(u))$ which do not contain the highest vector 1_μ is the unique maximal submodule. We let $L(\mu(u))$ denote the unique irreducible quotient of $M(\mu(u))$. It inherits the weight space decomposition

$$L(\mu(u)) = \bigoplus_{\eta} L(\mu(u))_{\eta}$$

with respect to \mathfrak{h} .

3 Irreducibility criterion for the Verma module

We present the proof of Theorem 1.1 as a sequence of propositions. We start by considering the key case of the theorem when $\mathfrak{a} = \mathfrak{sl}_2$. The algebra $Y(\mathfrak{sl}_2)$ has two more presentations in addition to the one used in the Introduction; see Drinfeld [6]. Following [9], we shall use the realization of $Y(\mathfrak{sl}_2)$ as a subalgebra of the Yangian

$Y(\mathfrak{gl}_2)$. The latter is an associative algebra with generators $t_{ij}^{(1)}, t_{ij}^{(2)}, \dots$ where $i, j \in \{1, 2\}$, and the defining relations

$$[t_{ij}^{(r+1)}, t_{kl}^{(s)}] - [t_{ij}^{(r)}, t_{kl}^{(s+1)}] = t_{kj}^{(r)} t_{il}^{(s)} - t_{kj}^{(s)} t_{il}^{(r)}, \quad (3.1)$$

where $r, s = 0, 1, \dots$ and $t_{ij}^{(0)} = \delta_{ij}$. Introducing the formal generating series

$$t_{ij}(u) = \delta_{ij} + t_{ij}^{(1)} u^{-1} + t_{ij}^{(2)} u^{-2} + \dots \in Y(\mathfrak{gl}_2)[[u^{-1}]], \quad (3.2)$$

we can write (3.1) in the equivalent form

$$(u - v) [t_{ij}(u), t_{kl}(v)] = t_{kj}(u) t_{il}(v) - t_{kj}(v) t_{il}(u). \quad (3.3)$$

The system of relations (3.1) is also equivalent to the system

$$[t_{ij}^{(r)}, t_{kl}^{(s)}] = \sum_{a=1}^{\min(r,s)} \left(t_{kj}^{(a-1)} t_{il}^{(r+s-a)} - t_{kj}^{(r+s-a)} t_{il}^{(a-1)} \right). \quad (3.4)$$

Observe that the upper summation index $\min(r, s)$ can be replaced with r or s because the additional sum which may occur is automatically zero.

Let $\varphi(u)$ be any formal power series in u^{-1} with the leading term 1,

$$\varphi(u) = 1 + \varphi_1 u^{-1} + \varphi_2 u^{-2} + \dots, \quad \varphi_i \in \mathbb{C}.$$

Then the mapping

$$t_{ij}(u) \mapsto \varphi(u) t_{ij}(u) \quad (3.5)$$

defines an automorphism of the algebra $Y(\mathfrak{gl}_2)$. The Yangian $Y(\mathfrak{sl}_2)$ is isomorphic to the subalgebra of $Y(\mathfrak{gl}_2)$ which consists of the elements stable under all automorphisms (3.5). We shall identify $Y(\mathfrak{sl}_2)$ with this subalgebra by the following formulas,

$$\begin{aligned} e(u) &= t_{22}(u)^{-1} t_{12}(u), \\ f(u) &= t_{21}(u) t_{22}(u)^{-1}, \\ h(u) &= t_{11}(u) t_{22}(u)^{-1} - t_{21}(u) t_{22}(u)^{-1} t_{12}(u) t_{22}(u)^{-1}, \end{aligned} \quad (3.6)$$

where

$$\begin{aligned} e(u) &= \sum_{r=0}^{\infty} e_1^{(r)} u^{-r-1}, \\ f(u) &= \sum_{r=0}^{\infty} f_1^{(r)} u^{-r-1}, \\ h(u) &= 1 + \sum_{r=0}^{\infty} h_1^{(r)} u^{-r-1}. \end{aligned}$$

We have the following tensor product decomposition

$$Y(\mathfrak{gl}_2) = ZY(\mathfrak{gl}_2) \otimes Y(\mathfrak{sl}_2), \quad (3.7)$$

where $ZY(\mathfrak{gl}_2)$ denotes the center of $Y(\mathfrak{gl}_2)$. The series

$$\begin{aligned} \partial(u) &= t_{11}(u)t_{22}(u-1) - t_{21}(u)t_{12}(u-1) \\ &= 1 + \partial_1 u^{-1} + \partial_2 u^{-2} + \dots \end{aligned}$$

is called the *quantum determinant*. All its coefficients $\partial_1, \partial_2, \dots$ are central in $Y(\mathfrak{gl}_2)$, they are algebraically independent and generate the center [9, Theorem 2.13]. Note that the series $h(u)$ can also be written as

$$h(u) = t_{22}(u)^{-1} t_{22}(u-1)^{-1} \partial(u). \quad (3.8)$$

The Poincaré–Birkhoff–Witt basis for the algebra $Y(\mathfrak{gl}_2)$ has the following form [9, Theorem 1.22]: given an arbitrary total ordering on the set of generators $t_{ij}^{(r)}$, any element of the algebra $Y(\mathfrak{gl}_2)$ can be uniquely written as a linear combination of the ordered monomials in the generators.

