

Representations of toroidal extended affine Lie algebras.

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Abstract. We show that the representation theory for the toroidal extended affine Lie algebra is controlled by a VOA which is a tensor product of four VOAs: a sub-VOA V_{Hyp}^+ of a hyperbolic lattice VOA, affine $\widehat{\mathfrak{g}}$ and \widehat{sl}_N VOAs and a Virasoro VOA. A tensor product of irreducible modules for these VOAs admits the structure of an irreducible module for the toroidal extended affine Lie algebra. We also show that for $N = 12$, V_{Hyp}^+ becomes an exceptional irreducible module for the extended affine Lie algebra of rank 0.

Keywords: toroidal Lie algebras, extended affine Lie algebras.

0. Introduction.

In this paper we study representations of toroidal extended affine Lie algebras using recently developed representation theory for the full toroidal Lie algebras [B3]. Extended affine Lie algebras (EALAs) have been extensively studied during the last decade (see [N], [ABFP], [AABGP] and references therein). The main features of an extended affine Lie algebra is that it is graded by a finite root system and possesses a non-degenerate symmetric invariant bilinear form.

The construction of toroidal Lie algebras parallels one for affine algebras. We start with the Lie algebra of maps from an $N + 1$ -dimensional torus into a finite-dimensional simple Lie algebra \mathfrak{g} . This multi-loop algebra may be written as a tensor product $\mathcal{R} \otimes \mathfrak{g}$ of the algebra \mathcal{R} of Laurent polynomials in $N + 1$ variables with \mathfrak{g} . Next, we take the universal central extension $\mathcal{R} \otimes \mathfrak{g} \oplus \mathcal{K}$ of this multi-loop algebra and add a Lie algebra of vector fields on the torus, possibly twisted with a 2-cocycle (see section 1 for details). If we add all vector fields, we get the full toroidal Lie algebra. However, the full toroidal Lie algebra does not possess a non-degenerate invariant form, whereas its subalgebra with the divergence zero vector fields

$$\mathfrak{g}_{\text{div}} = (\mathcal{R} \otimes \mathfrak{g}) \oplus \mathcal{K} \oplus \mathcal{D}_{\text{div}}$$

does. We call this last algebra the toroidal extended affine Lie algebra.

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The representation theory of toroidal Lie algebras is best described in the framework of vertex operator algebras (VOAs). We prove in this paper that the vertex operator algebra that controls the representation theory of $\mathfrak{g}_{\text{div}}$ is a tensor product of an affine $\widehat{\mathfrak{g}}$ VOA, a sub-VOA of a hyperbolic lattice VOA V_{Hyp}^+ , affine $\widehat{\mathfrak{sl}}_N$ VOA and a Virasoro VOA. By taking a tensor product of irreducible modules for each of these VOAs we get irreducible modules for the toroidal extended affine algebra $\mathfrak{g}_{\text{div}}$ (Theorem 5.5). This gives an explicit realization for these modules and allows us to compute their characters.

The results of this paper can be used to construct irreducible modules for virtually all other EALAs [BL].

If we set $\mathfrak{g} = (0)$, we will get representations for the Lie algebra $\mathcal{D}_{\text{div}} \oplus \mathcal{K}$. This Lie algebra still possesses a non-degenerate symmetric invariant bilinear form and may be viewed as an EALA of rank 0. This Lie algebra plays an important role in magnetic hydrodynamics [VD], [B2]. We establish one curious fact about the representation theory of $\mathcal{D}_{\text{div}} \oplus \mathcal{K}$. When $N = 12$, this Lie algebra has an exceptional module with a particularly simple structure. Only for this value of N a hyperbolic lattice sub-VOA V_{Hyp}^+ can be given a structure of an irreducible module for $\mathcal{D}_{\text{div}} \oplus \mathcal{K}$. The character of this module is given by the -24 -th power of the Dedekind η -function and has nice modular properties.

This paper is a sequel to [B3], and we will be using constructions and notations from that paper. A weaker form of the results of this paper was presented in [B1] (unpublished).

This paper is structured as follows. In section 1 we review the construction of toroidal Lie algebras. In sections 2 and 3 we discuss vertex operator algebras that we will need in order to build the representation theory of the toroidal EALAs. In section 4 we recall the results on the representations of the full toroidal Lie algebras. The final section 5 contains the main results of this paper, there we develop the representation theory for toroidal extended affine Lie algebras.

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1. Toroidal Lie algebras.

Toroidal Lie algebras are the natural multi-variable generalizations of affine Lie algebras. In this review of the toroidal Lie algebras we follow the work [BB]. Let \mathfrak{g} be a simple finite-dimensional Lie algebra over \mathbb{C} with a non-degenerate invariant bilinear form $(\cdot|\cdot)$ and let $N \geq 1$ be an integer. We consider the Lie algebra $\mathcal{R} \otimes \mathfrak{g}$ of maps of an $N + 1$ dimensional torus into \mathfrak{g} , where $\mathcal{R} = \mathbb{C}[t_0^\pm, t_1^\pm, \dots, t_N^\pm]$ is the algebra of functions on a torus (in the Fourier basis). The universal central extension of this Lie algebra may be described by means of the following construction which is due to Kassel [Kas]. Let $\Omega_{\mathcal{R}}$ be the space of 1-forms on a torus: $\Omega_{\mathcal{R}} = \bigoplus_{p=0}^N \mathcal{R} dt_p$. We will choose the forms $\{k_p = t_p^{-1} dt_p \mid p = 0, \dots, N\}$ as a basis of this free \mathcal{R} module. There is a natural map d from the space of functions \mathcal{R} into $\Omega_{\mathcal{R}}$: $d(f) = \sum_{p=0}^N \frac{\partial f}{\partial t_p} dt_p = \sum_{p=0}^N t_p \frac{\partial f}{\partial t_p} k_p$. The center \mathcal{K} for the universal central extension $(\mathcal{R} \otimes \mathfrak{g}) \oplus \mathcal{K}$

of $\mathcal{R} \otimes \mathfrak{g}$ is realized as

$$\mathcal{K} = \Omega_{\mathcal{R}}/d(\mathcal{R}),$$

and the Lie bracket is given by the formula

$$[f_1(t)g_1, f_2(t)g_2] = f_1(t)f_2(t)[g_1, g_2] + (g_1|g_2)f_2d(f_1).$$

Here and in the rest of the paper we will denote elements of \mathcal{K} by the same symbols as elements of $\Omega_{\mathcal{R}}$, keeping in mind the canonical projection $\Omega_{\mathcal{R}} \rightarrow \Omega_{\mathcal{R}}/d(\mathcal{R})$.

Just as in affine case, we add to $(\mathcal{R} \otimes \mathfrak{g}) \oplus \mathcal{K}$ the algebra \mathcal{D} of outer derivations

$$\mathcal{D} = \bigoplus_{p=0}^N \mathcal{R}d_p,$$

where $d_p = t_p \frac{\partial}{\partial t_p}$. We will use the multi-index notations, $r = (r_0, r_1, \dots, r_N)$, $t^r = t_0^{r_0} t_1^{r_1} \dots t_N^{r_N}$.

The natural action of \mathcal{D} on $\mathcal{R} \otimes \mathfrak{g}$

$$[t^r d_a, t^m g] = m_a t^{r+m} g \tag{1.1}$$

uniquely extends to the action on the universal central extension $(\mathcal{R} \otimes \mathfrak{g}) \oplus \mathcal{K}$ by

$$[t^r d_a, t^m k_b] = m_a t^{r+m} k_b + \delta_{ab} \sum_{p=0}^N r_p t^{r+m} k_p. \tag{1.2}$$

This corresponds to the Lie derivative action of the vector fields on 1-forms.