Given any pair of formal series $\lambda_1(u), \lambda_2(u)$,

$$\lambda_i(u) = 1 + \lambda_i^{(1)} u^{-1} + \lambda_i^{(2)} u^{-2} + \dots, \quad \lambda_i^{(r)} \in \mathbb{C}, \quad (3.9)$$

the Verma module $M(\lambda_1(u), \lambda_2(u))$ over $Y(\mathfrak{gl}_2)$ is the quotient of $Y(\mathfrak{gl}_2)$ by the left ideal generated by the elements $t_{12}^{(r)}, t_{11}^{(r)} - \lambda_1^{(r)}$ and $t_{22}^{(r)} - \lambda_2^{(r)}$ for $r \geq 1$. By the Poincaré–Birkhoff–Witt theorem for the algebra $Y(\mathfrak{gl}_2)$, a basis of $M(\lambda_1(u), \lambda_2(u))$ is formed by the elements

$$t_{21}^{(r_1)} \cdots t_{21}^{(r_k)} 1_\lambda, \quad k \geq 0, \quad 1 \leq r_1 \leq \cdots \leq r_k, \quad (3.10)$$

where 1_λ is the image of the element $1 \in Y(\mathfrak{gl}_2)$ in $M(\lambda_1(u), \lambda_2(u))$. Note that the ordering of the factors $t_{21}^{(r_i)}$ in (3.10) is irrelevant, because, by the defining relations, $[t_{21}^{(r)}, t_{21}^{(s)}] = 0$ for any r and s .

Proposition 3.1. *The restriction of the $Y(\mathfrak{gl}_2)$ -module $M(\lambda_1(u), \lambda_2(u))$ to the subalgebra $Y(\mathfrak{sl}_2)$ is isomorphic to the Verma module $M(\mu(u))$, where the highest weight $\mu(u)$ is given by $\mu(u) = \lambda_1(u)/\lambda_2(u)$.*

Proof. By the formulas (3.6), we have

$$e(u) 1_\lambda = 0, \quad h(u) 1_\lambda = \mu(u) 1_\lambda$$

for $\mu(u) = \lambda_1(u)/\lambda_2(u)$. Therefore, we have a $Y(\mathfrak{sl}_2)$ -homomorphism $\Phi : M(\mu(u)) \rightarrow M(\lambda_1(u), \lambda_2(u))$ such that $\Phi(1_\mu) = 1_\lambda$. In order to see that Φ is surjective let us verify

that $M(\lambda_1(u), \lambda_2(u))$ is generated by 1_λ as a $Y(\mathfrak{sl}_2)$ -module. Indeed, this follows from the decomposition (3.7) because the elements of $ZY(\mathfrak{gl}_2)$ act on $M(\lambda_1(u), \lambda_2(u))$ as scalar operators. Furthermore, the Verma module $M(\mu(u))$ over $Y(\mathfrak{sl}_2)$ with $\mu(u) = \lambda_1(u)/\lambda_2(u)$ can be extended to a module over the Yangian $Y(\mathfrak{gl}_2)$ by defining the action of the quantum determinant $\partial(u)$ on $M(\mu(u))$ to be the scalar multiplication by the series $\lambda_1(u)\lambda_2(u-1)$. Then by (3.6) and (3.8) we have

$$t_{12}(u) 1_\mu = 0, \quad t_{ii}(u) 1_\mu = \lambda_i(u) 1_\mu, \quad i = 1, 2.$$

Hence, we have a $Y(\mathfrak{gl}_2)$ -homomorphism $\Psi : M(\lambda_1(u), \lambda_2(u)) \rightarrow M(\mu(u))$ such that $\Psi(1_\lambda) = 1_\mu$. The composition map $\Psi \circ \Phi$ is the identity map on $M(\mu(u))$ which shows that Φ is injective. \square

Corollary 3.2. *The Verma module $M(\lambda_1(u), \lambda_2(u))$ over $Y(\mathfrak{gl}_2)$ is irreducible if and only if the Verma module $M(\mu(u))$ with $\mu(u) = \lambda_1(u)/\lambda_2(u)$ over $Y(\mathfrak{sl}_2)$ is irreducible.*

Proof. This is immediate from Proposition 3.1. \square

Proposition 3.3. *Suppose that formal series $\lambda_1(u)$ and $\lambda_2(u)$ given by (3.9) are such that the ratio $\lambda_1(u)/\lambda_2(u)$ is the Laurent expansion at $u = \infty$ of a rational function in u . Then the Verma module $M(\lambda_1(u), \lambda_2(u))$ over $Y(\mathfrak{gl}_2)$ is reducible.*

Proof. We can write

$$\frac{\lambda_1(u)}{\lambda_2(u)} = \frac{P(u)}{Q(u)},$$

where $P(u)$ and $Q(u)$ are polynomials in u . Since the Laurent series $\lambda_1(u)/\lambda_2(u)$ does not contain positive powers of u and has constant term 1, we may assume that the polynomials $P(u)$ and $Q(u)$ are monic of the same degree, say, p . Note that the composition of the action of $Y(\mathfrak{gl}_2)$ on $M(\lambda_1(u), \lambda_2(u))$ with an automorphism of the form (3.5) defines a representation of $Y(\mathfrak{gl}_2)$ on the vector space $M(\lambda_1(u), \lambda_2(u))$ which is isomorphic to the Verma module $M(\varphi(u)\lambda_1(u), \varphi(u)\lambda_2(u))$. Obviously, this Verma module is reducible or irreducible simultaneously with $M(\lambda_1(u), \lambda_2(u))$. Hence, taking $\varphi(u) = \lambda_2(u)^{-1} u^{-p} Q(u)$, we may assume without loss of generality that both $\lambda_1(u)$ and $\lambda_2(u)$ are polynomials in u^{-1} of degree $\leq p$. Consider the vector subspace K of $M(\lambda_1(u), \lambda_2(u))$ which is spanned by the vectors of the form (3.10) where at least one of the indices r_i exceeds p . We claim that K is a submodule of $M(\lambda_1(u), \lambda_2(u))$. Indeed, it is obvious that K is stable under the action of the elements $t_{21}^{(r)}$. Let us now verify by induction on $k \geq 1$ that for any $r \geq 1$ and any positive integers r_i with $r_k \geq p + 1$ the element

$$t_{11}^{(r)} t_{21}^{(r_1)} \cdots t_{21}^{(r_k)} 1_\lambda \tag{3.11}$$

lies in K . By the defining relations (3.4), we have

$$[t_{11}^{(r)}, t_{21}^{(r_1)}] = \sum_{a=1}^{\min(r, r_1)} \left(t_{21}^{(a-1)} t_{11}^{(r+r_1-a)} - t_{21}^{(r+r_1-a)} t_{11}^{(a-1)} \right). \quad (3.12)$$

If $k = 1$ then $r_1 \geq p+1$ and so $r+r_1-a \geq p+1$. Since $t_{11}^{(r+r_1-a)} 1_\lambda = 0$, the statement is true. Suppose now that $k \geq 2$. The statement now follows from (3.12) and the induction hypothesis.

The same argument shows that the subspace K is stable under the action of the elements $t_{22}^{(r)}$ with $r \geq 1$. Here we use the following consequence of the relations (3.4) instead of (3.12):

$$[t_{22}^{(r)}, t_{21}^{(r_1)}] = \sum_{a=1}^{\min(r, r_1)} \left(t_{21}^{(r+r_1-a)} t_{22}^{(a-1)} - t_{21}^{(a-1)} t_{22}^{(r+r_1-a)} \right). \quad (3.13)$$

Finally, let us verify that the subspace K is stable under the action of the elements $t_{12}^{(r)}$ with $r \geq 1$. We shall show by induction on $k \geq 1$ that all elements of the form

$$t_{12}^{(r)} t_{21}^{(r_1)} \cdots t_{21}^{(r_k)} 1_\lambda, \quad (3.14)$$

where the r_i are positive integers and $r_k \geq p+1$, lie in K . By (3.4) we have

$$[t_{12}^{(r)}, t_{21}^{(r_1)}] = \sum_{a=1}^{\min(r, r_1)} \left(t_{22}^{(a-1)} t_{11}^{(r+r_1-a)} - t_{22}^{(r+r_1-a)} t_{11}^{(a-1)} \right). \quad (3.15)$$

Hence, if $k = 1$ the statement is true because $t_{12}^{(r)} 1_\lambda = 0$ and $t_{11}^{(s)} 1_\lambda = t_{22}^{(s)} 1_\lambda = 0$ for $s \geq p+1$. Suppose now that $k \geq 2$. Due to (3.15), the statement follows from the induction hypothesis and the fact that K is stable under the action of the elements $t_{11}^{(r)}$ and $t_{22}^{(r)}$ with $r \geq 1$. \square

Lemma 3.4. *Let*

$$\nu(u) = 1 + \nu^{(1)} u^{-1} + \nu^{(2)} u^{-2} + \cdots \quad (3.16)$$

be a formal series in u^{-1} with complex coefficients. Suppose that there exist a positive integer N and complex numbers c_0, \dots, c_m , not all zero, such that the coefficients $\nu^{(r)}$ of the formal series satisfy the recurrence relation

$$c_0 \nu^{(r)} + c_1 \nu^{(r+1)} + \cdots + c_m \nu^{(r+m)} = 0, \quad (3.17)$$

for all $r \geq N$. Then $\nu(u)$ is the Laurent expansion at $u = \infty$ of a rational function $P(u)/Q(u)$ in u , where $P(u)$ and $Q(u)$ are monic polynomials in u of the same degree.

Proof. Let us set

$$\tilde{\nu}(u) = \nu^{(N)} + \nu^{(N+1)}u^{-1} + \dots .$$

Then (3.17) implies

$$\tilde{\nu}(u) (c_0 + c_1 u + \dots + c_m u^m) = b_1 u + \dots + b_m u^m$$

for some coefficients b_i . Hence the series

$$\nu(u) = 1 + \nu^{(1)}u^{-1} + \nu^{(2)}u^{-2} + \dots + \nu^{(N-1)}u^{-N+1} + u^{-N} \tilde{\nu}(u)$$

clearly has the desired form. \square

Proposition 3.5. *Suppose that the Verma module $M(\lambda_1(u), \lambda_2(u))$ over $Y(\mathfrak{gl}_2)$ is reducible. Then the ratio $\lambda_1(u)/\lambda_2(u)$ is the Laurent expansion at $u = \infty$ of a rational function in u .*

Proof. By twisting the action of $Y(\mathfrak{gl}_2)$ on $M(\lambda_1(u), \lambda_2(u))$ by the automorphism (3.5) with $\varphi(u) = \lambda_1(u)^{-1}$, we obtain a module over $Y(\mathfrak{gl}_2)$ which is isomorphic to the Verma module $M(1, \nu(u))$ with $\nu(u) = \lambda_2(u)/\lambda_1(u)$. We shall be proving that if the Verma module $M(1, \nu(u))$ is reducible then the series $\nu(u)$ is the Laurent expansion at $u = \infty$ of a rational function in u .