It turns out that there is still an extra degree of freedom in defining the Lie algebra structure on $(\mathcal{R} \otimes \mathfrak{g}) \oplus \mathcal{K} \oplus \mathcal{D}$. The Lie bracket on \mathcal{D} may be twisted with a \mathcal{K} -valued 2-cocycle:

$$[t^r d_a, t^m d_b] = m_a t^{r+m} d_b - r_b t^{r+m} d_a + \mu m_a r_b \sum_{p=0}^N m_p t^{r+m} k_p. \tag{1.3}$$

As a result we get a family of full toroidal Lie algebras

$$\mathfrak{g}(\mu) = (\mathcal{R} \otimes \mathfrak{g}) \oplus \mathcal{K} \oplus \mathcal{D}.$$

Note that after adding the algebra of derivations \mathcal{D} , the center of the toroidal Lie \mathfrak{g} becomes finite-dimensional with the basis $\{k_0, k_1, \dots, k_N\}$. This can be seen from the action (1.2) of \mathcal{D} on \mathcal{K} , which is non-trivial.

The toroidal Lie algebra $\mathfrak{g}(\mu) = (\mathcal{R} \otimes \mathfrak{g}) \oplus \mathcal{K} \oplus \mathcal{D}$ has an important subalgebra $\mathfrak{g}_{\text{div}}(\mu)$ that has divergence free vector fields as the derivation part:

$$\mathfrak{g}_{\text{div}} = \mathfrak{g}_{\text{div}}(\mu) = (\mathcal{R} \otimes \mathfrak{g}) \oplus \mathcal{K} \oplus \mathcal{D}_{\text{div}},$$

where

$$\mathcal{D}_{\text{div}} = \left\{ \sum_{p=0}^N f_p(t) d_p \mid \sum_{p=0}^N t_p \frac{\partial f_p}{\partial t_p} = 0 \right\}.$$

The expression $i \sum_{p=0}^N t_p \frac{\partial f_p}{\partial t_p}$ becomes the divergence of a vector field in the angular coordinates (x_0, \dots, x_N) on a torus, where $t_j = e^{ix_j}$.

The importance of this subalgebra is explained by the fact that unlike the full toroidal Lie algebra, $\mathfrak{g}_{\text{div}}(\mu)$ is an extended affine Lie algebra [BGK], i.e., $\mathfrak{g}_{\text{div}}(\mu)$ has a non-degenerate symmetric invariant bilinear form. The restrictions of this form to both $\mathcal{R} \otimes \mathfrak{g}$ and to $\mathcal{D}_{\text{div}} \oplus \mathcal{K}$ are non-degenerate:

$$(t^r g_1 | t^m g_2) = \delta_{r,-m} (g_1 | g_2), \quad g_1, g_2 \in \mathfrak{g},$$

while the vector fields pair with the 1-forms:

$$\left(\sum_{p=0}^N a_p t^r d_p | t^m k_q \right) = \delta_{r,-m} a_q. \quad (1.4)$$

One can see that the above formula is ill-defined for the full \mathcal{D} , since $d(t^m) = \sum_{q=0}^N m_q t^m k_q$, being zero in \mathcal{K} , must be in the kernel of the form. For the subalgebra \mathcal{D}_{div} this is precisely the case since

$$\left(\sum_{p=0}^N a_p t^r d_p | \sum_{q=0}^N r_q t^{-r} k_q \right) = \sum_{q=0}^N a_q r_q = 0.$$

All other values of the bilinear form are trivial:

$$(\mathcal{R} \otimes \mathfrak{g} | \mathcal{D}_{\text{div}} \oplus \mathcal{K}) = 0, \quad (\mathcal{D}_{\text{div}} | \mathcal{D}_{\text{div}}) = 0, \quad (\mathcal{K} | \mathcal{K}) = 0.$$

It is easy to verify that the resulting symmetric bilinear form is invariant and non-degenerate.

The 2-cocycle that we consider here was originally introduced in [EM] and was denoted $\mu\tau_1$ in [BB], [B3]. There is another cocycle τ_2 used in [L], [BB], [B3]. However the cocycle τ_2 vanishes on \mathcal{D}_{div} , and thus plays no role in the present paper.

2. Hyperbolic lattice VOA.

Representations of toroidal Lie algebras are best described using the language of vertex algebras. Here we sketch the construction of a hyperbolic lattice VOA. The details can be found in [BBS] or [B3]. We refer to [K] for the definition and properties of vertex algebras.

Consider a hyperbolic lattice Hyp , which is a free abelian group on $2N$ generators $\{u_i, v_i | i = 1, \dots, N\}$ with the symmetric bilinear form

$$(\cdot | \cdot) : \quad Hyp \times Hyp \rightarrow \mathbb{Z},$$

defined by

$$(u_i | v_j) = \delta_{ij}, \quad (u_i | u_j) = (v_i | v_j) = 0.$$

Note that the form $(\cdot | \cdot)$ is non-degenerate and Hyp is an even lattice, i.e., $(x|x) \in 2\mathbb{Z}$.

Consider an infinite-dimensional Heisenberg algebra

$$\widehat{H} = (\mathbb{C}[t, t^{-1}] \otimes_{\mathbb{Z}} Hyp) \oplus \mathbb{C}K$$

with the bracket

$$[x(n), y(m)] = n(x|y)\delta_{n,-m}K, \quad x, y \in Hyp, \quad [\widehat{H}, K] = 0.$$

Here and in what follows, we are using the notation $x(n) = t^n \otimes x$. The natural \mathbb{Z} -grading of \widehat{H} yields the decomposition $\widehat{H} = \widehat{H}_- \oplus \widehat{H}_0 \oplus \widehat{H}_+$.

The hyperbolic lattice VOA V_{Hyp} is a tensor product of a twisted group algebra $\mathbb{C}[Hyp]$ of the lattice and the Fock space $S(\widehat{H}_-)$.

The elements of \widehat{H} of degree zero act on V_{Hyp} as follows:

$$x(0)e^y = (x|y)e^y, \quad Ke^y = e^y.$$

The Virasoro element in V_{Hyp} is $\omega_{Hyp} = \sum_{p=1}^N u_p(-1)v_p(-1)\mathbf{1}$, where $\mathbf{1} = e^0$ is the identity element of V_{Hyp} . The rank of V_{Hyp} is $2N$.

The Y -map is defined on the basis elements of $\mathbb{C}[Hyp]$ by

$$Y(e^x, z) = \exp\left(\sum_{j \geq 1} \frac{x(-j)}{j} z^j\right) \exp\left(-\sum_{j \geq 1} \frac{x(j)}{j} z^{-j}\right) e^x z^x,$$

where $z^x e^y = z^{(x|y)} e^y$.

For the generators of the Fock space, the Y -map is given by

$$Y(u_p(-1)\mathbf{1}, z) = u_p(z) = \sum_{j \in \mathbb{Z}} u_p(j) z^{-j-1}, \quad Y(v_p(-1)\mathbf{1}, z) = v_p(z) = \sum_{j \in \mathbb{Z}} v_p(j) z^{-j-1}.$$

In the construction of the toroidal VOAs we would need not V_{Hyp} itself, but its sub-VOA V_{Hyp}^+ :

$$V_{Hyp}^+ = S(\widehat{H}_-) \otimes \mathbb{C}[Hyp^+],$$

where Hyp^+ is the isotropic sublattice of Hyp generated by $\{u_i | i = 1, \dots, N\}$. Here $\mathbb{C}[Hyp^+]$ is the ordinary group algebra of Hyp^+ .

The Virasoro element of V_{Hyp}^+ is the same as in V_{Hyp} , and so the rank of V_{Hyp}^+ is also $2N$.

We will need a class of modules for V_{Hyp}^+ . Fix $\alpha \in \mathbb{C}^N, \beta \in \mathbb{Z}^N$. Then the space

$$M_{Hyp}^+(\alpha, \beta) = S(\widehat{H}_-) \otimes e^{\alpha u + \beta v} \mathbb{C}[Hyp^+]$$

has a structure of an irreducible VOA module for V_{Hyp}^+ [BBS], [B3]. Here we are using the notations $\alpha u = \alpha_1 u_1 + \dots + \alpha_N u_N$, etc.