Clearly, $M(1, \nu(u))$ is the direct sum of its \mathfrak{sl}_2 -weight subspaces,

$$M(1, \nu(u)) = \bigoplus_{\eta} M(1, \nu(u))_{\eta},$$

where

$$M(1, \nu(u))_{\eta} = \{y \in M(1, \nu(u)) \mid (t_{11}^{(1)} - t_{22}^{(1)})y = \eta y\}.$$

The nonzero weight subspaces correspond to the weights $\eta = -\nu^{(1)} - 2k$, where k is a nonnegative integer and the coefficients of the series $\nu(u)$ are defined as in (3.16). The monomials (3.10) with fixed k form a basis of $M(1, \nu(u))_{\eta}$ with $\eta = -\nu^{(1)} - 2k$. If K is a nontrivial submodule of $M(1, \nu(u))$ then K inherits the weight space decomposition,

$$K = \bigoplus_{\eta} K_{\eta}, \quad K_{\eta} = K \cap M(1, \nu(u))_{\eta}.$$

Take the minimum positive integer k such that $K_{\eta} \neq 0$ for $\eta = -\nu^{(1)} - 2k$. Then any nonzero vector $\zeta \in K_{\eta}$ has the property $t_{12}(u)\zeta = 0$ since otherwise $t_{12}^{(r)}\zeta$ would be a nonzero element of the subspace $K_{\eta+2}$ for certain $r \geq 1$. Write

$$\zeta = \sum_{\mathbf{r}} c_{\mathbf{r}} t_{21}^{(r_1)} \cdots t_{21}^{(r_k)} 1_{\lambda}, \quad (3.18)$$

summed over a finite set of k -tuples $\mathbf{r} = (r_1, \dots, r_k)$ with $1 \leq r_1 \leq \dots \leq r_k$. Calculating $t_{12}^{(r)} \zeta$ as a linear combination of the basis vectors of $M(1, \nu(u))$ we get a family of linear relations on the coefficients $\nu^{(s)}$. It will be sufficient to demonstrate that at least one of these relations has the form (3.17) where not all coefficients are zero. The application of Lemma 3.4 will then complete the argument.

The action of $t_{12}^{(r)}$ on a basis monomial is calculated by

$$\begin{aligned} t_{12}^{(r)} t_{21}^{(r_1)} \cdots t_{21}^{(r_k)} 1_\lambda &= \sum_{i=1}^k t_{21}^{(r_1)} \cdots t_{21}^{(r_{i-1})} \\ &\times \sum_{a_i=1}^{r_i} \left(t_{22}^{(a_i-1)} t_{11}^{(r+r_i-a_i)} - t_{22}^{(r+r_i-a_i)} t_{11}^{(a_i-1)} \right) t_{21}^{(r_{i+1})} \cdots t_{21}^{(r_k)} 1_\lambda. \end{aligned} \quad (3.19)$$

In order to write this expression as a linear combination of basis monomials, we proceed by induction on k with the use of similar formulas for the actions of $t_{11}^{(r)}$ and $t_{22}^{(r)}$ with $r \geq 1$ on the basis monomials,

$$\begin{aligned} t_{11}^{(r)} t_{21}^{(r_1)} \cdots t_{21}^{(r_k)} 1_\lambda &= \sum_{i=1}^k t_{21}^{(r_1)} \cdots t_{21}^{(r_{i-1})} \\ &\times \sum_{a_i=1}^{r_i} \left(t_{21}^{(a_i-1)} t_{11}^{(r+r_i-a_i)} - t_{21}^{(r+r_i-a_i)} t_{11}^{(a_i-1)} \right) t_{21}^{(r_{i+1})} \cdots t_{21}^{(r_k)} 1_\lambda \end{aligned}$$

and

$$\begin{aligned} t_{22}^{(r)} t_{21}^{(r_1)} \cdots t_{21}^{(r_k)} 1_\lambda &= \nu^{(r)} t_{21}^{(r_1)} \cdots t_{21}^{(r_k)} 1_\lambda + \sum_{i=1}^k t_{21}^{(r_1)} \cdots t_{21}^{(r_{i-1})} \\ &\times \sum_{a_i=1}^{r_i} \left(t_{21}^{(r+r_i-a_i)} t_{22}^{(a_i-1)} - t_{21}^{(a_i-1)} t_{22}^{(r+r_i-a_i)} \right) t_{21}^{(r_{i+1})} \cdots t_{21}^{(r_k)} 1_\lambda, \end{aligned}$$

where we have used our assumptions on the highest weight, so that

$$t_{11}^{(r)} 1_\lambda = 0 \quad \text{and} \quad t_{22}^{(r)} 1_\lambda = \nu^{(r)} 1_\lambda, \quad r \geq 1.$$

Using the above formulas, we write (3.19) as a linear combination of the monomials $t_{21}^{(s_1)} \cdots t_{21}^{(s_{k-1})} 1_\lambda$. Suppose that N is the maximum sum of the indices $r_1 + \dots + r_k$ such that the corresponding monomial occurs in (3.18) with a nonzero coefficient $c_{\mathbf{r}}$. It will be sufficient for our purposes to consider only values $r \geq N$ and take the coefficient in the expansion of $t_{12}^{(r)} \zeta$ at a monomial of the form $t_{21}^{(s_1)} \cdots t_{21}^{(s_{k-1})} 1_\lambda$ with $1 \leq s_1 \leq \dots \leq s_{k-1}$ and satisfying $s_1 + \dots + s_{k-1} \leq N - 1$. It is not difficult to see from the above formulas that this coefficient can be written as a linear combination

of $\nu^{(r)}, \nu^{(r+1)}, \dots, \nu^{(r+N-k)}$. Moreover, the coefficient at each $\nu^{(r+i)}$ is, in its turn, a linear combination of the coefficients $c_{\mathbf{r}}$, and this combination is independent of the value of the index $r \geq N$. In order to apply Lemma 3.4, we need to verify that at least one of the coefficients at the $\nu^{(r+i)}$ is nonzero. Let us calculate the coefficient at $\nu^{(r+s-1)}$ where s is such that $s_1 + \dots + s_{k-1} + s = N$. This coefficient can only arise from the expansion of (3.19) with $r_1 + \dots + r_k = N$. Furthermore, for each $i = 1, \dots, k$ the value of the summation index a_i in that formula must be equal to 1. Hence, keeping only the terms contributing to the desired coefficient, we can rewrite (3.19) as

$$t_{12}^{(r)} t_{21}^{(r_1)} \dots t_{21}^{(r_k)} 1_{\lambda} \equiv - \sum_{i=1}^k \nu^{(r+r_i-1)} t_{21}^{(r_1)} \dots t_{21}^{(r_{i-1})} t_{21}^{(r_{i+1})} \dots t_{21}^{(r_k)} 1_{\lambda}.$$

Thus, the coefficient in question is

$$-c_{\mathbf{s}^{(1)}} - c_{\mathbf{s}^{(2)}} - \dots - c_{\mathbf{s}^{(k)}}, \quad (3.20)$$

where $\mathbf{s}^{(i)} = (s_1, \dots, s_{i-1}, s, s_i, \dots, s_{k-1})$ with $s_{i-1} \leq s \leq s_i$. We may write (3.20) as $-(m+1)c_{\mathbf{s}^{(i)}}$, where m is the number of indices s_j equal to s . However, by our assumption, there exists a k -tuple $\mathbf{r} = (r_1, \dots, r_k)$ with $r_1 + \dots + r_k = N$ and $c_{\mathbf{r}} \neq 0$. So, choosing the parameters s and s_1, \dots, s_{k-1} in such a way that $c_{\mathbf{s}^{(i)}} \neq 0$ for some i , we complete the proof. \square

Corollary 3.2 and Propositions 3.3 and 3.5 imply the following corollary which is a particular case of Theorem 1.1 for $\mathfrak{a} = \mathfrak{sl}_2$.

Corollary 3.6. *The Verma module $M(\mu(u))$ over the Yangian $Y(\mathfrak{sl}_2)$ is reducible if and only if the series $\mu(u)$ is the Laurent expansion at $u = \infty$ of a rational function in u .* \square

Corollary 3.7. *Let $M(\mu(u))$ be the Verma module over the Yangian $Y(\mathfrak{sl}_2)$, where $\mu(u) = P(u)/Q(u)$ for monic polynomials $P(u)$ and $Q(u)$ in u of degree p . Then for any $s \geq p$ there exist constants c_0, \dots, c_{s-1} such that the vector*

$$\zeta = c_0 f_1^{(0)} 1_{\mu} + \dots + c_{s-1} f_1^{(s-1)} 1_{\mu} + f_1^{(s)} 1_{\mu},$$

satisfies $e(u)\zeta = 0$.

Proof. By Proposition 3.1, the $Y(\mathfrak{sl}_2)$ -module $M(\mu(u))$ is isomorphic to the restriction of a $Y(\mathfrak{gl}_2)$ -module $M(\lambda_1(u), \lambda_2(u))$ such that $\lambda_1(u)/\lambda_2(u) = \mu(u)$. As in the proof of Proposition 3.3, we can find a series $\varphi(u)$ such that $\varphi(u)\lambda_1(u)$ and $\varphi(u)\lambda_2(u)$ are polynomials in u^{-1} of degree $\leq p$. For any $s \geq p$ the vector $\zeta = t_{21}^{(s+1)} 1_{\lambda}$ belongs to the

submodule K introduced in that proof. Hence, $t_{12}^{(r)} \zeta = 0$ for all $r \geq 1$, since otherwise the highest vector of the Verma module would belong to K , which is impossible. The image of the generator $t_{21}^{(s+1)}$ under an automorphism of the form (3.5) is given by $t_{21}^{(s+1)} + \varphi_1 t_{21}^{(s)} + \cdots + \varphi_s t_{21}^{(1)}$. Thus, the Verma module $M(\lambda_1(u), \lambda_2(u))$ over $Y(\mathfrak{gl}_2)$ contains a nonzero vector of the form

$$\zeta = c_1 t_{21}^{(1)} 1_\lambda + \cdots + c_s t_{21}^{(s)} 1_\lambda + t_{21}^{(s+1)} 1_\lambda, \quad c_i \in \mathbb{C},$$

such that $t_{12}^{(r)} \zeta = 0$ for all $r \geq 1$. The proof is completed by writing this vector in terms of the elements $f_1^{(r)}$ with the use of the isomorphism (3.6) and noting that $t_{12}(u) \zeta = 0$ implies $e(u) \zeta = 0$. \square

Remark 3.8. Nonzero vectors $\zeta \in M(\mu(u))$ satisfying $e(u) \zeta = 0$ are analogous to the singular vectors in the Verma modules over semisimple Lie algebras. Note, however, that in contrast with the case of Lie algebras, if $M(\mu(u))$ is reducible then, due to Corollary 3.7, the subspace spanned by the vectors ζ is infinite-dimensional. Moreover, ζ does not have to be an eigenvector for the action of $h(u)$. Therefore, ζ does not have to generate a submodule in $M(\mu(u))$ isomorphic to a Verma module over the Yangian $Y(\mathfrak{sl}_2)$. \square

We now turn to the case of the Yangian $Y(\mathfrak{a})$ for an arbitrary simple Lie algebra \mathfrak{a} over \mathbb{C} . We start by proving the “if” part of Theorem 1.1.