3. VOA associated with the twisted Virasoro-affine algebra.

It has been shown in [B3] that the representation theory of the full toroidal Lie algebra is controlled by a VOA which is a tensor product of two VOAs – V_{Hyp}^+ described above and a twisted Virasoro-affine VOA $V_{\mathfrak{f}}(\gamma)$ which we are going to discuss now.

Let \mathfrak{f} be a finite-dimensional reductive Lie algebra.

Consider the semi-direct product of the Lie algebra of vector fields on a circle with a loop algebra:

$$\tilde{\mathfrak{f}} = \text{Der } \mathbb{C}[t_0, t_0^{-1}] \ltimes \left(\mathbb{C}[t_0, t_0^{-1}] \otimes \mathfrak{f} \right).$$

The Lie algebra $\tilde{\mathfrak{f}}$ is a perfect Lie algebra, and we consider its universal central extension \mathfrak{f} , which we call the *twisted Virasoro-affine algebra* corresponding to \mathfrak{f} .

In case when \mathfrak{f} is a trivial one dimensional Lie algebra, the structure of the universal central extension of $\tilde{\mathfrak{f}}$ was determined in [ACKP], Proposition 2.1.

In the present paper, we are interested in the case when $\mathfrak{f} = \mathfrak{g} \oplus gl_N$, which we fix for the rest of the paper. Using the well-known description of the universal central extensions for the loop algebras together with Theorem 1.5.2 of [F], one can modify the argument of Proposition 2.1 in [ACKP] to show that for $\tilde{\mathfrak{f}} = \mathfrak{g} \oplus gl_N$, the corresponding twisted Virasoro-affine algebra \mathfrak{f} is a 5-dimensional extension of $\tilde{\mathfrak{f}}$:

$$\mathfrak{f} = \tilde{\mathfrak{f}} \oplus \mathbb{C}C_{\mathfrak{g}} \oplus \mathbb{C}C_{sl_N} \oplus \mathbb{C}C_{\mathcal{H}ei} \oplus \mathbb{C}C_{\mathcal{V}\mathcal{H}} \oplus \mathbb{C}C_{\mathcal{V}ir}.$$

Let us describe this central extension in more detail.

The Lie algebra \mathfrak{f} contains four subalgebras – a Virasoro algebra $\mathcal{V}ir = \text{Der } \mathbb{C}[t_0, t_0^{-1}] \oplus \mathbb{C}C_{\mathcal{V}ir}$, with the bracket

$$[L(n), L(m)] = (n - m)L(n + m) + \frac{n^3 - n}{12} \delta_{n, -m} C_{\mathcal{V}ir},$$

two affine algebras, $\widehat{\mathfrak{g}} = \mathbb{C}[t_0, t_0^{-1}] \otimes \mathfrak{g} \oplus \mathbb{C}C_{\mathfrak{g}}$ and $\widehat{sl}_N = \mathbb{C}[t_0, t_0^{-1}] \otimes sl_N \oplus \mathbb{C}C_{sl_N}$, and an infinite-dimensional Heisenberg algebra $\mathcal{H}ei = \mathbb{C}[t_0, t_0^{-1}] \otimes I \oplus \mathbb{C}C_{\mathcal{H}ei}$, where I is the identity matrix in gl_N , and the bracket is given by

$$[I(n), I(m)] = n \delta_{n, -m} C_{\mathcal{H}ei}.$$

The action of the Virasoro algebra on affine subalgebras is the usual one, whereas for the Heisenberg algebra it is twisted with a cocycle:

$$[L(n), I(m)] = -mI(m + n) - (n^2 + n) \delta_{n, -m} C_{\mathcal{V}\mathcal{H}}.$$

Here $L(n) = -t_0^{n+1} \frac{d}{dt_0}$, $I(m) = t_0^m \otimes I$.

The Virasoro and the Heisenberg subalgebras together generate a subalgebra $\mathcal{H}\mathcal{V}ir$ in \mathfrak{f} called the twisted Virasoro-Heisenberg algebra:

$$\mathcal{H}\mathcal{V}ir = \text{Der } \mathbb{C}[t_0, t_0^{-1}] \oplus \left(\mathbb{C}[t_0, t_0^{-1}] \otimes I \right) \oplus \mathbb{C}C_{\mathcal{H}ei} \oplus \mathbb{C}C_{\mathcal{V}\mathcal{H}} \oplus \mathbb{C}C_{\mathcal{V}ir}.$$

In [B3] we have constructed a VOA $V_{\mathfrak{f}}(\gamma)$ associated with the twisted Virasoro-affine algebra \mathfrak{f} and a central character γ :

$$\gamma(C_{\mathfrak{g}}) = c_{\mathfrak{g}}, \quad \gamma(C_{sl_N}) = c_{sl_N}, \quad \gamma(C_{\mathcal{H}ei}) = c_{\mathcal{H}ei}, \quad \gamma(C_{\mathcal{V}\mathcal{H}}) = c_{\mathcal{V}\mathcal{H}}, \quad \gamma(C_{\mathcal{V}ir}) = c_{\mathcal{V}ir}.$$

For a finite-dimensional irreducible \mathfrak{g} -module V , a finite-dimensional irreducible sl_N -module W , and two constants $h_{\mathcal{V}ir}, h_{\mathcal{H}ei} \in \mathbb{C}$, we get a generalized Verma module $M_{\mathfrak{f}}(V, W, h_{\mathcal{H}ei}, h_{\mathcal{V}ir}, \gamma)$ and its irreducible quotient $L_{\mathfrak{f}}(V, W, h_{\mathcal{H}ei}, h_{\mathcal{V}ir}, \gamma)$ as modules for both \mathfrak{f} and the VOA $V_{\mathfrak{f}}(\gamma)$ (for details see section 3.4 in [B3]).

Let $\overline{\mathcal{V}ir}$ be another copy of the Virasoro algebra with the basis $\{\overline{L}(n), \overline{C}_{\mathcal{V}ir} \mid n \in \mathbb{Z}\}$. Consider a semidirect product $\overline{\mathfrak{f}}$ of $\overline{\mathcal{V}ir}$ with affine algebras $\widehat{\mathfrak{g}} \oplus \widehat{sl}_N$. We want to distinguish between $\mathcal{V}ir$ and $\overline{\mathcal{V}ir}$ because we will be using non-trivial embeddings of $\overline{\mathfrak{f}}$ into \mathfrak{f} (Lemma 5.1 below). A similar construction applied to $\overline{\mathfrak{f}}$ instead of \mathfrak{f} results in a VOA $V_{\overline{\mathfrak{f}}}(c_{\mathfrak{g}}, c_{sl_N}, \overline{c}_{\mathcal{V}ir})$, a generalized Verma module $M_{\overline{\mathfrak{f}}}(V, W, \overline{h}_{\mathcal{V}ir}, c_{\mathfrak{g}}, c_{sl_N}, \overline{c}_{\mathcal{V}ir})$ and the irreducible module $L_{\overline{\mathfrak{f}}}(V, W, \overline{h}_{\mathcal{V}ir}, c_{\mathfrak{g}}, c_{sl_N}, \overline{c}_{\mathcal{V}ir})$ for this VOA.

We will also be using the affine vertex algebras $V_{\widehat{\mathfrak{g}}}(c_{\mathfrak{g}})$, $V_{\widehat{sl}_N}(c_{sl_N})$, the Virasoro VOA $V_{\mathcal{V}ir}(c_{\mathcal{V}ir})$, twisted Virasoro-Heisenberg VOA $V_{\mathcal{H}\mathcal{V}ir}(c_{\mathcal{H}ei}, c_{\mathcal{V}\mathcal{H}}, c_{\mathcal{V}ir})$, their Verma modules and their irreducible highest weight modules $L_{\widehat{\mathfrak{g}}}(V, c_{\mathfrak{g}})$, $L_{\widehat{sl}_N}(W, c_{sl_N})$, $L_{\mathcal{V}ir}(h_{\mathcal{V}ir}, c_{\mathcal{V}ir})$ and $L_{\mathcal{H}\mathcal{V}ir}(h_{\mathcal{H}ei}, h_{\mathcal{V}ir}, c_{\mathcal{H}ei}, c_{\mathcal{V}\mathcal{H}}, c_{\mathcal{V}ir})$.