Proposition 3.9. *Let $\mu(u) = (\mu_1(u), \dots, \mu_n(u))$ be the highest weight of the Verma module $M(\mu(u))$ over the Yangian $Y(\mathfrak{a})$ such that for some $i \in \{1, \dots, n\}$ the series $\mu_i(u)$ is the Laurent expansion at $u = \infty$ of a rational function in u . Then the module $M(\mu(u))$ is reducible.*

Proof. Consider the subalgebra of $Y(\mathfrak{a})$ generated by the elements $e_i^{(r)}, h_i^{(r)}, f_i^{(r)}$ with $r = 0, 1, 2, \dots$. By the Yangian defining relations and the Poincaré–Birkhoff–Witt theorem for $Y(\mathfrak{a})$, this subalgebra is isomorphic to the Yangian $Y(\mathfrak{sl}_2)$. Moreover, the $Y(\mathfrak{sl}_2)$ -span of the highest vector 1_μ of $M(\mu(u))$ is isomorphic to the Verma module $M(\mu_i(u))$ over $Y(\mathfrak{sl}_2)$. By Corollaries 3.6 and 3.7, this module contains a nonzero vector of the form

$$\zeta = c_0 f_i^{(0)} 1_\mu + \cdots + c_p f_i^{(p)} 1_\mu,$$

such that $e_i(u) \zeta = 0$. By the relations (2.1), we have $e_j(u) \zeta = 0$ for all $j = 1, \dots, n$. The Poincaré–Birkhoff–Witt theorem for $Y(\mathfrak{a})$ implies that the submodule $Y(\mathfrak{a}) \zeta$ of $M(\mu(u))$ does not contain the highest vector 1_μ . Thus, $M(\mu(u))$ is reducible. \square

The definition of the Yangian $Y(\mathfrak{a})$ can be extended to the case of semisimple Lie algebras \mathfrak{a} ; see Section 2. If $\mathfrak{a} = \mathfrak{a}_1 \oplus \cdots \oplus \mathfrak{a}_p$ is a decomposition of \mathfrak{a} into the

direct sum of simple ideals, then extending the Poincaré–Birkhoff–Witt theorem to the Yangian $Y(\mathfrak{a})$ as in [7], one can show that $Y(\mathfrak{a})$ is isomorphic to the tensor product of the Yangians $Y(\mathfrak{a}_1), \dots, Y(\mathfrak{a}_p)$.

The proof of Theorem 1.1 is completed by the following proposition which is the “only if” part of the theorem. We assume here that \mathfrak{a} is a semisimple Lie algebra.

Proposition 3.10. *Suppose that the Verma module $M(\mu(u))$ over the Yangian $Y(\mathfrak{a})$ is reducible. Then there exists an index $i \in \{1, \dots, n\}$ such that the i -th component $\mu_i(u)$ of the highest weight is the Laurent expansion at $u = \infty$ of a rational function in u .*

Proof. Consider the weight subspace decomposition (2.3) of $M(\mu(u))$ with respect to the Cartan subalgebra \mathfrak{h} of \mathfrak{a} . Let us equip the set of weights with the standard partial ordering: η precedes η' if $\eta' - \eta$ is a linear combination of the positive roots with nonnegative integer coefficients. Any nontrivial submodule K of $M(\mu(u))$ inherits the weight space decomposition. Taking a nonzero element of the weight subspace of K with a maximal weight η we conclude that $M(\mu(u))$ must contain a nonzero weight vector ζ such that

$$e_i(u)\zeta = 0, \quad i = 1, \dots, n.$$

Let us write the weight η in the form (2.4) and suppose that $k_1 = \dots = k_{i-1} = 0$ and $k_i > 0$ for some index i . We shall be proving that the i -th component $\mu_i(u)$ of the highest weight has the desired property. We may assume without loss of generality that $i = 1$. Indeed, otherwise we can consider the Yangian for the semisimple Lie algebra \mathfrak{a}' associated with the Cartan matrix obtained from (a_{ij}) by deleting the first $i-1$ rows and columns. The vector ζ may then be regarded as an element of the Verma module $M(\mu'(u))$ over $Y(\mathfrak{a}')$ with the highest weight $\mu'(u) = (\mu_i(u), \dots, \mu_n(u))$.

Let us fix a total ordering \prec on the set of positive roots Δ^+ such that α_i precedes α_j if $i > j$ and any composite root precedes any simple root. Now consider the ordering on the set $\{f_\alpha^{(r)} \mid \alpha \in \Delta^+, r \geq 0\}$ defined by the rule: $f_\alpha^{(r)}$ precedes $f_\beta^{(s)}$ if $\alpha \prec \beta$, or $\alpha = \beta$ and $r < s$. Let us write ζ as a linear combination of the basis vectors (2.2) with the chosen ordering. We claim that none of the generators $f_\alpha^{(r)}$ corresponding to a composite root α can occur in the expansion of ζ . In other words, our claim is that ζ is a linear combination of monomials of the form

$$f_{\alpha_n}^{(p_1)} \dots f_{\alpha_n}^{(p_{k_n})} \dots f_{\alpha_1}^{(q_1)} \dots f_{\alpha_1}^{(q_{k_1})} 1_\mu. \quad (3.21)$$

Write $\zeta = \zeta_0 + \dots + \zeta_m$, where ζ_k is a linear combination of basis monomials (2.2) of degree k . We shall prove by a reverse induction on k that ζ_k is a linear combination of monomials of the form (3.21). Let β be a positive root such that $\gamma := \beta - \alpha_1$ is a

root. Suppose that s is the maximum nonnegative integer such that the element $f_\beta^{(s)}$ occurs in the expansion of ζ_k . By the construction of the elements $f_\alpha^{(r)}$, we have

$$[e_{\alpha_1}^{(r)}, f_\beta^{(s)}] \equiv c f_\gamma^{(r+s)},$$

modulo elements of degree smaller than $r+s$, where c is a nonzero constant. Therefore, the expansion of $e_{\alpha_1}^{(r)} \zeta$ into a linear combination of basis monomials will contain a monomial of degree $r+k$ where $f_\gamma^{(r+s)}$ occurs as a factor. By the induction hypothesis, the components $\zeta_{k+1}, \dots, \zeta_m$ of ζ are linear combinations of monomials of the form (3.21). Hence, for a sufficiently large r the expansion of $e_{\alpha_1}^{(r)} \zeta_j$ with $j = k+1, \dots, m$ will not contain $f_\gamma^{(r+s)}$. Since $e_{\alpha_1}^{(r)} \zeta = 0$ for all $r \geq 0$, we come to a contradiction, as the monomial containing $f_\gamma^{(r+s)}$ with a sufficiently large r will occur in the expansion of $e_{\alpha_1}^{(r)} \zeta$ with a nonzero coefficient. So, if β is a positive root such that $\beta - \alpha_1$ is a root then none of the elements $f_\beta^{(s)}$ can occur in the expansion of ζ_k .

Similarly, if β is a composite positive root such that $\beta - \alpha_1$ is not a root, but $\beta - \alpha_2$ is a root, then we use the relations $e_{\alpha_2}^{(r)} \zeta = 0$ to show that $f_\beta^{(s)}$ cannot occur in the expansion of ζ_k either. Continuing in the same manner, we conclude that ζ_k may only contain the generators $f_\beta^{(s)}$ where β is a simple root thus proving the claim.

Thus, we may write the vector ζ as the sum of vectors of the form

$$f_{\alpha_n}^{(p_1)} \dots f_{\alpha_n}^{(p_{k_n})} \dots f_{\alpha_2}^{(s_1)} \dots f_{\alpha_2}^{(s_{k_2})} \zeta',$$

where each vector ζ' is a linear combination of monomials $f_{\alpha_1}^{(q_1)} \dots f_{\alpha_1}^{(q_{k_1})} 1_\mu$ and at least one of the vectors is nonzero. However, the relations $e_{\alpha_1}^{(r)} \zeta = 0$ imply $e_{\alpha_1}^{(r)} \zeta' = 0$ for each vector ζ' . Recall that the elements $e_{\alpha_1}^{(r)}, f_{\alpha_1}^{(r)}$ and $h_1^{(r)}$ generate a subalgebra of $Y(\mathfrak{a})$ isomorphic to the Yangian $Y(\mathfrak{sl}_2)$. So, the Verma module $M(\mu_1(u))$ over $Y(\mathfrak{sl}_2)$ is reducible. By Corollary 3.7, the series $\mu_1(u)$ is the Laurent expansion at $u = \infty$ of a rational function in u . \square

4 Weight subspaces of the irreducible quotient

We now prove Theorem 1.2. The following is the “only if” part of the theorem.

Proposition 4.1. *Suppose that all weight subspaces of the \mathfrak{a} -module $L(\mu(u))$ are finite-dimensional. Then each component $\mu_i(u)$ of the highest weight is the Laurent expansion at $u = \infty$ of a rational function in u .*

Proof. Given index $i \in \{1, \dots, n\}$ identify the subalgebra of $Y(\mathfrak{a})$ generated by the elements $e_i^{(r)}, f_i^{(r)}$ and $h_i^{(r)}$ for $r \geq 0$ with the Yangian $Y(\mathfrak{sl}_2)$. The $Y(\mathfrak{sl}_2)$ -span of the highest vector 1_μ in $L(\mu(u))$ is isomorphic to a quotient L of the Verma module

$M(\mu_i(u))$ over $Y(\mathfrak{sl}_2)$. By our assumptions, all weight subspaces of L , regarded as an \mathfrak{sl}_2 -module, are finite-dimensional. Therefore, the module $M(\mu_i(u))$ is reducible. The proof is completed by the application of Corollary 3.6. \square

The following proposition completes the proof of Theorem 1.2.