Applying the well-known Sugawara construction, we can decompose the VOA $V_{\overline{\mathfrak{f}}}(c_{\mathfrak{g}}, c_{sl_N}, \overline{c}_{\mathcal{V}ir})$ and its irreducible modules into tensor products:

Proposition 3.1. *Let $c_{\mathfrak{g}} \neq -h^{\vee}$, $c_{sl_N} \neq -N$, where h^{\vee} is the dual Coxeter number for $\widehat{\mathfrak{g}}$. Then*

(i) *the VOA $V_{\overline{\mathfrak{f}}}(c_{\mathfrak{g}}, c_{sl_N}, \overline{c}_{\mathcal{V}ir})$ decomposes into a tensor product of three VOAs:*

$$V_{\overline{\mathfrak{f}}}(c_{\mathfrak{g}}, c_{sl_N}, \overline{c}_{\mathcal{V}ir}) \cong V_{\widehat{\mathfrak{g}}}(c_{\mathfrak{g}}) \otimes V_{\widehat{sl}_N}(c_{sl_N}) \otimes V_{\mathcal{V}ir}(c'_{\mathcal{V}ir}),$$

where

$$c'_{\mathcal{V}ir} = \overline{c}_{\mathcal{V}ir} - \frac{c_{\mathfrak{g}} \dim(\mathfrak{g})}{c_{\mathfrak{g}} + h^{\vee}} - \frac{c_{sl_N}(N^2 - 1)}{c_{sl_N} + N}, \quad (3.1)$$

(ii) *the irreducible highest weight $\overline{\mathfrak{f}}$ -module $L_{\overline{\mathfrak{f}}}(V, W, \overline{h}_{\mathcal{V}ir}, c_{\mathfrak{g}}, c_{sl_N}, \overline{c}_{\mathcal{V}ir})$ decomposes into a tensor product of irreducible highest weight modules for the affine algebras $\widehat{\mathfrak{g}}$, \widehat{sl}_N and the Virasoro modules:*

$$L_{\overline{\mathfrak{f}}}(V, W, \overline{h}_{\mathcal{V}ir}, c_{\mathfrak{g}}, c_{sl_N}, \overline{c}_{\mathcal{V}ir}) \cong L_{\widehat{\mathfrak{g}}}(V, c_{\mathfrak{g}}) \otimes L_{\widehat{sl}_N}(W, c_{sl_N}) \otimes L_{\mathcal{V}ir}(h'_{\mathcal{V}ir}, c'_{\mathcal{V}ir}),$$

where $c'_{\mathcal{V}ir}$ is given by (3.1) and

$$h'_{\mathcal{V}ir} = \overline{h}_{\mathcal{V}ir} - \frac{\Omega_V}{2(c_{\mathfrak{g}} + h^{\vee})} - \frac{\Omega_W}{2(c_{sl_N} + N)}. \quad (3.2)$$

Here Ω_V and Ω_W are the eigenvalues of the Casimir operators of $\widehat{\mathfrak{g}}$ and sl_N on V and W respectively.

4. Irreducible modules for the full toroidal Lie algebra.

In [B3] we introduced a category of bounded modules for the full toroidal Lie algebras and described irreducible modules in that category. When we restrict these modules to the subalgebra $\mathfrak{g}_{\text{div}}$, they become reducible. The goal of this paper is to describe a class of irreducible modules for $\mathfrak{g}_{\text{div}}$ that occur as quotients in this reduction.

We begin by recalling a result of [B3]:

Theorem 4.1. [B3]. *Let $c \neq 0$. Let V_{Hyp}^+ be a sub-VOA of the hyperbolic lattice VOA and let $V_{\mathfrak{f}}(\gamma_0)$ be the enveloping vertex algebra for the twisted Virasoro-affine Lie algebra \mathfrak{f} with $\dot{\mathfrak{f}} = \dot{\mathfrak{g}} \oplus \mathfrak{gl}_N$, where the central character γ_0 given by the following values:*

$$\begin{aligned} c_{\dot{\mathfrak{g}}} &= c, & c_{sl_N} &= 1 - \mu c, & c_{\mathcal{H}ei} &= N(1 - \mu c), \\ c_{\mathcal{V}\mathcal{H}} &= \frac{N}{2}, & c_{\mathcal{V}ir} &= 12\mu c - 2N. \end{aligned} \quad (4.1)$$

Then the VOA

$$V_{\text{Hyp}}^+ \otimes V_{\mathfrak{f}}(\gamma_0)$$

has a structure of a module over the full toroidal Lie algebra $\mathfrak{g}(\mu)$ with the action given by

$$k_0(r, z) = \sum_{j \in \mathbb{Z}} t_0^j t^r k_0 z^{-j} \mapsto cY(e^{ru}, z), \quad r \in \mathbb{Z}^N, \quad (4.2)$$

$$k_a(r, z) = \sum_{j \in \mathbb{Z}} t_0^j t^r k_a z^{-j-1} \mapsto cu_a(z)Y(e^{ru}, z), \quad (4.3)$$

$$g(r, z) = \sum_{j \in \mathbb{Z}} t_0^j t^r g z^{-j-1} \mapsto g(z)Y(e^{ru}, z), \quad g \in \dot{\mathfrak{g}}, \quad (4.4)$$

$$d_a(r, z) = \sum_{j \in \mathbb{Z}} t_0^j t^r d_a z^{-j-1} \mapsto v_a(z)Y(e^{ru}, z) : + \sum_{p=1}^N r_p E_{pa}(z)Y(e^{ru}, z), \quad (4.5)$$

$$\tilde{d}_0(r, z) = \sum_{j \in \mathbb{Z}} t_0^j t^r \tilde{d}_0 z^{-j-2} \mapsto$$

$$: \omega(z)Y(e^{ru}, z) : + \sum_{p,s=1}^N r_p u_s(z) E_{ps}(z)Y(e^{ru}, z) + (\mu c - 1) \sum_{p=1}^N r_p \left(\frac{\partial}{\partial z} u_p(z) \right) Y(e^{ru}, z). \quad (4.6)$$

Here

$$t_0^j t^r \tilde{d}_0 = -t_0^j t^r d_0 + \mu(j + \frac{1}{2})t_0^j t^r k_0$$

and $\omega(z)$ is the Virasoro field in the VOA $V_{Hyp}^+ \otimes V_{\mathfrak{f}}(\gamma)$, which has rank $12\mu c$. In (4.4)-(4.6) above we are using the notations $g(z) = Y(g(-1)\mathbf{1}, z)$, $E_{ab}(z) = Y(E_{ab}(-1)\mathbf{1}, z)$, where E_{ab} are the basis elements of gl_N .

Taking simple modules for the VOAs in the previous theorem, we obtain irreducible representations of $\mathfrak{g}(\mu)$:

Theorem 4.2. [B3]. *Let $M_{Hyp}^+(\alpha, \beta)$ be a simple module for V_{Hyp}^+ and let $M_{\mathfrak{f}}$ be a VOA module for $V_{\mathfrak{f}}(\gamma_0)$. Then*

$$M_{Hyp}^+(\alpha, \beta) \otimes M_{\mathfrak{f}}$$

is a module for the full toroidal Lie algebra $\mathfrak{g}(\mu)$ with the action given by the formulas (4.2)-(4.6). This module is irreducible as a $\mathfrak{g}(\mu)$ -module whenever $M_{\mathfrak{f}}$ is an irreducible \mathfrak{f} -module.