Proposition 4.2. *Suppose that each component $\mu_i(u)$ of the highest weight of the $Y(\mathfrak{a})$ -module $L(\mu(u))$ is the Laurent expansion at $u = \infty$ of a rational function in u . Then all weight subspaces of the \mathfrak{a} -module $L(\mu(u))$ are finite-dimensional.*

Proof. Each component $\mu_i(u)$ is the Laurent expansion at $u = \infty$ of a rational function $P_i(u)/Q_i(u)$, where $P_i(u)$ and $Q_i(u)$ are monic polynomials in u of the same degree. Let p_i be the degree of these polynomials. For any positive root α denote by $[\alpha : \alpha_i]$ the multiplicity of the simple root α_i in α . We shall be proving that the monomials of the form (2.2) with the condition

$$r_j < \sum_{i=1}^n [\alpha^{(j)} : \alpha_i] p_i, \quad j = 1, \dots, k, \quad (4.1)$$

span the module $L(\mu(u))$. This clearly implies that each weight subspace $L(\mu(u))_\eta$ is finite-dimensional.

It suffices to show that each monomial (2.2) can be written in $L(\mu(u))$ as a linear combination of those monomials satisfying (4.1). We argue by induction on the degree r of the monomial. With this degree fixed, we also use induction on the length of the monomial. As the induction base, consider a monomial $f_\beta^{(r)} 1_\mu$ of degree r and length 1, where β is a positive root. There is nothing to prove unless r satisfies

$$r \geq \sum_{i=1}^n [\beta : \alpha_i] p_i. \quad (4.2)$$

In this case, by definition of the elements $f_\beta^{(r)}$, we can write, modulo terms of smaller degree,

$$f_\beta^{(r)} \equiv [f_{i_1}^{(r_{i_1})}, [f_{i_2}^{(r_{i_2})}, \dots, [f_{i_{l-1}}^{(r_{i_{l-1})}), f_{i_l}^{(r_{i_l})}] \dots]], \quad (4.3)$$

for a partition $r = r_{i_1} + \dots + r_{i_l}$ which due to (4.2) can be chosen in such a way that $r_{i_a} \geq p_{i_a}$ for all a . However, for any index $i \in \{1, \dots, n\}$ and any $r_i \geq p_i$ the monomial $f_i^{(r_i)} 1_\mu$ is a linear combination of monomials of smaller degree. Indeed, this follows from Corollary 3.7 by identifying the subalgebra of $Y(\mathfrak{a})$ generated by the elements $e_i^{(r)}$, $f_i^{(r)}$ and $h_i^{(r)}$ for $r \geq 0$ with the Yangian $Y(\mathfrak{sl}_2)$. Therefore, expanding the commutators in (4.3) we conclude that $f_\beta^{(r)} 1_\mu$ is a linear combination of monomials of smaller degree.

Now consider a monomial of the form (2.2) of degree r , where the condition (4.1) is violated for some j . It will suffice to assume that $j = 1$. Recall that modulo terms of smaller degree,

$$[f_\alpha^{(r)}, f_\beta^{(s)}] \equiv c f_{\alpha+\beta}^{(r+s)}, \quad c \in \mathbb{C},$$

if $\alpha + \beta$ is a root; otherwise the commutator is zero. Using this relation, we shall move the factor $f_{\alpha(1)}^{(r_1)}$ in (2.2) to the right so that the monomial will be equal, modulo terms of smaller degree or smaller length, to the monomial

$$f_{\alpha(2)}^{(r_2)} \cdots f_{\alpha(k)}^{(r_k)} f_{\alpha(1)}^{(r_1)} 1_\mu.$$

However, by the induction base, $f_{\alpha(1)}^{(r_1)} 1_\mu$ is zero, modulo terms of smaller degree. \square

Remark 4.3. In the case where all weight subspaces of $L(\mu(u))$ are finite-dimensional we can consider the character of the \mathfrak{a} -module $L(\mu(u))$,

$$\text{ch } L(\mu(u)) = \sum_{\eta} \dim L(\mu(u))_{\eta} \cdot e^{\eta},$$

where e^{η} is a formal exponential; see e.g. Dixmier [4, Chapter 7]. It would be interesting to find a formula for this character. Character formulas for finite-dimensional modules $L(\mu(u))$ are given e.g. in [1] and [11]. In the particular case $\mathfrak{a} = \mathfrak{sl}_2$, a character formula for $L(\mu(u))$ can be deduced from the tensor product decomposition for $L(\mu(u))$; see [10] and [8, Proposition 3.6]. More precisely, write the rational function $\mu(u)$ as

$$\mu(u) = \frac{(u + \alpha_1) \cdots (u + \alpha_k)}{(u + \beta_1) \cdots (u + \beta_k)}, \quad \alpha_i, \beta_i \in \mathbb{C}.$$

Renumbering the α_i and β_i if necessary, we may assume without loss of generality that for every $i = 1, \dots, k-1$ the following condition holds: if the multiset

$$\{\alpha_p - \beta_q \mid i \leq p, q \leq k\}$$

contains nonnegative integers, then $\alpha_i - \beta_i$ is minimal amongst them. Let l be the number of the indices i such that $\alpha_i - \beta_i$ is a nonnegative integer. Then

$$\text{ch } L(\mu(u)) = \prod_{i=1}^l \frac{x^{\alpha_i - \beta_i + 1} - x^{-\alpha_i + \beta_i - 1}}{x - x^{-1}} \prod_{i=l+1}^k \frac{x^{\alpha_i - \beta_i + 1}}{x - x^{-1}},$$

where $x = e^{\omega}$ and ω is the fundamental weight for \mathfrak{sl}_2 .

Remark 4.4. The results of this paper can be extended to obtain analogs of Theorems 1.1 and 1.2 for the quantum affine algebras $U_q(\widehat{\mathfrak{a}})$; see e.g. [5, 6] for their definition.

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