5. Irreducible representations for $\mathfrak{g}_{\text{div}}$.

Now we will study the restriction of the modules for the toroidal Lie algebra to the subalgebra $\mathfrak{g}_{\text{div}}(\mu) = (\mathcal{R} \otimes \mathfrak{g}) \oplus \mathcal{K} \oplus \mathcal{D}_{\text{div}}$. This subalgebra is spanned by the elements $t_0^j k_0$, $t_0^j k_p$, $t_0^j g$, d_0 , $t_0^j d_p$,

$$r_b t_0^j d_a - r_a t_0^j d_b, \quad (5.1)$$

and

$$t_0^j \widehat{d}_a = r_a t_0^j \widetilde{d}_0 + j t_0^j d_a + \frac{r_a}{2cN} (N - 1 + \mu c) t_0^j k_0. \quad (5.2)$$

The reason for adding the last term in (5.2) will become clear later.

The elements $t_0^j k_0$, $t_0^j k_p$, $t_0^j g$ correspond to the fields (4.2)-(4.4), while

$$d_p(z) = \sum_{j \in \mathbb{Z}} t_0^j d_p z^{-j-1} \mapsto v_p(z). \quad (5.3)$$

Collect the elements of the form (5.1) and (5.2) into the fields:

$$d_{ab}(r, z) = \sum_{j \in \mathbb{Z}} \left(r_b t_0^j d_a - r_a t_0^j d_b \right) z^{-j-1},$$

$$\widehat{d}_a(r, z) = \sum_{j \in \mathbb{Z}} t_0^j \widehat{d}_a z^{-j-2}.$$

It follows from (4.5) that $d_{ab}(r, z)$ acts on $V_{Hyp}^+ \otimes V_{\mathfrak{f}}(\gamma_0)$ by

$$\begin{aligned} & d_{ab}(r, z) \mapsto: (r_b v_a(z) - r_a v_b(z)) Y(e^{ru}, z) : \\ & + r_b \sum_{\substack{p=1 \\ p \neq a}}^N r_p E_{pa}(z) Y(e^{ru}, z) - r_a \sum_{\substack{p=1 \\ p \neq b}}^N r_p E_{pb}(z) Y(e^{ru}, z) + r_a r_b (E_{aa} - E_{bb})(z) Y(e^{ru}, z). \end{aligned} \quad (5.4)$$

It is easy to see that the moments of the fields (4.2)-(4.4), (5.3), (5.4) generate $V_{Hyp}^+, V_{\hat{\mathfrak{g}}}$ and $V_{sl_N}^-$ (see the proof of Theorem 5.1 in [BBS]).

Using (4.2)-(4.6), we obtain that $\widehat{d}_a(r, z)$ is represented on $V_{Hyp}^+ \otimes V_f(\gamma_0)$ in the following way:

$$\begin{aligned} \widehat{d}_a(r, z) \mapsto & r_a Y \left(\omega_{(-1)} e^{ru} + \sum_{p,s=1}^N r_p u_s(-1) e^{ru} \otimes E_{ps}(-1) + (\mu c - 1) \sum_{p=1}^N r_p u_p(-2) e^{ru}, z \right) \\ & - \left(z^{-1} + \frac{\partial}{\partial z} \right) Y \left(v_a(-1) e^{ru} + \sum_{p=1}^N r_p e^{ru} \otimes E_{pa}(-1), z \right) + \frac{r_a z^{-2}}{2N} (N - 1 + \mu c) Y(e^{ru}, z). \end{aligned} \quad (5.5)$$

Since $V_{Hyp}^+ \otimes V_{\hat{\mathfrak{g}}} \otimes V_{sl_N}^-$ is generated by \mathfrak{g}_{div} , we shall consider in (5.5) only the terms that involve $V_{\mathcal{HVir}}$, together with the last summand:

$$\begin{aligned} & r_a Y(e^{ru} \otimes \omega_{\mathcal{HVir}}, z) + \frac{r_a}{N} Y \left(\sum_{p=1}^N r_p u_p(-1) e^{ru} \otimes I(-1), z \right) \\ & - \frac{r_a}{N} Y(D(e^{ru} \otimes I(-1)), z) - z^{-1} \frac{r_a}{N} Y(e^{ru} \otimes I(-1), z) + \frac{r_a z^{-2}}{2N} (N - 1 + \mu c) Y(e^{ru}, z) \\ & = r_a \left(Y(e^{ru} \otimes \omega_{\mathcal{HVir}}, z) - \frac{1}{N} Y(e^{ru} \otimes D(I(-1)), z) \right. \\ & \quad \left. - z^{-1} \frac{1}{N} Y(e^{ru} \otimes I(-1), z) + \frac{z^{-2}}{2N} (N - 1 + \mu c) Y(e^{ru}, z) \right) \\ & = r_a Y(e^{ru}, z) \otimes \left(Y(\omega_{\mathcal{HVir}}, z) - \frac{1}{N} \left(z^{-1} + \frac{\partial}{\partial z} \right) Y(I(-1), z) + \frac{z^{-2}}{2N} (N - 1 + \mu c) \text{Id} \right). \end{aligned}$$

In the above calculation we used the fact that $D(e^{ru}) = \sum_{p=1}^N r_p u_p(-1) e^{ru}$ in V_{Hyp}^+ .

To understand the structure of the expression

$$Y(\omega_{\mathcal{HVir}}, z) - \frac{1}{N} \left(z^{-1} + \frac{\partial}{\partial z} \right) Y(I(-1), z) + \frac{z^{-2}}{2N} (N - 1 + \mu c) \text{Id}, \quad (5.6)$$

we consider the following

Lemma 5.1. *Let $\overline{\mathcal{V}ir}$ be the Virasoro algebra with the basis $\{\overline{L}(n), \overline{C}_{Vir}\}$. For any $\sigma \in \mathbb{C}$ the map*

$$\rho_\sigma : \overline{\mathcal{V}ir} \rightarrow \mathcal{HVir},$$

given by

$$\rho_\sigma(\overline{L}(n)) = L(n) + \sigma n I(n) + \delta_{n,0} (\sigma C_{\mathcal{V}\mathcal{H}} - \frac{\sigma^2}{2} C_{\mathcal{H}ei}),$$

$$\rho_\sigma(\overline{C}_{\mathcal{V}ir}) = C_{\mathcal{V}ir} + 24\sigma C_{\mathcal{V}\mathcal{H}} - 12\sigma^2 C_{\mathcal{H}ei},$$

is an embedding of Lie algebras.

Corollary 5.2. (i) Let $h_{\mathcal{V}ir}, h_{\mathcal{H}ei}, c_{\mathcal{V}ir}, c_{\mathcal{V}\mathcal{H}}, c_{\mathcal{H}ei} \in \mathbb{C}$, and let

$$\bar{h}_{\mathcal{V}ir} = h_{\mathcal{V}ir} + \sigma c_{\mathcal{V}\mathcal{H}} - \frac{\sigma^2}{2} c_{\mathcal{H}ei}, \quad \bar{c}_{\mathcal{V}ir} = c_{\mathcal{V}ir} + 24\sigma c_{\mathcal{V}\mathcal{H}} - 12\sigma^2 c_{\mathcal{H}ei}. \quad (5.7)$$

The homomorphism ρ_σ extends to the embedding of the Verma module $M_{\overline{\mathcal{V}ir}}(\bar{h}_{\mathcal{V}ir}, \bar{c}_{\mathcal{V}ir})$ for the Virasoro algebra $\overline{\mathcal{V}ir}$, into the Verma module $M_{\mathcal{H}\mathcal{V}ir}(h_{\mathcal{H}ei}, h_{\mathcal{V}ir}, c_{\mathcal{H}ei}, c_{\mathcal{V}\mathcal{H}}, c_{\mathcal{V}ir})$ for the twisted Virasoro-Heisenberg algebra $\mathcal{H}\mathcal{V}ir$.

(ii) The homomorphism ρ_σ also extends to the embedding of the Lie algebras

$$\rho_\sigma : \bar{\mathfrak{f}} \rightarrow \mathfrak{f},$$

with the identical action on the subalgebras $\hat{\mathfrak{g}}$ and \hat{sl}_N . This yields an embedding of the generalized Verma modules

$$M_{\bar{\mathfrak{f}}}(V, W, \bar{h}_{\mathcal{V}ir}, c_{\hat{\mathfrak{g}}}, c_{sl_N}, \bar{c}_{\mathcal{V}ir}) \hookrightarrow M_{\mathfrak{f}}(V, W, h_{\mathcal{H}ei}, h_{\mathcal{V}ir}, \gamma).$$

(iii) Let $c_{\mathcal{V}\mathcal{H}} = \frac{N}{2}, c_{\mathcal{H}ei} = N(1 - \mu c), \sigma = \frac{1}{N}$. Under the map ρ_σ we have the correspondence of the fields

$$\sum_{n \in \mathbb{Z}} \bar{L}(n) z^{-n-2} \mapsto \sum_{n \in \mathbb{Z}} L(n) z^{-n-2} - \frac{1}{N} \left(z^{-1} + \frac{\partial}{\partial z} \right) \sum_{n \in \mathbb{Z}} I(n) z^{-n-1} + z^{-2} \left(\frac{1}{2} - \frac{1 - \mu c}{2N} \right) \text{Id}. \quad (5.8)$$

The proof of the Lemma is a straightforward computation, and the Corollary is an immediate consequence. Note that the field in the right hand side of (5.8) coincides precisely with (5.6). Thus when $c_{\mathcal{V}\mathcal{H}} = \frac{N}{2}, c_{\mathcal{H}ei} = N(1 - \mu c)$, (as we have in Theorem 4.1), the components of this field satisfy the relations of the Virasoro algebra with the value of the central charge $\bar{c}_{\mathcal{V}ir} = c_{\mathcal{V}ir} + 12 - \frac{12}{N}(1 - \mu c)$.

Remark. Note that the map ρ_σ does not extend to the homomorphism of VOAs, since $\bar{L}(-1)\mathbf{1} = 0$, while $\rho_\sigma(\bar{L}(-1))\mathbf{1} \neq 0$ for $\sigma \neq 0$.

In order to construct representations, one needs the principle of ‘‘preservation of identities’’ for the vertex algebras ([Li], Lemma 2.3.5). In our situation, however, the field (5.6) involves vertex operators that are shifted by powers of z . To deal with such expressions we need to establish a generalization of this principle (Lemma 5.3 below).

A convenient set-up for working with the fields shifted by powers of z is given by the construction of the affinization of a vertex algebra. Let us review this construction.

We recall that for a commutative associative unital algebra R with a derivation D , one can define a vertex algebra structure on R by the formula

$$Y(a, z)b = \sum_{n=0}^{\infty} \frac{z^n}{n!} D^n(a)b.$$

Let us now take R to be the algebra of Laurent polynomials $\mathbb{C}[t, t^{-1}]$ with the derivation $D = \frac{d}{dt}$. This vertex algebra has a 1-dimensional module Z such that

$$Y_Z(t^k, z) = z^k \text{Id}_Z.$$

The affinization of a vertex algebra V is defined as a tensor product of vertex algebras $V \otimes \mathbb{C}[t, t^{-1}]$. Let M be a module for the vertex algebra V . Then we can view $M \cong M \otimes Z$ as a module for the affinization of V :

$$Y_{M \otimes Z}(a \otimes t^k, z) = z^k Y_M(a, z).$$

Lemma 5.3. *Let V be a VOA and let M be a VOA module for V . Let $a \in V \otimes \mathbb{C}[t, t^{-1}]$.*
(i) If M is a faithful VOA module for V and $Y_{M \otimes Z}(a, z) = 0$ then $a = 0$.
(ii) For $a, b, c^0, \dots, c^N \in V \otimes \mathbb{C}[t, t^{-1}]$, if

$$[Y_{V \otimes Z}(a, z), Y_{V \otimes Z}(b, w)] = \sum_{n=0}^N Y_{V \otimes Z}(c^n, w) \left[z^{-1} \left(\frac{\partial}{\partial w} \right)^n \delta \left(\frac{w}{z} \right) \right], \quad (5.9)$$

then

$$[Y_{M \otimes Z}(a, z), Y_{M \otimes Z}(b, w)] = \sum_{n=0}^N Y_{M \otimes Z}(c^n, w) \left[z^{-1} \left(\frac{\partial}{\partial w} \right)^n \delta \left(\frac{w}{z} \right) \right]. \quad (5.10)$$

(iii) If M is a faithful VOA module for V then (5.10) implies (5.9).

Proof. We begin by proving (i). Let D_M be the infinitesimal translation operator on M . Write $a = \sum_s a^s \otimes t^{-s}$, with $a^s \in V$. If $\sum_s Y_M(a^s, z) z^{-s} = 0$ then

$$\begin{aligned} 0 &= z \sum_s [D_M, Y_M(a^s, z)] z^{-s} = z \sum_s \left(\frac{\partial}{\partial z} Y_M(a^s, z) \right) z^{-s} \\ &= z \sum_s \left(\frac{\partial}{\partial z} Y_M(a^s, z) \right) z^{-s} - z \frac{\partial}{\partial z} \left(\sum_s Y_M(a^s, z) z^{-s} \right) \\ &= - \sum_s Y_M(a^s, z) z \frac{\partial}{\partial z} (z^{-s}) = \sum_s s Y_M(a^s, z) z^{-s}. \end{aligned}$$

Repeating this argument, we get that for any $m = 0, 1, 2, \dots$

$$\sum_s s^m Y_M(a^s, z) z^{-s} = 0.$$

Since the sum in s is finite, we can apply the Vandermonde determinant argument and derive that $Y_M(a^s, z) = 0$ for all s . By the definition of a faithful module, this implies that all $a^s = 0$.

To prove (ii), we use the commutator formula for the vertex algebras and the basic properties of the delta-function:

$$\left[\sum_s z^{-s} Y_V(a^s, z), \sum_k w^{-k} Y_V(b^k, w) \right]$$

$$= \sum_{n,i \geq 0} \sum_{s,k} \frac{1}{(n+i)!} \binom{-s}{i} w^{-k-s-i} Y_V(a_{(n+i)}^s b^k, w) \left[z^{-1} \left(\frac{\partial}{\partial w} \right)^n \delta \left(\frac{w}{z} \right) \right] \quad (\text{all sums finite}). \quad (5.11)$$

By Corollary 2.2 from [K], we obtain that for all $n \geq 0$,

$$Y_{V \otimes Z}(c^n, w) = \sum_j w^{-j} Y_V(c^{n,j}, w) = \sum_{i \geq 0} \sum_{s,k} \frac{1}{(n+i)!} \binom{-s}{i} w^{-k-s-i} Y_V(a_{(n+i)}^s b^k, w).$$

Since V is a faithful VOA module over itself, we get using part (i) of the Lemma that

$$c^{n,j} = \sum_{s,k} \sum_{\substack{i \geq 0 \\ s+k+i=j}} \frac{1}{(n+i)!} \binom{-s}{i} a_{(n+i)}^s b^k. \quad (5.12)$$

However the relation (5.11) holds in every VOA module M . Taking (5.12) into account, we see that (5.10) also holds.

The proof for part (iii) is similar. We first see that

$$\begin{aligned} & \sum_{n \geq 0} \sum_j w^{-j} Y_M(c^{n,j}, w) \left[z^{-1} \left(\frac{\partial}{\partial w} \right)^n \delta \left(\frac{w}{z} \right) \right] = \\ & \sum_{n,i \geq 0} \sum_{s,k} \frac{1}{(n+i)!} \binom{-s}{i} w^{-k-s-i} Y_M(a_{(n+i)}^s b^k, w) \left[z^{-1} \left(\frac{\partial}{\partial w} \right)^n \delta \left(\frac{w}{z} \right) \right]. \end{aligned}$$

Again using Corollary 2.2 from [K] and part (i) of the Lemma, we obtain that the relation (5.12) holds in V . Thus (5.9) also holds. This completes the proof of the Lemma.

Now we have done all the preparatory work and now ready to describe the representations for $\mathfrak{g}_{\text{div}}$.

Theorem 5.4. *Let $c \neq 0$. Let $\bar{\mathfrak{f}}$ be the semidirect product of the Virasoro algebra with the affine Lie algebra $\hat{\mathfrak{g}} \oplus \hat{\mathfrak{sl}}_N$, and let $V_{\bar{\mathfrak{f}}}(c_{\hat{\mathfrak{g}}}, c_{\mathfrak{sl}_N}, \bar{c}_{\mathcal{V}ir})$ be the enveloping VOA for $\bar{\mathfrak{f}}$ with the central charges*

$$c_{\hat{\mathfrak{g}}} = c, \quad c_{\mathfrak{sl}_N} = 1 - \mu c, \quad \bar{c}_{\mathcal{V}ir} = 12 \left(1 - \frac{1}{N} \right) + 12\mu c \left(1 + \frac{1}{N} \right) - 2N. \quad (5.13)$$

Then

(i) the VOA $V_{\mathfrak{g}_{\text{div}}} = V_{\text{Hyp}}^+ \otimes V_{\bar{\mathfrak{f}}}(c_{\hat{\mathfrak{g}}}, c_{\mathfrak{sl}_N}, \bar{c}_{\mathcal{V}ir})$ is a module for the Lie algebra $\mathfrak{g}_{\text{div}}(\mu)$. The action of the fields $k_0(r, z), k_a(r, z), g(r, z)$ is given by the formulas (4.2)-(4.4). The action of $d_p(z)$ is given by (5.3). The field $d_{ab}(r, z)$ acts according to (5.4). The action of d_0 is given by

$$d_0 \mapsto -\omega_{(1)}, \quad (5.14)$$

where ω is the Virasoro element of the VOA $V_{\mathfrak{g}_{\text{div}}}$. Finally, the field $\hat{d}_a(r, z)$ is represented by

$$r_a Y \left(\omega_{(-1)} e^{ru} + \sum_{p,s=1}^N r_p u_s (-1) e^{ru} \otimes \psi(E_{ps})(-1) + (\mu c - 1) \sum_{p=1}^N r_p u_p (-2) e^{ru}, z \right)$$

$$-\left(z^{-1} + \frac{\partial}{\partial z}\right) Y \left(v_a(-1)e^{ru} + \sum_{p=1}^N r_p e^{ru} \otimes \psi(E_{pa})(-1), z \right), \quad (5.15)$$

where ψ is the natural projection $gl_N \rightarrow sl_N$, $\psi(X) = X - \frac{1}{N}\text{tr}(X)I$.

(ii) Let $M_{Hyp}^+(\alpha, \beta)$ be a VOA module for the sub-VOA V_{Hyp}^+ of the hyperbolic lattice VOA, and let $M_{\bar{f}}$ be a module for the VOA $V_{\bar{f}}(c_{\mathfrak{g}}, c_{sl_N}, \bar{c}_{\mathcal{V}ir})$. Then

$$M_{Hyp}^+(\alpha, \beta) \otimes M_{\bar{f}}$$

is a module for the Lie algebra $\mathfrak{g}_{\text{div}}(\mu)$ with the action transferred from $V_{\mathfrak{g}_{\text{div}}}$.

Remark. Since degree zero derivations d_0, d_p do not belong to the commutant of $\mathfrak{g}_{\text{div}}$, their action may be modified by adding arbitrary multiples of the identity operator.

Proof. The Lie bracket in $\mathfrak{g}_{\text{div}}$ may be encoded in the commutator relations between the fields $k_0(r, z), k_a(r, z), g(r, z), d_p(z), d_{ab}(r, z), \widehat{d}_a(r, z)$, and the element d_0 . We need to show that the same commutator relations hold for their images (4.2)-(4.4), (5.3), (5.4), (5.14) and (5.15). It is easy to see that the relations involving d_0 in the left hand sides hold since $\omega_{(1)}$ acts on the VOA as a degree derivation. Also, d_0 does not belong to the commutant of $\mathfrak{g}_{\text{div}}$ and will not appear in the right hand sides of the commutator relations.

Actually, there is no need to write down the commutator relations involving the remaining fields explicitly. All we need to know is that these relations are of the form (5.9), which follows from the corresponding result for the full toroidal Lie algebra (Theorem 4.1 in [B3]). Our strategy is to embed one of the modules for $V_{\mathfrak{g}_{\text{div}}}$ into a module for the full toroidal Lie algebra $\mathfrak{g}(\mu)$. This embedding will have the property that the restriction of the action of $\mathfrak{g}(\mu)$ to subalgebra $\mathfrak{g}_{\text{div}}(\mu)$ will coincide with (4.2)-(4.4), (5.3), (5.4) and (5.15). This will imply that the necessary commutator relations hold in the chosen module for $V_{\mathfrak{g}_{\text{div}}}$. Since the module that we will consider will be faithful, then by preservation of identities (Lemma 5.3), the same required relations will hold in all VOA modules for $V_{\mathfrak{g}_{\text{div}}}$.

Let us carry out this plan. Let $h_{\mathcal{V}ir} = -\frac{1}{N}c_{\mathcal{V}\mathcal{H}} + \frac{1}{2N^2}c_{\mathcal{H}ei}$, $h_{\mathcal{H}ei} = 0$, and $c_{\mathfrak{g}}, c_{\mathcal{H}ei}, c_{\mathcal{V}\mathcal{H}}, c_{sl_N}, c_{\mathcal{V}ir}, \bar{c}_{\mathcal{V}ir}$ given by (4.1), (5.7) and let V, W be trivial 1-dimensional modules for \mathfrak{g} and sl_N . Consider the embedding given by Corollary 5.2(ii) with $\sigma = \frac{1}{N}$ of the generalized Verma \bar{f} -module into the generalized Verma module for \bar{f} :

$$M_{\bar{f}}(V, W, 0, c_{\mathfrak{g}}, c_{sl_N}, \bar{c}_{\mathcal{V}ir}) \hookrightarrow M_{\bar{f}}(V, W, 0, h_{\mathcal{V}ir}, \gamma_0).$$

Under this homomorphism

$$\bar{L}(z) = \sum_{n \in \mathbb{Z}} \bar{L}(n)z^{-n-2} \mapsto L(z) - \frac{1}{N} \left(z^{-1} + \frac{\partial}{\partial z} \right) I(z) + z^{-2} \left(\frac{1}{2} - \frac{1-\mu c}{2N} \right) \text{Id}.$$

This map extends to the embedding

$$V_{Hyp}^+ \otimes M_{\bar{f}}(V, W, 0, c_{\mathfrak{g}}, c_{sl_N}, \bar{c}_{\mathcal{V}ir}) \hookrightarrow V_{Hyp}^+ \otimes M_{\bar{f}}(V, W, 0, h_{\mathcal{V}ir}, \gamma_0).$$

By Theorem 4.2, the latter is a module for the full toroidal Lie algebra $\mathfrak{g}(\mu)$. We consider the restriction of this representation to the subalgebra $\mathfrak{g}_{\text{div}}(\mu)$ and claim that $V_{Hyp}^+ \otimes$

$M_{\bar{f}}(V, W, 0, c_{\bar{g}}, c_{sl_N}, \bar{c}_{\mathcal{V}ir})$ is invariant under the action of $\mathfrak{g}_{\text{div}}(\mu)$. The action of $k_0(r, z)$, $k_a(r, z)$, $g(r, z)$, $d_p(z)$ and $d_{ab}(r, z)$ is given by (4.2)-(4.4), (5.3), (5.4) and the invariance with respect to these fields is clear. Let us show that the action of $\widehat{d}_a(r, z)$ on $V_{Hyp}^+ \otimes M_{\bar{f}}(V, W, 0, c_{\bar{g}}, c_{sl_N}, \bar{c}_{\mathcal{V}ir})$ coincides with (5.15). Indeed, following the computations (5.5)–(5.8), we get:

$$\begin{aligned} \widehat{d}_a(r, z) &\mapsto r_a Y(e^{ru}, z) \otimes \left(Y(\omega_{\bar{f}}, z) - \frac{1}{N} \left(z^{-1} + \frac{\partial}{\partial z} \right) Y(I(-1), z) + \frac{z^{-2}}{2N} (N-1 + \mu c) \text{Id} \right) \\ &+ r_a Y \left(\omega_{Hyp(-1)} e^{ru} + \sum_{p,s=1}^N r_p u_s(-1) e^{ru} \otimes \psi(E_{ps})(-1) + (\mu c - 1) \sum_{p=1}^N r_p u_p(-2) e^{ru}, z \right) \\ &\quad - \left(z^{-1} + \frac{\partial}{\partial z} \right) Y \left(v_a(-1) e^{ru} + \sum_{p=1}^N r_p e^{ru} \otimes \psi(E_{pa})(-1), z \right) \\ &= r_a Y \left(\left(\omega_{Hyp} + \rho_{\sigma}(\omega_{\bar{f}}) \right)_{(-1)} e^{ru}, z \right) \\ &+ r_a Y \left(\sum_{p,s=1}^N r_p u_s(-1) e^{ru} \otimes \psi(E_{ps})(-1) + (\mu c - 1) \sum_{p=1}^N r_p u_p(-2) e^{ru}, z \right) \\ &\quad - \left(z^{-1} + \frac{\partial}{\partial z} \right) Y \left(v_a(-1) e^{ru} + \sum_{p=1}^N r_p e^{ru} \otimes \psi(E_{pa})(-1), z \right), \end{aligned}$$

which is the same as (5.15). Thus the specified action defines a representation of $\mathfrak{g}_{\text{div}}(\mu)$ on $V_{Hyp}^+ \otimes M_{\bar{f}}(V, W, 0, c_{\bar{g}}, c_{sl_N}, \bar{c}_{\mathcal{V}ir})$, and the fields (4.2)-(4.4), (5.3), (5.4) and (5.15) satisfy the relations that reflect the Lie bracket in $\mathfrak{g}_{\text{div}}(\mu)$. This module is a faithful VOA module for $V_{\mathfrak{g}_{\text{div}}}$, since $V_{\mathfrak{g}_{\text{div}}}$ itself is its factor module. Thus by the preservation of identities, Lemma 5.3, the required commutator relations hold in $V_{\mathfrak{g}_{\text{div}}}$ and in all VOA modules for $V_{\mathfrak{g}_{\text{div}}}$. This establishes the claim of the theorem.

In the next theorem we give the description of the irreducible modules for $\mathfrak{g}_{\text{div}}$.

Theorem 5.5. *Let $c \neq 0$ and let the constants $c_{\bar{g}}, c_{sl_N}, \bar{c}_{\mathcal{V}ir}$ be given by (5.13).*

(i) *Let V be a finite-dimensional irreducible \bar{g} -module, W be a finite-dimensional irreducible sl_N -module, $\bar{h}_{\mathcal{V}ir} \in \mathbb{C}$ and let $L_{\bar{f}}(V, W, \bar{h}_{\mathcal{V}ir}, c_{\bar{g}}, c_{sl_N}, \bar{c}_{\mathcal{V}ir})$ be an irreducible highest weight module for the Lie algebra \bar{f} . Let $\alpha \in \mathbb{C}^N, \beta \in \mathbb{Z}^N$, and let $M_{Hyp}^+(\alpha, \beta)$ be the irreducible VOA module for V_{Hyp}^+ . Then*

$$L_{\mathfrak{g}_{\text{div}}} = M_{Hyp}^+(\alpha, \beta) \otimes L_{\bar{f}}(V, W, \bar{h}_{\mathcal{V}ir}, c_{\bar{g}}, c_{sl_N}, \bar{c}_{\mathcal{V}ir})$$

is an irreducible module for the Lie algebra $\mathfrak{g}_{\text{div}}(\mu)$ with the action given by (4.2)-(4.4), (5.3), (5.4), (5.14) and (5.15).

(ii) If, in addition, $c_{\hat{\mathfrak{g}}} \neq -h^\vee$, where h^\vee is the dual Coxeter number of $\hat{\mathfrak{g}}$, and $c_{sl_N} = 1 - \mu c \neq -N$ then

$$L_{\mathfrak{g}_{\text{div}}} \cong M_{Hyp}^+(\alpha, \beta) \otimes L_{\hat{\mathfrak{g}}}(V, c_{\hat{\mathfrak{g}}}) \otimes L_{\widehat{sl}_N}(W, c_{sl_N}) \otimes L_{Vir}(h'_{Vir}, c'_{Vir}),$$

where h'_{Vir} is given by (3.2) and

$$c'_{Vir} = 12 \left(1 - \frac{1}{N}\right) + 12\mu c \left(1 + \frac{1}{N}\right) - 2N - \frac{c \dim(\hat{\mathfrak{g}})}{c + h^\vee} - \frac{(1 - \mu c)(N^2 - 1)}{1 - \mu c + N}. \quad (5.16)$$

The proof of this theorem is completely analogous to the proof of Theorem 5.3 in [B3] and will be omitted.

We conclude the paper with the following observation. Let us now set $\hat{\mathfrak{g}} = (0)$. Then Theorem 5.5 gives a family of irreducible representations for the Lie algebras

$$\mathcal{DK}(\mu) = \mathcal{D}_{\text{div}} \oplus \mathcal{K}.$$

These Lie algebras possess a non-degenerate symmetric invariant bilinear form and may be viewed as extended affine Lie algebras of rank 0. In fact, all such algebras with $\mu \neq 0$ are isomorphic to each other. We will thus normalize μ to be 1. If we take trivial modules for the Virasoro and for the affine algebra \widehat{sl}_N , then only when $N = 12$ we arrive at the irreducible representation of $\mathcal{DK}(\mu)$ just on the lattice part V_{Hyp}^+ . We get the following remarkable result:

Theorem 5.6. *Let $N = 12$ and let $\mu = c = 1$. Then V_{Hyp}^+ has a structure of an irreducible module for $\mathcal{DK}(1)$. The action of the Lie algebra is given by*

$$\begin{aligned} k_0(r, z) &\mapsto Y(e^{ru}, z), & k_p(r, z) &\mapsto u_p(z)Y(e^{ru}, z), \\ d_{ab}(r, z) &\mapsto: (r_b v_a(z) - r_a v_b(z)) Y(e^{ru}, z) :, \\ d_0 &\mapsto \text{Id} - \omega_{(1)}, & d_p(z) &\mapsto v_p(z), \\ \widehat{d}_a(r, z) &\mapsto r_a : \omega(z)Y(e^{ru}, z) : - \left(z^{-1} + \frac{\partial}{\partial z} \right) : v_a(z)Y(e^{ru}, z) :. \end{aligned}$$

The character of this module with respect to the diagonalizable operators d_0, d_1, \dots, d_N has nice modular properties – it is a product of 12 delta-functions with the -24 -th power of the Dedekind η -function:

$$\text{char } V_{Hyp}^+ = q_0 \prod_{k=1}^{\infty} (1 - q_0^{-k})^{-24} \times \prod_{p=1}^{12} \sum_{n \in \mathbb{Z}} q_p^n.$$

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